On an optimal consumption problem for p-integrable consumption plans

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Summary. A generalization is presented of the existence results for an optimal consumption problem of Aumann and Perles [4] and Cox and Huang [10]. In addition, we present a very general optimality principle.

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1 Introduction

In a seminal paper [4] Aumann and Perles gave existence results for optimal consumption problems with linear inequality and equality constraints that are special cases of two problems, (IC_p) and (EC_p), to be formulated in section 2. These are variational problems in a space of p-integrable functions, either for $p = 0$ (0-integrability being interpreted as mere measurability) or for $p \geq 1$, as is the case in [4, 10]. Problem (EC_p) generalizes a problem studied in [4] in the Main Theorem (p. 489) and in Theorem 6.2 (the former has $p = 1$ and the latter is for $p > 1$). A version of (IC_1) was considered in [4, Theorem 6.1]. More recently, Cox and Huang continued this work in [10], where they gave existence results for a dynamic consumption-portfolio problem. They did so by using the well-known fact [13] that such problems can be transformed into a static problem of the type (IC_p), $p \geq 1$, using Ito's calculus. The existence results in [10] show several differences with the results in [4]. As one practical limitation of the version of (IC_p) used in [10] we point out that it only allows for a single consumption good and one inequality constraint. This restriction play an important technical role in [10]. Closer inspection of [10] *vis à* [4] reveals a number of other substantial technical differences between [4] and [10] that affect certain comparisons with [4] that were claimed in [10]. Next to the already cited fact that [4] deals with a multi-good model, these differences are as follows. (i) In all of [10] the utility function $u(z, \omega)$ is concave in the decision variable z, but it is not so in any of the three above-mentioned existence results in [4]. (ii) On the other hand, in all of [4] the underlying measure space is nonatomic, whereas in [10] it is general. (iii) In all of [10], $u(z, \omega)$ is required to be increasing (by this we mean *strictly* increasing) in z, but this is not so in [4, Theorem 6.1] (which has no monotonicity requirement at all) and [4, Main Theorem] (which only requires $u(z, \omega)$ to be nondecreasing in z); however, Theorem 6.2 in [4] requires $u(z, \omega)$ to be increasing in z.

For these reasons, the totality of the results in [4] and [10] is intransparent. To subsume all of the cited results in [4] and [10] and to go beyond them, this work presents three central existence results. These offer several considerable improvements, in particular for the utility functions. For $p = 0$ (and also for $p = 1$ under additional conditions that turn out to be valid in [4] but not in [10]) our main existence results are Propositions 2.5 and 2.6, respectively for the inequality- and the equality-constrained problems. These propositions are immediate consequences of [6, Corollary 2], a result recapitulated here as Theorem 3.1. A growth property (γ_1) from [8] is used, as well as its logical extension (γ_p). We show that this unifies the different growth conditions used by Aumann and Perles [4] and Cox and Huang [10]. Our main existence result is Theorem 2.8; this is new, but it is obtained along the lines set out by Aumann and Perles in their proof of [4, Theorem 6.2]. First, for (IC_0) the propositions mentioned above yield existence of an optimal solution x_* in a space of measurable functions. Next, in Theorem 3.2 optimality is characterized by a pointwise optimality principle, which comes from [1, 2, 11] (see [4, Theorem 5.1]). It is essential that all

Lagrange multipliers of this optimality principle be strictly positive (Corollary 3.3); this forces x_* to be p-integrable, as a consequence of the optimality principle and the growth conditions for $u(z,\omega)$. In addition, such strict positivity causes the optimal solutions of (IC_p) and (EC_p) to coincide, because of complementary slackness.

2 Existence results

For $p = 0$ and $p \geq 1$ we consider the following optimal consumption problem with linear inequality constraints

$$(IC_p) \qquad \sup_{x \in \mathcal{L}^p_Z} \{U(x) : \int_\Omega x(\omega) \cdot \xi_i(\omega)\mu(d\omega) \leq \alpha_i, i = 1, \ldots, m\}$$

and its equality-constrained counterpart

$$(EC_p) \qquad \sup_{x \in \mathcal{L}^p_Z} \{U(x) : \int_\Omega x(\omega) \cdot \xi_i(\omega)\mu(d\omega) = \alpha_i, i = 1, \ldots, m\}.$$

As we shall see in section 5, this model can easily incorporate consumption over time as well. Here $(\Omega, \mathcal{F}, \mu)$ is a finite measure space and $\mathcal{L}^p_Z$ is shorthand for the set of all p-integrable consumption functions on $(\Omega, \mathcal{F}, \mu)$ with values in $Z := \mathbb{R}^d_+$. Here d is a fixed, given dimension. For $p = 0$ this definition has to be understood as follows: $\mathcal{L}^0_Z$ is the set of all *measurable* functions from Ω into $\mathbb{R}^d_+$. Also, $\alpha_1, \ldots, \alpha_m > 0$ are given constants. Further $\xi_1, \ldots, \xi_m$ are given functions in $\mathcal{L}^0_Z$, $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,d})$, with

$$\hat{\xi}(\omega) := \min_{1 \leq j \leq d} \sum_{i=1}^m \xi_{i,j}(\omega) > 0 \text{ for every } \omega \text{ in } \Omega. \tag{1}$$

By nonnegativity of $x \cdot \xi_i$, the meaning of $\int_\Omega x \cdot \xi_i \, d\mu$ is always clear (the integral is allowed to be $+\infty$).[1] Finally, above we denote

$$U(x) := \int_\Omega u(x(\omega), \omega)\mu(d\omega),$$

where $u : \mathbb{R}^d_+ \times \Omega \to [-\infty, \infty)$ is a $\mathcal{B}(\mathbb{R}^d_+) \times \mathcal{F}$-measurable utility function. Of course, the integrand $\omega \mapsto u(x(\omega), \omega)$ is $\mathcal{F}$-measurable for every $x \in \mathcal{L}^p_Z$, but it is not necessarily summable. However, growth property (γ_p) that is to follow will hold for all our existence results. This implies that $\int_\Omega \max(u(x(\omega), \omega), 0)\mu(d\omega)$ is finite for all $x \in \mathcal{L}^p_Z$, so, by allowing for $U(x) = -\infty$, the meaning of the integral is never in doubt; this means that we interpret the integral in the definition of $U(x)$ as a *quasi-integral* [15].

Extensions, examples and special cases of this model are discussed in sections 4 and 5. As one particular economic example of (IC_p) one could, for instance, think of a consumer, facing uncertainty about the true state of nature, who consults m experts. Each expert i suggests a random variable $\xi_i \in \mathcal{L}^0_Z$ to describe expert i's best guess for stochastic price behavior: should state ω in Ω arise under μ, then expert i predicts that this results in the price vector $\xi_i(\omega) \in \mathbb{R}^d$. If the consumer takes all expert opinions seriously, he/she could wish to use only state-contingent consumption plans $x \in \mathcal{L}^p_Z$ for which for each i the expectation $\int_\Omega \xi_i \cdot x \, d\mu$ does not exceed a certain budget value. As illustrated by Example 4.12, mechanical problems of the type (EC_p) were already studied by Newton.

The following special conditions will sometimes be imposed on $(\xi_{i,j})$. Of these, order-equivalence works in connection with $p \geq 1$, both for (IC_p) and (EC_p), and diagonal dominance serves to make all problems (EC_p), $p = 0$ or $p \geq 1$, automatically feasible.

[1] Thus, we dispense with the condition $\xi_i \in \mathcal{L}^q$ of [10], with q as specified in footnote 2. In retrospect, this justifies Cox and Huang's use of both $p = 1$ and $p > 1$ in [10], although their own restriction $\xi_i \in \mathcal{L}^q$ effectively rules out $p = 1$ (i.e., $q = \infty$) because of their formula (8).

