On the Boundary Behaviour of the Riemannian Structure of a Self-Concordant Barrier Function

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1 Introduction

Following Vinberg [18], we will define a *convex domain* in $\mathbf{R}^n$ as an open convex subset of $\mathbf{R}^n$ which does not contain a full straight line. A self-concordant barrier function for a convex domain Q is defined as a strongly convex smooth function f on Q, which tends to ∞ at the boundary and satisfies the estimates (iii) and (iv) of Definition 2.1 below, for the derivatives of f up to the third order. The strong convexity of f means that the Hessian $g_{ij}(x) = \partial_i \partial_j f(x)$ is positive definite and therefore defines a Riemannian structure on Q. (Such Riemannian structures on convex domains have been studied already by Koszul [13] and Vinberg [18], who refer further back to the theory of bounded domains in $\mathbf{C}^n$, with its Bergmann metric.) In this paper we investigate the asymptotic behaviour of this Riemannian structure, and of its geodesics and its curvature, near points of the boundary where the boundary is smooth and strongly convex, which means that its curvature, described by its second fundamental form, is positive definite.

In order to obtain good asymptotic expansions near such points, we introduce Assumption 2.1 below about the behaviour of the function f near the boundary, and argue that these assumptions are quite natural, cf. Remark 2.3.

Under these assumptions we will show in Section 5 that, after suitable reparametrization, the geodesics, near points of the boundary where the boundary is smooth and strongly convex, extend to a smooth family of smooth curves which cross the boundary in arbitrary directions, where the tangent ones curve out of Q. We will also compare the geodesics with the gradient curves of linear functions with respect to the Hessian Riemannian structure. These have a similar behaviour near the boundary, with the difference that the gradient curves which are tangent to the boundary remain in the boundary. See Proposition 5.1 for more details.

In Section 6 we draw some conclusions about the global behaviour of the geodesics and the aforementined gradient curves, in the case that Q is bounded and ∂Q is a smooth and strongly convex hypersurface in $\mathbf{R}^n$ (which automatically is compact and diffeomorphic to the $(n-1)$dimensional sphere). In this case all geodesics $\gamma(t)$ in Q converge for $t \to \pm\infty$ to a point $\gamma(\pm\infty) \in \partial Q$. Furthermore, for every $x \in Q$ and $y_+ \in \partial Q$ there exists a geodesic γ such that $\gamma(0) = x$ and $\gamma(+\infty) = y_+$. Also, for every $y_\pm \in \partial Q$ such that $y_- \neq y_+$ there exists a geodesic γ in Q such that $y_\pm = \gamma(\pm\infty)$. In contrast, the gradient curves $\delta(t)$ for linear functions which for $t \to -\infty$ converge to a given point $y_- \in \partial Q$ all are defined by the same linear function ξ which attains its minimum on ∂Q at y_-, and therefore all converge for $t \to \infty$ to the same, "opposite" point $y_+ \in \partial Q$ where ξ attains it maximum on ∂Q. See Theorem 6.1 for more details.

In Section 7 we prove that, still under the Assumption 2.1, all the sectional curvatures of the Hessian Riemannian structure converge to $-1/4$ if one approaches the boundary where the boundary is smooth and strongly convex.

In Section 8 we discuss the simple examples of the parabolic domain, the ball, the corner, and the triangle. The example of the corner in Section 8 shows that near points where the boundary is not smooth or not strongly convex, the behaviour can be (and probably always is) very different. The impression is that the boundary catches geodesics roughly in proportion to the Gaussian curvature at the boundary points. The sectional curvature is constant equal to $-\frac{1}{4}$ for the parabolic domain, negative for the ball except at the origin, equal to zero for the corner, and positive for the triangle.

2 The Assumption

Throughout this paper we will abbreviate the partial derivative $\partial f(x)/\partial x^i$ of a function f at the point x by means of $\partial_i f(x)$, such that the corresponding function of x is denoted by $\partial_i f$. The total derivative of f is denoted by $\mathrm{d}f$ or f' and second and third order total derivatives by f'' and f''', respectively.

Definition 2.1 A *self-concordant barrier function* for a convex domain Q in $\mathbf{R}^n$ is a smooth, real-valued function f on Q, which satisfies the following conditions, cf. Nesterov and Nemirovski [14].

(i) f is strongly convex on Q, which means that for every $x \in Q$ the Hessian $\partial_i \partial_j f(x)$, $1 \leq i,\, j \leq n$, of f at the point x is a positive definite symmetric matrix.

(ii) $f(x) \to \infty$ when $x \in Q$ converges to a point $y \in \partial Q$ of the boundary ∂Q of Q in $\mathbf{R}^n$.

(iii)

$$C_1(f) := \sup_{x \in Q,\, v \in \mathbf{R}^n,\, v \neq 0} f'(x)(v)^2 / f''(x)(v,\, v) < \infty.$$

(iv)

$$C_2(f) := \sup_{x \in Q,\, v \in \mathbf{R}^n,\, v \neq 0} \left[\tfrac{1}{2} f'''(x)(v,\, v,\, v)\right]^2 / f''(x)(v,\, v)^3 < \infty.$$

It is clear that if f is a self-concordant barrier function for Q and c is a strictly positive constant, then $c\, f$ is a self-concordant barrier function for Q and $C_1(c\, f) = c\, C_1(f)$, whereas $C_2(c\, f) = c^{-1}\, C_2(f)$. The number

$$\vartheta(f) := C_1(f)\, C_2(f)$$

is called the *parameter* of the barrier function f. cf. [14, Def. 2.3.1, Def. 2.1.1 and formula (2.2.1)]. One always has that $\vartheta(f) \geq 1$, and $\vartheta(f) \geq k$ if Q has boundary points in the neighborhood of which ∂Q is described by k independent linear equations, cf. [14, Remark 2.3.1 and Proposition 2.3.6]. $\oslash$

If f is a self-concordant barrier for Q, the we will write $\phi := \mathrm{e}^{-f}$, or equivalently $f = -\ln \phi$. Then $\phi(x) > 0$ for every $x \in Q$ and, because $f(x)$ tends to ∞ as $x \in Q$ tends to a boundary point of Q, ϕ extends to a continuous function on $\overline{Q}$ which is equal to zero on ∂Q. Here $\overline{Q} = Q \cup \partial Q$ and ∂Q denote the closure and the boundary of Q in $\mathbf{R}^n$, respectively. We denote the continuous extension of ϕ to $\overline{Q}$ also by ϕ.

The term *smooth* in this article means arbitrarily often differentiable. Smoothness up to the boundary means that all derivatives have limits if one approaches the boundary. For all applications smoothness can be replaced by the the condition that sufficiently many derivatives exist and are continuous up to the boundary. However, this would require a constant bookkeeping of the number of continuous derivatives, which would complicate the already quite technical presentation even further.

Let ∂Q be smooth in a neighborhood of the point $z \in Q$. By means of a suitable affine substitution of variables, one can arrange that $z = 0$, $\mathrm{T}_z(\partial Q) = \mathbf{R}^{n-1} \times \{0\}$ and then there exists a smooth real-valued function h of $n-1$ variables in an open neighborhood of the origin in $\mathbf{R}^{n-1}$, such that for x near z we have that $x \in Q$ if and only if

$$x^n > h\left(x^1, \ldots, x^n\right).$$

We say that Q is *strongly convex* at $z \in \partial Q$ if the Hessian $h''(0)$ of h at 0, which always is positive definite when Q is convex, actually is positive definite. This condition is independent of the choice of the affine substitution of variables.

Assumption 2.1 U is an open subset of $\mathbf{R}^n$, such that $U \cap \partial Q$ is a smooth and strongly convex hypersurface in U. We take $f = -\ln \phi$, where ϕ is a real-valued strictly positive function on $U \cap Q$, which is smooth up to $U \cap \partial Q$. The smooth extension of ϕ to $U \cap \overline{Q}$ is also denoted by ϕ. For every $y \in U \cap \partial Q$ we have that $\phi(y) = 0$ and the total derivative $\phi'(y) = \mathrm{d}\phi(y)$ of ϕ at y is not equal to zero. $\oslash$

Remark 2.1 For the concrete examples in the book [14] of self-concordant barrier functions, Assumption 2.1 holds near every boundary point where the boundary is a smooth and strongly convex hypersurface.

For the "universal barrier function" of [14, Section 2.5.1] they hold with an additional term in the function e^{-f}, cf. Proposition 3.1 below. When n

is sufficiently large, then the additional term is of such low order that it does not invalidate the results in the later sections of this paper, if smoothness is replaced by differentiability of order roughly equal to $\frac{n}{2} - 1$. It appears however that even in the worst case $n = 2$ the conclusions of Theorem 6.1 and Theorem 7.1 remain valid, cf. Remark 5.5, Remark 6.5 and Remark 7.3.

Moreover, if Q is bounded and ∂Q is a smooth and strongly convex hypersurface in $\mathbf{R}^n$, then the asymptotic description in Proposition 3.1 yields that by means of an asymptotically small modification near the boundary the universal barrier function can be changed into a self-concordant barrier function which satisfies Assumption 2.1. See Corollary 4.3 for the precise statement. $\oslash$

Remark 2.2 Assumption 2.1 holds with ϕ replaced by ψ if and only if $\psi = \lambda\,\phi$, where λ is smooth and strictly positive on $U \cap \overline{Q}$. In particular $-\ln\psi = -\ln\phi - \ln\lambda$, where $\ln\lambda$ is smooth on $U \cap \overline{Q}$, which implies that all derivatives of $\ln\lambda$ are bounded on every compact subset of $U \cap \overline{Q}$. This can be viewed as an illustration of how strong Assumption 2.1 really is. $\oslash$

Remark 2.3 It is not true that for every self-concordant barrier function f the derivatives have expansions in negative powers of the distance to the boundary. For instance, suppose that f satisfies Assumption 2.1. Then $\widetilde{f}(x) = f(x) + A\,\sin f(x)$ defines a self-concordant barrier function if $|A|\sqrt{1 + C_1(f)^2} < 1$. However, the m-th order derivatives of $\widetilde{f}$, multiplied with the m-th power of the distance to the boundary, are bounded but exhibit infinite oscillatory behaviour, without having a limit, when x tends to a boundary point. Such oscillatory behaviour would complicate the asymptotic analysis of the Riemannian structure near the boundary considerably. $\oslash$

3 The Universal Barrier Function

Let Q be a convex domain in $\mathbf{R}^n$. For each $x \in Q$, the bounded convex subset

$$Q^*(x) := \{\xi \in \mathbf{R}^n \mid y \in Q \Longrightarrow \langle y - x,\, \xi\rangle \leq 1\} \tag{3.1}$$

of the dual space is called the *polar set of Q with respect to the point x*. Let $I(x)$ denote the n-dimensional volume of $Q^*(x)$. (It follows from (8.27) that, up to a constant factor, the function $I(x)$ is equal to the *characteristic function* of Q as defined by Vinberg [18, Def. 10, p. 356].) According to [14, Thm. 2.5.1], the function $x \mapsto \ln I(x)$ is a self-concordant barrier function for Q, with parameter $\leq C\,n$, where C is a universal constant.

Proposition 3.1 *Let $I(x) = I_Q(x)$ denote the n-dimensional volume of $Q^*(x)$ and define $f(x) = f_Q(x) = \frac{2}{n+1}\ln I(x)$. For $x \in Q$ near points of the boundary ∂Q where ∂Q is smooth and strongly convex, we have the following conclusions.*

a) *If $n = 1$ then f satisfies Assumption 2.1.*

b) *If n is even then*

$$\phi(x) := \mathrm{e}^{-f(x)} = \alpha(x) \left[1 + \alpha(x)^{\frac{n+1}{2}} \beta(x) \right]^{-\frac{2}{n+1}},$$

where the functions α and β are smooth up to the boundary. Moreover, $\alpha = 0$ on ∂Q, $\alpha > 0$ in Q and $\mathrm{d}\alpha(y) \neq 0$ for every $y \in \partial Q$ where ∂Q is smooth and strongly convex.

c) *If n is odd and $n \geq 3$ then*

$$\phi(x) := \mathrm{e}^{-f(x)} = \alpha(x) \left[1 + \alpha(x)^{\frac{n+1}{2}} \beta(x) \left(\ln \phi(x) + \gamma(x) \right) \right]^{-\frac{2}{n+1}},$$

where the functions α, β and γ are smooth up to the boundary. Moreover, $\alpha = 0$ on ∂Q, $\alpha > 0$ in Q and $\mathrm{d}\alpha(y) \neq 0$ for every $y \in \partial Q$ where ∂Q is smooth and strongly convex.

Proof If $n = 1$, then it is easily verified that f satisfies Assumption 2.1. Therefore we assume from now on that $n \geq 2$.

Define the *supporting function* $p = p_Q : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ of Q by

$$p(\eta) := \sup_{y \in Q} \langle y, \eta \rangle, \quad \eta \in \mathbf{R}^n, \tag{3.2}$$

where $p(\eta) = \infty$ if the linear form $y \mapsto \langle y, \eta \rangle$ is not bounded from above on Q. Let S^{n-1} denote the unit sphere in $\mathbf{R}^n$. Then we have for any $r \geq 0$ and $\eta \in \mathrm{S}^{n-1}$ that $r\,\eta \in Q^*(x)$, if and only if $(p(\eta) - \langle x, \eta \rangle)\, r \leq 1$. It follows that

$$\begin{aligned} I(x) &:= \mathrm{vol}_n\left(Q^*(x)\right) = \int_{\mathrm{S}^{n-1}} \int_0^{1/(p(\eta) - \langle x, \eta \rangle)} r^{n-1}\, \mathrm{d}r\, \mathrm{d}_{n-1}\eta \\ &= \int_{\mathrm{S}^{n-1}} \tfrac{1}{n} \left(p(\eta) - \langle x, \eta \rangle \right)^{-n} \mathrm{d}_{n-1}\eta, \end{aligned} \tag{3.3}$$

where the integrand is taken to be equal to zero when $p(\eta) = \infty$.

Now assume that x is close to $y_0 \in \partial Q$ and that ∂Q is smooth and strongly convex near y_0. Write $\nu(y)$ for the exterior normal at points $y \in \partial Q$ near y_0. Then the positive number $p(\eta) - \langle x, \eta \rangle$ can only be small when η is close to $\eta_0 := \nu(y_0)$. Let ψ be a smooth function on S^{n-1} which is equal to 1 in a neighborhood of η_0 and equal to zero outside a somewhat larger neighborhood V of η_0, and write

$$I_\psi(x) := \tfrac{1}{n} \int_{\mathrm{S}^{n-1}} \left(p(\eta) - \langle x, \eta \rangle \right)^{-n} \psi(\eta)\, \mathrm{d}_{n-1}\eta. \tag{3.4}$$

Because $p(\eta) - \langle x, \eta \rangle$ is bounded away from zero when $\eta \in \mathrm{S}^{n-1} \setminus V$, the function $I - I_\psi$ is smooth up to the boundary.

The mapping $\nu : y \to \nu(y)$, which is sometimes called the *Gauss map* of the smooth hypersurface ∂Q near y_0, is a smooth diffeomorphism from $U \cap \partial Q$ onto an open subset of S^{n-1}. Its Jacobi determinant $\kappa(y)$ at the point $y \in U \cap \partial Q$ is called the *Gauss curvature* of ∂Q at y. The function κ is strictly positive and smooth on an open neighborhood of y_0 in ∂Q.

In the sequel we will arrange that V is contained in the open subset $\kappa(U \cap \partial Q)$ of S^{n-1}. If $\eta = \nu(y)$, for $y \in U \cap \partial Q$, we have that $p(\eta) = \langle y, \eta \rangle$. If we apply the substitution of variables $\eta = \nu(y)$ to the integral (3.4), then we obtain that

$$I_\psi(x) = \frac{1}{n} \int_{U \cap \partial Q} \langle y - x, \nu(y) \rangle^{-n} \kappa(y)\, \psi(\nu(y))\, \mathrm{d}_{n-1} y. \tag{3.5}$$

It is an application of the implicit function theorem that the points $x \in Q$ close to y_0 can be written as $x = z - \delta\, \nu(z)$, for a unique $z = z(x) \in \partial Q$ close to y_0 and a unique $\delta = \delta(x) > 0$, where the functions $x \mapsto z(x)$ and $x \mapsto \delta(x)$ are smooth up to the boundary ∂Q. With the substitution $x = z - \delta\, \nu(z)$, we have that

$$\langle y - x, \nu(y) \rangle = (\delta + \theta(y, z))\, \langle \nu(z), \nu(y) \rangle,$$

where

$$\theta(y, z) := \langle y - z, \nu(y) \rangle / \langle \nu(z), \nu(y) \rangle. \tag{3.6}$$

Note that $\langle \nu(z), \nu(y) \rangle = 1$ when $y = z$. Furthermore, $\theta(z) = 0$ and $\mathrm{d}\theta(z) = 0$.

