A remark on sheaf theory for non-Hausdorff manifolds*

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Much of sheaf theory can be developed for arbitrary topological spaces. This applies, for example, to the definition of 'sheaf' itself, to the existence of injective resolutions, to the properties of the operations f_* and f^* associated to a continuous map $f : Y \longrightarrow X$, etc, etc. On the other hand, there is a very basic part of the theory which seems to depend crucially on the Hausdorff property (together with local compactness and paracompactness). Here one could think of the properties of soft and fine sheaves, of compact supports, of the operation $f_!$ and its right adjoint $f^!$ ('Verdier duality'), etc. It is for this reason that, for a large part of the theory, all the standard text make the overall assumption that the underlying spaces must be locally compact, Hausdorff, and of finite cohomological dimension (cf. [7, 10, 2]).

There are geometric situations, however, such as foliation theory, where one naturally encounters sheaves on non-separated manifolds. For example, a central role is played by the holonomy groupoid of a foliation, and by Haefliger's classifying groupoid Γ^q. These groupoids are smooth and very well behaved in many respects, but they are not all Hausdorff. This fact impedes not only the (transverse) sheaf theory on foliations, but also the study of the convolution algebra and the (reduced) C^*-algebra associated to the foliation.

Motivated by foliations, we wish to indicate in this short note how sheaf theory can be extended, in an essentially unique way, to spaces which are locally sufficiently nice, but are not necessarily separated. The crucial step is a suitable adaptation of 'compact supports' in such spaces. After this adaptation, all the usual constructions and arguments (of [2], say) up to Verdier duality (which includes Poincaré duality) and beyond, remain valid for this more general class of spaces.

This note was originally written as an appendix to an earlier version of [6], where we apply the extended sheaf theory to the study of the cyclic type homologies of non-separated smooth groupoids, such as holonomy groupoids of a foliations.

1 . Overall assumptions. For any space X in this paper we do assume that X has an open cover by subsets $U \subset X$ which are each paracompact, Hausdorff, locally compact, and of cohomological dimension bounded by a number d (depending on X but not on U).

2 . c-soft sheaves. Let X be a space satisfying the general assumptions in 1. An abelian sheaf $\mathcal{A}$ on X is said to be *c-soft* if for any Hausdorff open $U \subset X$ its restriction $\mathcal{A}|_U$ is a c-soft sheaf on U in the usual sense. By the same property for Hausdorff spaces, it follows that c-softness is a local property, i.e., a sheaf $\mathcal{A}$ is c-soft iff there is an open cover $X = \bigcup U_i$ such that each $\mathcal{A}|_U$ is a c-soft sheaf on $\mathcal{A}$.

3 . The functor Γ_c. Let $\mathcal{A}$ be a c-soft sheaf on X and let $\mathcal{A}'$ be its Godement resolution (i.e. $\mathcal{A}'(U) = \Gamma(U_{\text{discr}}; \mathcal{A})$ is the set of all (not necessarily continuous) sections, for any open $U \subset X$). For any Hausdorff open set $W \subset X$, let $\Gamma_c(W, \mathcal{A})$ be the usual set of compactly supported sections. If $W \subset U$, there is an evident homomorphism, "extension by 0" $\Gamma_c(W, \mathcal{A}) \longrightarrow \Gamma_c(U, \mathcal{A}) \subset \Gamma(U, \mathcal{A}')$. For any (not necessarily Hausdorff) open set $U \subset X$, we define $\Gamma_c(U, \mathcal{A})$ to be the image of the map:

$$\bigoplus_{W} \Gamma_c(W, \mathcal{A}) \longrightarrow \Gamma(U, \mathcal{A}') \ ,$$

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where W ranges over all Hausdorff open subsets $W \subset U$. An alternative definition follows by choosing a Hausdorff open cover in Proposition 6 below.

Observe that $\Gamma_c(U, \mathcal{A})$ so defined is evidently functorial in $\mathcal{A}$, and that for any inclusion $U \subset U'$ we have an "extension by zero" monomorphism $\Gamma_c(U, \mathcal{A}) \longrightarrow \Gamma_c(U', \mathcal{A})$.