Definition 2.1 (i) The matrix function $(\xi_{i,j})$ is said to be *order-equivalent to* $\hat{\xi}$ if there exists $C > 0$ such that

$$\max_{1 \le j \le d} \sum_{i=1}^{m} \xi_{i,j}(\omega) \le C\hat{\xi}(\omega) \text{ for a.e. } \omega \text{ in } \Omega.$$

(ii) The matrix function $(\xi_{i,j})$ is said to have *diagonal structure* if $m = d$ and $\xi_{i,j} \equiv 0$ whenever $i \neq j$, $i, j = 1, \ldots, d$.

Observe already that diagonal structure implies $\xi_{i,i} > 0$ for every i, in view of (1). Note also that Aumann and Perles [4] use diagonal structure, with ξ_i identically equal to the i-th unit vector e_i. Hence, they also have order-equivalence with $\hat{\xi} \equiv 1$. In [10] one simply has $m = d = 1$, whence $\hat{\xi} = \xi_{1,1}$. The growth condition for u mentioned above is as follows; it is an obvious extension to $p \ge 1$ of the property introduced in [8] to unify the three different growth conditions used in [4].

Definition 2.2 u has *growth property* (γ_p) if for every $\epsilon > 0$ there exists $\psi_\epsilon \in \mathcal{L}^p_+$ such that for a.e. $\omega \in \Omega$

$$u(z, \omega) \le \epsilon\hat{\xi}(\omega)|z| + \hat{\xi}(\omega)\psi_\epsilon(\omega) \text{ for all } z \in \mathbb{R}^d_+.$$

In connection with the existence results for $p \ge 1$ the following nonsatiation condition is important:

Definition 2.3 The function u is said to be *essentially nonsatiated with respect to* $\xi_1, \ldots, \xi_m$ if there do not exist j, $1 \le j \le m$, and $\lambda_i \ge 0$, $i \neq j$, for which

$$\text{argmax}_{z \in \mathbb{R}^d_+} [u(z, \omega) - \sum_{i, i \neq j} \lambda_i z \cdot \xi_i(\omega)] \neq \emptyset \text{ for a.e. } \omega \text{ in } \Omega.$$

Remark 2.4 Obviously, if $(\xi_{i,j})$ has diagonal structure, then u in nonsatiated with respect to $\xi_1, \ldots, \xi_m$ if and only if there do not exist j, $1 \le j \le m$, and $\lambda_i \ge 0$, $i \neq j$, for which

$$\text{argmax}_{z \in \mathbb{R}^d_+} u(z, \omega) - \sum_{i, i \neq j} \lambda_i \xi_{i,i}(\omega) z^i \neq \emptyset \text{ a.e.}$$

So the above certainly holds if for every ω in some non-null subset B of Ω (i.e., $\mu(B) > 0$) and every j the function $u(z, \omega)$ is nonsatiated in each coordinate z^j of z (i.e., $\text{argmax}_{z^j \ge 0} u(z, \omega) = \emptyset$ for every $z^1, \ldots, z^{j-1}, z^{j+1}, \ldots, z^d \ge 0$). In particular, this holds when $u(z, \omega)$ is strictly increasing in each coordinate z^j for all ω in some subset non-null subset of Ω.

Proposition 2.5 (existence of optimal measurable plans) *Suppose that $u(z, \omega)$ is upper semicontinuous in z for a.e. ω in Ω. Suppose also that u satisfies growth condition (γ_1). Then problem (IC_0), provided that it is feasible, has an optimal solution x_* with $\int_\Omega |x_*|\hat{\xi} d\mu < +\infty$.*

Proposition 2.6 (existence of optimal measurable plans) *Suppose that $u(z, \omega)$ is upper semicontinuous and nondecreasing in z for a.e. ω in Ω and that $(\xi_{i,j})$ has diagonal structure. Suppose also that u satisfies growth condition (γ_1). Then problem (EC_0) has an optimal solution x_*, with $\int_\Omega |x_*|\hat{\xi} d\mu < +\infty$, that is simultaneously an optimal solution of (IC_0).*

Observe that Proposition 2.6 contains no explicit feasibility condition. Here $u(z, \omega)$ is said to be nondecreasing in z if $z' \ge z$ (coordinatewise) in $\mathbb{R}^d$ implies $u(z', \omega) \ge u(z, \omega)$.

Remark 2.7 Of course, if ess $\inf_\Omega \hat{\xi}$, the essential infimum of $\hat{\xi}$ over Ω, is strictly positive, the additional property $\int_\Omega |x_*|\hat{\xi} d\mu < +\infty$ of x_* implies $x_* \in \mathcal{L}^1_Z$, which causes the existence results for (IC_0) and (IC_1), as well as those for (EC_0) and (EC_1), to coincide. This observation applies in particular to [4], where $\hat{\xi} \equiv 1$; cf. section 4.

The following theorem is the main result of this work. It gives sufficient conditions for the existence of an optimal solution of (IC_p) and of (EC_p).

Theorem 2.8 (existence of optimal p-integrable plans) *Suppose for $p \geq 1$ that $u(z,\omega)$ is upper semicontinuous in z for a.e. ω in Ω, that $u(z,\omega)$ is concave in z for a.e. ω in the purely atomic part Ω^{pa} of $(\Omega, \mathcal{F}, \mu)$ and that u is essentially nonsatiated with respect to $\xi_1, \ldots, \xi_m$. Suppose also that u satisfies growth condition (γ_p), that $(\xi_{i,j})$ is order-equivalent to $\hat{\xi}$ and that there exists some $\tilde{x} \in \mathcal{L}^p_Z$ for which the function $\omega \mapsto u(\tilde{x}(\omega),\omega)/\hat{\xi}(\omega)$ is p-integrable. Then problem (IC_p), provided that it is feasible, has an optimal solution that is simultaneously an optimal solution of problem (EC_p).*

Recall here [14] that Ω can always be partitioned into a *purely atomic* part Ω^{pa} (this is the union of at most countably many non-null atoms) and a *nonatomic* part Ω^{na}.

Remark 2.9 Suppose that $(\xi_{i,j})$ has diagonal structure with $\hat{\xi}^{-1} \in \mathcal{L}^p$. Then in Theorem 2.8 problem (IC_p) is feasible. By $\hat{\xi}^{-1} \in \mathcal{L}^p$ and $\hat{\xi} \leq \xi_{i,i} \leq C\hat{\xi}$ we have $\hat{\xi}_{i,i}^{-1} \in \mathcal{L}^p_+$ for $i = 1, \ldots, d$. Hence, $(\alpha_1 \xi_{1,1}^{-1}, \ldots, \alpha_d \xi_{d,d}^{-1})$ in $(\mathcal{L}^p_+)^d$ defines a feasible solution of (EC_p), whence of (IC_p).

Even when $m = d = 1$ the essential nonsatiation condition that we use constitutes a considerable improvement over [4, Theorem 6.2] and [10], where $u(z,\omega)$ is required to be strictly increasing in each coordinate of z for all (or at least a.e.) ω in Ω. See Examples 4.8 and 4.9 below.

3 Auxiliary results and proofs

The proof of Proposition 2.5 is an immediate application of the following result from [6], where it was shown to extend [8, Proposition 1, p. 155] and the existence results of [3, 5]) to a general underlying measure space (all those references use a nonatomic measure).