In order to compute the Hessian of $y \mapsto \theta(y, z)$ at the point $y = z$, we parametrize ∂Q near z by means of the substitution $y = z + u - h(u)\, \nu(z)$, where u varies in a small open neighborhood of the origin in the tangent space $\mathrm{T}_z(\partial Q)$ of ∂Q at the point z. Here h is a smooth real-valued function, $h(0) = 0$, $h'(0) = 0$ and the Hessian $H(z) := h''(0)$ of h at the origin is a positive definite symmetric bilinear form on $\mathrm{T}_z(\partial Q)$. Classically, $H(z)$ is called the *second fundamental form* at z of the hypersurface ∂Q, and we note that

$$\kappa(z) = \det H(z). \tag{3.7}$$

For the computation of $\theta(y, z)$ it is convenient to arrange by means of a rigid motion, that $z = 0$ and $\mathrm{T}_z(\partial Q) = \mathbf{R}^{n-1} \times \{0\} \simeq \mathbf{R}^{n-1}$. Then h can be viewed as a function on an open neighborhood of the origin in $\mathbf{R}^{n-1}$ and $y \in \partial Q$ is parametrized by $u \in \mathbf{R}^{n-1}$ by means of $y = (u, h(u))$. In this situtation, we have that

$$\nu(y) = \left(1 + \|h'(u)\|^2\right)^{-1/2} (\mathrm{d}h(u), -1),$$

and it follows that

$$\theta(u, h(u), z) = \langle u, h'(u) \rangle - h(u).$$

Because

$$\begin{aligned} h'(u) &= h''(0)(u) + \mathcal{O}\left(\|u\|^2\right), \\ h(u) &= \tfrac{1}{2} h''(u, u) + \mathcal{O}\left(\|u\|^3\right), \end{aligned}$$

we arrive at the conclusion that

$$\frac{\partial^2 \theta(y, z)}{\partial y^2}\Big|_{y=z} = \tfrac{1}{2} H(z), \tag{3.8}$$

which is positive definite.

The Morse lemma with parameters of Hörmander [11, Lemma 3.2.3] implies that there exists a smooth substitution of variables $y = y(v, z)$, depending smoothly on the parameters z, such that $\theta(y(v, z), z) = \|v\|^2$. We have

$$j(z) := \det \frac{\partial y(v, z)}{\partial v}\Big|_{v=0} = \det \left[\tfrac{1}{2} H(z)\right]^{-1/2} = 2^{1-n}\, \kappa(x)^{-1/2}, \tag{3.9}$$

where the last identity follows from (3.7).

The substitution of variables $y = y(v, z)$ in the integral (3.5) leads to

$$I_\psi(x) = \int_{\mathbf{R}^{n-1}} \left(\delta + \|v\|^2\right)^{-n} a(v, z)\; \mathrm{d}_{n-1} v, \tag{3.10}$$

where a is a smooth function of all variables, equal to zero when $\|v\| \geq C$ for a suitable positive constant C, and satisfies

$$a(0, z) = \tfrac{1}{n}\, 2^{1-n}\, \kappa(z)^{1/2}. \tag{3.11}$$

With the substitution $v = r\, w$, where $r \geq 0$ and w on the $(n-2)$-dimensional unit sphere S^{n-2} in $\mathbf{R}^{n-1}$, the integral (3.10) transforms into

$$I_\psi(x) = \int_0^\infty \left(\delta + r^2\right)^{-n} r^{n-2} A(r, z)\; \mathrm{d}r, \tag{3.12}$$

in which

$$A(r, z) := \int_{\mathrm{S}^{n-2}} a(r\, w, z)\; \mathrm{d}_{n-2} w. \tag{3.13}$$

The right hand side is an even smooth function of r, because if we replace r by $-r$ then the substitution of $w \in \mathrm{S}^{n-2}$ by the antipodal point $-w$ shows that the integral remains the same. It follows that we have an asymptotic expansion for $r \to 0$ in even powers of r. More precisely,

$$A(r, z) \sim \sum_{k=0}^{\infty} c_k\, \Delta^k a(0, z)\, r^{2k}, \quad r \to 0, \tag{3.14}$$

where Δ denotes the Laplace operator with respect to the variable v and the c_k are universal constants, with c_0 equal to the $(n-2)$-dimensional volume of S^{n-2}.

We have reduced the asymptotic expansion for $I(x)$ as x tends to a smooth and strongly convex piece of ∂Q to the investigation of integrals of the form

$$\int_0^\infty \left(\delta + r^2\right)^{-n} r^{n-2+2k}\, \mu(r^2)\; \mathrm{d}r = \tfrac{1}{2} \int_0^\infty (\delta + s)^{-n}\, s^{\frac{n-3}{2}+k}\, \mu(s)\; \mathrm{d}s \tag{3.15}$$

as $\delta \downarrow 0$, where μ is a cut-off function at the origin, a compactly supported smooth function which is equal to 1 on an open neighborhood of the origin. Because we have to investigate the derivatives with respect to δ of (3.15) as well, we will study of the asymptotic behaviour for $\delta \downarrow 0$ of integrals of the form

$$I_{m,\,p}(\delta) := \int_0^\infty (\delta + s)^{-m}\, s^p\, \mu(\sigma)\; \mathrm{d}s, \tag{3.16}$$

where $m \geq n$ is an integer, $p = \frac{n-3}{2} + k$, and μ is a cut-off function at the origin.

When $-m+p < -1$, then the substitution of variables $s = \delta\,\sigma$ yields that

$$\begin{aligned} I_{m,p}(\delta) &= \int_0^\infty (\delta+s)^{-m}\, s^p\, \mathrm{d}s + \int_0^\infty (\delta+s)^{-m}\, s^p\, [\mu(s)-1]\, \mathrm{d}s \\ &= \delta^{-m+p+1} \int_0^\infty (1+\sigma)^{-m}\, \sigma^p\, \mathrm{d}\sigma + \int_0^\infty (\delta+s)^{-m}\, s^p\, [\mu(s)-1]\, \mathrm{d}s, \end{aligned}$$

where the last integral is an analytic function of δ in a neighborhood of $\delta = 0$.

When $-m+p \geq 1$, then we take a positive integer l such that $-(m+l)+p < -1$, write

$$I_{m,p}^{(l)}(\delta) = (-1)^l \tfrac{(m+l-1)!}{(m-1)!}\, I_{m+l,p}(\delta),$$

apply the previous result to $I_{m+l,p}$ and then integrate l times. If p is not an integer, which occurs when n is even, it follows that $I_{m,p}(\delta)$ is equal to a constant times δ^{-m+p+1} plus an analytic function of δ near $\delta = 0$. However, if p is an integer, which occurs when n is odd, then $I_{m,p}^{(-m+p+2)}(\delta)$ is equal to a constant times δ^{-1} plus an analytic function of δ. Integrating this $-m+p+2$ times, we obtain that $I_{m,p}(\delta)$ is equal to a constant times $\delta^{-m+p+1}\ln\delta$ plus an analytic function of δ near $\delta = 0$.

Because $-m+p+1 = -\frac{n+1}{2}$ when $m = n$ and $p = \frac{n-3}{2}$, we obtain that there exist functions J and K which are smooth up to the boundary, such that

$$\begin{aligned} I(x) &= J(x)\,\delta^{-\frac{n+1}{2}} + K(x), \quad \text{when } n \text{ is even, and} && (3.17) \\ I(x) &= J(x)\,\delta^{-\frac{n+1}{2}} + K(x)\,\ln\delta, \quad \text{when } n \text{ is odd.} && (3.18) \end{aligned}$$

Furthermore,

$$\lim_{x\to z} J(x) = c(n)\,\kappa(z)^{1/2}, \quad z \in \partial Q, \tag{3.19}$$

in which the constant

$$c(n) := \tfrac{2-n}{n}\, \mathrm{vol}_{n-2}\left(\mathrm{S}^{n-2}\right) \int_0^\infty (1+\sigma)^{-n}\, \sigma^{\frac{n-3}{2}}\, \mathrm{d}\sigma \tag{3.20}$$

only depends on the dimension n. Note that the limit in (3.19) is strictly positive.

It follows that $\mathrm{e}^{-f(x)} = I(x)^{-\frac{2}{n+1}}$ is equal to

$$\delta\left[J(x) + \delta^{\frac{n+1}{2}}\, K(x)\right]^{-\frac{2}{n+1}}$$

when n is even and which is equal to

$$\delta\left[J(x) + \delta^{\frac{n+1}{2}}\, K(x)\,\ln\delta\right]^{-\frac{2}{n+1}}$$

when n is odd. Therefore, the conclusions of the proposition follow with $\alpha(x) := \delta\, J(x)^{-\frac{2}{n+1}}$, $\beta(x) = K(x)$ and $\gamma(x) = \frac{2}{n+1}\ln J(x)$. The function α is equal to zero at the boundary, is strictly positive in the interior, and has nonzero derivative at every boundary point where the boundary is smooth and strongly convex. q.e.d.

Remark 3.1 In the proof of Proposition 3.1 we have found that for every $z \in \partial Q$ the linear form $\lambda_{\partial Q}(z) := \phi'(z) = \alpha'(z)$ is determined by the conditions that it is equal to zero on $\mathrm{T}_z\, Q$ and attains the value $c(n)^{-\frac{2}{n+1}}\, \kappa(z)^{-\frac{1}{n+1}}$ on the interior normal $-\nu(z)$. Here $c(n)$ is defined in (3.20).

If $T : x \mapsto A\, x + a$ is an affine transformation, where A is an invertible linear transformation and $a \in \mathbf{R}^n$, then it follows from the definition (3.1) that

$$T^*(Q^*(x)) = \left(T^{-1}(Q)\right) \left(T^{-1}(x)\right).$$

Because the volume of the left hand side is equal to $\det A^* = \det A$ times the volume of $Q^*(x)$, it follows that

$$f_{T^{-1}(Q)}\left(T^{-1}(x)\right) = f_Q(x) + \tfrac{2}{n+1}\, \ln\, \det A,$$

which in turn implies that for every $z \in \partial Q$ where ∂Q is smooth and strongly convex:

$$\lambda_{T^{-1}(Q)}\left(T^{-1}(z)\right) = (\det A)^{-\frac{2}{n+1}}\, \lambda_Q(z). \tag{3.21}$$

Therefore, although the Gaussian curvature $\kappa(z)$ and the interior normal $-\nu(z)$ are quantities which are only invariant under Euclidean isometries, the form λ_Q on ∂Q has the invariance property (3.21) for arbitrary affine transformations T.

The linear form

$$\nu_{\mathrm{aff}}(z) : v \mapsto \kappa(z)^{-\frac{1}{n+1}}\, \langle v,\, \nu(z)\rangle, \quad z \in \partial Q, \tag{3.22}$$

which is equal to $c(n)^{\frac{2}{n+1}}\, \lambda_{\partial Q}(z)$, is the *affinely invariant conormal form* of the strongly convex hypersurface ∂Q, discovered by Berwald and Blaschke, cf. [2, II, §65] and Calabi [3]. Here "affinely invariant" means invariant under affine transformations which preserve the n-dimensional volume form ω of $\mathbf{R}^n$. If

$$\omega_{\mathrm{aff}} := \kappa^{\frac{1}{n+1}}\, \mathrm{d}_{n-1}y = \omega / \nu_{\mathrm{aff}} \tag{3.23}$$

denotes the corresponding affinely invariant $(n-1)$-dimensional volume form on ∂Q, then the formula (3.5) can be written in the affinely invariant form

$$I_\chi(x) = \tfrac{1}{n} \int_{\partial Q} \langle y - x,\, \nu_{\mathrm{aff}}(y)\rangle^{-n}\, \chi(y)\, \omega_{\mathrm{aff}}\,(\mathrm{d}y)\,, \tag{3.24}$$

where χ is a smooth function with compact support in the open subset of ∂Q where ∂Q is smooth and strongly convex, and equal to 1 in a neighborhood of the points of ∂Q close to x. $\oslash$

4 Preparations

The partial derivatives of f of order one, two, and three are given by

$$\partial_i f = -\frac{\partial_i \phi}{\phi}, \tag{4.1}$$

$$g_{ij} := \partial_i \partial_j f = -\frac{\partial_i \partial_j \phi}{\phi} + \frac{\partial_i \phi \cdot \partial_j \phi}{\phi^2}, \quad \text{and} \tag{4.2}$$

$$\partial_i \partial_j \partial_k f = -\frac{\partial_i \partial_j \partial_k \phi}{\phi} + \frac{\partial_i \partial_j \phi \cdot \partial_k \phi + \partial_k \partial_i \phi \cdot \partial_j \phi + \partial_j \partial_k \phi \cdot \partial_i \phi}{\phi^2}$$
$$-2 \frac{\partial_i \phi \cdot \partial_j \phi \cdot \partial_k \phi}{\phi^3}, \tag{4.3}$$

respectively. Assumption 2.1 implies that the function ϕ can be used, near ∂Q, as an indicator of the distance to the boundary. The formulas (4.1), (4.2),(4.3) yield expansions in negative powers of ϕ of the derivatives of f up to the order three.

By shrinking U if necessary, we may assume that the unique point in Q where f attains its minimum does not belong to U. It follows that in U the level sets of f, which are equal to the level sets of ϕ, are smooth hypersurfaces, with tangent space at $x \in U \cap Q$ equal to the null space N_x of $\mathrm{d}f(x)$, which according to (4.1) is equal to the null space of $\mathrm{d}\phi(x)$. Because of the latter characterization, the N_x, $x \in U \cap Q$, extend smoothly to $u \cap \overline{Q}$, where for $y \in U \cap \partial Q$ we have $N_y = \mathrm{T}_y(\partial Q)$.

If $x \in U \cap Q$ and $u \in N_x$, then (4.2) implies that we have for every $v \in \mathbf{R}^n$ that

$$g(x)(u, v) = -\phi''(x)(u, v)/\phi(x).$$

Therefore, the $g(x)$-orthogonal complement $N_x^{\perp}$ of N_x in $\mathbf{R}^n$, which is one--dimensional, is equal to the $\phi''(x)$-orthogonal complement of N_x. In $N_x^{\perp}$ we have the unique vector $\nu_{\phi}(x)$, defined by the conditions that

$$\nu_{\phi}(x) \in N_x^{\perp} \quad \text{and} \tag{4.4}$$

$$\langle \nu_{\phi}(x), \mathrm{d}\phi(x) \rangle = 1. \tag{4.5}$$

Lemma 4.1 *For every $y \in U \cap \partial Q$, the restriction of $\phi''(y)$ to $N_y = \mathrm{T}_y(\partial Q)$ is negative definite. The mapping $\nu_{\phi} : U \cap Q \to \mathbf{R}^n$, defined by* (4.4) *and* (4.5)*, is smooth up to the boundary.*

Proof By means of an affine substitution of variables we may arrange that $y = 0$, $\mathrm{T}_y(\partial Q) = \mathbf{R}^{n-1} \times \{0\}$, and near the origin Q is equal to the domain

$$\left\{x \in U \mid x^n > h\left(x^1, \ldots, x^{n-1}\right)\right\} \tag{4.6}$$

above the graph of a smooth real-valued function h of $n-1$ variables. The fact that $y \in \partial Q$, $\mathrm{T}_y(\partial Q) = \mathbf{R}^{n-1} \times \{0\}$, and ∂Q is strongly convex at y imply that $h(0) = 0$, $\mathrm{d}h(0) = 0$ and $h''(0)$ is positive definite, respectively. Because near y the boundary ∂Q is equal to the graph of h, and $\phi = 0$ on ∂Q, we have

$$\phi(z, h(z)) = 0 \tag{4.7}$$

for all $z \in \mathbf{R}^{n-1}$ in a neighborhood of the origin in $\mathbf{R}^{n-1}$. If we differentiate the relation (4.7) twice with respect to z and then substitute $z = 0$, we obtain that

$$\partial_i \partial_j \phi(0) + \partial_n \phi(0) \cdot \partial_i \partial_j h(0) = 0, \quad 1 \le i, j \le n-1. \tag{4.8}$$

Because $\mathrm{d}\phi(0) \neq 0$ and $\partial_i \phi(0) = 0$ when $1 \leq i \leq n-1$, we have that $\partial_n \phi(0) \neq 0$, and because $\phi > 0$ in $U \cap Q$ it follows that $\partial_n \phi(0) > 0$. Therefore (4.8), in combination with $h''(0) > 0$ and $\partial_n \phi(0) > 0$, implies that $\phi''(y) = \phi''(0)$ is negative definite on $\mathrm{T}_y(\partial Q) = \mathbf{R}^{n-1} \times \{0\}$.