The following lemma shows that in the definition of $\Gamma_c(U, \mathcal{A})$ it is enough to let W range over a Hausdorff open cover of U; in particular, it shows that the definition agrees with the usual one if U itself is Hausdorff.

4 . Lemma *Let $\mathcal{A}$ be a c-soft sheaf on X. For any open cover $U = \bigcup W_i$ where each W_i is Hausdorff, the sequence $\bigoplus_i \Gamma_c(W_i, \mathcal{A}) \longrightarrow \Gamma_c(U, \mathcal{A}) \longrightarrow 0$ is exact.*

Proof: It suffices to show that for any Hausdorff open $W \subset U$, the map $\bigoplus_i \Gamma_c(W \bigcap W_i, \mathcal{A}) \longrightarrow \Gamma_c(W, \mathcal{A})$ is surjective. This is well known (see e.g. [7]). $\square$

5 . Example: If M is a manifold (not necessarily Hausdorff), we put $C_c^\infty(M) := \Gamma_c(M; \mathcal{C}_M^\infty)$, where $\mathcal{C}_M^\infty$ is the sheaf of smooth functions on M. Similarly we define $\Omega_c^k(M)$ by using the sheaf Ω_M^k of k-forms on M. These are the natural objects with the property that constructions performed in coordinate charts patch globally (as insured by the Mayer-Vietoris sequence below). For instance, this is the case for the DeRham differential.

If by $\bar{\mathcal{C}}_c^\infty(M)$ we denote Connes' version [4] of compactly supported functions, there is an obvious surjection $C_c^\infty(M) \longrightarrow \bar{\mathcal{C}}_c^\infty(M)$ which is not injective in general (take for instance $M = (0,2) \cup_X (0,2)$, $X = (0,2) - \{1\}$). The advantage of $C_c^\infty(M)$ over Connes' version consists on the existence of DeRham differential, better functorial properties, Poincaré duality, and the fact that the construction extends to arbitrary sheaves (and these are esentially used in [5, 6]). We remark that, in the case of étale groupoids $\mathcal{G}$, the two convolution algebras $C_c^\infty(\mathcal{G})$, and $\bar{\mathcal{C}}_c^\infty(\mathcal{G})$ define (by the construction in [4]) the same C^*-algebra.

6 . Proposition *(Mayer-Vietoris sequence) Let $X = \bigcup_i U_i$ be an open cover indexed by an ordered set I, and let $\mathcal{A}$ be a c-soft sheaf on X. Then there is a long exact sequence:*

$$\ldots \longrightarrow \bigoplus_{i_0 < i_1} \Gamma_c(U_{i_0 i_1}, \mathcal{A}) \longrightarrow \bigoplus_{i_0} \Gamma_c(U_{i_0}, \mathcal{A}) \longrightarrow \Gamma_c(X, \mathcal{A}) \longrightarrow 0 \tag{1}$$

Here $U_{i_0 \ldots i_n} = U_{i_0} \cap \ldots \cap U_{i_n}$, as usual. (There is of course a similar exact sequence if I is not ordered.)