Theorem 3.1 ([6, Corollary 2]) *Let $g_0, g_1, \ldots g_{m+1} : \mathbb{R}^d_+ \times \Omega \to (-\infty, +\infty]$ be product measurable functions. Also, let $\beta_1, \ldots, \beta_{m+1}$ be given real numbers. Suppose that $g_1(z,\omega), \ldots, g_m(z,\omega)$ are lower semicontinuous in the variable z and suppose that $g_{m+1}(z,\omega)$ is inf-compact in the variable z and nonnegative. Suppose also that for every $\epsilon > 0$ there exists $\psi_\epsilon \in \mathcal{L}^1$ such that for $i = 0, \ldots, m$*

$$\max(-g_i(z,\omega), 0) \leq \epsilon g_{m+1}(z,\omega) + \psi_\epsilon(\omega) \text{ for a.e. } \omega. \tag{2}$$

Then the optimization problem

$$\inf_{x \in \mathcal{L}^0_Z} \{ \int_\Omega g_0(x(\omega),\omega)\mu(d\omega) : \int_\Omega g_i(x(\omega),\omega)\mu(d\omega) \leq \beta_i, i = 1, \ldots, m+1 \}$$

has an optimal solution, provided that this problem is feasible.

In [6, Corollary 2] a more general space is taken instead of $\mathbb{R}^d_+$. Above the integrals $\int_\Omega g_i(x(\omega),\omega)\mu(d\omega)$, $x \in \mathcal{L}^0_Z$, should be interpreted as quasi-integrals (concretely, they can have values $+\infty$, but not $-\infty$).

Proof of Proposition 2.5. Let $\alpha_{m+1} := \sum_i \alpha_i$ and consider the following auxiliary optimization problem:

$$(Q) \qquad \inf_{x \in \mathcal{L}^0_Z} \{ \int_\Omega -u(x(\omega),\omega)\mu(d\omega) : \int_\Omega x \cdot \xi_i d\mu \leq \alpha_i, i = 1, \ldots, m, \int_\Omega \hat{\xi}|x| d\mu \leq \alpha_{m+1} \}.$$

Let us show that this problem is equivalent with (IC_0). First, the $m+1$-st constraint of the optimization problem is redundant (it is only introduced because it is formally required). To see its redundance, just observe that the elementary inequality $\hat{\xi}|z| \leq \hat{\xi} \sum_j z^j \leq \sum_i \xi_i \cdot z$ for all $z \in \mathbb{R}^d_+$ causes the first m constraints in (Q) to imply the $m+1$-st one. Secondly, the change into a minimization problem is explained by the additional minus sign. So (IC_0) is equivalent to (Q); hence, it is enough to prove existence of an optimal solution of (Q). We do this by a direct application of Theorem 3.1, setting $g_0(z,\omega) := -u(z,\omega)$, $g_{m+1}(z,\omega) := \hat{\xi}(\omega)|z|$, $g_i(z,\omega) := z \cdot \xi_i(\omega)$ and $\beta_i := \alpha_i$

for $i = 1, \ldots, m+1$. Before invoking Theorem 3.1 it remains to verify (2). For $i = 1, \ldots, m$ this is trivial by $g_i \geq 0$ and for $i = 0$ it is an immediate consequence of (γ_1). QED

PROOF OF PROPOSITION 2.6. Because of the additional diagonal structure, $(\alpha_1 \xi_{1,1}^{-1}, \ldots, \alpha_d \xi_{d,d}^{-1})$ in $\mathcal{L}_Z^0$ defines a feasible solution of (EC_0), whence of (IC_0). So Proposition 2.5 can be applied. This guarantees existence of an optimal solution x_{**} of (IC_0). Define $x_* \in \mathcal{L}_Z^0$ by

$$x_*^i(\omega) := x_{**}^i(\omega) + (\alpha_i - \alpha_i')\xi_{i,i}^{-1}(\omega), \tag{3}$$

for $\alpha_i' := \int_\Omega x_{**} \cdot \xi_i d\mu = \int_\Omega x_{**}^i \xi_{i,i} d\mu \leq \alpha_i$. Then $U(x_*) = U(x_{**})$, which causes x_* to be an optimal solution of both (EC_0) and (IC_0). The identity holds, because on the one hand x_* is obviously feasible for (EC_0) (whence for (IC_0), which implies $U(x_*) \leq U(x_{**})$) and on the other hand $x_*(\omega) \geq x_{**}(\omega)$ (coordinatewise), causing $u(x_*(\omega), \omega) \geq u(x_{**}(\omega), \omega)$ for all ω, whence $U(x_*) \geq U(x_{**})$. QED

We now prepare the proof of Theorem 2.8. We shall need the following theorem, which comes from [1, 2, 11]. Essentially, it is based on an application of Lyapunov's theorem (convexity of the range of a nonatomic vector measure) and the separating hyperplane theorem in $\mathbb{R}^{m+1}$, plus some measurable selection arguments.

Theorem 3.2 (optimality principle) *Suppose for any p, $p = 0$ or $p \geq 1$, that $u(z,\omega)$ is concave in z for for a.e. ω in the purely atomic part Ω^{pa}. Then $x_* \in \mathcal{L}_Z^p$ is an optimal solution of (IC_p) if and only if x_* is feasible for (IC_p) and there exist $\lambda_1, \ldots, \lambda_m \geq 0$ such that the following two conditions hold:*

$$x_*(\omega) \in \mathrm{argmax}_{z \in \mathbb{R}_+^d} u(z,\omega) - \sum_{i=1}^m \lambda_i z \cdot \xi_i(\omega) \text{ for a.e. } \omega \text{ in } \Omega \text{ (pointwise maximum principle).}$$

$$\lambda_i\left(\int_\Omega x_* \cdot \xi_i d\mu - \alpha_i\right) = 0,\ i = 1, \ldots, m \text{ (complementary slackness).}$$

Under additional conditions for $\xi_1, \ldots, \xi_m$, a similar result can be also given for (EC_p) [1], but for the present paper this is not very relevant.[2] The following Corollary 3.3 of the above theorem will play an essential role in establishing existence for $p \geq 1$. Its essential nonsatiatedness condition alone is responsible for the (strict) positivity of its multipliers; cf. Examples 4.8 and 4.9.

Corollary 3.3 *Suppose for any p, $p = 0$ or $p \geq 1$, that $u(z,\omega)$ is concave in z for for a.e. ω in Ω^{pa} and that u is essentially nonsatiated with respect to $\xi_1, \ldots, \xi_m$. Then $x_* \in \mathcal{L}_Z^p$ is an optimal solution of (IC_p) if and only if x_* is feasible for (EC_p) and there exist $\lambda_1, \ldots, \lambda_m > 0$ such that*

$$x_*(\omega) \in \mathrm{argmax}_{z \in \mathbb{R}_+^d} u(z,\omega) - \sum_{i=1}^m \lambda_i z \cdot \xi_i(\omega) \text{ for a.e. } \omega \text{ in } \Omega \text{ (pointwise maximum principle).}$$

In the above corollary any optimal solution of (IC_p) is also an optimal solution of (EC_p), but the converse implication need not hold:

Example 3.4 Let Ω be the unit interval, equipped with Lebesgue measure μ. Let $m = d = 1$, $\eta \in [0,1)$ and define the utility function as follows:

$$u(z,\omega) := \begin{cases} \frac{1}{z} & \text{if } 0 < z \leq 1 \\ 0 & \text{if } z = 0 \\ \infty & \text{if } z > 1 \end{cases}$$

Consider the problems (IC_p) and (EC_p) with $\xi_1 \equiv 1$ and $\alpha_1 = 1$. Then, apart from null sets, $x \equiv 1$ is the only feasible element of (EC_p) for which $U(x) > -\infty$. Hence, $x_* \equiv 1$ is the (essentially) unique optimal solution of (EC_p). However, (IC_p) clearly has no optimal solution, even though u meets all conditions of Corollary 3.3 (here $\Omega^{pa} = \emptyset$).