We now turn to the proof of the second statement of the lemma. The first statement implies that, for every $y \in \partial Q$, the $\phi''(y)$-orthogonal complement of N_y in $\mathbf{R}^n$, which we again denote by $N_y^{\perp}$, is one-dimensional. Moreover, $N_x^{\perp}$ depends smoothly on $x \in U \cap \overline{Q}$, which implies the second statement of the lemma. q.e.d.

When ϕ is replaced by the function α in Proposition 3.1, then $N_y^{\perp}$ is the direction of the *affine normal* of ∂Q at the point $y \in \partial Q$, as defined in [2, II, §65]. The geometric interpretation of the affine normal in [2, II, §43] is quite instructive.

For $y \in \partial Q$, the vector $\nu_{\phi}(y)$ plays the role of an *interior normal* to ∂Q, because

$$\langle \nu_{\phi}(y), \, \mathrm{d}\phi(y) \rangle = 1 > 0 \quad \text{and} \quad \phi > 0 \text{ in } U \cap Q$$

imply that $\nu_{\phi}(y)$ points in the direction of Q.

Corollary 4.2 *Assume that ϕ satisfies Assumption 2.1 and that $f = \mathrm{e}^{-\phi}$. Write, for every $x \in Q$ sufficiently close to $U \cap \partial Q$,*

$$\begin{aligned} C_1(f, \, x) &:= \sup_{v \in \mathbf{R}^n, \, v \neq 0} f'(x)(v)^2 / f''(x)(v, \, v) < \infty, \\ C_2(f, \, x) &:= \sup_{v \in \mathbf{R}^n, \, v \neq 0} \left[\tfrac{1}{2} f'''(x)(v, \, v, \, v) \right]^2 / f''(x)(v, \, v)^3 < \infty. \end{aligned}$$

Then both $C_1(f, \, x)$ and $C_2(f, \, x)$ converge to 1 as x converges to $U \cap \partial Q$, locally uniformly on any compact subset of $U \cap \overline{Q}$. In this sense, the parameter of a self-concordant barrier function which satisfies Assumption 2.1 is asymptotically equal to 1 near the smooth and strongly convex part of the boundary.

If Q is bounded and has a smooth and strongly convex boundary, Assumption 2.1 holds globally (for $U = \mathbf{R}^n$), and f is strongly convex on Q, then f is a self-concordant barrier function for Q.

Proof If v is a vector of unit Euclidean length $\|v\|$ in $\mathbf{R}^n$ then we have, writing $v = u + \tau \nu$ with $u \in N_x$, $\nu = \nu_{\phi}(x)$ and $\tau \in \mathbf{R}$, that $f'(x)(v) = \tau / \phi(x)$ and

$$f''(x)(v, \, v) = -\frac{\phi''(x)(u, \, u)}{\phi(x)} + \frac{\tau^2}{\phi(x)^2} \left[1 - \phi(x) \, \phi''(\nu, \, \nu) \right].$$

Because $-\phi''(u, \, u) \geq 0$, it follows that

$$C_1(f, \, x) = \left[1 - \phi(x) \, \phi'' \left(\nu_{\phi}(x), \, \nu_{\phi}(x) \right) \right]^{-1},$$

which converges to 1 as x converges to $U \cap \partial Q$, locally uniformly on any compact subset of $U \cap \overline{Q}$. Note also that there is a positive constant C such that $f''(v, \, v) \geq C / \phi(x)$ for x sufficiently close to any compact subset of $U \cap \partial Q$.

On the other hand,

$$f'''(x)(v,\,v,\,v) = -\frac{\phi'''(x)(v,\,v,\,v)}{\phi(x)} + 3\frac{\phi''(x)(v,\,v)\,\phi'(x)(v)}{\phi(x)^2} - 2\frac{\phi'(x)(v)^3}{\phi(x)^3}.$$

If we take $v = \pm\nu_\phi(x)$, then the leading term in $f'''(x)(v,\,v,\,v)$ as $\phi(x) \downarrow 0$ is equal to $\mp 2/\phi(x)^3$, whereas the leading term in $f''(x)(v,\,v)$ is equal to $1/\phi(x)^2$, and we conclude that the limes inferior of $C_2(f,\,x)$, as x converges to $U \cap \partial Q$, is ≥ 1, locally uniformly on any compact subset of $U \cap \overline{Q}$.

Furthermore

$$f'''(x)(v,\,v,\,v) = -\frac{\phi'''(x)(v,\,v,\,v)}{\phi(x)} + f'(x)(v)\,\left[3f''(x)(v,\,v) - f'(x)(v)^2\right].$$

If $b > 0$, then the function $a \mapsto a\,\left[3b - a^2\right]$ on the interval $\left[-\sqrt{3b},\,\sqrt{3b}\right]$ attains its minimum and maximum at $a = -\sqrt{b}$ and $a = \sqrt{b}$, where it takes the value $-2b^{3/2}$ and $2b^{3/2}$, respectively. Because $C_1(f,\,x)$ is close to one, and therefore certainly

$$|f'(x)(v)| \leq \sqrt{3f''(x)(v,\,v)},$$

we obtain that

$$\left|f'(x)(v)\,\left[3f''(x)(v,\,v) - f'(x)(v)^2\right]\right| \leq 2\,f''(x)(v,\,v)^{3/2}$$

when x is close to a compact subset of $U \cap \partial Q$. Also using that $\phi(x)^{-1}/f''(x)(v,\,v)^{3/2} \leq C^{-3/2}\phi(x)^{1/2}$, which converges to zero as x approaches the boundary, we obtain that the limes superior of $C_2(f,\,x)$, as x converges to $U \cap \partial Q$, is ≤ 1, locally uniformly on any compact subset of $U \cap \overline{Q}$. q.e.d.

Corollary 4.3 *Let Q be a bounded open convex subset of $\mathbf{R}^n$, such that the boundary ∂Q is a smooth and strongly convex hypersurface in $\mathbf{R}^n$. Let f be as in Proposition 3.1.*

Let χ be a smooth function of one variable, equal to 1 on a neighborhood of 0 and equal to zero outside some bounded interval. Write, for any $\epsilon > 0$, $\chi_\epsilon(\phi) := \chi\left(\phi/\epsilon\right)$, and

$$\phi_\epsilon(x) = \left[1 - \chi_\epsilon(\phi(x))\right]\,\phi(x) + \chi_\epsilon(\phi(x))\,\alpha(x).$$

Then, if $\epsilon > 0$ is sufficiently small, $f_\epsilon(x) = -\ln\phi_\epsilon(x)$ is a self-concordant barrier function for Q which satisfies Assumption 2.1.

Proof The function $\chi_\epsilon(x)$ is only nonzero when the distance from x to ∂Q is of order ϵ. There we have that

$$\phi_\epsilon(x) - \alpha(x) = \left[1 - \chi_\epsilon(\phi(x))\right]\left[\phi(x) - \alpha(x)\right].$$

Because $1 + \frac{n+1}{2} > 2$ when $n \geq 2$, the description of ϕ in Proposition 3.1 implies that for every $\delta > 0$ there exists an $\epsilon_0 > 0$ such that, for every $1 \leq i,\, j \leq n$,

$$|\partial_i\partial_j\phi_\epsilon(x) - \partial_i\partial_j\alpha(x)| < \delta$$

when $0 < \epsilon < \epsilon_0$ and $\chi_\epsilon(x) \neq 0$. Because the matrix $\partial_i \partial_j \alpha(x)$ is negative definite when $x \in \partial Q$, it follows from (4.2) with ϕ replaced by ϕ_ϵ that, when ϵ is sufficiently small the matrix $\partial_i \partial_j f_\epsilon(x)$ is positive definite for every $x \in Q$. Because near ∂Q the function ϕ_ϵ is equal to α, the function f_ϵ satsifies Assumption 2.1. Because of Corollary 4.2 it therefore also satifies the estimates (iii) and (iv) for a self-concordant barrier function. q.e.d.

Using the implicit function theorem, we obtain that the mapping $(y,\, \epsilon) \mapsto y + \epsilon\, \nu_\phi(y)$ is a smooth diffeomorphism from V to $\widetilde{U}$, where V is an open neighborhood of $(U \cap \partial Q) \times \{0\}$ in $(U \cap \partial Q) \times \mathbf{R}$ and $\widetilde{U}$ is an open neighborhood of $U \cap \partial Q$ in U. This implies that for every $x \in \widetilde{U} \cap \overline{Q}$, we have a unique $y = y(x) \in U \cap \partial Q$ and $\epsilon = \epsilon(x) \geq 0$, such that $x = y + \epsilon\, \nu_\phi(y)$. Moreover, $y(x)$ and $\epsilon(x)$ depend smoothly on $x \in \widetilde{U} \cap \overline{Q}$, and $\epsilon(x) > 0$ corresponds to $x \in Q$. Actually ϵ and ϕ are of the same order of magnitude near $U \cap \partial Q$, because $\epsilon = \phi = 0$ and $\mathrm{d}\epsilon = \mathrm{d}\phi$ on $U \cap \partial Q$.

In the sequel, we will fix $y \in U \cap \partial Q$ and obtain asymptotic expansions of the various quantities at the point $y + \epsilon\, \nu_\phi(y)$ in powers of ϵ (including negative powers) as $\epsilon \downarrow 0$. In these expansions, the estimates for the remainders will be locally uniform when y varies in a compact subset of $U \cap \partial Q$. Because of the smooth dependence of $y(x)$ on x, this will lead to expansions of the various quantities at the point x in powers of ϕ (including negative powers), when $x \in U \cap Q$ approaches $U \cap \partial Q$, where the estimates for the remainders will be locally uniform when x stays in a compact subset of $U \cap \overline{Q}$.

If $x = A(y)$ is an affine substitution of variables, then the Hessian Riemannian structure $(f \circ A)''$ of the function $f \circ A$ is equal to the pull-back under A of the Hessian Riemannian structure f'' of the function f; for this reason all the quantities which we will study here are invariant under affine substitutions of variables.

For the given point $y \in U \cap \partial Q$, we can arrange by means of an affine substitution of variables (which can locally be taken to depend smoothly on y) that $y = 0$, $\mathrm{T}_y(\partial Q) = \mathbf{R}^{n-1} \times \{0\}$ and, in addition to the requirements in the proof of Lemma 4.1, $\nu_\phi(y) = e_n$, the n-th standard basis vector in $\mathbf{R}^n$. In this situation, (4.4) and (4.5) mean that

$$\begin{aligned} \partial_i \partial_n \phi(0) &= 0, \quad 1 \leq i \leq n-1 \quad \text{and} && (4.9) \\ \partial_i \phi(0) &= \delta_{in}, \quad 1 \leq i \leq n, && (4.10) \end{aligned}$$

respectively.

By a subsequent linear substitution of variables in the first $n-1$ coordinates (which again can be taken locally to depend smoothly on the point $y \in \partial Q$, we can also arrange that $h''(0)$ is equal to the identity matrix, which in view of (4.8) and (4.10) means that

$$\partial_i \partial_j \phi(0) = -\delta_{ij}, \quad 1 \leq i,\, j \leq n-1. \tag{4.11}$$

The only freedom in the affine substitution of variables which is left is an orthogonal linear transformation in the first $n-1$ variables.

If we differentiate the relation (4.7) three times with respect to z and then substitute $z = 0$, we obtain, using (4.9) and (4.10), that

$$\phi_{ijk} := \partial_i \partial_j \partial_k \phi(0) = -\partial_i \partial_j \partial_k h(0), \quad 1 \le i,\, j,\, k \le n-1. \tag{4.12}$$

In terms of the geometry of the boundary, we have no further control over the partial derivatives

$$\begin{aligned} \phi_{nn} &:= \partial_n{}^2 \phi(0) \quad \text{and} && (4.13) \\ \phi_{ijn} &:= \partial_i \partial_j \partial_n \phi(0), \quad 1 \le i,\, j \le n && (4.14) \end{aligned}$$

of ϕ at the origin.

Our next goal is to determine the asymptotic behaviour near the boundary of the Hessian Riemannian structure and its inverse. These quantities appear in almost every formula in Riemannian geometry.

The formulas (4.2), (4.9), (4.11), (4.12), (4.13) and (4.14) imply that the Riemannian structure defined by the Hessian of f at $x = (0,\, \epsilon) = y + \epsilon\, \nu_\phi(y)$ is given by

$$\begin{aligned} g_{ij} &= \epsilon^{-1}\, \delta_{ij} - \tfrac{1}{2}\, \phi_{nn}\, \delta_{ij} - \phi_{ijn} + \mathcal{O}(\epsilon), \quad 1 \le i,\, j \le n-1, && (4.15) \\ g_{in} &= g_{ni} = -\tfrac{1}{2}\, \phi_{inn} + \mathcal{O}(\epsilon), \quad 1 \le i \le n-1, && (4.16) \\ g_{nn} &= \epsilon^{-2} + \tfrac{1}{4}\, \phi_{nn}{}^2 - \tfrac{1}{3}\, \phi_{nnn} + \mathcal{O}(\epsilon). && (4.17) \end{aligned}$$

The fact that in the right hand side of (4.17) no term with $1/\epsilon$ appears can be explained by the fact that $\phi = \epsilon\, \psi$ where ψ is smooth up to the boundary and equal to 1 on the boundary. Therefore

$$\partial_n{}^2 f = \frac{\partial^2}{\partial \epsilon^2} \left(-\ln \left(\epsilon\, \psi \right) \right) = -\frac{\partial^2}{\partial \epsilon^2} \ln \epsilon - \frac{\partial^2}{\partial \epsilon^2} \ln \psi = \epsilon^{-2} + \chi,$$

where χ is smooth up to the boundary.

Let $\widehat{g}$ denote the leading term of g, viz.

$$\begin{aligned} \widehat{g}_{ij} &:= \epsilon^{-1}\, \delta_{ij}, \quad 1 \le i,\, j \le n-1, && (4.18) \\ \widehat{g}_{in} &= \widehat{g}_{ni} := 0, \quad 1 \le i \le n-1, && (4.19) \\ \widehat{g}_{nn} &:= \epsilon^{-2}. && (4.20) \end{aligned}$$

Using that $\widehat{g}^{-1}\, g = 1 - \left(1 - \widehat{g}^{-1}\, g\right)$, we obtain the series expansion

$$g^{-1} = \sum_{m=0}^{\infty} \left(1 - \widehat{g}^{-1}\, g\right)^m\, \widehat{g}^{-1} \tag{4.21}$$

for the inverse $g^{ij}(x)$ of the matrix $g_{ij}(x)$.