Proof: The proposition is of course well known in the case where X is a paracompact Hausdorff space. We first reduce the proof to the case where each of the U_i is Hausdorff, as follows. Let $X = \bigcup_{j \in J} W_j$ be a cover by Hausdorff open sets, and consider the double complex $C_{p,q} = \bigoplus \Gamma_c(W_{j_0 \ldots j_p} \cap U_{i_0 .. i_q}, \mathcal{A})$, where the sum is over all $j_0 < \ldots < j_p$, $i_0 < \ldots < i_q$. For a fixed $p \geq 0$, the column $C_{p,\bullet}$ is a sum of exact Mayer-Vietoris sequences for the Hausdorff open sets $W_{j_0 \ldots j_p}$, augmented by $C_{p,-1} = \bigoplus_{j_0 < \ldots < j_p} \Gamma_c(W_{j_0 \ldots j_p}, \mathcal{A})$. Keeping the notation $U_{i_0 .. i_q} = X = W_{j_0 \ldots j_p}$ if $q = -1 = p$, we observe that for a fixed $q \geq -1$, the row $C_{\bullet,q}$ is a sum of Mayer-Vietoris sequences for the spaces $U_{i_0 .. i_q}$ with respect to the open covers $\{W_j \cap U_{i_0 .. i_q}\}$. So, if the proposition would hold for covers by Hausdorff sets, each row $C_{\bullet,q}$ $(q \geq -1)$ is also exact. By a standard double complex argument it follows that the augumentation column $C_{-1,\bullet}$ is also exact, and this column is precisely the sequence in the statement of the proposition. This shows that it suffices to show the proposition in the special case where each U_i is Hausdorff.

So assume each $U_i \subset X$ is Hausdorff. Observe first that exactness of the sequence (1) at $\Gamma_c(X, \mathcal{A})$ now follows by Lemma 4. To show exactness elsewhere, consider for each finite subset $I_0 \subset I$ the space $U^{I_0} = \bigcup_{i \in I_0} U_i$ and the subsequence:

$$\ldots \longrightarrow \bigoplus_{i_0 < i_1 \text{ in } I_0} \Gamma_c(U_{i_0 i_1}, \mathcal{A}) \longrightarrow \bigoplus_{i_0 \text{ in } I_0} \Gamma_c(U_{i_0}, \mathcal{A}) \longrightarrow \Gamma_c(U^{I_0}, \mathcal{A}) \longrightarrow 0 \tag{2}$$

of (1). Clearly (1) is the directed union of the sequences of the form (2), where $I_0 \subset I$ ranges over all finite subsets of I. So exactness of (1) follows from exactness of each such sequence of the form (2). Thus, it remains to prove the proposition in the special case of a *finite* cover $\{U_i\}$ of X by Haudorff open sets.

So assume $X = U_1 \cup \ldots \cup U_n$ where each U_i is Hausdorff. For $n = 1$, there is nothing to prove. For $n = 2$, the sequence has the form

$$0 \longrightarrow \Gamma_c(U_1 \cap U_2, \mathcal{A}) \longrightarrow \Gamma_c(U_1, \mathcal{A}) \bigoplus \Gamma_c(U_2, \mathcal{A}) \longrightarrow \Gamma_c(U_1 \cap U_2, \mathcal{A}) \longrightarrow 0 \ .$$

This sequence is exact at $\Gamma_c(X, \mathcal{A})$ by 4, and evidently exact at other places. Exactness for $n = 3$ can be proved using exactness for $n = 2$. Indeed, consider the following diagram, whose upper two rows are the sequences for $n = 2, 3$ and whose third row is constructed by taking vertical cokernels, so that all columns are exact (we delete the sheaf $\mathcal{A}$ from the notation)(compare to pp. 187 in [1]):

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] \arrow[d] & \Gamma_c(U_{12}) \arrow[r] \arrow[d] & \Gamma_c(U_1) \oplus \Gamma_c(U_2) \arrow[r] \arrow[d] & \Gamma_c(U_1 \cup U_2) \arrow[r] \arrow[d] & 0 \\
\Gamma_c(U_{123}) \arrow[r] \arrow[d] & \oplus_{1 \le i < j \le 3} \Gamma_c(U_{ij}) \arrow[r] \arrow[d] & \Gamma_c(U_1) \oplus \Gamma_c(U_2) \oplus \Gamma_c(U_3) \arrow[r] \arrow[d] & \Gamma_c(U_1 \cup U_2 \cup U_3) \arrow[r] \arrow[d] & 0 \\
\Gamma_c(U_{123}) \arrow[r] \arrow[d] & \Gamma_c(U_{13}) \oplus \Gamma_c(U_{23}) \arrow[r] \arrow[d] & \Gamma_c(U_3) \arrow[r, "\pi"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 & 0 &
\end{tikzcd}$$

