On Hessian Riemannian Structures

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Abstract In Proposition 4.1 a characterization is given of Hessian Riemannian structures in terms of a natural connection in the general linear group $\mathrm{GL}(n, \mathbf{R})^+$, which is viewed as a principal $\mathrm{SO}(n)$-bundle over the space of positive definite symmetric $n \times n$-matrices. For $n = 2$, Proposition 5.3 contains an interpretation of the curvature of a Hessian Riemannian structure at a given point, in terms of an umbilic point of a related surface in $\mathbf{R}^3$.

0 Introduction

In convex programming, one makes use of a so-called self-concordant barrier function f on an open convex subset Q of $\mathbf{R}^n$, cf. the book [7] of Nesterov and Nemirovskii, and one is interested in the behaviour of the geodesics of the Riemannian structure defined by the Hessian of f. I got acquainted with the subject when I was asked to give an introduction to Riemannian geometry at the conference HPOPT'99 at the Erasmus University Rotterdam, in June 1999.

The formula for the curvature tensor of a Hessian Riemannian structure, cf. (1.7) below, involves only second and third order derivatives of the function f, and no fourth order ones as one would a priori expect. In my attempt to understand this, I arrived at the characterization of Hessian Riemannian structures in Proposition 4.1. In the case $n = 2$ there is also an interpretation of the curvature of a Hessian Riemannian structure in terms of umbilic points of surfaces, see Section 5.

The study of Hessian Riemannian structures on convex domains goes back at least to Koszul [6] and Vinberg [11], who were inspired by the theory of bounded domains in $\mathbf{C}^n$ with its Bergmann metric. Closely related to our subject is Shima's theory of Hessian manifolds, cf. [10]. Ruuska [8] characterized Hessian Riemannian structures as those which admit an abelian Lie algebra of gradient vector fields, where the local action is simply transitive. Hitchin [4] characterized Hessian Riemannian structures in term of a Lagrangean submanifolds of the cotangent bundle. I am grateful to Nigel Hitchin and Lieven Vanhecke for getting me started with the literature on Hessian Riemannian structures.

As a general reference on differential geometry one may use [5].

1 Hessian Riemannian Structures

Suppose that f is a smooth strongly convex function, defined on an open subset Q of $\mathbf{R}^n$. The strong convexity of f means that for every $x \in Q$ the Hessian $\partial_i \partial_j f(x)$ is positive definite, which implies that $g_{ij}(x) := \partial_i \partial_j f(x)$ defines a Riemannian structure. Here, and in the sequel, we use the abbreviation $\partial_i \phi(x)$ for the partial derivative $\partial \phi(x)/\partial x^i$ of any function ϕ with respect to the i-th coordinate.

For a general Riemannian structure $g_{ij}(x)$ the inverse matrix is denoted by $g^{kl}(x)$. The Christoffel symbols then are defined by

$$\Gamma^k_{ij}(x) := \sum_{l=1}^{n} g^{kl}(x)\, \Gamma_{lij}(x), \tag{1.1}$$

in which

$$\Gamma_{lij}(x) := \frac{1}{2}\left[\partial_i g_{jl}(x) - \partial_l g_{ij}(x) + \partial_j g_{li}(x)\right]. \tag{1.2}$$

A straightforward computation yields that for a Hessian Riemannian structure the Christoffel symbols are given by

$$\Gamma_{kij}(x) = \frac{1}{2}\, \partial_k \partial_i \partial_j f(x), \tag{1.3}$$

These symbols are used to define the covariant derivative of a vector field Y, at the point x and in the direction of the tangent vector X, by

$$(\nabla_X Y(x))^i := \sum_{k=1}^{n} X_k \left[\partial_k Y^i(x) + \sum_{j=1}^{n} \Gamma^i_{kj}(x)\, Y^j(x)\right]. \tag{1.4}$$

The horizontal space $H_{(x,Y)}$ at (x, Y) is defined as the linear subspace of the tangent bundle of Q which is equal to graph of the linear mapping

$$X \mapsto -\sum_{k,\,j=1}^{n} X^k\, \Gamma^i_{kj}(x)\, Y^j.$$

In view of (1.4) this means that $\nabla Y(x) = 0$ if and only if the tangent space at $(x, Y(x))$ to the graph of Y is equal to the horizontal space $H_{(x,Y(x))}$. The distribution of horizontal spaces $H_{(x,Y)}$, for (x, Y) in the tangent bundle of U, is called the *Levi-Civita connection* defined by the Riemannian structure g.

More generally, any x-dependent symbols $\Gamma^i_{kj}(x)$ define in the above way what is called a *linear connection* in the tangent bundle. The corresponding covariant dervative $\nabla_X Y$ of a vector field Y in the direction of the vector field X then is linear in X and Y. The linear connection is called *torsion-free* when $\Gamma^i_{kj}(x) = \Gamma^i_{jk}(x)$, which is the case for the Levi-Civita connection of a Riemannian structure.

If X, Y, U are smooth vector fields on Q then, for any $x \in Q$, the expression in the right hand side of

$$R(x)(X(x), Y(x))(U(x)) = \left(\nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X,Y]} U\right)(x) \tag{1.5}$$

depends only on $X(x)$, $Y(x)$, $U(x)$, and not on higher order derivatives of X, Y and U at x, as one would a priori expect. Therefore (1.5) defines a $\mathrm{T}_x Q$--valued trilinear form $(X, Y, U) \mapsto R(x)(X, Y)(U)$ on $\mathrm{T}_x Q$, antisymmetric with respect to the first two variables, which is called the *Riemannian curvature tensor* of the connection. If the k-coordinate of $R(x)(X, Y)(U)$ is written as

$$\sum_{l,\, i,\, j=1}^{n} R^k_{lij}(x)\, U^l\, X^i\, Y^j,$$

then the coordinates of the curvature tensor are given by

$$R^k_{lij}(x) = \partial_i \Gamma^k_{lj}(x) - \partial_j \Gamma^k_{li}(x) + \sum_{m=1}^{n} \left(\Gamma^k_{im}(x)\, \Gamma^m_{jl}(x) - \Gamma^k_{jm}(x)\, \Gamma^m_{il}(x) \right) \tag{1.6}$$

in terms of the Christoffel symbols.