With induction over m we obtain that

$$\begin{aligned} \left(\left(1 - \widehat{g}^{-1}\, g\right)^m\right)_{ij} &= \mathcal{O}\left(\epsilon^m\right), \quad 1 \le i \le n-1,\, 1 \le j \le n, \\ \left(\left(1 - \widehat{g}^{-1}\, g\right)^m\right)_{nj} &= \mathcal{O}\left(\epsilon^{m+1}\right), \quad 1 \le j \le n, \end{aligned}$$

which in turn implies that

$$\begin{aligned}
\left(\left(1-\widehat{g}^{-1}\,g\right)^m\,\widehat{g}^{-1}\right)_{ij} &= \mathcal{O}\left(\epsilon^{m+1}\right), \quad 1\le i,\,j\le n-1,\\
\left(\left(1-\widehat{g}^{-1}\,g\right)^m\,\widehat{g}^{-1}\right)_{in} &= \mathcal{O}\left(\epsilon^{m+2}\right), \quad 1\le i\le n-1,\\
\left(\left(1-\widehat{g}^{-1}\,g\right)^m\,\widehat{g}^{-1}\right)_{nj} &= \mathcal{O}\left(\epsilon^{m+2}\right), \quad 1\le j\le n-1,\\
\left(\left(1-\widehat{g}^{-1}\,g\right)^m\,\widehat{g}^{-1}\right)_{nn} &= \mathcal{O}\left(\epsilon^{m+3}\right), \quad 1\le j\le n-1
\end{aligned}$$

This leads to the conclusion that

$$\begin{aligned}
g^{ij} &= \epsilon\,\delta_{ij} + \left(\tfrac{1}{2}\,\phi_{nn}\,\delta_{ij} + \phi_{ijn}\right)\,\epsilon^2 + \mathcal{O}\left(\epsilon^3\right), \quad 1\le i,\,j\le n-1, && (4.22)\\
g^{in} &= g^{ni} = \tfrac{1}{2}\,\phi_{inn}\,\epsilon^3 + \mathcal{O}\left(\epsilon^4\right), \quad 1\le i\le n-1, && (4.23)\\
g^{nn} &= \epsilon^2 - \left(\tfrac{1}{4}\,{\phi_{nn}}^2 - \tfrac{1}{3}\,\phi_{nnn}\right)\,\epsilon^4 + \mathcal{O}\left(\epsilon^5\right). && (4.24)
\end{aligned}$$

For the applications in the sections 5 and 7, we will also need the asymptotic behaviour of the third order derivatives of f near the boundary. The formulas (4.3), (4.9), (4.11), (4.12), (4.13) and (4.14) imply that

$$\begin{aligned}
\partial_i\partial_j\partial_k f &= -\epsilon^{-1}\,\phi_{ijk} + \mathcal{O}(1), \quad 1\le i,\,j,\,k\le n-1, && (4.25)\\
\partial_i\partial_j\partial_n f &= -\epsilon^{-2}\,\delta_{ij} - \epsilon^{-1}\,\tfrac{1}{2}\phi_{nn}\,\delta_{ij} + \mathcal{O}(1), \quad 1\le i,\,j\le n-1, && (4.26)\\
\partial_i{\partial_n}^2 f &= \mathcal{O}(1), \quad 1\le i\le n-1, && (4.27)\\
{\partial_n}^3 f &= -2\epsilon^{-3} + \mathcal{O}(1). && (4.28)
\end{aligned}$$

I obtained the expansions (4.25), (4.26) and (4.27) by means of a direct calculation. For (4.28) I used that $\phi = \epsilon\,\psi$ where ψ is smooth up to the boundary and equal to 1 on the boundary. Therefore

$${\partial_n}^3 f = \frac{\partial^3}{\partial\epsilon^3}\,\left(-\ln\left(\epsilon\,\psi\right)\right) = -\frac{\partial^3}{\partial\epsilon^3}\ln\epsilon - \frac{\partial^3}{\partial\epsilon^3}\ln\psi = -2\epsilon^{-3} + \widetilde{\psi},$$

where $\widetilde{\psi}$ is smooth up to the boundary.

5 Geodesics near the Boundary

As a background reference for the differential geometry used in this paper, one may use the book [12] of Kobayashi and Nomizu.

The *Christoffel symbols* of a Riemannian structure $g_{ij}(x)$ are defined by

$$\Gamma_{lij}(x) := \tfrac{1}{2}\left[\partial_i g_{jl}(x) - \partial_l g_{ij}(x) + \partial_j g_{li}(x)\right] \qquad (5.1)$$

and

$$\Gamma^k_{ij}(x) := \sum_{l=1}^n g^{kl}(x)\,\Gamma_{lij}(x), \qquad (5.2)$$

where $g^{kl}(x)$ denotes the inverse of the matrix $g_{ij}(x)$. Note that $\Gamma_{lij}(x) = \Gamma_{lji}(x)$ and $\Gamma^k_{ij}(x) = \Gamma^k_{ji}(x)$, for all $1\le i,\,j,\,k,\,l\le n$.

Any collection of smooth functions $\Gamma_{ij}^k(x)$ with indices $1 \le i,\, j,\, k \le n$ and such that $\Gamma_{ij}^k(x) = \Gamma_{ji}^k(x)$ defines an *infinitesimal connection* in the tangent bundle of the manifold which is linear and torsionfree, via the definition of the *covariant derivative* of a vector field $Y(x)$ in the direction of the vector field $X(x)$ by means of the formula

$$(\nabla_X Y(x))^i = \sum_{k=1}^{n} X^k(x)\,\partial_k Y^i(x) + \sum_{j=1}^{n} \Gamma_{kj}^i(x)\, X^k(x)\, Y^j(x), \tag{5.3}$$

or, in shorthand,

$$\nabla_X Y(x) = Y'(x)(X(x)) + \Gamma(x)\,(X(x)\, Y(x)). \tag{5.4}$$

It is usual to denote the connection by the symbol ∇ of its covariant derivative. The connection defined by (5.1) and (5.2) in terms of a Riemannian structure g is called the *Levi-Civita connection of* g.

A twice differentiable curve $t \mapsto \gamma(t)$ is called a *geodesic* for the connection ∇ if it satisfies the second order system of differential equations

$$\frac{d^2}{dt^2}\gamma^k(t) + \sum_{i,j=1}^{n} \Gamma_{ij}^k(\gamma(t))\,\frac{d\gamma^i(t)}{dt}\,\frac{d\gamma^j(t)}{dt} = 0, \quad 1 \le k \le n, \tag{5.5}$$

in shorthand also written as $\nabla_{\gamma'}\gamma' = 0$. (This shorthand notation expresses the fact that the velocity field of the curve is covariantly constant with respect to the induced connection in the pullback of the tangent bundle by means of the mapping $\gamma : I \to \mathbf{R}^n$, where I is the interval of definition of γ. The pullback bundle is a vector bundle over I, where the fiber over $t \in I$ is identified with the tangent space at the point $\gamma(t)$.) If ∇ is the Levi-Civita connection of the Riemannian structure g, then the equations (5.5) are equivalent to the Euler-Lagrange equations for the kinetic energy function defined by g, and the geodesics are locally the shortest paths for the corresponding Riemannian distance function, parametrized by a constant factor times the arclength.

When $g_{ij}(x) = \partial_i\partial_j f(x)$ is the Riemannian structure defined by the Hessian of a smooth strongly convex function f, then its Christoffel symbols take the form

$$\Gamma_{lij}(x) = \tfrac{1}{2}\,\partial_l\partial_i\partial_j f(x), \quad \Gamma_{ij}^k(x) = \tfrac{1}{2}\sum_{l=1}^{n} g^{kl}(x)\,\partial_k\partial_i\partial_j f(x), \tag{5.6}$$

where the $g^{kl}(x)$ denote the inverse of the Hessian matrix $\partial_i\partial_j f(x)$ of f at the point x.

It will be instructive to consider the one-parameter family ${}^{\mu}\nabla$, $\mu \in \mathbf{R}$, of torsionfree linear infinitesimal connections defined by the Christoffel symbols

$${}^{\mu}\Gamma_{ij}^k(x) = \mu \sum_{l=1}^{n} g^{kl}(x)\,\partial_k\partial_i\partial_j f(x), \quad 1 \le i,\, j,\, k \le n. \tag{5.7}$$

For $\mu = 0$ the geodesics are the straight lines, parametrized with constant velocity. For $\mu = 1/2$ we have the Levi-Civita connection of the Hessian Riemannian structure, with its corresponding geodesics.

For $\mu = 1$ the geodesics are the gradient curves, with respect to the Hessian Riemannian structure, of arbitrary linear functions on $\mathbf{R}^n$. Indeed, if $\xi \in \mathbf{R}^n$ is a covector, then the gradient curves $t \mapsto \gamma(t)$ of the linear function $x \mapsto \langle x, \xi \rangle$ are determined by the first order system

$$\frac{\mathrm{d}}{\mathrm{d}t} \gamma^k(t) = \sum_{l=1}^{n} g^{kl}(\gamma(t))\, \xi_l, \quad 1 \leq k \leq n. \tag{5.8}$$

Differentiating (5.8) with respect to t and using that

$$\partial_i \left(g^{-1} \right) = -g^{-1}\, \partial_i g\, g^{-1},$$

one arrives at the equations (5.5) with $\Gamma =^1 \Gamma$. In this sense, the Levi-Civita connection of the Hessian Riemannian structure is in the middle between the standard affine connection, of which the geodesics are straight lines, and the connection for which the geodesics are the gradient curves of linear functions, where the gradient is taken with respect to the Hessian Riemannian structure.

For any connection the Riemannian curvature tensor is given by

$$R^k_{lij}(x) := \partial_i \Gamma^k_{lj}(x) - \partial_j \Gamma^k_{li}(x) + \sum_{m=1}^{n} \left(\Gamma^k_{im}(x)\, \Gamma^m_{jl}(x) - \Gamma^k_{jm}(x)\, \Gamma^m_{il}(x) \right). \tag{5.9}$$

The curvature of the connection with Christoffel symbols ${}^{\mu}\Gamma^k_{ij}(x)$ will be denoted by ${}^{\mu}R^k_{lij}(x)$. A direct calculation,in which it is used that $\partial_i \left(g^{-1} \right) = -g^{-1}\, \partial_i g\, g^{-1}$, yields that

$${}^{\mu}R^k_{lij}(x) := \mu\,(\mu - 1) \sum_{m=1}^{n} \left({}^1\Gamma^k_{im}(x)\, {}^1\Gamma^m_{jl}(x) -{}^1 \Gamma^k_{jm}(x)\, {}^1\Gamma^m_{il}(x) \right) = 4\mu\,(1 - \mu)\, {}^{\frac{1}{2}}R^k_{lij}(x). \tag{5.10}$$

Therefore the curvature does not interpolate — instead it takes its extreme values for $\mu = \frac{1}{2}$. That the curvature is equal to zero for $\mu = 0$ is obvious, because then $\Gamma = 0$ and we have the straight line system of geodesics. That is the connection is also flat for $\mu = 1$ is made clear by the observation that the mapping $x \mapsto \mathrm{d}f(x)$ to the dual space maps the gradient curves of the linear function ξ to the curves $t \mapsto \alpha + t\,\xi$, where α is an arbitrary constant covector. In Section 7 we will discuss the curvature near the boundary of Q, when f is a self-concordant barrier function for the convex domain Q which satisfies Assumption 2.1.

Remark 5.1 The fact that $x \mapsto \mathrm{d}f(x)$ maps the gradient curves for linear functions to the curves $\xi'' = 0$ implies that the gradient vector fields of the linear functions commute with each other. Ruuska [15] observed that a Riemannian structure is of Hessian type if and only if it admits an abelian Lie algebra of gradient vector fields, the local action of which being simply transitive. I learned this reference from Hitchin [10]. $\oslash$

In order to study the asymptotic behaviour of the geodesics near the boundary, we will reparametrize the geodesics $t \mapsto \gamma(t)$, preserving their orientation, in such a way that their velocity vectors have length equal to one with respect to a suitable smooth Riemannian structure b, defined on an open neighborhood of $U \cap \partial Q$ in $\mathbf{R}^n$. That is, we will write

$$\gamma(t) = \delta(s(t)), \quad s'(t) > 0, \quad b(\gamma(t))\,(\delta'(s(t)),\, \delta'(s(t))) = 1. \tag{5.11}$$

It follows that

$$0 = \gamma'' + \Gamma\left(\gamma',\, \gamma'\right) = \left(s'\right)^2 \delta'' + s''\,\delta' + \Gamma\left(s'\,\delta',\, s'\,\delta'\right),$$

which upon division by $(s')^2$ yields that

$$0 = \delta'' + \alpha\,\delta' + \Gamma(\delta)\left(\delta',\, \delta'\right), \tag{5.12}$$

in which $\alpha = s''\,\left(s'\right)^{-2}$.

On the other hand, differentiation of

$$1 = b(\delta(s))\,(\delta'(s),\, \delta'(s))$$

with respect to s yields that

$$0 = 2b(x)(v,\, a) + \sum_{i,\,j,\,k=1}^{n} \partial_k h_{ij}(x)\, v^i\, v^j\, v^k, \tag{5.13}$$

if we write $x = \delta(s)$, $v = \delta'(s)$ and $a = \delta''(s)$. If we take the $b(x)$-inner product of (5.12) with v, then we obtain that

$$0 = b(x)(v,\, a) + \alpha + h(x)\left(\Gamma(x)(v,\, v),\, v\right),$$

which in combination with (5.13) yields that

$$\alpha = -b(x)\left(\Gamma(x)(v,\, v),\, v\right) + \tfrac{1}{2} \sum_{i,\,j,\,k=1}^{n} \partial_k b_{ij}(x)\, v^i\, v^j\, v^k, \quad x = \delta(s),\ v = \delta'(s) \tag{5.14}$$

Because the sum over i, j, k in (5.14) is smooth, the second order system of differential equations for the reparametrized geodesics $s \mapsto \delta(s)$, defined by (5.12), (5.14), is smooth in an open neighborhood of $U \cap \partial Q$ if and only if the vector-valued function

$$\widetilde{\Gamma}(x,\, v) := \Gamma(x)(v,\, v) - b(x)\left(\Gamma(x)(v,\, v),\, v\right)\, v \tag{5.15}$$

of x and v extends smoothly over the boundary. Note that the second order system (5.12), (5.14) is not the system of second order differential equations for geodesics of a torsionfree linear connection, because of the appearance of the fourth order terms with respect to the velocity vector v, coming from the third order factor α in (5.14).

Proposition 5.1 *For $x = y + \epsilon\,\nu_\phi(y)$, $y \in U \cap \partial Q$, and $|\epsilon|$ sufficiently small, define*

$$b(x)(u,\, v) := \begin{cases} -\phi''(y)(u,\, v) & \text{when} \quad u,\, v \in \mathrm{T}_y(\partial Q), \\ 0 & \text{when} \quad u \in \mathrm{T}_y(\partial Q),\ v = \nu_\phi(y), \\ 1 & \text{when} \quad u = v = \nu_\phi(y). \end{cases} \tag{5.16}$$

Then these equations define a smooth Riemannian structure b on an open neighborhood of $U \cap \partial Q$ in $\mathbf{R}^n$.

Moreover, with this b and for every $\mu \in \mathbf{R}$, the system of second order differential equations (5.12), (5.14), *with $\Gamma = {}^{\mu}\,\Gamma$, has a smooth extension to an open neighborhood of $U \cap \partial Q$ in $\mathbf{R}^n$.*

When $\mu < 1$, its solution curves which are tangential to $U \cap \partial Q$ curve out of Q, whereas for $\mu > 1$ they curve into Q. For $\mu = 1$ the solution curves which are tangential to $U \cap \partial Q$ remain in $U \cap \partial Q$. The orbits of the solution curves in $U \cap \partial Q$ coincide with the orbits of the gradient curves, with respect to the restriction of b to $U \cap \partial Q$, of the restrictions to $U \cap \partial Q$ of the linear functions (except for the critical points of the latter functions on $U \cap \partial Q$).

If $\mu \geq \frac{1}{2}$, then no geodesics reach $U \cap \partial Q$ in a finite time. If $\mu < \frac{1}{2}$, then the geodesics which after reparametrization intersect $U \cap \partial Q$ transversally (= not tangentially) reach $U \cap \partial Q$ after finite time.

Proof As in Section 4, we write $x = y + \epsilon\,\nu_\phi(y)$, with $y \in U \cap \partial Q$ and $\epsilon > 0$ small. With a suitable affine substitution of variables, we can arrange that $y = 0$, $\nu_\phi(y) = e_n$ and (4.11). This implies the expansions (4.22), (4.23), (4.24), (4.25), (4.26), (4.27), (4.28), and we also have that $h_{ij}(x) = \delta_{ij}$.

With $\Gamma = {}^{\mu}\,\Gamma$, we obtain that the p-th component of (5.15) is equal to $\mu\ (A^p - B^p)$, in which

$$\begin{aligned} A^p &:= \sum_{i,\,j,\,k} g^{pi}\,\partial_i\partial_j\partial_k f\, v^j\, v^k, \quad \text{and} \\ B^p &:= \sum_{h,\,i,\,j,\,k} v^h\, g^{hi}\,\partial_i\partial_j\partial_k f\, v^j\, v^k\, v^p. \end{aligned}$$

If $p < n$, then

$$A^p = \sum_{i<n}\left(\epsilon\,\delta_{pi} + \mathcal{O}\left(\epsilon^2\right)\right)\left(-2\sum_{j<n}\epsilon^{-2}\,\delta_{ij}\,v^j\,v^n + \mathcal{O}\left(\epsilon^{-1}\right)\right) + \mathcal{O}\left(\epsilon^3\right)\,\mathcal{O}\left(\epsilon^{-3}\right),$$

from which we conclude that

$$A^p = -2\epsilon^{-1}\,v^p\,v^n + \mathcal{O}(1), \quad 1 \leq p \leq n-1. \tag{5.17}$$

Similarly, we have that

$$A^n = \sum_{i<n}\mathcal{O}\left(\epsilon^3\right)\,\mathcal{O}\left(\epsilon^{-3}\right) + \left(\epsilon^2 + \mathcal{O}\left(\epsilon^4\right)\right)\left(-2\epsilon^{-3}\,v^n\,v^n + \mathcal{O}\left(\epsilon^{-2}\right)\right)$$

and therefore

$$A^n = -2\epsilon^{-1}\,(v^n)^2 + \mathcal{O}(1). \tag{5.18}$$

Using (5.17), (5.18) and the fact that the sum of the squares of the coordinates of v is equal to one, we obtain that

$$B^p = \sum_h v^h\,A^h\,v^p = -2\epsilon^{-1}\sum_h v^h\,v^h\,v^n\,v^p + \mathcal{O}(1) = -2\epsilon^{-1}\,v^n\,v^p + \mathcal{O}(1) = A^p + \mathcal{O}(1).$$

It follows that $A^p - B^p$ is of order one and we conclude that the system (5.12), (5.14), with $\Gamma = ^\mu \Gamma$, has a smooth extension to an open neighborhood of $U \cap \partial Q$ in $\mathbf{R}^n$.