To show that the middle row is exact, it thus suffices to prove that the lower row is exact. This row can be decomposed into a Mayer-Vietoris sequence for the case $n = 2$, already shown to be exact,

$$0 \longrightarrow \Gamma_c(U_{123}) \longrightarrow \Gamma_c(U_{13}) \oplus \Gamma_c(U_{23}) \longrightarrow \Gamma_c(U_3 \cap (U_1 \cup U_2)) \longrightarrow 0$$

and the sequence $0 \longrightarrow \Gamma_c(U_3 \cap (U_1 \cup U_2)) \longrightarrow \Gamma_c(U_3) \longrightarrow C \longrightarrow 0$. The exactness of the latter sequence is easily proved by a diagram chase, using exactness of the right-hand column.

An identical argument will show that the exactness for a cover by $n + 1$ opens follows from exactness for one by n opens, so the proof is completed by induction. $\square$

Proposition 6 is our main tool for transfering standard facts from sheaf theory on Hausdorff spaces to the non-Hausdorff case, as illustrated by the following corollaries.

7 . Corollary *Let $Y \subset X$ be a closed subspace, and let $\mathcal{A}$ be a c-soft sheaf on X. There is an exact sequence $0 \longrightarrow \Gamma_c(X - Y, \mathcal{A}) \xrightarrow{i} \Gamma_c(X, \mathcal{A}) \xrightarrow{r} \Gamma_c(Y, \mathcal{A}) \longrightarrow 0$ (i is extension by zero, r is the restriction).*

Proof: This (including the fact that the map r is well defined) follows by elementary homological algebra from the fact that the Corollary holds for Hausdorff spaces, by using 6 for a cover of X by Hausdorff open sets U_i and for the induced covers of Y by $\{U_i \cap Y\}$ and $X - Y$ by $\{U_i - Y\}$. $\square$

8 . Corollary *For a family $\mathcal{A}_i$ of c-soft sheaves on X the direct sum $\oplus \mathcal{A}_i$ is again c-soft, and $\Gamma_c(X, \oplus \mathcal{A}_i) \cong \oplus \Gamma_c(X, \mathcal{A}_i)$. In particular, when working over $\mathbb{R}$, we have for any c-soft sheaf $\mathcal{S}$ of $\mathbb{R}$-vector spaces and any vector space V that the tensor product $\mathcal{S} \otimes_{\mathbb{R}} V$ (here V is the constant sheaf) is again c-soft, and the familiar formula $\Gamma_c(X, \mathcal{S} \otimes_{\mathbb{R}} V) \cong \Gamma_c(X, \mathcal{S}) \otimes_{\mathbb{R}} V$.*

9 . Corollary *Let $\mathcal{A}_\bullet \longrightarrow \mathcal{B}_\bullet$ be a quasi-isomorphism between chain complexes of c-soft sheaves on X. Then $\Gamma_c(X, \mathcal{A}_\bullet) \longrightarrow \Gamma_c(X, \mathcal{B}_\bullet)$ is again a quasi-isomorphism.*

Proof: By a "mapping cone argument" we may assume that $\mathcal{B}_\bullet = 0$. In other words, we have to show that $\Gamma_c(X, \mathcal{A}_\bullet)$ is acyclic whenever $\mathcal{A}_\bullet$ is. This follows from the Mayer-Vietoris sequence 6 together with the Hausdorff case.

(We remark that it is necessary to assume that the chain complexes are bounded below if X does not have locally finite cohomological dimension, as in 1). $\square$

The following Corollary is included for application in [5].

10 . Corollary *Let $Y \subset X$ be a closed subspace, and let $\theta : X \longrightarrow \mathbb{R}$ be a continuous map such that $\theta^{-1}(0) = Y$. Let $\mathcal{A}$ be a c-soft sheaf on X. Then for any $\alpha \in \Gamma_c(X, \mathcal{A})$,*

$$\alpha|_Y = 0 \ \text{ iff } \ \exists\, \varepsilon > 0 : \quad \alpha|_{\theta^{-1}(-\varepsilon,\varepsilon)} = 0$$

(here $\alpha|_Y$ is the restriction $r(\alpha)$ as in 7).