[2] By [1, 4.3.3] an analogous characterization holds for (EC_p) if $\xi_1, \ldots, \xi_m$ are additionally q-integrable, with $q := p/(1-p)$ if $p > 1$ and $q := \infty$ if $p = 0$.

PROOF OF THEOREM 3.2. When Ω^{na} is equipped with $\mathcal{F} \cap \Omega^{na}$ and $\mu(\cdot \cap \Omega^{na})$, it forms a nonatomic measure space. Denote by $\mathcal{V}$ $[\mathcal{W}]$ the space of all p-integrable functions from Ω^{na} $[\Omega^{pa}]$ into $\mathbb{R}^d_+$. Every $x \in \mathcal{L}^p_Z$ can be identified with the pair (v, w) in $\mathcal{V} \times \mathcal{W}$, where $v := x \mid_{\Omega^{na}}$ is the restriction of x to the nonatomic part Ω^{na} and where $w := x \mid_{\Omega^{pa}}$ is the restriction of x to the purely atomic part Ω^{pa}. Then x_* is an optimal solution of (IC_p) if and only if (v_*, w_*), with $v_* := x_* \mid_{\Omega^{na}}$ and $w_* := x_* \mid_{\Omega^{pa}}$, is an optimal solution of the following optimization problem

$$(L) \qquad \inf_{v \in \mathcal{V}, w \in \mathcal{W}} \{- \int_{\Omega^{na}} u(v(\omega), \omega)\mu(d\omega) + a_0(w) : \int_{\Omega^{na}} v \cdot \xi_i d\mu + a_i(w) \leq \alpha_i, i = 1, \ldots, m\}.$$

Here $a_0(w) := - \int_{\Omega^{pa}} u(w(\omega), \omega)\mu(d\omega)$ is convex in the variable w (by the given concavity property of u). Each $a_i(w) := \int_{\Omega^{pa}} w \cdot \xi_i d\mu$ is also obviously convex in w. In the terminology of [1, 4.3.3], problem (L) is a *Lyapunov-type optimization problem*. By the main theorem of section 4.3.3 in [1, p. 240-241] it follows that, corresponding to the optimal pair (v_*, w_*), there exist nonnegative multipliers $\lambda_0, \lambda_1, \ldots, \lambda_m$, not all zero, such that the two minimum principles

$$v_*(\omega) \in \text{argmin}_{z \in \mathbb{R}^d_+} - \lambda_0 u(z, \omega) + \sum_{i=1}^{m} \lambda_i z \cdot \xi_i(\omega) \text{ for a.e. } \omega \text{ in } \Omega^{na}$$

and

$$w_* \in \text{argmin}_{w \in \mathcal{W}} \sum_{i=0}^{m} \lambda_i a_i(w),$$

hold, as well as the complementary slackness relationships

$$\lambda_i(\int_{\Omega^{na}} v_* \cdot \xi_i d\mu + a_i(w_*) - \alpha_i) = 0, i = 1, \ldots, m.$$

Writing out the definition of the $a_i(w)$ immediately gives that the above complementary slackness relationship is equivalent to the one stated in Theorem 3.2. Since $\alpha_1, \ldots, \alpha_m > 0$, a Slater type constraint qualification holds, which causes $\lambda_0 \neq 0$. This can also be seen directly: if λ_0 were 0, then obviously $\sum_i \lambda_i x_*(\omega) \cdot \xi_i(\omega) = 0$ for a.e. ω (set $z = x_*(\omega)/2$ and $z = 2x_*(\omega)$ respectively). This would result in $\sum_i \lambda_i \int_\Omega x_* \cdot \xi_i d\mu = 0$. By complementarity, $\sum_i \lambda_i \int_\Omega x_* \cdot \xi_i d\mu = \sum_i \lambda_i \alpha_i$, so we would have $\sum_i \lambda_i \alpha_i = 0$. By $\lambda_i \geq 0$ and $\alpha_i > 0$ for all i, this would mean that also the multipliers $\lambda_1, \ldots, \lambda_m$ are zero. This gives a contradiction. So $\lambda_0 \neq 0$, and, rather than dividing all λ_i by λ_0, we can suppose without loss of generality $\lambda_0 = 1$. Also, because Ω^{pa} consists of at most countably many atoms and because each function in $\mathcal{W}$ is a.e. constant on such an atom, it is easy to see that the second minimum principle is equivalent to the following:

$$w_*(\omega) \in \text{argmin}_{z \in \mathbb{R}^d_+} - u(z, \omega) + \sum_{i=1}^{m} \lambda_i z \cdot \xi_i(\omega) \text{ for a.e. } \omega \text{ in } \Omega^{pa}$$

Combined, the above two mimimum principles (with $\lambda_0 = 1$) are precisely equivalent to the pointwise maximum principle that is stated in Theorem 3.2. QED

Some comments should be added to justify the application above of the main theorem of [1, section 4.3.3]. Formally speaking, the conditions of [1] require Ω to be a Lebesgue interval of $\mathbb{R}$ and the functions $u(z, \omega)$ and $z \cdot \xi_i(\omega)$ to be jointly continuous in (z, ω). However, from the proof in [1] it is evident that the only reason for this is the rather crude Lemma D) on p. 244, which is known to hold in much more general forms for functions $u(z, \omega)$ and $z \cdot \xi_i(\omega)$ that are just jointly measurable in (z, ω) and for a *decomposable* class of measurable functions, such as $\mathcal{L}^p_Z$. This is the so-called reduction theorem; e.g., see [16, Theorem 3A], [7, Theorem B.1] and [12]. Actually, the approach taken in [1] can already be found in more general terms in [2].

PROOF OF COROLLARY 3.3. Clearly, all that has to be done is to demonstrate that $\lambda_i > 0$ for $i = 1, \ldots, m$ in the pointwise maximum principle of Theorem 3.2 (because complementary slackness then implies feasibility for (EC_p)). If there were j with $\lambda_j = 0$, then the pointwise maximum principle would imply that $x_*(\omega)$ belongs to $\mathrm{argmax}_{z\in\mathbb{R}^d_+}[u(z,\omega)-\sum_{i,i\neq j}\lambda_i z\cdot\xi_i(\omega)]$ for a.e. ω. But this contradicts the definition of essential nonsatiation. QED

Proof of Theorem 2.8. Since the conditions of Proposition 2.5 clearly hold, we certainly have existence of an optimal solution $x_* \in \mathcal{L}^0_Z$ of (IC_0). We can apply Corollary 3.3 (for $p=0$) to (IC_0). Observe already that this already gives feasibility of x_* for (EC_0). Setting $z := \tilde{x}(\omega)$ in the pointwise maximum principle, we obtain

$$u(x_*(\omega),\omega) - \sum_{i=1}^m \lambda_i x_*(\omega)\cdot\xi_i(\omega) \geq u(\tilde{x}(\omega),\omega) - \sum_{i=1}^m \lambda_i \tilde{x}(\omega)\cdot\xi_i(\omega) \text{ a.e.}$$

where $\tilde{x} \in \mathcal{L}^p_Z$ is as postulated in Theorem 2.8. Let $\epsilon := \min_{1\leq i\leq m}\lambda_i$; then $\epsilon > 0$ by Corollary 3.3. By using (γ_p) and order-equivalence of $(\xi_{i,j})$ we obtain from the above

$$\frac{\epsilon}{2}\hat{\xi}(\omega)|x_*(\omega)| + \hat{\xi}(\omega)\psi_{\epsilon/2}(\omega) - u(\tilde{x}(\omega),\omega) + C\sqrt{d}\max_i \lambda_i\hat{\xi}(\omega)|\tilde{x}(\omega)| \geq \epsilon\hat{\xi}(\omega)|x_*(\omega)|.$$