A straightforward computation yields that for a Hessian Riemannian structure the Riemannian curvature tensor is given by

$$\begin{aligned} R_{klij}(x) &:= \sum_{m=1}^{n} g_{km}(x)\, R^m_{lij}(x) = -\frac{1}{4} \sum_{p,\, q=1}^{n} g^{pq}(x) \\ &[\partial_k \partial_i \partial_p f(x) \cdot \partial_q \partial_j \partial_l f(x) - \partial_k \partial_j \partial_p f(x) \cdot \partial_q \partial_i \partial_l f(x)], \end{aligned} \tag{1.7}$$

in which $g^{pq}(x)$ is the inverse of the matrix $g_{ij}(x) = \partial_i \partial_j f(x)$. It is a bit surprising that this formula involves only the derivatives of f of order two and three, and no derivatives of order four as one would expect a priori.

When $n = 2$, then the (Gaussian) curvature $K(x)$ at the point x of a Riemannian structure g is defined as the number

$$K(x) := g(x)\, (R(x)\, (f_1,\, f_2)\, (f_2)\, ,\, f_1)\, , \tag{1.8}$$

in which f_1, f_2 denotes an orthonormal basis of tangent vectors at x with respect to the inner product $g(x)$. (Such a basis can be obtained from the standard basis e_1, e_2 by means of Gram-Schmidt's orthogonalization procedure. The right hand side of (1.8) is the same for every $g(x)$-orthonormal basis f_1, f_2.) For a Hessian Riemannian structure we obtain, using (1.7), that

$$K(x) = \frac{-c\ (\alpha\, \gamma - \beta^2) + b\ (\alpha\, \delta - \beta\, \gamma) - a\ (\beta\, \delta - \gamma^2)}{4\, (a\, c - b^2)^2}, \tag{1.9}$$

in which

$$\begin{aligned} &a := {\partial_1}^2 f(x),\ b := \partial_1 \partial_2 f(x),\ c := {\partial_2}^2 f(x), \\ &\alpha := {\partial_1}^3 f(x),\ \beta := {\partial_1}^2 \partial_2 f(x),\ \gamma := \partial_1 {\partial_2}^2 f(x),\ \delta = {\partial_2}^3 f(x). \end{aligned}$$

2 A Connection in a Lie Group with Involution

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let σ be an involutory automorphism of G, which means that $\sigma : G \to G$ is an automorphism of Lie groups and $\sigma \circ \sigma$

is equal to the identity. We denote the tangent map of σ at the identity element $1 \in G$ by σ'. Then $\sigma' : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of Lie algebras and $(\sigma')^2 = 1$. The set

$$H := \{h \in G \mid \sigma(h) = h\} \tag{2.1}$$

of fixed points of σ in G is a closed Lie subgroup of G, with Lie algebra equal to

$$\mathfrak{h} := \{X \in \mathfrak{g} \mid \sigma'(X) = X\}, \tag{2.2}$$

the eigenspace of $\sigma' : \mathfrak{g} \to \mathfrak{g}$ for the eigenvalue 1. If

$$\mathfrak{s} := \{X \in \mathfrak{g} \mid \sigma'(X) = -X\} \tag{2.3}$$

denotes the eigenspace of $\sigma' : \mathfrak{g} \to \mathfrak{g}$ for the eigenvalue -1, then $\mathfrak{g}$ is equal to the direct sum of $\mathfrak{h}$ and $\mathfrak{s}$, in formula:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}. \tag{2.4}$$

With respect to the Lie brackets, this splitting has the following behaviour:

$$X, Y \in \mathfrak{h} \implies [X, Y] \in \mathfrak{h}, \tag{2.5}$$
$$X \in \mathfrak{h},\ Y \in \mathfrak{s} \implies [X, Y] \in \mathfrak{s}, \tag{2.6}$$
$$X, Y \in \mathfrak{s} \implies [X, Y] \in \mathfrak{h}. \tag{2.7}$$

The property (2.5) expresses the fact that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. The properties (2.5) and (2.6) follow from the slightly stronger fact that the splitting (2.4) is invariant under the adjoint action of H on $\mathfrak{g}$:

$$h \in H,\ X \in \mathfrak{h} \implies \operatorname{Ad} h(X) \in \mathfrak{h}, \tag{2.8}$$
$$h \in H,\ X \in \mathfrak{s} \implies \operatorname{Ad} h(X) \in \mathfrak{s}. \tag{2.9}$$

The right action of H on G, where $h \in H$ acts on G by sending $g \in G$ to $g\,h$, is free and proper, and therefore the orbit space

$$G/H := \{g\,H \mid g \in G\} \tag{2.10}$$

has a unique structure of an analytic manifold, such that the projection

$$\pi_{G/H} : g \mapsto g\,H : G \to G/H \tag{2.11}$$

exhibits G as an H-principal fiber bundle over G/H. The manifold G/H is called the *symmetric space* associated to G and σ.

On G we also have the left action of G, where $\gamma \in G$ acts on G by sending $g \in G$ to $\mathrm{L}_\gamma(g) := \gamma\, g$. Because the left G-action commutes with the right H-action, this action passes to an action of G on G/H, where $\gamma \in G$ acts on G/H by sending $g\,H \in G/H$ to $\gamma\, g\, H$. The bundle projection $\pi_{G/H}$ intertwines the left action of G on G with the action of G on G/H. The action of G on G/H is transitive, with stabilizer group at the point $1\,H$ equal to H. If we identify the tangent space of G/H at $1\,H$ with $\mathfrak{s}$, then the infinitesimal action of H on $\mathrm{T}_{1\,H}(G/H)$ is given by the adjoint representation of H on $\mathfrak{s}$, cf. (2.9).

It is natural to use the left G-action to extend the linear subspace $\mathfrak{s}$ of $\mathrm{T}_1 G$ to a left G-invariant vector subbundle of $\mathrm{T}(G)$, by defining

$$\mathfrak{s}_g := \mathrm{T}_1 \mathrm{L}_g(\mathfrak{s}), \quad g \in G, \tag{2.12}$$

where L_g denotes the multiplication $x \mapsto g\,x$ by g from the left and $\mathrm{T}_1 \mathrm{L}_g : \mathrm{T}_1 G \to \mathrm{T}_g G$ is its tangent mapping. Noting that $\mathrm{T}_1 \mathrm{L}_g(\mathfrak{h})$ is equal to the tangent space at g of the orbit $g\,H$ of the right H-action, we obtain that $\mathfrak{s}_g$ is a complementary linear subspace in $\mathrm{T}_g G$ of $\mathrm{T}_g(g\,H)$. Therefore the $\mathfrak{s}_g$, $g \in G$, can be taken as the horizontal spaces of an infinitesimal connection in the bundle $\pi_{G/H} : G \to G/H$. Because of (2.9), this connection is also invariant under the right H-action on G, and therefore it is a so-called *H-principal bundle connection* in the bundle $\pi_{G/H} : G \to G/H$.