In order to determine the curvature of the reparametrized geodesics which are tangent to the boundary, we observe that if $v = \delta'$, $v^n = 0$, then the n-th component of (5.12) yields that

$$0 = (\delta^n)'' + \mu \sum_i \sum_{j,\, k<n} g^{ni}\, \partial_i \partial_j \partial_k \, v^j \, v^k .$$

For $i < n$ we have $g^{ni} = \mathcal{O}\left(\epsilon^3\right)$ and $\partial_i \partial_j \partial_k f = \mathcal{O}\left(\epsilon^{-1}\right)$, which leads to a zero contribution for $\epsilon \to 0$. For $i = n$ the term after the sum signs is equal to

$$\left(\epsilon^2 + \mathcal{O}\left(\epsilon^4\right)\right) \left(-\epsilon^{-2}\, \delta_{jk} + \mathcal{O}\left(\epsilon^{-1}\right)\right) v^j \, v^k ,$$

which leads to the conclusion that for $\epsilon = 0$ we have that $(\delta^n)'' = \mu$. Because the boundary is equal to the graph of a function h such that $h(0) = 0$, $h'(0) = 0$ and $h''(0)$ is equal to the identity matrix, where Q lies above this graph, we conclude that the tangent reparametrized geodesics curve out of Q when $\mu < 1$ and into Q when $\mu > 1$.

In order to discuss the situation for $\mu = 1$, we consider the first order system of differential equations $\mathrm{d}x/\mathrm{d}s = v$, $\mathrm{d}v/\mathrm{d}s = a(x,\, v)$, which corresponds to the second order system (5.12), (5.14) with $\Gamma = ^1 \Gamma$. Let A be the vector field in the right hand side of the first order system, viewed as a vector field on the unit tangent bundle of an open neighborhood U of $U \cap \partial Q$ in $\mathbf{R}^n$. Then the fact that $(\delta^n)'' = 1$ just means that A is everywhere tangent to the unit tangent bundle of $U \cap \partial Q$, which is a smooth submanifold of codimension two in the unit tangent bundle of U. It follows that the solutions of the first order system which start in the unit tangent bundle of $U \cap \partial Q$, remain in the unit tangent bundle of $U \cap \partial Q$. But this just means that the solutions of (5.12), (5.14) with $\Gamma = ^1 \Gamma$, which are tangent to $U \cap \partial Q$ remain in $U \cap \partial Q$.

The solution curves in $U \cap Q$ of (5.12), (5.14) with $\Gamma = ^1 \Gamma$ are equal to reparametrized gradient curves of linear functions ξ. For a given nonzero covector ξ, the gradient vector field G has its i-th component equal to

$$G^i := \sum_{j=1}^n g^{ij}\, \xi_j .$$

When $i < n$ we have that

$$G^i = \sum_{j<n} \left(\epsilon\, \delta_{ij} + \mathcal{O}\left(\epsilon^2\right)\right) \xi_j + \left(\epsilon^2 + \mathcal{O}\left(\epsilon^4\right)\right) \xi_n = \epsilon\, \xi_i + \mathcal{O}\left(\epsilon^2\right),$$

whereas $G^n = \mathcal{O}\left(\epsilon^2\right)$. Therefore $\epsilon^{-1}\, G$ converges, when $\epsilon \downarrow 0$, to the orthogonal projection of ξ onto $\mathrm{T}_y(\partial Q)$, with respect to the inner product $b(y)$. Note that the multiplication with ϵ^{-1} corresponds to a reparametrization of the gradient curves which is different from the one of the system (5.12), (5.14)). In particular the gradient vector field of the restriction to $U \cap \partial Q$ of ξ, with respect to the restriction to $U \cap \partial Q$ of the Riemannian structure b, is equal to zero at a critical point of the restriction of ξ to $U \cap \partial Q$, whereas the solution δ of (5.12), (5.14)) satisfy

$$b(\delta(s))\,(\delta'(s),\,\delta'(s)) = 1,$$

and therefore their velocity never is equal to zero.

For the last statements in the proposition, we observe that the computation (5.18) of the n-th component of the Christoffel symbol implies that along a geodesic we have asymptotically, without reparametrization of the time,

$$\epsilon'' = 2\mu\,\epsilon^{-1}\,(\epsilon')^2 .$$

This equation is equivalent to

$$\left(\epsilon^{-2\mu}\,\epsilon'\right)' = 0,$$

or

$$\epsilon' = c\,\epsilon^{2\mu},$$

where c is a constant. The positive solutions of this equation can reach zero in a finite time if and only if $2\mu < 1$. q.e.d.

Remark 5.2 For the reparametrized geodesics which intersect $U \cap \partial Q$ transversally, we have the following conclusions about the distance $d(t)$ to the boundary as a function of the original time. When $\mu > \frac{1}{2}$, then $d(t)$ is of order $t^{1/(1-2\mu)}$ as $t \to \infty$. When $\mu = \frac{1}{2}$, which is the case of the geodesics of the Hessian Riemannian structure, then $d(t)$ is of order e^{-t} for geodesics with unit velocity with respect to the Hessian Riemannian structure. This exponential decrease of $d(t)$ is faster than the power law which we have for $\mu > \frac{1}{2}$. When $\mu < \frac{1}{2}$, then the geodesic reaches the boundary at a finite time T and $d(t)$ is of order $(T-t)^{1/(1-2\mu)}$ as $t \uparrow T$. $\oslash$

Remark 5.3 When f is as in Proposititon 3.1, then the restriction of b to $U \cap \partial Q$ is, up to a constant factor, equal to the affinely invariant Riemannian structure on $U \cap \partial Q$ which has been introduced by Berwald and Blaschke, cf. [2, II, §65]. $\oslash$

Remark 5.4 A Riemannian structure g is called *conformal* to the Riemannian structure $\widetilde{g}$ if there exists a positive real-valued function λ such that $g(x) = \lambda(x)\,\widetilde{g}(x)$ for every x. If we parametrize the points in Q near $U \cap \partial Q$ by $(y,\,\eta)$ via $x = y + \frac{1}{4}\,\eta^2\,\nu_\phi(y)$, then $g_{ij}(x)$ is asymptotically equal to $\frac{4}{\eta^2}\,\delta_{ij}$. In other words, in the coordinates $(y,\,\eta)$ the Riemannian structure g is asymptotically conformal to the standard Euclidean structure. In these coordinates, all the geodesic curves near the boundary $\eta = 0$ come in orthogonally to the boundary, and those with the same limit point are distinguished by their curvature, rather than by their direction. This is the familiar picture for the behaviour of the geodesics near the boundary of the Poincaré upper half space, or the Poincaré sphere. Cf. Wolf [19, Cor. 2.4.13]. $\oslash$

Remark 5.5 When f is as in Proposition 3.1 then the conclusions of Proposition 5.1 remain valid but with the word "smooth" replaced by a finite degree of differentiability, somewhat less than $\frac{n}{2}$. More precisely, my calculations indicated that the additional term $r(x)$ in $f(x) = -\ln \alpha(x) + r(x)$ leads to an additional term in the acceleration function $a(x, v)$ in $\delta''(s) = a(x, v)$, $x = \delta(s)$, $v = \delta'(s)$, which is of order $\epsilon^{\frac{n-1}{2}}$ when n is even and of order $\epsilon^{\frac{n-1}{2}}(-\ln \epsilon)$ when n is odd. This would imply that for $n \geq 4$ the acceleration function $a(x, v)$ for the reparametrized geodesics is continuously differentiable up to any order $k < \frac{n-1}{2}$, but that for $n = 2$ and $n = 3$ the acceleration function is not differentiable. $\oslash$

6 Global Results

In the case that Q is bounded and the whole boundary ∂Q (which is a compact subset of $\mathbf{R}^n$) is smooth and strongly convex, then we can use topological arguments to draw some rather strong conclusions about the behaviour of the geodesics with respect to the boundary.

Theorem 6.1 *Let Q be a bounded convex open subset of $\mathbf{R}^n$ with a smooth and strongly convex boundary ∂Q. Assume that f is a smooth and strongly convex function on Q which satisfies Assumption 2.1 along the whole boundary ∂Q. We will consider the geodesics defined by the Christoffel symbols (5.7), with the factor μ in front.*

Let $\mu \leq 1$. Then every geodesic, after suitable reparametrization near the boundary as in Proposition 5.1, eventually intersects the boundary, and the intersection is transversal. Let S denote the Euclidean unit sphere of direction vectors. For every $v \in \mathrm{S}$, let $\sigma_x(v) \in \partial Q$ denote the point of the boundary where the geodesic, which starts at x in the direction v, hits the boundary. Then σ_x is a smooth mapping from S to ∂Q, and the mapping degree of σ_x is equal to one. In particular, for every $x \in Q$ and $y \in \partial Q$ there exists a $v \in \mathrm{S}$ such that $\sigma_x(v) = y$. If $\mu = 0$ or $\mu = 1$, then σ_x is a diffeomorphism from S onto ∂Q.

For every $y \in \partial Q$, let IS_y denote the half sphere of the direction vectors $v \in \mathrm{S}_y$ such that v points inwards Q. For every $v \in \mathrm{IS}_y$, the reparametrized geodesic geodesic which starts at y in the direction v eventually hits the boundary again, and transversally, in a point $\sigma_y(v)$. σ_y is a smooth mapping from IS_y to $\partial Q \setminus \{y\}$.

Let $\widehat{\mathrm{SI}_y}$ denote the one point compactification of IS_y, which is obtained from the closure of IS_y in S_y by contracting the boundary equator to a point p_y. Now assume that $\mu < 1$. Then the definition $\sigma_y(p_y) = y$ leads to an extension of σ_y to a continuous mapping from $\widehat{\mathrm{IS}}_y$ to ∂Q, which we again denote by σ_y. This mapping has degree equal to one. In particular, for every y, $z \in \partial Q$ such that $z \neq y$ there exists a (reparametrized) geodesic in Q which starts at y and ends at z.

Proof If $\mu = 1$, then the conclusions about the mapping σ_x from S to ∂Q follow from the fact that the geodesics in this case are the gradient curves (with respect to the Hessian Riemannian structure) of the linear functions ξ. The strong convexity of ∂Q implies that the Gauss mapping ν, which assigns to $y \in \partial Q$ the exterior normal $\nu(y)$ of ∂Q at y, is a diffeomorphism from ∂Q onto S. For every $\xi \in$ S, $\nu^{-1}(\xi)$ is equal to the point of ∂Q where the restriction to ∂Q of ξ attains is maximum. Let η_x denote the mapping which assigns to $\xi \in$ S the direction of the gradient vector $g(x)^{-1}(\xi)$ at the point x. Then η_x is a diffeomorphism from S onto S. For every $\xi \in$ S, every gradient curve of ξ in Q converges to the boundary point $\nu^{-1}(\xi)$. Therefore $\sigma_x = \nu^{-1} \circ {\eta_x}^{-1}$, which is a diffeomorphism from S onto ∂Q. This conclusion is trivial in the case that $\mu = 0$, and all conclusions are trivial if $n = 1$.

Therefore, from now on in the proof, we may assume that $\mu < 1$ and $n \geq 2$. The fact that in this case the reparametrized geodesics which are tangential to ∂Q are curving out of Q, cf. Proposition 5.1, implies that if a geodesic enters Q at $y \in \partial Q$, at a small $b(y)$-angle α with $\mathrm{T}_y(\partial Q)$, then it will curve back to ∂Q, at a b-distance of order $\frac{\alpha}{1-\mu}$. The maximal b-distance to ∂Q of the part in Q of this geodesic is of order $\frac{\alpha^2}{2(1-\mu)}$. This implies that if K is a compact subset of Q (which has a positive distance to ∂Q), then there exists an $\alpha_0 > 0$, such that if γ is a geodesic which starts in K and reaches $y \in \partial Q$, then its direction vector at y has a $b(y)$-angle $> \alpha_0$ with $\mathrm{T}_y(\partial Q)$.

Let U denote the set of $(x,\, v) \in Q \times$ S such that the geodesic which starts at x in the direction v eventually (after suitable reparametrization) intersects ∂Q transversally, with first intersection point equal to $\sigma_x(v)$. It follows from the implicit function theorem that U is an open subset of $Q \times$ S and that $(x,\, v) \mapsto \sigma_x(v)$ defines a smooth mapping from U to ∂Q.

Suppose that $(x_j,\, v_j)$ is an infinite sequence in U which converges to $(x,\, v) \in Q \times$ S as $j \to \infty$. Let $y_j = \sigma_{x_j}(v_j) \in \partial Q$ and denote by w_j the direction at which the (reparametrized) geodesic arrives at y_j. Because $\partial Q \times$ S is compact, we can arrange, by passing to a subsequence if necessary, that the $(y_j,\, w_j)$ converge to some $(y,\, w) \in \partial Q \times$ S. Because the $(x_j,\, v_j)$ remain in a compact subset of $Q \times$ S, the angle of w_j with $\mathrm{T}_{y_j}(\partial Q)$ stays bounded away from zero, and the conclusion is that $w \notin \mathrm{T}_y(\partial Q)$, which in turn implies that $(x,\, v) \in U$. We therefore have proved that U is also closed in $Q \times$ S. Because $Q \times$ S is connected, the conclusion is that $U = Q \times$ S.

The mapping $\sigma_x : \mathrm{S} \to \partial Q$ also depends continuously on $\mu \in \,]-\infty,\, 1[$, and therefore its degree does not depend on μ, see for instance Schwartz [16, Thm. 1'A on p.27]. Because its degree is equal to one for $\mu = 0$, it is equal to one for every $\mu < 1$. The conclusions about the mapping σ_y from $\widehat{\mathrm{IS}}_y$ to ∂Q are obtained in a similar manner. q.e.d.

Remark 6.1 The assumptions imply that f is a self-concordant barrier function for Q, cf. Corollary 4.2. For $\mu \geq \frac{1}{2}$ we have that the (not reparametrized!) geodesics $\gamma(t)$ in Q are complete in the sense that they exist (remain in Q) for all $t \in \mathbf{R}$, cf. 5.1. For $\mu = \frac{1}{2}$ these are the geodesics for the Riemannian structure and the theorem of Hopf-Rinow states that this notion of completeness is equivalent to the completeness of Q as a (Riemannian) metric space. This confirms the theorem of Nemirovskii, cf. [5], that for any self-concordant barrier function the Hessian Riemannian metric space Q is complete. $\oslash$

Remark 6.2 The geodesics for the Hessian Riemannian structure g share with the gradient curves of linear functions the property that for every $x \in Q$ and $y \in \partial Q$ there exists a unit velocity geodesic γ such that $\gamma(0) = x$ and

$$\lim_{t \to \infty} \gamma(t) = y. \tag{6.1}$$

Moreover, the convergence in (6.1) is exponentially fast, whereas for the gradient curves of linear functions it is only of order $1/t$, cf. Proposition 5.1 and Remark 5.2.

The problem with the geodesics however is, that in general the geodesic γ from x to y is not uniquely determined. This corresponds to the phenomenon that the mapping $\sigma_x : \mathrm{S} \to \partial X$, which is surjective and has topological degree equal to one, need not be injective. Moreover, even if the geodesic γ is unique, then in general there is no simple formula for determining $\gamma'(0)$ in terms of x and y. $\oslash$

Remark 6.3 Let us look at the geodesics for the Hessian Riemannian structure defined by the barrier function f, which means that we take $\mu = \frac{1}{2}$. Let $\gamma_i(t)$, $i = 1, 2$ be geodesics with unit velocity vector with respect to the Hessian Riemannian structure. If the Riemannian distance between $\gamma_1(t)$ and $\gamma_2(t)$ remains bounded as $t \to \infty$, then the fact that near the boundary the Riemannian structure is large compared to the Euclidean one, implies that $\gamma_1(t)$ and $\gamma_2(t)$ converge to the same point $y \in \partial Q$ (with respect to the Euclidean metric) as $t \to \infty$.