Here we use the elementary inequalities $\epsilon|x_*|\hat{\xi} \leq \sum_i \lambda_i x_*\cdot\xi_i \leq Cd^{1/2}\max_i\lambda_i\hat{\xi}|x_*|$. After division by $\hat{\xi}(\omega)$, the resulting majorization of $\epsilon|x_*|/2$ by the p-integrable expression $\psi_{\epsilon/2} - u(\tilde{x}(\cdot),\cdot)/\hat{\xi} + Cd^{1/2}\max_i\lambda_i|\tilde{x}|$ immediately implies the p-integrability of $|x_*|$. Finally, (EC_p)-feasibility of x_* now follows simply from our earlier observation about its (EC_0)-feasibility. So x_* is also an optimal solution of problem (EC_p). QED

4 Applications

In this section we show how the existence results in [4] and [10] all follow from the results developed in section 2. We also give some examples to show that Theorem 2.8 also applies to new situations, not covered by [4, 10]. To begin with, we prepare the conversion of the following growth properties used in [4, 10] for $p \geq 1$:

Definition 4.1 *(i)* u has *growth property* (δ_p) if for every $\epsilon > 0$ there exists $\phi_\epsilon \in \mathcal{L}^p_+$ such that for a.e. ω

$$u(z,\omega) \leq \epsilon\hat{\xi}(\omega)|z| \text{ for all } z \in \mathbb{R}^d_+ \text{ with } |z| \geq \phi_\epsilon(\omega).$$

(ii) u has *growth property* (δ'_p) if for every $\epsilon > 0$ there exists $\phi'_\epsilon \in \mathcal{L}^p_+$ such that for a.e. ω

$$u(z,\omega) \leq \epsilon\hat{\xi}(\omega)|z| \text{ for all } z \in \mathbb{R}^d_+ \text{ with } \min_{1\leq i\leq d} z_i \geq \phi'_\epsilon(\omega).$$

Because of $d=1$, in [10] one has $|z| = z$ for all $z \in \mathbb{R}_+$, which causes the growth properties (δ_p) and (δ'_p) to be indistinguishable. Growth property (δ'_p), for $p \geq 1$, can already be found in [4], and also property (δ_1). Growth property (δ'_p) is also used (but just for $m = d = 1$) in [10, Definition 4.1, Lemma 4.2, ff.], as can be seen by means of the following example.

Example 4.2 *(i)* Suppose that there exist $b \in (0,1)$, $\beta_1 \geq 0$ and $\beta_2 > 0$ such that for a.e. ω

$$u(z,\omega) \leq \beta_1 + \beta_2|z|^{1-b} \text{ for all } z \in \mathbb{R}^d_+.$$

Suppose also that $\hat{\xi}^{-1}$ belongs to $\mathcal{L}^{p/b}$. Then growth condition (δ_p) holds: similar to [10, Lemma 4.2], we simply observe that $u(z,\omega) \leq \beta_1 + \epsilon\hat{\xi}(\omega)|z|$ for a.e. ω and for all z with $|z| \geq (\epsilon\hat{\xi}(\omega)/\beta_2)^{-1/b}$. Hence, $u(z,\omega) \leq 2\epsilon\hat{\xi}(\omega)|z|$ if $|z| \geq \phi_{2\epsilon}(\omega)$, where $\phi_{2\epsilon} := \max[(\epsilon\hat{\xi}/\beta_2)^{-1/b}, \beta_1\hat{\xi}^{-1}]$ defines a function in $\mathcal{L}^p_+$. This shows (δ_p) to hold.

(ii) If $\beta_2 = 0$ in part *(i)*, then condition (γ_p) holds trivially. This implies that condition (δ_p) then holds as well (by Proposition 4.3*a* below), without the above condition for $\hat{\xi}^{-1}$.

Proposition 4.3 *a. For any $p \geq 1$, (γ_p) implies (δ_p) implies (δ'_p).*
b. Suppose that

$$u(z,\omega) \text{ is nondecreasing in } z \text{ for a.e. } \omega \text{ in } \Omega.$$

Then for any $p \geq 1$ the three growth properties (γ_p), (δ_p) and (δ'_p) are equivalent.

PROOF *a.* ($(\gamma_p) \Rightarrow (\delta_p)$): For any $\epsilon > 0$ we have $u(z,\omega)/\hat{\xi}(\omega) \leq \epsilon|z|/2 + \psi_{\epsilon/2}(\omega)$ for a.e. ω and all z. Define $\phi_\epsilon := 2\epsilon^{-1}\psi_{\epsilon/2} \in \mathcal{L}^p_+$. Then $|z| \geq \phi_\epsilon(\omega)$ is easily seen to imply $u(z,\omega)/\hat{\xi}(\omega) \leq \epsilon|z|$.

($(\delta_p) \Rightarrow (\delta'_p)$): This follows simply from the implication $\min_i z_i \geq \phi_\epsilon(\omega) \Rightarrow |z| \geq \phi_\epsilon(\omega)$.

b. ($(\delta'_p) \Rightarrow (\delta_p)$): For any $\epsilon > 0$ let ϕ'_ϵ be as in the definition of (δ'_p). Set $\phi_\epsilon := d\phi'_{\epsilon'}$ with $\epsilon' := \epsilon/d^{1/2}$. Then, for any $z \in \mathbb{R}^d_+$, let $z' := (\hat{z},\ldots,\hat{z})$, where $\hat{z} := \max_i z_i$. Then $|z| \geq \phi_\epsilon(\omega) \Rightarrow |z'| = d^{1/2}\hat{z} \geq \phi'_{\epsilon'}(\omega)$, which causes $u(z',\omega)/\hat{\xi}(\omega) \leq \epsilon'|z'| = \epsilon\hat{z} \leq \epsilon|z|$. Finally, observe that $u(z,\omega) \leq u(z',\omega)$ by monotonicity of u, since obviously $z' \geq z$.

($(\delta_p) \Rightarrow (\gamma_p)$): Define $\psi_\epsilon := d^{1/2}(\phi_1 + \phi_\epsilon) \in \mathcal{L}^p_+$, with ϕ_1 (for $\epsilon := 1$) and ϕ_ϵ as in the definition of condition (δ_p). Then $\psi_\epsilon(\omega) = |z_\epsilon(\omega)|$, where $z_\epsilon(\omega) \in \mathbb{R}^d_+$ is the vector all of whose components are equal to $\phi_1(\omega) + \phi_\epsilon(\omega)$. Observe that $u(z_\epsilon(\omega),\omega) \leq \hat{\xi}(\omega)\psi_\epsilon(\omega)$ by (δ_p) (for $\epsilon := 1$), in view of $\psi_\epsilon(\omega) = |z_\epsilon(\omega)| \geq \phi_1(\omega)$. Let $\omega \in \Omega$ be arbitrary and nonexceptional and let $z \in \mathbb{R}^d_+$ be arbitrary. Now either $z \leq z_\epsilon(\omega)$ (i.e., componentwise) or not. In the latter case one has $|z| \geq \phi_\epsilon(\omega)$ (since at least one coordinate must be greater than $\psi_\epsilon(\omega)$), which implies $u(z,\omega) \leq \epsilon\hat{\xi}(\omega)|z|$. In the former case one has $u(z,\omega) \leq u(z_\epsilon(\omega),\omega)$ by monotonicity of u, which gives $u(z,\omega) \leq \hat{\xi}(\omega)\psi_\epsilon(\omega)$ when it is combined with the earlier inequality for $u(z_\epsilon(\omega),\omega)$. We conclude that in either case $u(z,\omega) \leq \epsilon\hat{\xi}(\omega)|z| + \hat{\xi}(\omega)\psi_\epsilon(\omega)$. That is to say, (γ_p) has been shown to hold. QED

It is intuitively obvious that the global growth control of u, as excercised by (γ_p), cannot be maintained under (δ_p) and (δ'_p), which only exercise such control outside a certain radius from the origin. This is confirmed by the following example, which shows that the implications in Proposition 4.3*a* cannot be reverted without additional conditions such as monotonicity.