Proof: For $\varepsilon \geq 0$, write $Y_\varepsilon = \{x \in X : |\theta(x)| \leq \varepsilon\}$, and for each open $U \subset X$ write

$$\Gamma_c^\varepsilon(U, \mathcal{A}) = \{\alpha \in \Gamma_c(U, \mathcal{A}) : \alpha|_{U \cap Y_\varepsilon} = 0\}\ .$$

It suffices to show that $\bigoplus_{\varepsilon > 0} \Gamma_c^\varepsilon(X, \mathcal{A}) \longrightarrow \Gamma_c^0(X, \mathcal{A})$ is epi. Let $\{U_i\}$ be a cover of X by Hausdorff open sets, and consider the diagram:

$$\begin{tikzcd}
\bigoplus_{i,\varepsilon>0} \Gamma_c^\varepsilon(U_i, \mathcal{A}) \arrow[r, "u"] \arrow[d] & \bigoplus_i \Gamma_c^0(U_i, \mathcal{A}) \arrow[r, "\sim"] \arrow[d, "\pi"] & \bigoplus_i \Gamma_c(U_i - Y, \mathcal{A}) \arrow[d, "\pi'"] \\
\bigoplus_{\varepsilon>0} \Gamma_c^\varepsilon(X, \mathcal{A}) \arrow[r, "v"] & \Gamma_c^0(X, \mathcal{A}) \arrow[r, "\sim"] & \Gamma_c(X - Y, \mathcal{A})
\end{tikzcd}$$

where the isomorphisms on the right come from 7. We wish to show that v is epi. Since u is epi by the Hausdorff case, it suffices to show that π is epi, or, equivalently, that π' is epi. This is indeed the case by 6. $\square$

It is quite clear that using c-soft resolutions one can define compactly supported cohomology $H_c^*(X, \mathcal{A})$ for any $\mathcal{A} \in \underline{Ab}(X)$. In particular, we get an extension $H_c^0(X, -)$ of $\Gamma_c(X, -)$ to all sheaves; this extension is still denoted by $\Gamma_c(X, -)$.

11 . Proposition *Let $f : Y \longrightarrow X$ be a continuous map. There is a functor $f_! : \underline{Ab}(Y) \longrightarrow \underline{Ab}(X)$ with the following properties:*
(i) For any open $U \subset X$ and any $\mathcal{B} \in \underline{Ab}(Y)$, $\Gamma_c(U, f_!\mathcal{B}) = \Gamma_c(f^{-1}(U), \mathcal{B})$.
(ii) For any point $x \in X$ and any $\mathcal{B} \in \underline{Ab}(Y)$, $f_!(\mathcal{B})_x = \Gamma_c(f^{-1}(x), \mathcal{B})$.
(iii) $f_!$ is left exact and maps c-soft sheaves into c-soft sheaves.
(iv) For any fibered product

$$\begin{tikzcd}
Z \times_X Y \arrow[r, "p"] \arrow[d, "q"'] & Y \arrow[d, "f"] \\
Z \arrow[r, "e"] & X
\end{tikzcd}$$

along an étale map e and for any c-soft $\mathcal{B} \in \underline{Ab}(Y)$, there is a canonical isomorphism $q_!p^*\mathcal{B} \cong e^*f_!\mathcal{B}$. (see 13 below for the case where e is not étale).