A G-invariant connection of the H-principal fiber bundle $G \to G/H$ arises as soon as we have an $\mathrm{Ad}\,H$-invariant linear complement $\mathfrak{s}$ of $\mathfrak{h}$ in $\mathfrak{g}$. However, the additional property (2.7), which is typical for symmetric spaces, leads to a particularly simple formula for the curvature of the connection.

Proposition 2.1 *The curvature form Ω of the connection in G, which is left G-invariant, is determined by*

$$\begin{aligned} \Omega_1(X, Y) &= 0 \quad \text{when} \quad X \in \mathfrak{h},\ Y \in \mathfrak{g}, && (2.13)\\ \Omega_1(X, Y) &= -[X, Y] \quad \text{when} \quad X, Y \in \mathfrak{s}. && (2.14) \end{aligned}$$

Proof The *connection form* of the connection is defined as the $\mathfrak{h}$-valued one--form θ on the bundle which is equal to zero on the horizontal spaces and, for any $X \in \mathfrak{h}$, takes the value X on the tangent vector $\mathrm{T}_1 \mathrm{L}_g(X) \in \mathrm{T}_g(g\,H)$. The *curvature form* is defined as the $\mathfrak{h}$-valued two-form Ω on the bundle defined by É. Cartan's structural equation

$$\Omega(v,\, w) = (\mathrm{d}\theta)(v,\, w) + [\theta(v),\, \theta(w)]\,, \tag{2.15}$$

for any pair of vector fields v, w on the bundle. The term with the Lie brackets is added in order to make Ω equivariant with respect to the right H-action.

Because θ and Ω are clearly invariant under the left G-action, it suffices to compute Ω at the identity element $1 \in G$. The equation (2.13) holds for any principal bundle connection, therefore we only need to verify (2.14).

It follows from (2.15) and the general formula

$$(\mathrm{d}\theta)(v,\, w) = v\,\theta(w) - w\,\theta(v) - \theta\,([v,\, w])$$

for the exterior deriviative of any one-form θ evaluated at any pair of vector fields v and w, that

$$\Omega(v,\, w) = -\theta\,([v,\, w]) \quad \text{when} \quad v \quad \text{and} \quad w \quad \text{are horizontal.} \tag{2.16}$$

Now let $X, Y \in \mathfrak{s}$ and denote by X_{L} and Y_{L} their respective extensions to left invariant vector fields on G. Because the connection is left G-invariant, the vector fields X_{L} and Y_{L} are horizontal. Because X_{L} and Y_{L} are left invariant, we have that $[X_{\mathrm{L}}, Y_{\mathrm{L}}] = [X, Y]_{\mathrm{L}}$. Because of (2.7) we have that $[X, Y] \in \mathfrak{h}$, and therefore (2.14) follows from (2.16). q.e.d.

Remarks

i) É. Cartan introduced a connection in the tangent bundle bundle of the symmetric space G/H, of which the covariant derivative of the curvature is equal to zero, cf. [5, Vol. 2, Ch. XI, §3]. Our connection in the bundle $\pi_{G/H} : G \to G/H$ is quite different from Cartan's connection in $\mathrm{T}(G/H)$. For instance, the curvature of Cartan's connection at $1\,H$ is given by $R(X, Y)\,Z = -\,[[X, Y]\,,\,Z]$.

ii) When f is an $\mathrm{Ad}\,H$-invariant polynomial on $\mathfrak{h}$, homogeneous of degree m, then substitution of the curvature form in it yields a differential form of degree $2p$, which is equal to the pull-back under $\pi_{G/H}$ of a unique G-invariant $2p$-form ω_f on the symmetric space G/H. The form ω_f is called the *characteristic form* defined by f. The vector space spanned by the forms of even degree form a commutative ring with respect to the exterior product, of which the characteristic forms constitute an interesting subring. For example, the Killing form of $\mathfrak{h}$ leads to a characteristic form of degree four. In Section 5 we will meet the rather exceptional case that $\mathfrak{h} \simeq \mathbf{R}$ and the adjoint representation of H on $\mathfrak{h}$ is trivial, in which case $\Omega = {\pi_{G/H}}^{*}\,\omega$ for a unique G-invariant two–form ω on G/H, which automatically is G-invariant.

3 The General Linear Group

The case to which we will apply the above theory is when $G = \mathrm{GL}\,(n,\,\mathbf{R})^{+}$ is the group of all $n \times n$-matrices A with $\det A > 0$, the orientation-preserving linear transformations of $\mathbf{R}^{n}$, with the involution $\sigma(A) = {}^{\mathrm{t}}\,A^{-1}$, the inverse of the the transposed of A. Then the fixed point subgroup H of G is equal to the group $\mathrm{SO}(n)$ of all the rotations. The condition $\det A > 0$ has been imposed in order to arrange that the groups G and H are connected.

The Lie algebra $\mathfrak{g}$ is equal to the space of all $n \times n$-matrices, with Lie bracket equal to the commutator: $[X, Y] = X\,Y - Y\,X$. We have that $\sigma'(A) = -\,{}^{\mathrm{t}}A$, and therefore the Lie algebra $\mathfrak{h}$ of H is equal to the Lie subalgebra of all the anti-symmetric $n \times n$-matrices. The complementary subspace $\mathfrak{s}$ is equal to the space of all symmetric $n \times n$-matrices, and (2.7) reflects the familiar fact that the commutator of two symmetric matrices is anti-symmetric.

The role of the "abstract" symmetric space G/H in this case will be played by the space $\mathcal{P}$ of all positive definite symmetric $n \times n$-matrices. The mapping $\pi : A \mapsto \sigma(A) \circ A^{-1}$ is surjective and in fact exhibits G as an analytic fiber bundle over $\mathcal{P}$, where the fiber $\pi^{-1}\,(\pi(A))$ over $P = \pi(A)$, the set of all $B \in G$ such that $\pi(B) = \pi(A)$, is equal to the set of all AC with $C \in \mathrm{SO}(n)$, the right $\mathrm{SO}(n)$-orbit of A in $\mathrm{GL}(n,\,\mathbf{R})^{+}$. Therefore the mapping π induces a diffeomorphism from G/H onto $\mathcal{P}$. If $A,\, B \in G$ then

$$\pi(A\,B) = \sigma(A)\,\pi(B)\,A^{-1}. \tag{3.1}$$

If we define the action of $A \in G$ on $\mathcal{P}$ by sending $P \in \mathcal{P}$ to $\sigma(A)\,P\,A^{-1}$, then (3.1) implies that π intertwines the left action of G on G with the just defined transitive action of G on $\mathcal{P}$.