Example 4.4 Let $d = 1$ and consider $\Omega = (0,1)$ with the Lebesgue measure. Let $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be as follows:

$$u(z,\omega) := \begin{cases} \sqrt{z-1} & \text{if } z \geq 1, \\ \omega^{-1}(1-z) & \text{if } z < 1 \end{cases}$$

Also, let $\hat{\xi} \equiv 1$. Then (γ_p) cannot hold (since $1/\omega = u(0,\omega) \leq \psi_\epsilon(\omega)$ would force non-integrability of ψ_ϵ). However, for $z \geq \phi_\epsilon(\omega) := \max(1, \epsilon^{-2})$ one has $u(z,\omega) \leq \epsilon|z|$.

We begin to apply our results of section 2 to situations – rather they are *generalizations* of such situations – considered in [4].

Corollary 4.5 ([4, Main Theorem]) *Suppose that $(\xi_{i,j})$ has diagonal structure with* $\operatorname{ess\,inf}_\Omega \hat{\xi} > 0$ *and that $u(z,\omega)$ is upper semicontinuous and nondecreasing in z for a.e. ω. Suppose also that u has growth property (δ'_1). Then problem (EC_1) has an optimal solution that is also an optimal solution of (IC_1).*

PROOF. Proposition 4.3 implies that (γ_1) holds. We may now apply Proposition 2.6, which gives the existence of an optimal solution x_* of (EC_0), with $\int_\Omega x_*\hat{\xi}d\mu < +\infty$, that is optimal for (IC_0) at the same time. By remark 2.7, x_* is also an optimal solution of (EC_1) and (IC_1). QED

Corollary 4.6 ([4, Theorem 6.1]) *Suppose that $u(z,\omega)$ is upper semicontinuous in z for a.e. ω in Ω. Suppose also that* $\operatorname{ess\,inf}_\Omega \hat{\xi} > 0$ *and that u has growth property (δ_1), together with the following additional property: for every $\eta \in \mathcal{L}^1_+$ there exists $\zeta \in \mathcal{L}^1_+$ such that $|z| \leq \eta(\omega)$ implies $u(z,\omega) \leq \zeta(\omega)\hat{\xi}(\omega)$. Then problem (IC_1) has an optimal solution.*

PROOF. To prove that u has growth property (γ_1), let $\epsilon > 0$ be arbitrary. By (δ_1) there exists $\phi_\epsilon \in \mathcal{L}^1_+$ such that $|z| \geq \phi_\epsilon(\omega)$ implies $u(z,\omega)/\hat{\xi}(\omega) \leq \epsilon|z|$. By the additional property there exists $\zeta_\epsilon \in \mathcal{L}^1_+$ such that $|z| < \phi_\epsilon(\omega)$ implies $u(z,\omega)/\hat{\xi}(\omega) \leq \zeta_\epsilon(\omega)$. Together, this means that $u(z,\omega)/\hat{\xi}(\omega) \leq \epsilon|z| + \zeta_\epsilon(\omega)$ for all z. This proves (γ_1). All conditions of Proposition 2.5 are now fulfilled, so there exists an optimal solution x_* of problem (IC_0), with $\int_\Omega x_*\hat{\xi}d\mu < +\infty$. By Remark 2.7 x_* is also an optimal solution of (IC_1). QED

Corollary 4.7 ([4, Theorem 6.2]) *Suppose that $(\Omega, \mathcal{F}, \mu)$ is nonatomic, that $u(z,\omega)$ is upper semicontinuous and nondecreasing in z for a.e. ω in Ω and that $u(z,\omega)$ is increasing in z for all ω in some non-null subset of Ω. Suppose also that $(\xi_{i,j})$ has diagonal structure, is order equivalent to $\hat{\xi}$, with $\hat{\xi}^{-1} \in \mathcal{L}^p$. Suppose further that u is nonnegative and has growth property (δ'_p). Then problem (EC_p) has an optimal solution that is also an optimal solution of (IC_p).*

Proof. Let us check that the conditions of Theorem 2.8 hold. Here we have $\Omega^{pa} = \emptyset$, so that the concavity condition holds vacuously. Also, by Remark 2.4, u is clearly nonsatiated with respect to $(\xi_{i,j})$. By Proposition 4.3, u has property (γ_p), since $u(z,\omega)$ is certainly nondecreasing in z. By (γ_p), we get for $\tilde{x} \equiv 0$ that $0 \leq u(\tilde{x}(\cdot),\cdot)/\hat{\xi} \leq \psi_1$ (take $\epsilon = 1$). By $u \geq 0$, this proves that $u(\tilde{x}(\cdot),\cdot)$ belongs to $\mathcal{L}^p$. So all conditions of Theorem 2.8 hold. It follows that there exists an optimal solution of (EC_p) that is also an optimal solution of (IC_p). QED

Even as specializations of Theorem 2.8, the above corollaries still improve the corresponding results in [4] in a number of respects. For instance, Corollaries 4.5 and 4.6 do not require $(\Omega, \mathcal{F}, \mu)$ to be nonatomic, Corollary 4.7 does not require $u(z,\omega)$ to be increasing for a.e. ω and none of the three corollaries requires $\xi_i \equiv e_i$. Besides, they allow for easy improvements that have not been considered in [4]. For instance, in Corollary 4.7 one could also consider a general measure space instead of a nonatomic one by introducing for $\omega \in \Omega^{pa}$ extra concavity for $u(z,\omega)$ in the variable z, just as in Theorem 2.8. Also, in that same corollary, one could omit the nondecreasingness of $u(z,\omega)$ in z for most ω (except for those ω that are in the non-null set mentioned in the statement) by requiring (γ_p) to hold instead of (δ'_p). This is illustrated by the following examples:

Example 4.8 Let Ω be the unit interval, equipped with Lebesgue measure μ. Let $m = d = 1$, $\eta \in (0,1]$ and define the utility function as follows:

$$u(z,\omega) := \begin{cases} -z^2 & \text{if } \omega \leq 1-\eta \\ \sqrt{z\omega} & \text{if } \omega > 1-\eta \end{cases}$$