Conversely, if $\gamma_1(t)$ and $\gamma_2(t)$ converge to the same point $y \in \partial Q$ as $t \to \infty$, then the fact that the b-distances from $\gamma_1(t)$ and $\gamma_2(t)$ to ∂Q are of the same order, cf. Remark 5.2, implies that for large t the difference vector $\gamma_2(t) - \gamma_1(t)$ is asymptotically parallel to the boundary. Moreover, the orbits of γ_1 and γ_2 are smooth curves which intersect ∂Q transversally at y. Because the Riemannian distance parallel to the boundary is asymptotically inversely proportional to the Euclidean distance to the boundary, it follows that the Riemannian distance of $\gamma_1(t)$ and $\gamma_2(t)$ remains bounded as $t \to \infty$.

In any complete Riemannian manifold Q where all geodesics leave every compact subset of Q, two unit velocity geodesics γ_1 and γ_2 are called *equivalent at infinity* if the Riemannian distance of $\gamma_1(t)$ and $\gamma_2(t)$ remains bounded as $t \to \infty$. The set of the equivalence classes of unit velocity geodesics is called the *sphere at infinity* of the Riemannian manifold, cf. [1]. The above observations lead, in the situation of Theoerm 6.1, to an identification of the sphere at infinity of Q with ∂Q.

The manifold Q is called a *visibility manifold*, cf. Eberlein and O'Neill, [7], if for every pair of distinct elements α, β of the sphere at infinity there is a unique unit geodesic γ such that $\gamma \in \alpha$ and $-\gamma \in \beta$, where $-\gamma(t) := \gamma(-t)$. Here the uniqueness is modulo a translation in the parametrization of the geodesics. In the situation of Theorem 6.1, we have the existence of such a geodesic γ, whereas the uniqueness corresponds to the statement that, for every $y \in \partial Q$, the mapping σ_y is injective from IS_y to $\partial Q \setminus \{y\}$. Because the mapping σ_y has degree equal to one, one could argue that *in a topological sense Q, provided with the Hessian Riemannian structure, is a visibility manifold.*

In general the uniqueness of the geodesic connecting two points of the sphere at inifinity holds if the sectional curvatures of the Riemannian structure all are negative. Therefore, if in addition to the assumptions of Theorem 6.1, all the sectional curvatures of the Hessian Riemannian structure are negative, then Q is a visibility manifold in the strict sense of the word. $\oslash$

Remark 6.4 If $\mu = 1$, when the geodesics are the gradient curves (with respect to the Hessian Riemannian structure) of the linear functions ξ, the behaviour of the mapping σ_y from IS_y to ∂Q is very different from what happens in the case $\mu < 1$. Indeed, if $\nu : \partial Q \to \mathrm{S}$ denotes the Gauss mapping, then we have for $\mu = 1$ that

$$\sigma_y(v) = \iota(y) := \nu^{-1}\left(-\nu(y)\right), \tag{6.2}$$

which is independent of $v \in \mathrm{IS}_y$. That is, all geodesics for $\mu = 1$ which leave the the boundary at y converge to the boundary at the "opposite" point $\iota(y)$.

The limit geodesics in ∂Q, cf. Proposition 5.1, have the property that those starting at y all meet again at $\iota(y)$. Riemannian manifolds for which the geodesics have this property are called *wiedersehen manifolds* after Green [8]. For surfaces this property has been studied by Blaschke [2, I, §86]. It should be emphasized however that the geodesics for $\mu = 1$ are not defined as the geodesics of a Riemannian structure on ∂Q, but rather as the gradient curves, with respect to the Riemannian structure b, of the restrictions to ∂Q of the linear functions on $\mathbf{R}^n$. Yang [20] completed the proof of the theorem that every wiedersehen manifold is isometric to the round sphere. Therefore, if the limit geodesics on ∂Q for $\mu = 1$ would be geodesics for a Riemannian structure β, then β must be isometric to the Riemannian structure of the round sphere. $\oslash$

Remark 6.5 When f is as in Proposition 3.1 then it follows from Remark 5.5 that the conclusions of Theorem 6.1 remain valid, but with the word "smooth" replaced by continuously differentiable up to an order somewhat smaller than $\frac{n}{2}$. In the case that $n = 2$ or $n = 3$, when the acceleration function may be not differentiable, we may use ϕ as the time variable for the reparametrized geodesics which intersect the boundary transversally. It is known that, for a first order system which depends in a continuously differentiable fashion on the phase space variables and in a continuous way on the time parameter, the solutions depend in a continuously differentiable fashion on the initial values. This appears to lead to a proof that for any dimension n the mappings $\sigma_x : \mathrm{S} \to \partial Q$ and $\sigma_y : \mathrm{IS}_y \to \partial Q$ are continuously differentiable, or maybe even smooth. $\oslash$

7 Curvature near the Boundary

Let ∇ be a connection (= covariant derivative) on Q. If X, Y, V are smooth vector fields on Q, then, for any $x \in Q$, the expression in the right hand side of

$$R(x)(X(x), Y(x))(V(x)) = \left(\nabla_X \nabla_Y V - \nabla_Y \nabla_X V - \nabla_{[X,Y]} V\right)(x) \tag{7.1}$$

only depends on $X(x)$, $Y(x)$, $V(x)$, and not on the derivtives of order one or two of X, Y and V at x, as one would a priori expect. Therefore (7.1) defines a $\mathrm{T}_x Q$-valued trilinear form

$$R(x) : (X, Y, V) \mapsto R(x)(X, Y)(V)$$

on $\mathrm{T}_x Q$, which is called the *Riemannian curvature tensor* of the connection ∇. Its coordinates $R^k_{lij}(x)$ are given by

$$R(x)(X, Y)(V)^k = \sum_{l,i,j=1}^{n} R^k_{lij}(x) V^l X^i Y^j, \tag{7.2}$$

and expressed in terms of the Christoffel symbols of ∇ by means of the formula (5.9).

One has the corresponding quadrilinear real-valued form

$$(X, Y, U, V) \mapsto g(x)\left(R(x)(X, Y)(V), U\right),$$

with its coordinates

$$R_{klij} := g(x)\left(R(x)\left(e_i, e_j\right)\left(e_l\right), e_k\right) = \sum_{m=1}^{n} g_{km}(x)\, R^m_{lij}(x). \tag{7.3}$$

Here $e_1, \ldots, e_n$ denotes the standard basis in $\mathbf{R}^n$.

A straightforward computation yields that for a Hessian Riemannian structure the Riemannian curvature tensor is given by

$$\begin{aligned} R_{klij}(x) = -\tfrac{1}{4} \sum_{p,q=1}^{n} & g^{pq}(x) \\ & \left[\partial_k \partial_i \partial_p f(x) \cdot \partial_j \partial_l \partial_q f(x) - \partial_k \partial_j \partial_p f(x) \cdot \partial_i \partial_l \partial_q f(x)\right], \end{aligned} \tag{7.4}$$

in which $g^{pq}(x)$ is the inverse of the matrix $g_{ij}(x) = \partial_i \partial_j f(x)$. It is a bit surprising that this formula involves only the derivatives of f of order two and three, and no derivatives of order four as one would expect a priori. An interpretation of (7.4) in terms of the general linear group, viewed as an orthonormal frame bundle over the space of positive definite symmetric matrices, has been given in [6].

A more natural system of coordinates for the curvature tensor is obtained by choosing an orthonormal basis $F = (f_1, \ldots, f_n)$ with respect to the inner product $g(x)$, and then writing

$$R^F_{klij} := g(x)\left(R(x)\left(f_i, f_j\right)\left(f_l\right), f_k\right). \tag{7.5}$$

One may write $f_i = \sum_{j=1}^{n} F_i^j \, e_j$ for a unique $n \times n$-matrix F, and the orthonormality of F with respect to $g(x)$ then is expressed by the equation

$$ {}^{\mathrm{t}}F \, g(x) \, F = 1, \quad \text{or} \quad g(x) = {}^{\mathrm{t}} F^{-1} \circ F^{-1}. $$

The matrix $\widetilde{F}$ represents another orthonormal basis with respect to $g(x)$, if and only if $\widetilde{F} = F \circ C$ for an orthogonal $n \times n$-matrix C with respect to the standard Euclidean inner product. If the orthonormal basis $F = F(x)$ depends (smoothly) on the point X, then it is called a (smoothly) *moving frame*. Moving frames have been introduced as a very useful tool in differential geometry by Élie Cartan.

If P is a two-dimensional linear subspace of $\mathrm{T}_x \, Q$, a tangent plane at the point x, and U, V is a $g(x)$-orthonormal basis in P, then the real number

$$ K(x, \, P) = g(x)(R(x)(U, \, V)(V), \, U) \tag{7.6} $$

only depends on x and P and not on the choice of U and V, and is called the *sectional curvature of the plane* P. If the Riemannian manifold is complete and simply connected (which means that every closed curve is contractible in the manifold), and all the sectional curvatures are nonpositive, then every pair of points in the manifold is joined by a *unique* geodesic. In the case of a two--dimensional Riemannian manifold, this theorem is due to Hadamard [9]. The generalization to manifolds of an arbitrary dimension, indicated by Hadamard, has been proved by Élie Cartan in [4, Note III, pp. 254–267].

For a given x, all tangent planes P at x have the same sectional curvature $K(x, \, P) = K(x)$, if and only if for some (and hence every) $g(x)$-orthonormal basis F we have that

$$ R^F_{klij} = K \, [\delta_{ki} \, \delta_{lj} - \delta_{kj} \, \delta_{li}] \, , \tag{7.7} $$

where $R^F_{klij} = R^F_{klij}(x)$ and $K = K(x)$. The right hand side of (7.7) will be referred to as a *curvature tensor with constant curvature equal to* K. See Wolf [19, Cor. 2.2.5]. It follows that, when $y \in \partial Q$, the following conditions (i), (ii) are equivalent.

(i) All sectional curvatures $K(x, \, P)$, where P is a tangent plane at x, converge to K as $x \to y$.

(ii) For some (and hence every) moving frame $F(x)$ we have that $R^{F(x)}_{klij}(x)$ converges to the right hand side of (7.7) as $x \to y$.

We will say that *the curvature tensor at x converges to the curvature tensor with constant curvature equal to K* as $x \to y$, when (ii), or equivalently (i), holds.

Theorem 7.1 *Let Q be a convex domain in $\mathbf{R}^n$ and f a self-concordant barrier function for Q which satisfies Assumption 2.1 at the open part $U \cap \partial Q$ of the boundary ∂Q of Q in $\mathbf{R}^n$. Then the curvature tensor at x of the Hessian Riemannian structure $\partial_i \partial_j f(x)$ converges to the curvature tensor with constant curvature equal to $-\frac{1}{4}$, as x tends to $U \cap \partial Q$.*

Proof As in Section 4, we write $x = y + \epsilon\,\nu_\phi(y)$, with $y \in U \cap \partial Q$ and $\epsilon > 0$ small. With a suitable affine substitution of variables, we can arrange that $y = 0$, $\nu_\phi(y) = e_n$ and (4.11). This implies the expansions (4.22), (4.23), (4.24), (4.25), (4.26), (4.27) and (4.28). Using these we obtain the following expansions for (7.3).

If $k,\, l,\, i,\, j < n$, then we split the sum over all $p,\, q$ in (7.3) in the sum over all $p,\, q < n$, over all $p < n$, $q = n$, over all $q < n$, $p = n$ and the term with $p = q = n$. This leads to

$$\begin{aligned} R_{klij} &= \sum_{p,\,q<n} \mathcal{O}(\epsilon)\,\mathcal{O}\left(\epsilon^{-1}\right)\,\mathcal{O}\left(\epsilon^{-1}\right) \\ &\quad + \left(\sum_{p<n} + \sum_{q<n}\right)\,\mathcal{O}\left(\epsilon^{3}\right)\,\mathcal{O}\left(\epsilon^{-1}\right)\,\mathcal{O}\left(\epsilon^{-2}\right) \\ &\quad -\tfrac{1}{4}\left(\epsilon^2 + \mathcal{O}\left(\epsilon^4\right)\right)\,\left[\delta_{ki}\,\delta_{lj} - \delta_{kj}\,\delta_{li} + \mathcal{O}(\epsilon)\right]\,\epsilon^{-4} \\ &= -\tfrac{1}{4}\left[\delta_{ki}\,\delta_{lj} - \delta_{kj}\,\delta_{li}\right]\,\epsilon^{-2} + \mathcal{O}\left(\epsilon^{-1}\right). \end{aligned} \tag{7.8}$$

If $k,\, l,\, i < n$ and $j = n$, then a similar computation yields that

$$\begin{aligned} R_{klin} &= -\tfrac{1}{4}\sum_{p,\,q<n}\left(\epsilon\,\delta_{pq} + \mathcal{O}\left(\epsilon^2\right)\right) \\ &\quad \Big[\left(-\epsilon^{-1}\,\phi_{kip} + \mathcal{O}(1)\right)\,\left(-\epsilon^{-2}\,\delta_{ql} + \mathcal{O}\left(\epsilon^{-1}\right)\right) \\ &\quad - \left(-\epsilon^{-2}\,\delta_{kp} + \mathcal{O}\left(\epsilon^{-1}\right)\right)\,\left(-\epsilon^{-1}\,\phi_{liq} + \mathcal{O}(1)\right)\Big] \\ &\quad + \left(\sum_{p<n} + \sum_{q<n}\right)\,\mathcal{O}\left(\epsilon^{3}\right)\,\mathcal{O}\left(\epsilon^{-2}\right)\,\mathcal{O}\left(\epsilon^{-2}\right) \\ &\quad + \mathcal{O}\left(\epsilon^{2}\right)\,\mathcal{O}\left(\epsilon^{-2}\right)\,\mathcal{O}(1) \\ &= \mathcal{O}\left(\epsilon^{-1}\right), \end{aligned} \tag{7.9}$$

because the coefficient of ϵ^{-2} in the sum over $p,\, q < n$ is equal to zero.

Finally, if $k,\, i < n$ and $l = j = n$, then we obtain

$$\begin{aligned} R_{knin} &= -\tfrac{1}{4}\sum_{p,\,q<n}\left(\epsilon\,\delta_{pq} + \mathcal{O}\left(\epsilon^2\right)\right) \\ &\quad \left[\mathcal{O}\left(\epsilon^{-1}\right)\,\mathcal{O}(1) - \left(-\epsilon^{-2}\,\delta_{kp} + \mathcal{O}\left(\epsilon^{-1}\right)\right)\,\left(-\epsilon^{-2}\delta_{iq} + \mathcal{O}\left(\epsilon^{-1}\right)\right)\right] \\ &\quad + \left(\sum_{p<n} + \sum_{q<n}\right)\,\mathcal{O}\left(\epsilon^{3}\right)\,\mathcal{O}\left(\epsilon^{-1}\right)\,\mathcal{O}\left(\epsilon^{-3}\right) \\ &\quad -\tfrac{1}{4}\left(\epsilon^2 + \mathcal{O}\left(\epsilon^4\right)\right)\,\left[\left(-\epsilon^{-2}\,\delta_{ki} + \mathcal{O}\left(\epsilon^{-1}\right)\right)\,\left(-2\epsilon^{-3} + \mathcal{O}(1)\right) - \mathcal{O}(1)\,\mathcal{O}(1)\right] \\ &= -\tfrac{1}{4}\,\epsilon^{-3}\,\delta_{ki} + \mathcal{O}\left(\epsilon^{-2}\right). \end{aligned} \tag{7.10}$$

The other coefficients can be expressed in the above, because $R_{klji} = -R_{klij}$ (which implies that $R_{klii} = 0$) and $R_{klij} = R_{ijkl}$.

Let $f_n, f_{n-1}, \ldots, f_1$ be the $g(x)$-orthonormal basis (depending smoothly on x), which is obtained from the standard basis $e_n, e_{n-1}, \ldots, e_1$ by means of the Gram-Schmidt orthogonalization procedure. Then

$$f_n = g_{nn}(x)^{-1/2}\, e_n = \epsilon \left(1 + \mathcal{O}\left(\epsilon^2\right)\right)\, e_n. \tag{7.11}$$

For $i < n$ we have that

$$f_i = \sum_{j \geq i} c_j\, e_j,$$

where the coefficients c_j are chosen in such a way that

$$g(x)\,(f_i,\, e_k) = 0, \quad k > i,$$

and $g(x)\,(f_i,\, f_i) = 1$. The equation for $k = n$ yields that

$$c_n = \epsilon^2 \sum_{i \leq j < n} c_j\, \mathcal{O}(1), .$$

With downward induction one obtains that, for every $i < j < n$,

$$c_k = \epsilon \sum_{i \leq j < k} c_j\, \mathcal{O}(1),$$

which in turn implies that

$$c_n = c_i\, \mathcal{O}\left(\epsilon^2\right), \quad \text{and} \quad c_k = c_i\, \mathcal{O}(\epsilon), \quad i < k < n.$$

Substituting this in the equation $g(x)\,(f_i,\, f_i) = 1$, we obtain that

$$\begin{aligned}
c_i &= \epsilon^{1/2}\,(1 + \mathcal{O}(\epsilon)), \\
c_j &= \mathcal{O}\left(\epsilon^{3/2}\right), \quad i < j < n, \\
c_n &= \mathcal{O}\left(\epsilon^{5/2}\right),
\end{aligned}$$

which in turn implies that

$$f_i = \epsilon^{1/2} \left[e_i + \sum_{i \leq j < n} \mathcal{O}(\epsilon)\; e_j + \mathcal{O}\left(\epsilon^2\right)\; e_n \right]. \tag{7.12}$$

when $i < n$.