We denote by e_i the standard basis in $\mathbf{R}^n$. Let $P \in \mathcal{P}$. Then $\pi(A) = P$ means that ${}^{\mathrm{t}}A\,P\,A = 1$, or the vectors $A\,(e_i)$, $1 \leq i \leq n$, form a positively oriented orthonormal basis (= orthonormal frame) with respect to the inner product $(u,\, v) \mapsto \langle P\,u,\, v\rangle$. Therefore π may also be viewed as the mapping which assigns to a frame $A\,(e_i)$ the inner product with respect to which $A\,(e_i)$ is an orthonormal frame.

When X is a symmetrix matrix, then the symmetric matrix which is mapped by $\mathrm{T}_1\,\pi$ to X is equal to $-\frac{1}{2}X$. Therefore, if X is viewed as an element of $\mathrm{T}_P\,\mathcal{P}$ and $\pi(A) = P$, then the horizontal vector which is mapped to X is equal to

$$X_{\mathrm{hor},\,A} = -\frac{1}{2}\,A\,\left({}^{\mathrm{t}}A\,X\,A\right) = -\frac{1}{2}\,P^{-1}\,X\,A. \tag{3.2}$$

For the (left G-invariant) curvature form the formula (2.14) leads to

$$\Omega_A\left(X_{\mathrm{hor},\,A},\, Y_{\mathrm{hor},\,A}\right) = -\frac{1}{4}\,\left[{}^{\mathrm{t}}A\,X\,A,\; {}^{\mathrm{t}}A\,Y\,A\right], \tag{3.3}$$

where the brackets in the right hand side denote the commutator of the symmetric $n \times n$-matrices in question, which is an anti-symmetric $n \times n$-matrix.

4 A Characterization of Hessian Riemannian Structures

Let g be a Riemannian structure on an open subset Q of $\mathbf{R}^n$. This means that g is a smooth mapping from Q to the space $\mathcal{P}$ of positive definite symmetric matrices. Loosely speaking, the *pull-back of the* $\mathrm{SO}(n)$*-principal bundle* $\mathrm{GL}(n,\, \mathbf{R})^+$ *over* $\mathcal{P}$ *by means of the mapping* $g : Q \to \mathcal{P}$ is the $\mathrm{SO}(n)$-principal bundle over Q, such that the fiber over $x \in Q$ is equal to the fiber over $g(x) \in \mathcal{P}$. Because the latter fiber has been identified in Section 3 as the set of orthonormal frames with respect to the inner product $g(x)$, this pulled back bundle is equal to the *orthonormal frame bundle* $\mathrm{OF}(Q)$ *of the tangent bundle* $\mathrm{T}(Q)$ *of* Q, with respect to the Riemannian structure g on Q.

In general, when $\pi : B \to N$ is a smooth fiber bundle and $\phi : M \to N$ is a smooth mapping from a manifold M to the base manifold N of the fiber bundle B, then the pull-back ϕ^*B of B by means of ϕ is formally defined as the set of $(x,\, b) \in M \times B$ such that $\phi(x) = \pi(b)$. It follows immediately from the assumptions that ϕ^*B is a smooth submanifold of $M \times B$ of codimension equal to the dimension of N. The projection $(x,\, b) \mapsto x$ induces a mapping $\pi_1 : \phi^*B \to X$, which exhibits ϕ^*B as a smooth fiber bundle over X, where the fiber $\left(\phi^*B\right)_x$ over the point $x \in M$ is identified with the fiber $B_{\phi(x)}$ of B over the image point $\phi(x) \in N$. If H_b is the horizontal linear subspace of $\mathrm{T}_b\,B$ of an infinitesimal connection for $\pi : B \to N$, then we define $H_{(x,\,b)}$ as the set of $(\delta x,\, \delta b) \in \mathrm{T}_{(x,\,b)}\,(\phi^* B)$, such that $\delta b \in H_b$. Because $(\delta x,\, \delta b) \in \mathrm{T}_{(x,\,b)}\,(\phi^* B)$ if and only if $\mathrm{T}_x\,\phi\,(\delta x) = \mathrm{T}_b\,\pi\,(\delta b)$, we obtain that the horizontal vector $v_{\mathrm{hor},\,(x,\,b)}$ at $(x,\, b) \in \phi^* B$, which is mapped by $\mathrm{T}_{(x,\,b)}\,\pi_1$ to $v \in \mathrm{T}_x\,M$, is given by

$$v_{\mathrm{hor},\,(x,\,b)} = \left(v,\; (\mathrm{T}_x\,\phi(v))_{\mathrm{hor},\,b}\right), \tag{4.1}$$

where $w_{\mathrm{hor},\,b}$ denotes the horizontal vector which by $\mathrm{T}_b\,\pi$ is mapped to $w \in \mathrm{T}_{\pi(b)}\,N$. In this way, the orthonormal frame bundle $\mathrm{OF}(Q)$ is provided with an $\mathrm{SO}(n)$-principal connection, which is obtained by pulling back the connection in $\pi : \mathrm{GL}(n,\,\mathbf{R})^+ \to \mathcal{P}$ by means of the mapping $g : Q \to \mathcal{P}$.

In order to obtain the associated affine connection in the tangent bundle $\mathrm{T}(Q)$ of the Riemannian manifold Q, we observe that $\mathrm{T}(Q)$ is equal to the vector bundle which is associated to $\mathrm{OF}(Q)$ by means of the standard representation of $\mathrm{SO}(n)$ in $\mathbf{R}^n$. This means concretely that to the tangent vector $v \in \mathrm{T}_x\,Q$ we associate the mapping $\widetilde{v} : \mathrm{OF}_x\,Q \to \mathbf{R}^n$, defined by

$$\widetilde{v}(A)^k := g(x)\,(A\,(e_k)\,,\, v)\,, \quad 1 \leq k \leq n.$$

Here $A \in \mathrm{GL}(n,\,\mathbf{R})^+$ is identified with the oriented $g(x)$-orthonormal frame $A\,(e_k)$ in $\mathrm{T}_x\,Q \subset \mathbf{R}^n$. Using that $g(x) = {}^{\mathrm{t}}\,A^{-1}\,A^{-1}$, this formula can also be written as

$$\widetilde{v}(A) = A^{-1}\,v. \tag{4.2}$$

The mapping $\widetilde{v}$ is $\mathrm{SO}(n)$-equivariant in the sense that $\widetilde{v}(A\,C) = C^{-1}\,\widetilde{v}(A)$ for every $C \in \mathrm{SO}(n)$, and every $\mathrm{SO}(n)$-equivariant mapping from $\mathrm{OF}_x\,Q$ to $\mathbf{R}^n$ is of the form $\widetilde{v}$ for a unique $v \in \mathrm{T}_x\,Q$.