Proof: Of course the proposition is well known in the Hausdorff case. For the more general case, recall first from [3] the correspondence for any *Hausdorff* space Z between c-soft sheaves $\mathcal{S}$ on Z and flabby cosheaves $\mathcal{C}$ on Z, given by $\Gamma_c(W,\mathcal{S}) = \mathcal{C}(W)$ (natural with respect to the opens $W \subset Z$). Given the cosheaf $\mathcal{C}$, the stalk of the corresponding sheaf $\mathcal{S}$ at a point $z \in Z$ is given by the exact sequence:

$$0 \longrightarrow \mathcal{C}(Z-z) \longrightarrow \mathcal{C}(Z) \longrightarrow \mathcal{S}_z \longrightarrow 0\ . \tag{3}$$

We use this correspondence in the construction of $f_!$. (However, see remark 12 below for a description of $f_!$ which doesn't use this correspondence).

We discuss first the construction of $f_!$ on c-soft sheaves. Let $\mathcal{B} \in \underline{Ab}(Y)$ be c-soft. First, assume X is Hausdorff. Let $\mathcal{B}$ be a c-soft sheaf on Y, and define a cosheaf $\mathcal{C} = c(\mathcal{B})$ by $\mathcal{C}(U) = \Gamma_c(f^{-1}(U),\mathcal{B})$. Note that $\mathcal{C}$ is indeed a flabby cosheaf, by 6. Hence there exists a c-soft sheaf $\mathcal{S}$ on X, uniquely determined up to isomorphism by the identity $\Gamma_c(U,\mathcal{S}) = \mathcal{C}(U)$ for any open $U \subset X$. Thus, if X is Hausdorff, we can define $f_!\mathcal{B}$ to be this sheaf $\mathcal{S}$.

In the general case, cover X by Hausdorff opens U_i, and define in this way for each i a c-soft sheaf $\mathcal{S}_i$ on U_i by $\Gamma_c(V,\mathcal{S}_i) = \Gamma_c(f^{-1}(V),\mathcal{B})$. Then (again by the equivalence between sheaves and cosheaves) there is a canonical isomorphism $\theta_{ij} : \mathcal{S}_j|_{U_{ij}} \longrightarrow \mathcal{S}_i|_{U_{ij}}$ satisfying the cocycle condition. Therefore the sheaves $\mathcal{S}_i$ patch together to a sheaf $\mathcal{S}$ on X, uniquely determined up to isomorphism by the condition that $\mathcal{S}|_{U_i} = \mathcal{S}_i$ (by an isomorphism compatible with θ_{ij}). Thus we can define $f_!\mathcal{B}$ to be $\mathcal{S}$.

We prove the properties $(i)-(iv)$ in the statement of the proposition for $\mathcal{B} \in \underline{Ab}(Y)$ c-soft. Property (i) clearly holds for any open set U contained in some U_i. For general U, property (i) then follows by the Mayer-Vietoris sequence. Next, identity (3) yields for any point $x \in X$ an exact sequence $0 \longrightarrow \Gamma_c(Y - f^{-1}(x),\ \mathcal{B}) \longrightarrow \Gamma_c(Y,\mathcal{B}) \longrightarrow f_!(\mathcal{B})_x \longrightarrow 0$, and hence, by 7 the isomorphism (ii) of the Proposition. Finally, (iv) is clear from the local nature of the construction of $f_!$.

For general $\mathcal{A} \in \underline{Ab}(Y)$ we define $f_!(\mathcal{A}) \in \underline{Ab}(X)$ as the kernel of the map $f_!(\mathcal{S}^0) \longrightarrow f_!(\mathcal{S}^1)$ where $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{S}^0 \longrightarrow \mathcal{S}^1 \longrightarrow ...$ is a c-soft resolution of $\mathcal{A}$ (from the first part it follows that it is well defined up to isomorphisms). The properties (i) and (ii) are now immediate consequences of the definition and of the previous case. Using 9 and (ii) it easily follows that $f_!$ transforms acyclic complexes of c-soft sheaves on $\underline{Ab}(Y)$ into acyclic complexes on $\underline{Ab}(X)$. This immediately implies that $f_!$ is left exact. $\square$

12 . Remark. We outline an alternative construction and proof of Proposition 11, which does not use the correspondence between sheaves and cosheaves. This construction will be used in the proof of 13 below. We will assume that $\mathcal{B}$ is c-soft and X is Hausdorff. (As in the proof of 11, the construction of $f_!$ for general X is then obtained by glueing the constructions over a cover by Hausdorff opens $U_i \subset X$.)

