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NONCONFORMING FINITE ELEMENTS AND THE CASCADE ITERATION

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ABSTRACT. We derive sufficient conditions under which the Cascade iteration applied to nonconforming finite element discretizations yields an optimal solver. Key ingredients are optimal error estimates of such discretizations, which we therefore study in detail. We derive a new, efficient modified Morley finite element method. Optimal Cascade iterations are obtained for problems of second, and using a new smoother, of fourth order as well as for the Stokes problem.

1. INTRODUCTION

The Cascade multi-level iteration has been defined and analyzed for solving symmetric elliptic scalar problems of second order discretized on conforming finite element spaces by Bornemann and Deuflhard [BD96] and Shaidurov [Sha96]. As the full multi-grid method, cf. e.g. [Hac85, §5], the Cascade iteration is based on the use of a hierarchy of corresponding auxiliary discretizations on coarser meshes. Going from the lowest level to the highest one, on each level the obtained approximate solution from the previous level is used as a starting value of a number of iterations of a simple iterative solver (a smoother) like Conjugate Gradients. However, since in contrast to multi-grid, this smoother is not capable to significantly reduce the (smooth) algebraic error from the previous level, this error should already be strictly less than the final error that one permits. This is achieved by applying an increasing number of smoothing iterations on lower levels. On the other hand, since the problems on lower levels have a smaller dimension, the complete algorithm can be shown to be optimal, i.e., using a number of operations proportional to the number of unknowns, on the finest level an approximate solution is obtained with an algebraic error that, in energy norm, is of the same order as the discretization error.

In a recent paper [BD99], Braess and Dahmen analyzed the Cascade iteration applied to the discretized Stokes equations. The velocity component of the solution can be characterized as the solution of a symmetric elliptic variational problem on the space of discretely divergence-free velocities. Such spaces are non-nested, which means that explicit prolongations had to be introduced in the Cascade iteration. A difficulty is that these prolongations have an energy-norm that is larger than one. Yet, since only oscillating parts of the prolongated function are responsible for an increase in energy norm, and these oscillating

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parts are reduced by subsequent smoothing iterations, it could be shown that the Cascade iteration yields an approximate velocity vector having an algebraic error that, in energy norm, is of the same order as the discretization error.

In the present paper, we use the idea behind the analysis from [BD99] to analyze the Cascade iteration in an abstract setting of general nonconforming finite element discretizations of symmetric elliptic problems. As applications, we construct optimal Cascade iterations in the following three situations:

(a). Laplace equation and nonconforming P_1 elements,
(b). Biharmonic equation and Morley finite elements,
(c). Stokes equations and nonconforming P_1 elements for the velocity and P_0 elements for the pressure.

Key ingredients to the analysis of the Cascade iteration are optimal discretization error estimates. In one framework, we derive such estimates for less regular problems, and for fully regular problems, where in the latter case a reduced set of conditions suffices. Our analysis for less regular problems follows the one from Brenner in [Bre99] quite closely. For less regular problems it may be necessary to apply the nonconforming finite element discretization with a modified right-hand side, in order to make the discrete system well-defined. We show how cheap modifications can be constructed that yield optimal error estimates without needing additional regularity conditions. In particular, we construct a new efficient modified Morley method.

Standard smoothers for problems (b) and (c) yield a Cascade iteration that is only suboptimal. To obtain an iteration that is optimal, we construct more powerful smoothers, that involve a call of simple conforming multi-grid method applied to a scalar problem of second order.

The paper is organized as follows: In §2, we state the abstract variational problem and its discretization. In §3, we formulate the Cascade iteration and give sufficient conditions under which it is (sub)optimal. The smoothing property of (preconditioned) Conjugate Gradients is studied in §4. In Section 5, we derive error estimates for nonconforming finite elements. Finally, in §6, we discuss aforementioned applications.

2. Basic assumptions

Let $\mathcal{V}$ be a closed subspace of a Hilbert space $\mathcal{H}^1$ over $I\!R$, and let $a(\ ,\ )$ be a symmetric bilinear form on $\mathcal{V}$ satisfying

$$a(v,v) \eqsim \|v\|_{\mathcal{H}^1}^2 \qquad (v \in \mathcal{V}). \tag{A}$$

In order to avoid the repeated use of generic but unspecified constants, here by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ when both $C \lesssim D$ and $C \gtrsim D$. We consider the following symmetric and elliptic variational problem: Given $f \in \mathcal{V}'$, search $u \in \mathcal{V}$ such that