Replacing e_i, e_j, e_k, e_l in (7.3) by f_i, f_j, f_k, f_l, respectively, and applying the above expansions, we arrive at

$$R^F_{klij} = -\tfrac{1}{4}\,[\delta_{ki}\,\delta_{lj} - \delta_{kj}\,\delta_{li}] + \mathcal{O}(\epsilon), \tag{7.13}$$

which completes the proof of the theorem. q.e.d.

Remark 7.1 Combining Theorem 7.1 with the theorem of Hadamard and É. Cartan mentioned after (7.6), one obtains that near $U \cap \partial Q$ the geodesics cannot intersect more than once. Of course, this property also follows from the descriprion of the geodesic orbits near $U \cap \partial Q$ in Proposition 5.1.

Related to the question of uniqueness of geodesics are the questions of injectivity of the mappings $\sigma_x : \mathrm{S} \to \partial Q$, $x \in Q$ and of the mappings $\sigma_y : \widehat{\mathrm{IS}}_y \to \partial Q$, $y \in \partial Q$, introduced in Theorem 6.1. I have the impression that curvture estimates which imply the injectivity of $\sigma_y : \widehat{\mathrm{IS}}_y \to \partial Q$ for each $y \in Q$ are stronger than those which would lead to the injectivity of $\sigma_x : \mathrm{S} \to \partial Q$ for each $x \in Q$, which in turn would be stronger than estimates which imply unique joining of geodesics.

I have no example of a self-concordant barrier function for which joining of geodesics is not unique, but I admit that I have not searched for it very systematically. It would be interesting to have some insight into the somewhat more specific question whether geodesic joining is unique for the Hessian Riemannian structure of the universal barrier functions of an arbitrary convex domain. The example of the triangle in Section 8 shows that such a Riemannian structure can have positive curvature, which shows that in general unique joining of geodesics for the universal barrier function of arbitrary convex domains cannot be concluded on the basis of only applying the theorem of Hadamard and É. Cartan. $\oslash$

Remark 7.2 With more work, it is probably also feasible to compute the linear term in the distance to $U \cap \partial Q$ of the curvature tensor. In the case of the universal barrier function, there may be a relationship between this linear term and properties of the affinely invariant metric on $U \cap \partial Q$, introduced by Berwald and Blaschke, cf. [2, II, §65]. $\oslash$

Remark 7.3 When f is as in Proposition 3.1 then the calculations behind Remark 5.5 indicate that the conclusion of Theorem 7.1 remains valid. The only difference in the proof appears to be that in (7.13) we have to replace $\mathcal{O}(\epsilon)$ by $\mathcal{O}\left(\epsilon^{1/2}\right)$ when $n = 2$ and by $\mathcal{O}\left(-\epsilon \ln \epsilon\right)$ when $n = 3$. $\oslash$

8 Some Simple Examples

8.1 The Parabolic Domain

Consider the parabolic domain P in $\mathbf{R}^n$ defined by

$$P = \{(u, v) \in \mathbf{R}^{n-1} \mid v > \tfrac{1}{2} \langle u, u \rangle\}. \tag{8.1}$$

The supporting function of P, cf. (3.2), is given by

$$p(\alpha, \beta) = \begin{cases} \infty & \text{when} \quad \beta \geq 0, \\ -\frac{1}{2\beta} \langle \alpha, \alpha \rangle & \text{when} \quad \beta < 0. \end{cases}, \quad (\alpha, \beta) \in \mathbf{R}^{n-1} \times \mathbf{R}.$$

The polar set $P^*(u, v)$ with respect to the point $(u, v) \in P$, cf. (3.1), is equal to the set of $(\alpha, \beta) \in \mathbf{R}^{n-1} \times \mathbf{R}$ such that

$$\begin{aligned} 0 &\geq \langle \alpha, \alpha \rangle + 2\langle u, \alpha \rangle \beta + 2v\,\beta^2 + 2\beta \\ &= \langle \alpha + \beta\, u, \alpha + \beta\, u \rangle + (2v - \langle u, u \rangle) \left(\beta - \frac{1}{2v - \langle u, u \rangle} \right)^2 - \frac{1}{2v - \langle u, u \rangle}. \end{aligned}$$

By means of the substitution of variables

$$\alpha = \alpha' - \beta\, u, \quad \beta = (2v - \langle u, u \rangle)^{-1/2}\, \beta'$$

the ellipsoid $P^*(u, v)$ corresponds to the sphere in the (α', β')-space with center at the origin and radius equal to r, where

$$r = (2v - \langle u, u \rangle)^{-1/2}.$$

It follows that

$$I(u, v) := \mathrm{vol}_n\left(P^*(u, v)\right) = (2v - \langle u, u \rangle)^{-\frac{n+1}{2}}\, \mathrm{vol}_n\left(\mathrm{B}^n\right),$$

if B^n denotes the ball in $\mathbf{R}^n$ with radius equal to one. Therefore the function $f = f_P$ in Proposition 3.1 is equal to

$$f(u, v) = -\ln\left(v - \tfrac{1}{2}\langle u, u \rangle\right) + c, \tag{8.2}$$

where c is a constant. In particular f satisfies Assumption 2.1.

The Hessian Riemannian structure is given by

$$\begin{aligned} g_{ij} &:= \partial_i \partial_j f = \phi^{-1}\, \delta_{ij} + \phi^{-2}\, u^i\, u^j, \quad 1 \leq i, j \leq n-1, \\ g_{in} &= g_{ni} := \partial_i \partial_n f = -\phi^{-2}\, u^i, \quad 1 \leq i \leq n-1 \\ g_{nn} &:= {\partial_n}^2 f = \phi^{-2}, \end{aligned}$$

where

$$\phi = \phi(u, v) = v - \tfrac{1}{2}\langle u, u \rangle.$$

For every $a \in \mathbf{R}^{n-1}$ the affine substitution of variables $u = a + u'$, $v = \frac{1}{2}\langle a, a \rangle + v' + \langle a, u' \rangle$ transforms P onto itself. Because at the origin the n-th standard basis vector e_n obviously is equal to the interior affine normal of ∂P, it follows that at any point of ∂P the affine normal of ∂P is equal to e_n. Inspired by Remark 5.4, we apply the substitution of variables

$$u = u, \quad v = \tfrac{1}{2}\langle u, u \rangle + \tfrac{1}{4}\, w^2. \tag{8.3}$$

In the variables $(u, w) \in \mathbf{R}^{n-1} \times \mathbf{R}_{>0}$, the Riemannian structure then turns out to be equal to $\frac{4}{w^2}$ times the standard Euclidean Riemannian structure. This diffeomorphism with the Poincaré upper half space with all sectional curvatures constant equal to $-1/4$ has been found before by Shima [17, §2, Case B].

The geodesic orbits in P are the images, under the mapping $(u, w) \mapsto (u, v)$ defined by (8.3), of the half circles in the upper half space which are orthogonal to the boundary, cf. Wolf [19, Cor. 2.4.13]. The geodesic orbits through the origin are contained in the plane spanned by e_n and the direction vector of the geodesic orbit at the origin. For the description of these geodesic orbits we may assume that $n = 2$. In this case a circle in the (u, w)-upper half plane, which is orthogonal to the boundary, is determined by an equation of the form $(u + a)^2 + w^2 = a^2$, which is equivalent to

$$v = \tfrac{1}{4}\,(u - a)^2 - \tfrac{1}{4}\,a^2.$$

This equation describes a parabola in the (u, v)-plane with vertical asymptotes, lowest point at $u = a$, $v = -\frac{1}{4}\,a^2$, and factor $\frac{1}{4}$ in front of the quadratic term instead of the factor $\frac{1}{2}$ for the boundary. Therefore the parabola intersects the boundary not only at the origin, but also at a second point $u = -2a$, $v = 2a^2$.

The only geodesic orbit which is missing in this description, is the vertical ray emanating from the origin. This is also the only one which does not intersect the boundary for a second time. The geodesic orbits emanating from other boundary points are obtained from the ones through the origin by applying the affine transformations which map P onto itself. These orbits therefore too are either vertical rays or pieces in P of parabolas which intersect the boundary twice. I learned this description of the geodesics in P from a lecture of professor Nesterov at the conference HPOPT'99 at the Erasmus University in Rotterdam, June 17, 1999.

The parabolic domain can be viewed as the prototype for the properties described in Proposition 5.1 and Theorem 7.1 — and also for Theorem 6.1 if we add one point at infinity to the boundary of P.

8.2 The Ball

Consider the unit ball

$$B = \{x \in \mathbf{R}^n \mid \|x\| < 1\}$$

in $\mathbf{R}^n$, where $\|x\| = \langle x,\, x\rangle^{1/2}$. Its polar set $B^*(x)$ with respect to the point $x \in B$, cf. (3.1), is equal to the set of $\xi \in \mathbf{R}^n$ such that

$$\|\xi\| - \langle x,\, \xi\rangle \leq 1 \Longleftrightarrow \langle \xi,\, \xi\rangle \leq 1 + 2\langle x,\, \xi + \langle x,\, \xi\rangle^2.$$

If $x = r\, e_n$ and $\xi = (\eta,\, \zeta) \in \mathbf{R}^{n-1} \times \mathbf{R}$, then the latter inequality is equivalent to

$$\|\eta\|^2 + \left(\sqrt{1 - r^2}\,\zeta - \frac{r}{\sqrt{1 - r^2}}\right)^2 \leq \frac{r^2}{1 - r^2} + 1 = \frac{1}{1 - r^2},$$

and it follows that

$$\mathrm{vol}_n\,(B^*(x)) = \left(1 - \|x\|^2\right)^{-\frac{n+1}{2}}\ \mathrm{vol}_n(B).$$

Therefore the function f_B in Proposition 3.1 is equal to

$$f_B(x) = -\ln\left(1 - \|x\|^2\right) \tag{8.4}$$

plus a constant. In particular, it satisfies Assumption 2.1.

A large part of the analysis can be given in the somewhat more general situation of a rotationally symmetric barrier function f, which means that

$$f(x) = F(r), \quad r = \|x\|, \tag{8.5}$$

where F is an even smooth function of one variable. Using that $\partial_i r = x^i/r$, we obtain that

$$\begin{aligned}
\partial_i f(x) &= F'(r)\,\frac{x^i}{r}, \\
\partial_i\partial_j f(x) &= \left(F''(r) - \frac{F'(r)}{r}\right)\frac{x^i\,x^j}{r^2} + \frac{F'(r)}{r}\,\delta_{ij}, \\
\partial_i\partial_j\partial_k f(x) &= \left[\frac{\mathrm{d}}{\mathrm{d}r}\left(F''(r) - \frac{F'(r)}{r}\right) - \frac{2}{r}\left(F''(r) - \frac{F'(r)}{r}\right)\right]\frac{x^i\,x^j\,x^k}{r^3} \\
&\quad + \left(F''(r) - \frac{F'(r)}{r}\right)\frac{x^i\,\delta_{jk} + x^j\,\delta_{ki} + x^k\,\delta_{ij}}{r^2}.
\end{aligned}$$

At the point x such that $x^n = r$ and $x^j = 0$ for $j < n$, these formulas simplify to

$$\begin{aligned}
\partial_n f(x) &= F'(r), & (8.6)\\
\partial_i\partial_j f(x) &= \frac{F'(r)}{r}\,\delta_{ij} \quad \text{when} \quad i,\, j < n, & (8.7)\\
\partial_i\partial_n f(x) &= F''(r), & (8.8)\\
\partial_i\partial_j\partial_n f(x) &= \left(F''(r) - \frac{F'(r)}{r}\right)\frac{1}{r}\,\delta_{ij} \quad \text{when} \quad i,\, j < n, & (8.9)\\
{\partial_n}^3 f(x) &= F'''(r), & (8.10)
\end{aligned}$$

whereas all the other partial derivatives up to the order three are equal to zero.

The strong convexity of f, needed in order that the Hessian of f defines a Riemannian structure $g_{ij}(x)$ in B, is equivalent to the conditions that $F'(r) > 0$ for every $r > 0$ and $F''(r) > 0$ for every r. Note that the limit of $F'(r)/r$ as $r \downarrow 0$ is equal to $F''(0)$, and therefore is strictly positive too.

It is easy to verify that if γ is a geodesic and $x := \gamma'(0)$ and $v := \gamma(0)$ are linearly independent, then $\gamma(t)$ stays in the plane spanned by x and v. If x and v are linearly dependent, then $\gamma(t)$ stays on a straight line through the origin. These facts can be proved as a consequence of the invariance under rotations of the kinetic energy function, which in view of Noether's principle implies the constancy of the corresponding angular momentum. It can also be proved by verifying that the acceleration $\gamma''(t)$ is equal to a linear combination of $\gamma(t)$ and $\gamma'(t)$, by substituting (5.6) and (8.7) — (8.10) into the equation (5.5) for the geodesics. For the analysis of the geodesics we may therefore restrict ourselves to the case that $n = 2$.

In polar coordinates the kinetic energy with respect to the Hessian Riemannian structure (8.7) — (8.8) is given by

$$T = \tfrac{1}{2}\left[F''(r)\,\dot{r}^2 + \frac{F'(r)}{r}\left(r\,\dot{\phi}\right)^2\right]. \tag{8.11}$$

The geodesics are the solutions of the Euler-Lagrange equations for the kinetic energy function; as consequence T is a constant of motion. Because T does not depend on ϕ, the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi}$$

implies that the angular momentum

$$I := \frac{\partial T}{\partial \dot{\phi}} = F'(r)\, r\, \dot{\phi} \tag{8.12}$$

is a second constant of motion. This can be substituted in (8.11), from which we subsequently solve $\dot{r}^2$:

$$\dot{r}^2 = \left[2T - \frac{I^2}{F'(r)\, r}\right] / F''(r).$$

It follows from the other Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(F''(r)\,\dot{r}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{r}} = \frac{\partial T}{\partial r} = \tfrac{1}{2}\left[F'''(r)\,\dot{r}^2 + \left(F''(r)\, r + F'(r)\right)\dot{\phi}^2\right]$$

that $\ddot{r} > 0$ when $\dot{r} = 0$ and $r > 0$. We conclude that there is a time t_0 such that $\dot{r}(t)$ has the same sign as $t - t_0$; actually $r(t)$ is symmetric with respect to the reflection of t around t_0. We have

$$\tau := \frac{2T}{I^2} = \frac{1}{F'(r_0)\; r_0}, \tag{8.13}$$

if $r_0 := r(t_0)$ denotes the minimal value of the distance $r(t)$ to the origin of the geodesic.

For $t > t_0$ we have

$$\dot{r} = \left[2T - \frac{I^2}{F'(r)\, r}\right]^{1/2} F''(r)^{-1/2}. \tag{8.14}$$

The quantity between the square brackets is an increasing function of r and therefore is strictly positive between r_0 and 1. It follows that $r(t)$ increases to 1 for $t > t_0$. Combining (8.14) with $\dot{\phi} = I/F'(r)\, r$, we obtain that the increase of the angle along the geodesic, as a function of r, is given by

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{\dot{\phi}}{\dot{r}} = \frac{I}{F'(r)\, r}\left[2T - \frac{I^2}{F'(r)\, r}\right]^{-1/2} F''(r)^{1/2}. \tag{8.15}$$

We will assume in the subsequent calculations that $I > 0$.