So, let $\mathcal{B}$ be a c-soft sheaf on Y. For any open set $V \subset Y$, denote by $\mathcal{B}_V$ the sheaf on Y obtained by extending $\mathcal{B}|_V$ by zero. Thus $\mathcal{B}_V$ is evidently c-soft, and $\Gamma_c(Y,\mathcal{B}_V) = \Gamma_c(V,\mathcal{B})$. Moreover, an inclusion $V \subset W$ induces an evident map $\mathcal{B}_V \hookrightarrow \mathcal{B}_W$.

Now let $Y = \bigcup W_i$ be a cover by Hausdorff open sets. This cover induces a long exact sequence:

$$\ldots \longrightarrow \bigoplus_{i_0<i_1} \mathcal{B}_{W_{i_0 i_1}} \longrightarrow \bigoplus_{i_0} \mathcal{B}_{W_{i_0}} \longrightarrow \mathcal{B} \longrightarrow 0$$

of c-soft sheaves on Y. By Corollary 9, the functor $\Gamma_c(Y,-)$ applied to this long exact sequence again yields an exact sequence, and this is precisely the Mayer-Vietoris sequence of 6. For each $i_0,...,i_n$ let $f_{i_0,\ldots,i_n} : W_{i_0,\ldots,i_n} \longrightarrow X$ be the restriction of f; this is a map between Hausdorff spaces, so we have $(f_{i_0,\ldots,i_n})_!(\mathcal{B}_{W_{i_0,\ldots,i_n}})$ defined as usual. Define $f_!(\mathcal{B})$ as the cokernel fitting into a long exact sequence:

$$\ldots \longrightarrow \bigoplus_{i_0<i_1} (f_{i_0 i_1})_!(\mathcal{B}_{W_{i_0 i_1}}) \longrightarrow \bigoplus_{i_0} (f_{i_0})_!(\mathcal{B}_{W_{i_0}}) \longrightarrow f_!(\mathcal{B}) \longrightarrow 0\ . \tag{4}$$

For $x \in X$, we have $(f_{i_0})_!(\mathcal{B}_{W_{i_0}})_x = \Gamma_c(f^{-1}(x) \cap W_{i_0};\mathcal{B})$ by the Hausdorff case. So taking stalks of the long exact sequence in (4) at x and using the Mayer-Vietoris sequence 6 for the space $f^{-1}(x)$ we find $f_!(\mathcal{B})_x = \Gamma_c(f^{-1}(x),\mathcal{B})$ as in 11 (ii). Property 11 (i) is proved in a similar way (using 7).

The functor $f_!$ can be extended to the derived category $D(Y)$ by taking a c-soft resolution $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{S}^0 \longrightarrow \mathcal{S}^1 \longrightarrow \ldots$ and defining $\mathcal{R}f_!(\mathcal{A})$ as the complex $f_!(\mathcal{S}^\bullet)$. Up to quasi-isomorphisms, this complex is well defined and does not depend on the resolution $\mathcal{S}^\bullet$, by 7. (In this way, we obtaine in fact a well defined functor $\mathcal{R}f_! : D(Y) \longrightarrow D(X)$ at the level of derived categories, which is sometimes simply denoted by $f_!$ again). In particular, $\mathcal{H}^*(\mathcal{R}\varphi_!(\mathcal{A}))$ gives in fact the right derived functors $R^*f_!$ of $f_!$.

13 . Proposition*('Change of base') For any pullback diagram:*

$$\begin{array}{ccc} Z \times_X Y & \xrightarrow{p} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{e} & X \end{array}$$

and any sheaf $\mathcal{B}$ on Y, there is a canonical quasi-isomorphism $(\mathcal{R}q_!)p^*\mathcal{B} \simeq e^*(\mathcal{R}f_!)\mathcal{B}$.