[Here one could think of $1-\eta$ as some critical value; if the state of nature ω is less than this value, the benefit of consumption is completely reversed.] Consider the problems (IC_p) and (EC_p) with $\xi_1 \equiv 1$. It is obvious that u satisfies growth condition (γ_p) for any $p \geq 1$ and that $u(\tilde{x}(\omega),\omega) = 0$ on Ω for $\tilde{x} \equiv 0$. Even though $u(z,\omega)$ is *decreasing* in z for $\omega \in [0, 1-\eta]$, the conditions of Theorem 2.8, and in particular essential nonsatiation, are valid. This theorem therefore establishes existence of an optimal solution of (IC_p) and (EC_p) for every $p \geq 1$ (note that (IC_p) always has $x \equiv \alpha_1$ as a feasible solution – cf. Remark 2.9). It is illuminating to inspect this result by a more complete analysis of this example, based on an application of Theorem 3.2 (or Corollary 3.3). By this result the optimal solution x_* of (IC_p) must be feasible and must satisfy $x_*(\omega) \in \text{argmax}_{z\geq 0} u(z,\omega) - \lambda_1 z$ a.e. for some $\lambda_1 \geq 0$. If $\lambda_1 = 0$, then for $\omega > 1-\eta$ the above "argmax set" would be empty, which would give a contradiction. So the only possibility is $\lambda_1 > 0$ (note that this is in agreement with Corollary 3.3). For a.e. $\omega \in [0, 1-\eta]$ this gives $x_*(\omega) = 0$. For a.e. $\omega \in (1-\eta, 1]$ the above pointwise maximum principle gives $x_*(\omega) = \omega/4\lambda_1^2$. To satisfy complementary slackness we also need $\int_0^1 x_* = \alpha_1$, and this is easily seen to be solved for $\lambda_1 = [(2\eta - \eta^2)/8\alpha_1]^{1/2}$. The sufficiency part of Theorem 3.2 now also guarantees that the above x_* is an optimal solution of (IC_p). In fact, the above derivation shows that it is essentially (i.e., apart from null sets) the unique optimal solution of (IC_p) and $EC_p)$.

Example 4.9 Let Ω be the unit interval, equipped with Lebesgue measure μ. Let $m = d = 1$, $\eta \in [0,1]$ and define the utility function as follows:

$$u(z,\omega) := \begin{cases} \min(z\sqrt{\omega}, 1) & \text{if } \omega \leq 1-\eta \\ \sqrt{z\omega} & \text{if } \omega > 1-\eta \end{cases}$$

Consider the problems (IC_p) and (EC_p) with $\xi_1 \equiv 1$. It is not hard to check that u satisfies growth condition (γ_p) for any $p \geq 1$ and that $u(\tilde{x}(\omega), \omega) = 0$ on Ω for $\tilde{x} \equiv 0$. However, in Case 1 below the essential satiation condition is violated:

Case 1: $\eta = 0$, $\alpha_1 = 2$. This is precisely the example stated in [4, p. 502]. Although in this case the problem is completely elementary, we give a formal derivation for reasons of comparison with case 2 below. First of all, because $u(z,\omega)$ is nondecreasing in z, any optimal solution of (IC_p) also leads to an optimal solution of (EC_p) (see the proof of Proposition 2.6 – it turns out that this time we cannot use complementary slackness). So it makes sense to start looking for an optimal solution of (IC_p). By Theorem 3.2, to find an optimal solution x_* of (IC_p) we must find a multiplier $\lambda_1 \geq 0$ such that $x_*(\omega) \in \mathrm{argmax}_{z \geq 0} u(z,\omega) - \lambda_1 z$ a.e. If $\lambda_1 > 0$, then the pointwise maximum principle implies $x_*(\omega) = 0$ if $\sqrt{\omega} < \lambda_1$ and $x_*(\omega) = 1/\sqrt{\omega}$ if $\sqrt{\omega} > \lambda_1$. This clearly violates $\int_0^1 x_* = 2$, which must hold by complementary slackness in this case. So $\lambda_1 > 0$ is impossible, and we are left with $\lambda_1 = 0$. In this case the pointwise maximum principle implies $x_*(\omega) \geq 1/\sqrt{\omega}$ a.e. Together with the feasibility constraint $\int_0^1 x_* \leq 2$, this implies $x_*(\omega) = 1/\sqrt{\omega}$ a.e. Observe that $x_* \in \mathcal{L}^1_Z$, but $x_* \notin \mathcal{L}^2_Z$. So, by the sufficiency part of Theorem 3.2, x_* is the essentially unique optimal solution of (IC_0), (EC_0), (IC_1) and (EC_1), but *not* of (IC_2) or (EC_2). In fact, it follows that (IC_2) does not have an optimal solution at all, since the preceding application of the necessary conditions in Theorem 3.2 gave us the above x_* as its only candidate for optimality. Similar nonexistence can be proven for (EC_2) by considering an analogue of Theorem 3.2, mentioned in footnote 2.

Case 2: $\eta = 0.19$, $\alpha_1 = 5.89875$. This time the essential nonsatiation condition is valid (see Remark 2.4), so Theorem 2.8 applies: we know in advance that there exists an optimal solution of (IC_p) and (EC_p) for any $p \geq 1$. This is confirmed by determining the optimal solution explicitly. Again, by Theorem 3.2, the optimal solution x_* of (IC_p) must be feasible and satisfy $x_*(\omega) \in \mathrm{argmax}_{z \geq 0} u(z,\omega) - \lambda_1 z$ a.e. for some $\lambda_1 \geq 0$. If $\lambda_1 = 0$, then for $\omega > 0.81$ the pointwise maximum principle would be self-contradictory, its "argmax set" being empty. So we are left with $\lambda_1 > 0$. For $\omega > 0.81$, the set $\mathrm{argmax}_{z \geq 0} \sqrt{z\omega} - \lambda_1 z$ is the singleton $\{\omega/4\lambda_1^2\}$ (see Example 4.8). For $\omega \leq .81$, the set $\mathrm{argmax}_{z \geq 0} \min(z\sqrt{\omega}, 1) - \lambda_1 z$ is the singleton $\{1/\sqrt{\omega}\}$ if $\lambda_1 < \sqrt{\omega}$, but if $\lambda_1 > \sqrt{\omega}$ it is the singleton $\{0\}$. We now distinguish (a) $\lambda_1 \geq 0.9$ and (b) $0 < \lambda_1 < 0.9$. In case (a) we find, by the pointwise maximum principle, $x_*(\omega) = 0$ for a.e. $\omega \leq 0.81$, by $\omega < \lambda_1^2$. In case (b) we find (a.e.), by the same principle, $x_*(\omega) = 0$ if $\omega \in [0, \lambda_1^2)$ and $x_*(\omega) = 1/\sqrt{\omega}$ if $\omega \in (\lambda_1^2, 0.81]$. In both cases the equation $\int_0^1 x_* = 5.89875$ is forced by complementary slackness, since $\lambda_1 > 0$. In case (a) this equation gives immediately $\lambda_1 = 0.0853\ldots$, which is in conflict with the underlying inequality (a). In case (b) that same equation is the cubic equation $1.8 - 2\lambda_1 + 0.0429875\lambda_1^{-2} = 5.89875$, of which $\lambda_1 = 0.1$ is the only root complying with (b). By the sufficiency part of Theorem 3.2, $x_*(\omega) = 0$ if $\omega \in [0, 0.01)$, $x_*(\omega) = 1/\sqrt{\omega}$ if $\omega \in (0.01, 0.81]$ and $x_*(\omega) = 2.5\,\omega$ if $\omega \in (0.81, 1]$ is an optimal solution of (IC_p) and (EC_p) for any $p = 0$ or $p \geq 1$ (observe that $x_* \in \mathcal{L}^p_Z$ for any $p \geq 1$). Moreover, our derivation shows x_* to be the essentially unique optimal solution of (IC_p) and (EC_p).

Next, we turn to the existence results in [10].