In this way a vector field v on Q is associated to an $\mathrm{SO}(n)$-equivariant mapping $\widetilde{v} : \mathrm{OF}(Q) \to \mathbf{R}^n$, and the covariant derivative $\nabla_v w$ of a vector field w with respect to a vector field w can now be defined by

$$\widetilde{\nabla_v w} = v_{\mathrm{hor}}\,\widetilde{w}. \tag{4.3}$$

Here the right hand side denotes the "ordinary" derivative of the $\mathbf{R}^n$-valued function $\widetilde{w}$ on $\mathrm{OF}(Q)$ in the direction of the horizontal lift v_{hor} in $\mathrm{OF}(Q)$ of the vector field v.

When v is constantly equal to e_i, then the k-th component of the right hand side in (4.3) is, in view of (3.2) and (4.1), equal to

$$\begin{aligned}
&\frac{\partial}{\partial x^i} g(x)\,[A\,(e_k)\,,\, w(x)] - \frac{1}{2}\,g(x)\,\left[g(x)^{-1}\,\partial_i g(x)\,A\,(e_k)\,,\, w(x)\right] \\
&= g(x)\,[A\,(e_k)\,,\, \partial_i w(x)] + \frac{1}{2}\,\left(\partial_i g(x)\right)[A\,(e_k)\,,\, w(x)] \\
&= g(x)\,\Big[A\,(e_k)\,,\, \partial_i w(x) + \frac{1}{2}\,g(x)^{-1}\,\partial_i g(x) w(x)\Big)].
\end{aligned}$$

In view of (1.4) this means that the covariant derivative is defined by the Christoffel symbols

$$\Gamma^k_{ij}(x) = \frac{1}{2}\,\sum_{l=1}^{n} g^{kl}(x)\,\partial_i g_{lj}(x), \tag{4.4}$$

or, in view of (1.1), by

$$\Gamma_{kij}(x) = \frac{1}{2}\,\partial_i g_{kj}(x). \tag{4.5}$$

This connection leaves the Riemannian structure g invariant in the sense that $\nabla g = 0$. In general it is not torsion-free.

It is also not coordinate-invariant, because the torsion is a linear combination of the first order derivatives of g, and the first theorem of Christoffel states that one can always find local coordinates such that all the first order derivatives of the Riemannian structure vanish at the given point. This is to be contrasted with the characterization of the Levi-Civita connection of g as the unique one which is torsion-free and leaves g invariant, cf. [5, Vol. 1, Ch. IV, Thm. 2.2].

Of course, the connection is invariant under arbitrary affine substitutions of variables. These can be used to bring, at any given point x, the Riemannian structure $g_{ij}(x)$ into the standard form δ_{ij}.

The equivalence between a) in Proposition 4.1 below and c), where ∇ is taken as the connection with Christoffel symbols (4.5), has been observed before by Shima [9].

Proposition 4.1 *Let Q be a connected open subset of $\mathbf{R}^n$ such that $\mathrm{H}^1(Q,\,\mathbf{R}) = 0$, and let g be a Riemannian structure on Q. Let $\mathcal{P}$ be the space of all positive definite symmetric $n \times n$-matrices. Let $\pi : \mathrm{GL}(n,\,\mathbf{R})^+ \to \mathcal{P}$ be the $\mathrm{SO}(n)$--bundle with connection as defined in Section 3. Let ∇ be the connection in $\mathrm{T}(Q)$ which is associated to the pull-back under $g : Q \to \mathcal{P}$ of the connection of the $\mathrm{SO}(n)$-bundle $\pi : \mathrm{GL}(n,\,\mathbf{R})^+ \to \mathcal{P}$.*

Then the Christoffel symbols of ∇ are given by (4.5). *Moreover, the following conditions* a), b), c) *are equivalent.*

a) *There exists a real-valued smooth function f on Q (uniquely determined modulo a polynomial of degree ≤ 1) such that, for every $x \in Q$ and $1 \leq i,\, j \leq n$, we have that $g_{ij}(x) = \partial_i \partial_j f(x)$.*

b) *The Levi-Civita connection of g is equal to ∇.*

c) *∇ is torsion-free.*

Proof a) $\Longrightarrow$ b) follows from (1.3) and (4.5). We have b) $\Longrightarrow$ c), because the Levi-Civita connection of any Riemannian structure is torsion-free.

For c) $\Longrightarrow$ a) we begin by observing that the condition that ∇ is torsion-free means that the Christoffel symbols in (4.5) satisfy the symmetry condition that $\Gamma_{kij}(x) = \Gamma_{kji}(x)$. This means that $\partial_i g_{kj}(x) = \partial_j g_{kj}(x)$, which in combination with the assumption that $\mathrm{H}^1(Q,\,\mathbf{R}) = 0$ is equivalent to the existence of a smooth function g_k on Q, such that $g_{ki}(x) = \partial_i g_k(x)$. From the symmetry $g_{ki}(x) = g_{ik}(x)$ we now obtain, again using that $\mathrm{H}^1(Q,\,\mathbf{R}) = 0$, the existence of a smooth function f on Q, such that $g_k(x) = \partial_k f(x)$, which in turn implies that $g_{ki}(x) = \partial_i g_k(x) = \partial_i \partial_k f(x)$. This proves c) $\Longrightarrow$ a). q.e.d.