Proof: Using Mayer-Vietoris for covers of X and Z by Hausdorff open sets, it suffices to consider the case where X and Z are both Hausdorff. Clearly it also suffices to prove the lemma in the special case where $\mathcal{B}$ is c-soft.

Let $Y = \bigcup W_i$ as in 12, so that $f_!(\mathcal{B})$ fits into a long exact sequence (4) of c-soft sheaves on X. Applying the exact functor e^* to this sequence and using the lemma in the Hausdorff case, one obtains a long exact sequence of the form:

$$\ldots \longrightarrow \bigoplus_{i_0<i_1} q_!p^*(\mathcal{B}|_{W_{i_0i_1}}) \longrightarrow \bigoplus_{i_0} q_!p^*(\mathcal{B}|_{W_{i_0}}) \longrightarrow e^*f_!(\mathcal{B}) \longrightarrow 0 \ . \qquad (5)$$

Now let $p^*(\mathcal{B}) \longrightarrow \mathcal{S}^\bullet$ be a c-soft resolution over the pullback $Z \times_X Y$. Then for any open $U \subset Y$, $\mathcal{S}^\bullet_{p^{-1}(W)}$ is a c-soft resolution of $p^*(\mathcal{B}_W)$, so $q_!(\mathcal{S}^\bullet_{p^{-1}(W)})$ is a c-soft resolution of $q_!p^*(\mathcal{B})$. The lemma now follows by comparing the sequence (5) to the defining sequence

$$\ldots \longrightarrow \bigoplus_{i_0<i_1} q_!(\mathcal{S}_{p^{-1}W_{i_0i_1}}) \longrightarrow \bigoplus_{i_0} q_!(\mathcal{S}_{p^{-1}W_{i_0}}) \longrightarrow q_!(\mathcal{S}) \longrightarrow 0$$

for $q_!(p^*(\mathcal{B})) \stackrel{def}{=} q_!(\mathcal{S})$. $\square$

14 . $f_!$ on étale maps. Let $f : Y \longrightarrow X$ be an étale map, i.e. a local homeomorphism. It is well known that the pullback functor $f^* : \underline{Ab}(X) \longrightarrow \underline{Ab}(Y)$ has an exact left-adjoint $f_! : \underline{Ab}(Y) \longrightarrow \underline{Ab}(X)$, described on the stalks by $f_!(\mathcal{B})_x = \oplus_{y \in f^{-1}(x)} \mathcal{B}_y$. This construction agrees with the one in 11. In particular, for étale f, the counit of the adjunction defines a map $\Sigma_f : f_!f^*(\mathcal{A}) \longrightarrow \mathcal{A}$, "summation along the fiber", for any sheaf $\mathcal{A}$ on X.

15 . $f_!$ on proper maps. Define a map $f : Y \longrightarrow X$ between (non-necessarily Hausdorff) spaces to be *proper* if:

(i) the diagonal $Y \longrightarrow Y \times_X Y$ is closed.

(ii) for any Hausdorff open $U \subset X$ and any compact $K \subset U$, the set $f^{-1}(K)$ is compact.

It is easy to see that if f is proper then $f_! = f_*$, as in the Hausdorff case. Furthermore, for any c-soft sheaf $\mathcal{A}$ on X, there is a natural map $\Gamma_c(X, \mathcal{A}) \longrightarrow \Gamma_c(Y, f^*\mathcal{A})$ defined by pullback, as in the Hausdorff case.

16 . Verdier duality: Given a map $f : Y \longrightarrow X$, the functor $f_!$ has a right adjoint $f^!$ at the level of the derived categories. This Verdier duality is a very special case of Section 5 in [6]; alternatively, one remarks that the proof in the Hausdorff case (see e.g. [2]) extends to our setting since the basic properties of Γ_c, and $f_!$ are preserved. Let us point out Poincaré duality as a special case: $H^*(M; or) \cong H_c^{n-*}(M)^\vee$ for any $n-$dimensional manifold M (where *or* denotes the orientation sheaf of M).

References

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