Because $F(r) \to \infty$ as $r \uparrow 1$, $F'(r)$ cannot remain bounded. Because $F'(r)$ is increasing, it follows that $F'(r) \uparrow \infty$ as $r \uparrow 1$, and the expression between the square brackets converges to $2T$. It follows that, for r close to 1, $\mathrm{d}\phi/\,\mathrm{d}r$ is of the order

$$\tau^{-1/2}\,\frac{F''(r)^{1/2}}{F'(r)} = \tau^{-1/2}\,\frac{F'(r)}{F''(r)^{1/2}}\,\frac{F''(r)}{F'(r)^2} \le \tau^{-1/2}\, {C_1}^{1/2}\,\frac{F''(r)}{F'(r)^2},$$

if we assume the estimate (iii) for a self-concordant barrier function, cf. Definition 2.1. Because

$$\int_{r_0}^{r_1} \frac{F''(r)}{F'(r)^2}\,\mathrm{d}r = \frac{1}{F'(r_0)} - \frac{1}{F'(r_1)}$$

converges to $1/F'(r_0)$ as $r_1 \uparrow 1$, it follows that $\phi(r)$ converges when $r \uparrow 1$.

We have therefore proved that *every geodesic $\gamma(t)$ in B converges for $t \to \pm\infty$ to a point on the boundary ∂B.* The conditions for F which we needed in the proof are much weaker than Assumption 2.1; we did not even need the estimate (iv) in Definition 2.1 for a self-concordant barrier function.

For the universal barrier function of the ball we have $F(r) = -\ln(1-r^2)$, in which case

$$F'(r) = \frac{2r}{1-r^2}, \quad F''(r) = 2\frac{1+r^2}{(1-r^2)^2}, \tag{8.16}$$

and therefore (8.15) takes the form

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{1}{r}\left[\frac{1+r^2}{(2\tau+1)\,r^2-1}\right]^{1/2}, \tag{8.17}$$

with τ as in (8.13). Note that the right hand side converges to $\tau^{-1/2} = I/\sqrt{2T}$ when $r \uparrow 1$.

The integral of the right hand side of (8.17) can be computed by making the substitution of variables $r = s^{1/2}$, followed by the substitution

$$\frac{1+s}{(2\tau+1)\,s-1} = y^2 \Longleftrightarrow s = \frac{1+y^2}{(2\tau+1)\,y^2-1}.$$

The integrand then is a rational function of y, with the factors $1+y^2$ and $(2\tau+1)\,y^2-1$ in the denominator. This leads to the formula

$$\begin{aligned}\phi(r) - \phi(r_0) &= \frac{\pi}{2} - \arctan\left(r_0\left[\frac{1+r^2}{r^2-{r_0}^2}\right]^{1/2}\right)\\ &\quad + \frac{r_0}{2}\ln\frac{(1+r^2)^{1/2} + (r^2-{r_0}^2)^{1/2}}{(1+r^2)^{1/2} - (r^2-{r_0}^2)^{1/2}}. \end{aligned}\tag{8.18}$$

for the increase of the angle along the geodesic.

The arc α on the boundary between the intersection points of the geodesic with the boundary has length equal to $2(\phi(1) - \phi(r_0))$, for which (8.18) yields an explicit formula in terms of the minimal distance r_0 of the geodesic to the origin. On the other hand, the angle β of the geodesic at the boundary with the normal is equal to $\tau^{-1/2}$, where $\tau = (1-{r_0}^2)/2{r_0}^2$. This leads to an explicit formula for α in terms of β, or equivalently, for the mapping $\sigma_y : \mathrm{IS}_y \to \partial B$ which is discussed in Theorem 6.1.

Using (8.7) — (8.8) we obtain that at a point $(0, r)$ an orthonormal basis is formed by the vectors

$$f_i = \left(\frac{F'(r)}{r}\right)^{-1/2} e_n$$

for $1 \le i \le n-1$, and

$$f_n = F''(r)^{-1/2}\, e_n.$$

Using also (8.9), (8.10), we obtain for the curvature R^F with respect to this orthonormal basis, as defined in (7.5), that

$$R^F_{klij}(x) = -\tfrac{1}{4}\, F'(r)^{-2}\, F''(r)^{-1} \left(F''(r) - \frac{F'(r)}{r}\right)^2 (\delta_{ki}\,\delta_{lj} - \delta_{kj}\,\delta_{li})$$

when $i,\ j,\ k,\ l < n$. This implies that all sectional curvatures in planes orthogonal to the vector x (which occur when $n \ge 3$) are nonpositive. Furthermore,

$$R^F_{knin}(x) = -\tfrac{1}{4}\, F'(r)^{-1}\, F''(r)^{-1} \left(F''(r) - \frac{F'(r)}{r}\right) \left(\frac{F'''(r)}{F''(r)} - \frac{F''(r)}{F'(r)} + \frac{1}{r}\right) \delta_{ki}.$$

Finally, $R^F_{klij}(x) = 0$ when one, three, or four of the $i,\ j,\ k,\ l$ are equal to n.

In the case of the universal barrier function, when $F(r) = -\ln\left(1 - r^2\right)$, we have in view of (8.16) that the sectional curvatures in the planes orthogonal to x are equal to $-\frac{r^2}{2(1+r^2)}$. The sectional curvature of each plane which contains the vector x (which is the only one if $n = 2$) is equal to $-\frac{r^2}{(1+r^2)^2}$. In both cases the sectional curvature is equal to zero at the origin and decreases to $-\frac{1}{4}$ when $r \uparrow 1$. The sectional curvatures of the other planes are in between. In particular, because all sectional curvatures are nonpositive, an application of the theorem of Hadamard and É. Cartan yields that *geodesic joining is unique for the unit ball, provided with the Hessian Riemannian structure of its universal barrier function.*

For any even function f the curvature has to vanish at the origin, because the third order derivatives of an even function vanish at the origin. Still, one could try to find a function F for which the sectional curvatures are constant in some part of the ball. At the radius r the sectional curvatures of the different planes are equal to each other, if and only if

$$F'(r)^{-1} \left(F''(r) - \frac{F'(r)}{r}\right) = \frac{F'''(r)}{F''(r)} - \frac{F''(r)}{F'(r)} + \frac{1}{r}.$$

The solutions of this differential equation are given by

$$F(r) = a\,\left(-r - b\,\ln(b - r)\right) + c,$$

where a, b and c are arbitrary constants. If we want that $F(r) \to \infty$ as $r \uparrow 1$, then we have to take $b = 1$ and $a > 0$. In that case we also have that $F'(r) > 0$ and $F''(r) > 0$ for $r > 0$. When $a = 1$, the function $f(x) = F\left(\|x\|\right)$ satisfies Assumption 2.1 near the boundary. However, f is not differentiable at the origin.

8.3 The Corner

A simple example of a completely different nature is the corner

$$C = \left\{x \in \mathbf{R}^n \mid \forall 1 \le i \le n:\ x^i > 0\right\}. \tag{8.19}$$

Here the boundary is flat at each smooth point, and therefore there are no points where Assumption 2.1 holds.

The polar set $C^*(x)$ of C with respect to the point $x \in C$, cf. (3.1), is equal to the set of $\xi \in \mathbf{R}^n$ such that $\xi_i \leq 0$ for each i and $\langle x, \xi \rangle \geq -1$. The substitution of variables $\xi_i = \eta_i / x^i$ yields that the n-dimensional volume $I_C(x)$ of $C^*(x)$ is equal to a positive constant times the product of the x^i, and therefore the universal barrier function $\ln I_C(x)$ of C is equal to

$$f(x) = -\sum_{i=1}^{n} \ln x^i \quad \text{plus a constant.} \tag{8.20}$$

In the notation of Corollary 4.2 we have for every $x \in C$ that $C_1(f, x) = n$ and $C_2(f, x) = 1$, and therefore the parameter $\theta(f)$ of f is equal to n, the minimal value for any self-concordant barrier function of a convex domain with corners.

The Hessian Riemannian structure of (8.20) is given by

$$\partial_i \partial_j f(x) = \left(x^i\right)^{-2} \delta_{ij}. \tag{8.21}$$

The corresponding kinetic energy function is equal to

$$T = \tfrac{1}{2} \sum_{i=1}^{n} \left(x^i\right)^{-2} \left(\dot{x}^i\right)^2. \tag{8.22}$$

The Euler-Lagrange equations for the geodesics therefore decouple into the second order equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\left(x^i\right)^{-2} \dot{x}^i \right) = -\left(x^i\right)^{-3} \left(\dot{x}^i\right)^2. \tag{8.23}$$

for each of the coordinates x^i. In turn this implies that the kinetic energy

$$T_i := \tfrac{1}{2} \left(x^i\right)^{-2} \left(\dot{x}^i\right)^2$$

of each of the coordinates is constant, hence $\dot{x}^i / x^i$ is constant, which means that the geoedsics are given by

$$x^i(t) = x^i(0)\, \mathrm{e}^{c^i t}, \quad c^i = \dot{x}^i(0)/x^i(0), \quad 1 \leq i \leq n. \tag{8.24}$$

The behaviour of the geodesics is very different from what happens in the presence of a smooth and strongly convex boundary. For $t \to \infty$ the geodesic (8.24) runs away to infinity unless we have for each i that $\dot{x}^i(0) \leq 0$. In the latter case the geodesic converges to the boundary point y, where $y^i = 0$ when $\dot{x}^i(0) < 0$ and $y^i = x^i(0)$ when $\dot{x}^i(0) = 0$. In particular, if y lies in a k-dimensional face F of the boundary, then the geodesics which converge to y are lying in the intersection with C of the $(n-k)$-dimensional affine subspace of $\mathbf{R}^n$ through y which is orthogonal to F. In other words, the larger the dimension of the face, the smaller is the dimension of the manifolds of geodesics which converge to a given point of the face. The origin "catches" an open set of geodesics.

In view of (8.24) it will not be surprising that with the substitution of variables $x^i = \mathrm{e}^{t^i}$ the Hessian Riemannian structure (8.21) corresponds to the standard Euclidean Riemannian structure in the t^i-space $\mathbf{R}^n$. In particular it follows that *the curvature of the Hessian Riemannian structure (8.21) is equal to zero.*

Similar conclusions hold for the Cartesian product of n intervals, bounded or semi-bounded, and for their images under affine transformations, the (semi-)bounded parallepipeda.

8.4 The Triangle

If Q is a simplex in $\mathbf{R}^n$, defined by the inequalities $\lambda_i(x) > 0$, $1 \leq i \leq n = 1$, where the λ_i are suitable affine functions (= polynomials of degree one) on $\mathbf{R}^n$, then the universal barrier function of Q is equal to

$$f(x) = -\sum_{i=1}^{n+1} \ln \lambda_i(x) \quad \text{plus a constant.} \tag{8.25}$$

In other words, for the corner and the simplex the universal barrier function is equal to the *standard logarithmic barrier for a convex polytope* as defined in [14, Example 2 on p. 34]. (This is not true for arbitrary convex polytopes.)

The easiest way to prove (8.25) is probably to use the relation with the characteristic function of the cone over Q, cf. Vinberg [18, Def. 10, p. 356]. In order to explain this, we begin with the identity

$$\mathrm{vol}_n\, Q^*(x) = \frac{1}{n!} \int_{\mathbf{R}^n} \mathrm{e}^{\langle x,\, \xi \rangle - p_Q(\xi)}\, \mathrm{d}_n\xi, \tag{8.26}$$

which can be proved by performing the substitution of variables $\xi = \tau\, \eta$, with $\tau \geq 0$, $\eta \in \mathrm{S}^{n-1}$, and using the formula (3.3).

The *cone K over Q* and its *polar cone K^** are defined by

$$\begin{aligned} K &:= \left\{ (t\, x,\, t) \in \mathbf{R}^{n+1} \mid x \in Q, \quad t > 0 \right\} \quad \text{and} \\ K^* &:= \left\{ \zeta \in \mathbf{R}^{n+1} \mid z \in K \Longrightarrow \langle z,\, \zeta \rangle \leq 0 \right\}, \end{aligned}$$

respectively. Noting that $(\xi,\, \tau) \in K^*$ if and only if $\tau \leq p_Q(\xi)$, that e^{-p_K} is equal to the characteristic function of K^*, and that $\mathrm{e}^{-p_Q(\xi)}$ is equal to the integral of e^τ from $-\infty$ to $-p_Q(\xi)$, we obtain from (8.26) that

$$\mathrm{vol}_n\, Q^*(x) = \frac{1}{n!} \int_{\mathbf{R}^{n+1}} \mathrm{e}^{\langle x,\, \xi \rangle + \tau - p_K(\xi,\, \tau)}\, \mathrm{d}_{n+1}(\xi,\, \tau) = (n+1)\, \mathrm{vol}_{n+1}\, K^*(x,\, 1), \tag{8.27}$$

where in the last identity we used (8.26) with Q replaced by K.

If Q is a simplex then the cone over Q is a corner, which by a linear transformation can be brought into the form (8.19), with n replaced by $n+1$. Therefore (8.25) follows from (8.20).

Let $T = T(a, b, c)$ be an open triangle in the plane $\mathbf{R}^2$, with vertices a, b, c. Let λ_a be the affine function on $\mathbf{R}^2$ which is equal to one at a and equal to zero on b and c. Define λ_b and λ_c by means of a cyclic permutation of a, b, c. Then it follows from (8.25) that the universal barrier function $f(x) = \ln I_T(x)$ of T is equal to

$$f(x) = -\ln(\lambda_a(x)\,\lambda_b(x)\,\lambda_c(x)) \quad \text{plus a constant.} \tag{8.28}$$

We conjecture that an asymptotic analysis near the boundary will show that the geodesics have a similar behaviour as in the case of the corner: to each point on a side only one geodesic converges, whereas all the other geodesics converge to one of the vertices. However, instead of pursuing this matter here, we turn to the computation of the curvature.

In general the sectional (= Gaussian) curvature of a Hessian Riemannian structure in the plane is given by the formula

$$K(x) = \frac{-c\,(\alpha\,\gamma - \beta^2) + b\,(\alpha\,\delta - \beta\,\gamma) - a\,(\beta\,\delta - \gamma^2)}{4\,(a\,c - b^2)^2}, \tag{8.29}$$

in which

$$a := {\partial_1}^2 f(x),\ b := \partial_1\partial_2 f(x),\ c := {\partial_2}^2 f(x),$$
$$\alpha := {\partial_1}^3 f(x),\ \beta := {\partial_1}^2\partial_2 f(x),\ \gamma := \partial_1{\partial_2}^2 f(x),\ \delta := {\partial_2}^3 f(x).$$

The formula (8.29) follows from (7.4). In the computation, a $g(x)$-orthonormal basis in $\mathbf{R}^2$ can be obtained by means of Gram-Schmidt's orthogonalization procedure.

Consider the special case when $a = (1, 0)$, $b = (0, 1)$, $c = (0, 0)$, so that $\lambda_a(u, v) = u$, $\lambda_b(u, v) = v$, $\lambda_c(u, v) = 1 - u - v$. A straightforward computation then leads to

$$K(u, v) = \frac{u\,v\,(1 - u - v)}{(u^2 + v^2 + (1 - u - v)^2)^2}, \quad u > 0,\ v > 0,\ u + v < 1.$$

It follows that for the general triangle $T(a, b, c)$ with vertices a, b, c, we have that

$$K(x) = \frac{\lambda_a(x)\,\lambda_b(x)\,\lambda_c(x)}{(\lambda_a(x)^2 + \lambda_b(x)^2 + \lambda_c(x)^2)^2}, \quad x \in T(a, b, c). \tag{8.30}$$

Here the affine functions λ_a, λ_b, λ_c have been defined in front of (8.28). Conclusions: *The curvature K is strictly positive in $T(a, b, c)$, with its maximum equal to $\frac{1}{3}$, attained at the center $\frac{1}{3}(a + b + c)$ of $T(a, b, c)$. Furthermore, K converges to zero at the boundary.*

The triangle T can be approximated by a domain $\widetilde{T}$ with a smooth and strongly convex boundary, for instance by taking the set of $x \in T$ such that $\ln I_T(x) < C$, where C is a large positive constant. Then, on compact subsets which stay away from the boundary, the derivatives up to any order the function $\ln I_{\widetilde{T}}(x)$ will be close to the corresponding derivatives of $\ln I_T(x)$. It follows that the curvature of the barrier function $\ln I_{\widetilde{T}}(x)$ will be close to the curvature of $\ln_T(x)$, and therefore strictly positive, in a large part of $\widetilde{T}$.

The barrier function for $\widetilde{T}$ of Proposition 3.1 is equal to $\frac{2}{3}\ln I_{\widetilde{T}}(x)$. According to Remark 7.3, the curvature of the latter converges to $-1/4$ at the boundary of $\widetilde{T}$; one may also apply Theorem 7.1 to the modification of the barrier function which is described in Corollary 4.3. Because the curvature of $c\, g(x)$ is equal to $\frac{1}{c}$ times the curvature of $g(x)$, it follows that the curvature of the barrier function $\ln I_{\widetilde{T}}(x)$ turns sharply from the positive values, which it attains at some distance from the boundary, to its limit $-\frac{1}{6}$ at $\partial\widetilde{T}$.

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