The implication a) $\Longrightarrow$ b), for which the condition $\mathrm{H}^1(Q, \mathbf{R}) = 0$ is not needed, can be used in order to give an "explanation" of the formula (1.7) for the curvature of a Hessian Riemannian structure. For this purpose we use (1.5), and several times (4.3), in order to obtain that

$$\widetilde{R(X, Y)\, U} = ([X_{\mathrm{hor}}, Y_{\mathrm{hor}}] - [X, Y]_{\mathrm{hor}})\, \widetilde{U}.$$

Because of (4.2), we have that, for any $C \in \mathfrak{h} = \mathfrak{o}(n)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \widetilde{U}\left(A\, \mathrm{e}^{t\, C}\right)|_{t=0} = -C\, A^{-1} U.$$

Therefore, also using (2.16), we obtain that

$$R(X, Y) = A\, \Omega_A \left(X_{\mathrm{hor}, A}, Y_{\mathrm{hor}, A}\right)\, A^{-1}. \tag{4.6}$$

The curvature form in $\mathrm{OF}(Q)$ is equal to the pull-back under g of the curvature form in $\mathrm{GL}(n, \mathbf{R})^+$. This means that if in the left hand side we take $X = e_i$ and $Y = e_j$, then we can take in the right hand side for Ω the curvature form in $\mathrm{GL}(n, \mathbf{R})^+$, but with $X = \partial_i g$ and $Y = \partial_j g$. In view of (3.3), the right hand side of (4.6), with Ω equal to the curvature form in $\mathrm{GL}(n, \mathbf{R})^+$, is equal to

$$\begin{aligned} & -\frac{1}{4}\, A\, {}^{\mathrm{t}}\!A\, \left(X\, A\, {}^{\mathrm{t}}\!A\, Y - Y\, A\, {}^{\mathrm{t}}\!A\, X\right) \\ = \;& -\frac{1}{4}\, g(x)^{-1}\, \left(X\, g(x)^{-1} Y - Y\, g(x)^{-1}\, X\right), \end{aligned}$$

where we have used that $g(x) = \pi(A) = {}^{\mathrm{t}}A^{-1}\, A^{-1}$. Because in these formulas $X_{pq} = \partial_i g_{pq} = \partial_i \partial_p \partial_q f$ and $Y_{pq} = \partial_j g_{pq} = \partial_j \partial_p \partial_q f$, this exhibits (1.7) as a consequence of the formula (3.3) for the curvature of the connection of the $\mathrm{SO}(n)$-bundle $\pi : \mathrm{GL}(n, \mathbf{R})^+ \to \mathcal{P}$.

In particular this explains why in (1.7) no fourth order derivatives of f occur: the pull-back under the mapping $g : Q \to \mathcal{P}$ involves only first order derivatives of g, or third order derivatives of f, and the curvature in $\mathrm{GL}(n, \mathbf{R})^+$ is given algebraically, by just taking the commutator of two matrices.

5 Curvature and Umbilic Points

In this section we assume that $n = 2$. We begin with a description of the (scalar) curvature $K(x)$ of a Hessian Riemannian structure in terms of a universal two-form ω on the space $\mathcal{P}$ of all positive definite symmetric 2×2-matrices.

When $n = 2$, the rotation group $\mathrm{SO}(n) = \mathrm{SO}(2)$ is commutative, and its Lie algebra consists of the scalar multiples of $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. As observed in Remark ii) at the end of Section 2, it follows that the curvature form Ω on $G = \mathrm{GL}(2, \mathbf{R})^+$ is equal to $\pi^* \omega$ for a unique G-invariant two-form ω on the three-dimensional space $\mathcal{P}$ of all positive definite symmetric 2×2-matrices.

If a two-form μ on $\mathcal{P}$ is G-invariant, then the two-form μ_1 on $\mathrm{T}_1 \mathcal{P} = \mathfrak{s}$, the space of all symmetric 2×2-matrices, is $\mathrm{SO}(2)$-invariant, where the actions of $C \in \mathrm{SO}(2)$ on $\mathfrak{s}$ is given by $S \mapsto C\,S\,C^{-1}$. Conversely, any SO(2)-invariant two-form on $\mathrm{T}_1\,\mathcal{P} = \mathfrak{s}$ has a unique extension to a G-invariant two-form on $\mathcal{P}$.

The three-dimensional space $\mathfrak{s}$ has an SO(n)-invariant splitting

$$\mathfrak{s} = \mathfrak{s}_0 + \mathbf{R}\,I, \tag{5.1}$$

in which $\mathfrak{s}_0 = \{S \in \mathfrak{s} \mid \operatorname{tr} S = 0\}$ denotes the two-dimensional linear subspace of all traceless symmetric 2×2-matrices and $\mathbf{R}\,I$ denotes the one-dimensional subspace of all multiples of the identity matrix. The action of SO(2) on $\mathfrak{s}_0$ is nontrivial, whereas the action on the one-dimensional component $\mathbf{R}\,I$ is trivial. If the two-form μ_1 on $\mathfrak{s}$ is nonzero and SO(2)-invariant, then its kernel is one--dimensional and SO(2)-invariant, and therefore has to be equal to $\mathbf{R}\,I$. But then μ_1 is determined by its restriction to $\mathfrak{s}_0$. Because $\dim \mathfrak{s}_0 = 2$, it follows that μ_1, and therefore μ, is uniquely determined up to a multiplicative constant.

A natural candidate for an SO(2)-invariant two-form ω_1 on $\mathfrak{s}$ is given by the equation

$$-\frac{1}{4}\,[X,\,Y] = \omega_1(X,\,Y)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X,\,Y \in \mathfrak{s}, \tag{5.2}$$

or, more explicitly, by

$$4\,\omega_1\left(\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix},\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{22} \end{pmatrix}\right) = (X_{11} - X_{22})\,Y_{12} - X_{12}\,(Y_{11} - Y_{22})\,. \tag{5.3}$$

In view of (3.3), we arrive at the following conclusion.

Proposition 5.1 *Assume that ω is the $\mathrm{GL}(2,\,\mathbf{R})^+$-invariant two-form on $\mathcal{P}$ which is determined by* (5.3). *Then the curvature form Ω on the* SO(2)*-bundle $\pi : \mathrm{GL}(2,\,\mathbf{R})^+ \to \mathcal{P}$ is given by $\Omega = \pi^*\omega$.*

Assume that $g : Q \to \mathcal{P}$ is a Hessian Riemannian structure, $g_{ij}(x) = \partial_i\partial_j f(x)$, and $x \in Q$ is a point where $g_{ij}(x) = \delta_{ij}$. If we combine (1.8) with (4.6) where $A = 1$, and the fact that the curvature in $\mathrm{OF}(Q)$ is equal to the pull-back under g of the curvature in $\mathrm{GL}(2,\,\mathbf{R})^+$, we obtain that the scalar curvature $K(x)$ is given by

$$K(x) = \omega_1\left(\partial_2 g(x),\,\partial_1 g(x)\right), \quad \text{when} \quad g_{ij}(x) = \delta_{ij}. \tag{5.4}$$

In order to evaluate the right hand side of (5.4), we have to substitute $X_{11} = {\partial_1}^2\partial_2 f(x)$, $X_{12} = \partial_1{\partial_2}^2 f(x)$, $X_{22} = {\partial_2}^3 f(x)$, and $Y_{11} = {\partial_1}^3 f(x)$, $Y_{12} = {\partial_1}^2\partial_2 f(x)$, $Y_{22} = \partial_1{\partial_2}^2 f(x)$ in (5.1). The resulting quantity is equal to the right hand side (1.9), with $a = 1$, $b = 0$, $c = 1$.