$$a(u,v) = f(v) \qquad (v \in \mathcal{V}). \tag{2.1}$$

Let $\mathcal{H}^0$ be a Hilbert space such that

$$\mathcal{H}^1 \hookrightarrow \mathcal{H}^0$$

with dense embedding. For $s \in [0,1]$, we define $\mathcal{H}^s = [\mathcal{H}^0, \mathcal{H}^1]_s$ being the interpolation space obtained from $\mathcal{H}^0$ and $\mathcal{H}^1$ by the method of complex interpolation. We assume the existence of an $\alpha \in (0,1]$, and another Hilbert space that we denote by $\mathcal{H}^{1+\alpha}$, for which

$$\mathcal{H}^{1+\alpha} \hookrightarrow \mathcal{H}^1$$

with dense embedding, such that for $f \in (\mathcal{H}^{1-\alpha})'$, the solution u of (2.1) is in $\mathcal{H}^{1+\alpha}$ and satisfies

$$\text{(B)} \qquad \|u\|_{\mathcal{H}^{1+\alpha}} \lesssim \|f\|_{(\mathcal{H}^{1-\alpha})'} \qquad \textit{(regularity)}.$$

Example 2.1. A typical application is characterized by $\mathcal{H}^0 = L^2(\Omega)$ for some domain $\Omega \subset I\!R^d$, $\mathcal{H}^1 = H^r(\Omega)$ or $\mathcal{H}^1 = H^r_0(\Omega)$, in which case (2.1) is a problem of order $2r$, and $\mathcal{H}^{1+\alpha} = \mathcal{H}^1 \cap H^{(1+\alpha)r}(\Omega)$, or products of these spaces. We will consider applications where $\mathcal{V} = \mathcal{H}^1$, as well as an application concerning the Stokes problem where $\mathcal{V}$ is a proper subspace of $\mathcal{H}^1$.

Let $(\mathcal{V}_k)_{k\geq 0}$ be a sequence of finite dimensional subspaces of $\mathcal{H}^0$, which are not necessarily nested, or contained in $\mathcal{V}$. Let $\mathcal{V}_{-1} = \{0\}$. For each $k \geq 0$, we assume that we have a scalar product $a_k(\ ,\ )$ on $\mathcal{V} + \mathcal{V}_{k-1} + \mathcal{V}_k$ that coincides with $a(\ ,\ )$ on $\mathcal{V}$, and, for $k \geq 1$, with $a_{k-1}(\ ,\ )$ on $\mathcal{V}_{k-1}$. We put

$$\|\ \|_{a_k} := a_k(\ ,\ )^{\frac{1}{2}}.$$

For $s \in [0,1]$, we equip $\mathcal{V}_k$ with norms $\|\ \|_{s,k}$ defined by interpolation between $\|\ \|_{1,k} := \|\ \|_{a_k}$ and $\|\ \|_{0,k} := \|\ \|_{\mathcal{H}^0}$. We define $\beta_k := \inf_{0\neq v_k \in \mathcal{V}_k} \frac{\|v_k\|_{0,k}}{\|v_k\|_{1,k}}$. Note that β_k^{-1} is the smallest constant such that

$$\text{(2.2)} \qquad \|\ \|_{1,k} \leq \beta_k^{-1} \|\ \|_{0,k} \qquad \textit{(inverse inequality)}.$$

We assume that

$$\text{(C)} \qquad \beta_{k-1}/\beta_k \lesssim 1.$$

Example 2.2. In the situation from Example 2.1, and for $\mathcal{V}_k$ being a standard (nonconforming) finite element space with respect to a shape-regular, quasi-uniform mesh with meshsize h_k, and for $a_k(\ ,\ )$ being uniformly equivalent to the sum of the squared H^r-norms on the elements, we have $\beta_k \eqsim h_k^r$.

For some $f_k \in \mathcal{V}_k'$, we approximate the solution u of (2.1) by the solution $u_k \in \mathcal{V}_k$ of

$$\text{(2.3)} \qquad a_k(u_k, v_k) = f_k(v_k) \qquad (v_k \in \mathcal{V}_k).$$

As we will demonstrate in §5, for suitable choices of f_k error estimates of the following type are valid: For some $s \in [0, 1-\alpha]$,

$$\text{(2.4)} \qquad \|u - u_k\|_{a_k} \lesssim \beta_k^{\alpha} \|f\|_{(\mathcal{H}^{1-\alpha})'} + \beta_k^{1-s} \|f\|_{(\mathcal{H}^s)'} \qquad (f \in (\mathcal{H}^s)').$$

Remark 2.3. Compared to the $s = 1-\alpha$ case, (2.4) for $s < 1-\alpha$ requires additional smoothness of f without yielding a qualitatively better error estimate as function of β_k. On the other hand, taking $f \in (\mathcal{H}^s)'$ for $s < 1-\alpha$ generally allows for simpler constructions of f_k. Note that taking $f_k = f|_{\mathcal{V}_k}$ is only possible if $\mathcal{V}_k \subset \mathcal{H}^s$.