Corollary 4.10 ([10, Proposition 4.2]) *Suppose that $u(z,\omega)$ is upper semicontinuous and nondecreasing in z for a.e. ω in Ω and concave in z for a.e. ω in Ω^{pa}. Suppose also that u has growth property (δ'_1) and that $(\xi_{i,j})$ has diagonal structure. Then problem (IC_0) has an optimal solution x_*, $\int_\Omega \hat{\xi}|x_*|d\mu < +\infty$, that is also an optimal solution of (EC_0).*

Proof. Condition (γ_1) holds by Proposition 4.3, since $u(z,\omega)$ is nondecreasing in z. The conditions of Proposition 2.6 are thus fulfilled. This gives the existence result. QED

Corollary 4.11 ([10, Theorems 4.1, 4.2]) *Suppose that $u(z,\omega)$ is upper semicontinuous and nondecreasing in z for a.e. ω in Ω, concave in z for a.e. ω in Ω^{pa} and increasing for a.e. ω in some non-null subset of Ω. Suppose also that u has growth property (δ'_p) and that $(\xi_{i,j})$ has diagonal structure and is order-equivalent to $\hat{\xi}$ with $\hat{\xi}^{-1} \in \mathcal{L}^p$. Suppose also that there exists some $\tilde{x} \in \mathcal{L}^p$ for which $\omega \mapsto u(\tilde{x}(\omega), \omega)$ is essentially bounded. Then problem (EC_p) has an optimal solution that is also an optimal solution of (IC_p).*

PROOF. Again, by Proposition 4.3 u has property (γ_p) in view of the given monotonicity of $u(z,\omega)$ in z. Since $\hat{\xi}^{-1} \in \mathcal{L}^p$, it is evident that $\omega \mapsto u(\tilde{x}(\omega),\omega)/\hat{\xi}(\omega)$ is p-integrable. So all the conditions of Theorem 2.8 are valid and the result follows. QED

Observe that, by Example 4.2, the upper bounds for u in Theorems 4.1, 4.2 of [10] both imply the validity of (δ'_p), as used in the above corollary. Other improvements over the conditions used for the utility u in [10] are also quite evident; for instance, our concavity and monotonicity conditions are considerably weaker. We conclude this section by giving a very historical application of Theorem 2.8:

Example 4.12 Let Ω be the unit interval, equipped with Lebesgue measure μ. The following formulation can be given of Newton's classical problem of least resistance [1, p. 17].

$$\inf_{y \in \mathcal{Y}^p} \{ \int_0^1 \frac{\omega}{1+\dot{y}^2(\omega)} \mu(d\omega) : y(0) = 0, y(1) = \alpha_1, \dot{y} \geq 0 \}.$$

Here $\alpha_1 > 0$ and $\mathcal{Y}^p$ stands for the class of all p-absolutely continuous functions, i.e., the set of all functions $y : [0,1] \to \mathbb{R}$ for which there exists $\dot{y} \in \mathcal{L}^p$ such that $y(\omega) = y(0) + \int_0^\omega \dot{y} d\mu$ for every $\omega \in \Omega$. In [1] this problem is only studied for $p = 1$, but we wish to consider it also for $p \geq 1$. By substitution of $x := \dot{y}$, Newton's problem is seen to be precisely of the form (EC_p), with $m = d = 1$, $u(z,\omega) := -\omega/(1+z^2)$, $\hat{\xi} = \xi_{1,1} \equiv 1$ (observe that $\int_0^1 x = \int_0^1 \dot{y} = y(1) - y(0) = \alpha_1$). It is easy to check that all conditions of Theorem 2.8 hold in this example for any $p \geq 1$ (use Remark 2.4). Thus, for any $p \geq 1$ the above problem has an optimal solution. See [1, p. 60 ff.] for a complete description of this optimal solution. Just as in Examples 4.8 and 4.9, it could also be derived via Theorem 3.2.

5 Extensions

5.1 State-contingent consumption sets

The fact that $u(z,\omega)$ is allowed to be $-\infty$ can be exploited to absorb pointwise constraints on consumption of the type

$$x(\omega) \in X(\omega) \text{ for a.e. } \omega \text{ in } \Omega$$

in a very simple and direct way into the model. Here $X : \Omega \to 2^{\mathbb{R}^d_+}$ denotes a multifunction with a $\mathcal{F} \times \mathcal{B}(\mathbb{R}^d_+)$-measurable graph. Such absorption comes about very simply by introducing

$$\tilde{u}(z,\omega) := \begin{cases} u(z,\omega) & \text{if } z \in X(\omega) \\ -\infty & \text{if } z \notin X(\omega) \end{cases}$$

Of course now the conditions for X must be such that $\tilde{u}$ can be substituted for u in the various conditions. Observe that for $\tilde{u}(z,\omega)$ to be upper semicontinuous [concave] in the variable z, it is sufficient to have $X(\omega)$ closed [convex]. The reformulation of (γ_p) for $\tilde{u}$ obviously yields a version that is easier to satisfy than the one used previously, and in Definition 2.3 one must simply replace the maximization domain $\mathbb{R}^d_+$ by $X(\omega)$.

5.2 Optimal consumption over time

Other extensions and applications are to a time-dependent situation. First of all, one can specialize (IC_p) and (EC_p) to deterministic variational problems by setting $\Omega := [0,T]$ and taking $\mathcal{F}$ equal to the Lebesgue σ-algebra and μ equal to the Lebesgue measure on $[0,T]$. This is the situation of optimal consumption or resource allocation over time, as considered by Aumann and Perles [4] and several others (e.g., see [17]).

Secondly, as in [10], one can *automatically* extend the main results of this paper to a stochastic time-dependent situation, simply by a suitable choice of the underlying measure space. In addition to the space Ω of states of nature, whose distribution is given by the (probability) measure μ, there is now also a time interval $[0,T]$ and a filtration $\{\mathcal{F}_t : t \in [0,T]\}$ of information σ-algebras (e.g., this could be the natural filtration with respect to some stochastic process of signals). Equip $\tilde{\Omega} := [0,T] \times \Omega$ with the σ-algebra $\tilde{F}$ of progressively measurable sets (i.e., $A \in \tilde{\mathcal{F}}$ if and only if the section of A at t belongs to $\mathcal{F}_t$ for each t). If, moreover, a final wealth term is added to the objective function, then problem (IC_p) gets the following form (of course, the same can be done for (EC_p)):

$$(\tilde{IC_p}) \qquad \sup_{x \in \tilde{\mathcal{L}}^p_Z} \{\tilde{U}(x) : \int_\Omega \int_0^T x_t(\omega) \cdot \xi_{i,t}(\omega) dt \mu(d\omega) \leq \alpha_i, i = 1, \ldots, m\}.$$

Here

$$\tilde{U}(x) := \int_\Omega \int_0^T u_t(x_t(\omega), \omega) dt \mu(d\omega) + \int_\Omega u_T(x_T(\omega), \omega) \mu(d\omega)$$

and $\tilde{\mathcal{L}}^p_Z$ stands for $(\mathcal{L}^p_+(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mu}))^d$, where $\tilde{\mu} := \tilde{\mu}_1 + \tilde{\mu}_2$, with $\tilde{\mu}_1$ the product of the Lebesgue measure on $[0,T]$ and μ, and $\tilde{\mu}_2$ the measure on $[0,T] \times \Omega$ that is entirely concentrated on the subset $\{T\} \times \Omega$ and coincides there with μ (i.e., $\tilde{\mu}_2(A \times B) := 1_A(T)\mu(B)$). Observe that the strip $\{T\} \times \Omega$ has $\tilde{\mu}_1$-measure zero, which makes it possible to treat the restrictions $x\mid_{[0,T)\times\Omega}$ and $x\mid_{\{T\}\times\Omega}$ as separate functions. The reformulated problem (10) of Cox and Huang [10], an optimal consumption-portfolio problem in static form, is a special case of $(\tilde{IC_p})$.

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