In order to explain the relation between the curvature $K(x)$ and umbilic points of certain surfaces in $\mathbf{R}^3$, we start by recalling what umbilic points are. Let S be an oriented smooth surface in $\mathbf{R}^3$. For any $p \in S$ one can describe S in a neighborhood of p as the set of $p + v + \varphi(v)\,n(p)$, where v varies in a neighborhood U of the origin in the tangent plane $\mathrm{T}_p\,S \subset \mathbf{R}^3$ of S at p, $n(p)$ is the normal of S at p defined by the orientation, and φ is a smooth real-valued function on U. We have $\mathrm{d}\varphi(0) = 0$, and the Hessian of φ at 0 is a well-defined symmetric bilinear form $h(p)$ on $\mathrm{T}_p\, S$, which is called the *second fundamental form of S at the point p*. The restriction to $\mathrm{T}_p\, S$ of the standard inner product of $\mathbf{R}^3$ defines a positive definite symmetric bilinear form $g(p)$ on $\mathrm{T}_p\, S$, which is called the *first fundamental form of S at the point p*. Viewing a bilinear form on $\mathrm{T}_p\, S$ as a linear mapping from $\mathrm{T}_p\, S$ to its dual space $\mathrm{T}_p^*\, S$, we have $h(p) = g(p) \circ H(p)$ for a uniquely defined linear mapping $H(p) : \mathrm{T}_p\, S \to \mathrm{T}_p\, S$, which moreover is symmetric with respect to $g(p)$. It follows that $H(p)$ has two real eigenvalues $\kappa_1(p)$ and $\kappa_2(p)$, which are called the *principal curvatures of S at the point p*. One can always arrange, as we will, that $\kappa_1(p) \geq \kappa_2(p)$.

Gauss defined the curvature $K_S(p)$ of a surface S at the point p by $K_{\mathrm{Gauss}}(p) = \det H(p) = \kappa_1(p)\, \kappa_2(p)$. His *Theorema Egregium* states that this curvature, which is defined in terms of the immersion of the surface S into $\mathbf{R}^3$, can be expressed entirely in terms of the first fundamental form. More precisely, K_{Gauss} is equal to the curvature K, defined in (1.8), if we take the first fundamental form as the Riemannian structure on S.

When $\kappa_1(p) > \kappa_2(p)$, then we have the eigenspaces $L_i(p)$ of $H(p)$, which are one-dimensional linear subspaces of $\mathrm{T}_p\, S$, orthogonal to each other with respect to $g(p)$. The $L_i(p)$ are called the *lines of principal curvature of S at the point p*. When $\kappa_1(p) = \kappa_2(p)$, which happens if and only if $h(p)$ is a scalar multiple of $g(p)$, then the lines of principal curvature are not well-defined. In this case p is called an *umbilic point of the surfaces S*. The set U of umbilic points is a closed subset of S. In its complement the principal curvatures $\kappa_i(x)$ and the corresponding lines of principal curvature $L_i(x)$ depend smoothly on $x \in S \setminus U$.

If p is an isolated umbilic point of S, then the *index of the umbilic point p* is defined as $\frac{1}{\pi}$ times the increase of the angle of $L_1(x)$ or $L_2(x)$, when x runs once around p, along a small loop and in the positive direction with respect to the orientation of S. See Blaschke [1, p. 287]; the original definition of Hamburger [3] is slightly different. Hamburger [3] proved that if S is a compact oriented surface, immersed in $\mathbf{R}^3$ with only isolated umbilic points, then the sum of the indices of the umbilic points is equal to $2\,\chi = 2\,(2 - 2\,g)$, where χ and g denote the Euler characteristic and the genus of S, respectively.

For our purpose, of relating the curvature of a Hessian Riemannian structure to umbilic points, let us discuss the slightly more general situation that M is a two-dimensional oriented smooth manifold with a Riemannian structure g, provided with a second symmetric bilinear form $h(p)$ on every tangent space $\mathrm{T}_p\, M$, where we assume that $g(p)$ and $h(p)$ depend smoothly on p. For each $p \in M$, let $\mathrm{S}^2\, \mathrm{T}_p\, M$ denote the three-dimensional vector space of all symmetric bilinear forms on $\mathrm{T}_p\, M$. These spaces constitute a three-dimensional vector bundle $\mathrm{S}^2\, \mathrm{T}(M)$ over M, of which g and h are smooth sections. The multiples of $g(p)$ form a one-dimensional linear subspace of $\mathrm{S}^2\, \mathrm{T}_p\, M$, and we denote the two-dimensional quotient space $\mathrm{S}^2\, \mathrm{T}_p\, M / \mathbf{R}\, g(p)$ by $\mathrm{S}^2_0\, \mathrm{T}_p\, M$. This space can also be identified with the space of linear mappings $H : \mathrm{T}_p\, M \to \mathrm{T}_p\, M$ which are symmetric with respect to $g(p)$ and have trace equal to zero, whence the subscript 0. The section h of $\mathrm{S}^2\, \mathrm{T}(M)$ leads to a smooth section h_0 of $\mathrm{S}^2_0\, \mathrm{T}(M)$. The zeros of h_0 are the points p where $h(p)$ is equal to a multiple of $g(p)$.

Note that $h_0(M)$ and the zero section of $\mathrm{S}_0^2\,\mathrm{T}(M)$ are two-dimensional oriented submanifolds of $\mathrm{S}_0^2\,\mathrm{T}(M)$, and the zeros of h_0 correspond to the intersection points of $h_0(M)$ with the zero section.

In order to provide $\mathrm{S}_0^2\,\mathrm{T}(M)$ with an orientation, we use the two-form ω_p on $\mathrm{S}_0^2\,\mathrm{T}_p\,M$ which in analogy to (5.2) is defined by

$$-\frac{1}{4}\,[X,\,Y] = \omega_p(X,\,Y)\,J_p. \tag{5.5}$$

Here X and Y are two traceless $g(p)$-symmetric linear mappings from $\mathrm{T}_p\,M$ to itself, and J_p is the $g(p)$-rotation over $\frac{\pi}{2}$ in $\mathrm{T}_p\,M$, where the direction is determined by the given orientation of $\mathrm{T}_p\,M$. At a zero p of h_0, one has a well-defined tangent mapping of h_0, which is a linear mapping $\mathrm{T}_p\,h_0$ from $\mathrm{T}_p\,M$ to $\mathrm{S}_0^2\,\mathrm{T}_p\,M$. The pull-back of ω_p under $\mathrm{T}_p\,h_0$ is a two-form on $\mathrm{T}_p\,M$, and we have a real number $\Delta(p)$ such that

$$(\mathrm{T}_p\,h_0)^*\,\omega_p = \Delta(p)\,\alpha_p, \tag{5.6}$$

where α_p denotes the area form on $\mathrm{T}_p\,M$ defined by the inner product $g(p)$ and the given orientation of $\mathrm{T}_p\,M$. The number $\Delta(p)$ can be viewed as a sort of Jacobi-determinant of h_0 at the point p.

The zero p of h_0 is called *simple* if the intersection of $h_0(M)$ and the zero section at p is transversal. This condition is equivalent to the condition that the linear mapping $\mathrm{T}_p\,h_0$ from $\mathrm{T}_p\,M$ to $\mathrm{S}_0^2\,\mathrm{T}_p\,M$ is invertible, which in turn is equivalent to the condition that $\Delta(p) \neq 0$. When h is the second fundamental form of a surface S in $\mathbf{R}^3$, then p is called a *simple umbilic point of S* if h_0 has a simple zero at p.

If M is compact then, with the orientation of $\mathrm{S}_0^2\,\mathrm{T}(M)$ introduced above, we have a well-defined topological inetrsection number of $h_0(M)$ and the zero section. (This number changes sign if we change the orientation of the fibers of $\mathrm{S}_0^2\,\mathrm{T}(M)$.) I learned the following version of Hamburger's theorem from the Ph.D. thesis of Carlos Valero in Oxford.

Proposition 5.2 *When p is a simple zero of h_0, then the index of p is equal to $+1$ and -1 if $\Delta(p) > 0$ and $\Delta(p) < 0$, respectively. For compact M, the topological intersection number of $h_0(M)$ with the zero section of $\mathrm{S}_0^2\,\mathrm{T}(M)$ equals twice the Euler characteristic of M.*

Proof We start with a computation where $M = \mathbf{R}^2$ and g is equal to the standard (Euclidean) Riemannian structure. If $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and C is the rotation over the angle θ, then $C\,A\,C^{-1}$, which has $C\,(e_1)$ and $C\,(e_2)$ as eigenvectors, is equal to

$$C\,A\,C^{-1} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

Therefore the mapping

$$h_0 : x \mapsto \begin{pmatrix} x^1 & x^2 \\ x^2 & -x^1 \end{pmatrix}$$

has index equal to $+1$ at $x = 0$. On the other hand $\Delta(0) = \frac{1}{2}$. An interchange of x^1 and x^2 leads to a change of sign both in the index and in $\Delta(0)$. Because both the index and the sign of $\Delta(0)$ are invariant under homotopy within the class of sections h_0 with a simple zero, this leads to a proof of the first statement in the proposition.

The above computation of $C\,A\,C^{-1}$ exhibits $\mathrm{S}^2\,\mathrm{T}(M)$, when viewed as a complex line bundle over M, as the square of $\mathrm{T}(M)$, and therefore its Chern class is equal to $2\chi(M)$, because the Chern class of $\mathrm{T}(M)$ is equal to the Euler characteristic of M. On the other hand, the Chern class of a complex line bundle is equal to the topological intersection number of any continuous section of it with the zero section, cf. [2, Thm. 11.17 and (20.10.6)]. q.e.d.

Proposition 5.3 *Let f be a strongly convex smooth function on an open neighborhood Q of the point x in $\mathbf{R}^2$ and let $K(x)$ denote the scalar curvature at the point x of the Riemannian structure which is defined by the hessian of f. After substracting a polynomial of degree ≤ 1 from f (which does not change the Hessian Riemannian structure) and after a suitable affine substitution of variables (which does not change $K(x)$) we can arrange that $\mathrm{d}f(x) = 0$ and $\partial_i \partial_j f(x) = \delta_{ij}$.*

In this situation, the graph of f, viewed as a surface S in $\mathbf{R}^3$, has an umbilic point at $p = (x,\, f(x))$, and the claim is that $K(x) = -\Delta(p)$. This implies that $K(x) \neq 0$ if and only if the umbilic point p of S is simple. Furthermore, it follows from Proposition 5.2 that if $K(x) < 0$ and $K(x) > 0$, then the index of the umbilic point p of S is equal to $+1$ and -1, respectively.

Proof We may assume that $x = 0$, $f(0) = 0$, $\mathrm{d}f(0) = 0$ and $\partial_i \partial_j f(0) = \delta_{ij}$. In order to compute $\mathrm{T}_p\, h_0$, where h denotes the second fundamental form of the surface S, we write $y = \epsilon\, e_k$ with small $|\epsilon|$, and use a rotation R_ϵ in $\mathbf{R}^3$ which turns the tangent space at $(y,\, f(y))$ of S horizontal. This can be done with a family of rotations R_ϵ such that $R_\epsilon = I + \epsilon\, R' + O\,(\epsilon^2)$, for some anti-symmetric 3×3-matrix R'. Then $R_\epsilon(S)$ is equal to the graph of a function f_ϵ, which plays the role of the function φ in the definition of the second fundamental form. If

$$R_\epsilon(y,\, f(y)) = (x_\epsilon,\, f_\epsilon\,(x_\epsilon))\,,$$

then f_ϵ has a critical point at x_ϵ.

The function f_ϵ is determined by the equation

$$R_\epsilon(x,\, f(x))^3 = f_\epsilon\left(R_\epsilon(x,\, f(x))^1,\, R_\epsilon(x,\, f(x))^2\right),$$

where the superscripts denote the coordinate indices. Differentiating this equation twice with respect to x and collecting the linear terms in ϵ, one arrives at the conclusion that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} \partial_i \partial_j f_\epsilon\,(x_\epsilon)\,|_{\epsilon=0} = \partial_k \partial_i \partial_j f(0).$$

It follows therefore now from (5.6) that

$$\Delta(0) = \omega_0 \left(\partial_1 g(0), \, \partial_2 g(0)\right),$$

if $g(x)$ denotes the Hessian of the function f at the point x. Because (5.4) implies that

$$K(0) = \omega_0 \left(\partial_2 g(0), \, \partial_1 g(0)\right) = -\Delta(0),$$

the proof of the proposition is complete. q.e.d.

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