Remark 2.4. If $\mathcal{V} = \mathcal{H}^1$, then (2.1) defines a homeomorphism between $f \in (\mathcal{H}^1)'$ and $u \in \mathcal{H}^1$. If, in addition

$$|a(u,v)| \lesssim \|u\|_{\mathcal{H}^{1+\alpha}} \|v\|_{\mathcal{H}^{1-\alpha}} \qquad (u \in \mathcal{H}^{1+\alpha},\ v \in \mathcal{H}^1), \tag{2.5}$$

then from the fact that $\mathcal{H}^1 \hookrightarrow \mathcal{H}^{1-\alpha}$ is dense, it follows that for $u \in \mathcal{H}^{1+\alpha}$ the mapping $v \mapsto a(u,v)$ on $\mathcal{H}^1$ has a unique extension to a bounded linear functional on $\mathcal{H}^{1-\alpha}$, with norm that can be bounded on some multiple of $\|u\|_{\mathcal{H}^{1+\alpha}}$. Together with (B), this means that $f \leftrightarrow u$ defines a homeomorphism between $(\mathcal{H}^{1-\alpha})'$ and $\mathcal{H}^{1+\alpha}$. So when $s = 1-\alpha$, the meaning of the error estimate (2.4) does not change if we replace (both terms) $\beta_k^\alpha \|f\|_{(\mathcal{H}^{1-\alpha})'}$ by $\beta_k^\alpha \|u\|_{\mathcal{H}^{1+\alpha}}$ and quantify over $u \in \mathcal{H}^{1+\alpha}$ instead of over $f \in (\mathcal{H}^{1-\alpha})'$. A similar remark applies to other error estimates that we are going to derive (e.g. (3.4), (3.5) and (5.13)).

Similarly for $s \in [0, 1-\alpha)$, if $|a(u,v)| \lesssim \|u\|_{\mathcal{H}^{2-s}} \|v\|_{\mathcal{H}^s}$ $(u \in \mathcal{H}^{2-s},\ v \in \mathcal{H}^1)$, then $f(v) := a(u,v)$ satisfies $\|f\|_{(\mathcal{H}^s)'} \lesssim \|u\|_{\mathcal{H}^{2-s}}$. However, since $f \in (\mathcal{H}^s)'$ does not imply that $u \in \mathcal{H}^{2-s}$, (2.4) for $s \in [0, 1-\alpha)$ cannot be written in terms of u only.

For comparison, if $\mathcal{V}$ is a proper subspace of $\mathcal{H}^1$, then the mapping $(\mathcal{H}^1)' \to \mathcal{H}^1 : f \mapsto u$ defined by (2.1) is not injective. If moreover $\mathcal{V} \hookrightarrow \mathcal{H}^{1-\alpha}$ is not dense, then even $f \mapsto u$ restricted to $(\mathcal{H}^{1-\alpha})'$ is not injective. Normally, f_k will be a function of f, and so will be u_k. Yet, if $f \mapsto u$ is not invertible it cannot be concluded that u_k is a function of u. Indeed, in our application concerning the Stokes problem, it will turn out that this is not the case, which means that $\|u - u_k\|_{a_k}$ cannot be bounded in terms of norms of u only.

Assuming an error estimate of type (2.4), in the next section we derive sufficient conditions under which the Cascade iteration solves (2.3) with an algebraic error in the $\|\ \|_{1,k}$-norm which, as function of β_k, has the same order as (this bound on) the discretization error, while taking a number of operations that is proportional, or almost proportional to the number of unknowns.

3. The Cascade iteration

To solve the discrete system (2.3) on some level j, we assume the availability of some basic (semi-)iterative method (a "smoother") on all levels $1 \le k \le j$. The (algebraic) error after m iterations of this method on level k starting with an initial error v_k will be denoted by $S_{k,m}(v_k)$. As in [BD96], we assume that there exists a $\gamma > 0$, and *linear* operators $\hat{S}_{k,m} : \mathcal{V}_k \to \mathcal{V}_k$ such that

$$\|S_{k,m}(v_k)\|_{1,k} \le \|\hat{S}_{k,m} v_k\|_{1,k} \qquad (v_k \in \mathcal{V}_k), \tag{D}$$

$$\|\hat{S}_{k,m}\|_{1,k \leftarrow 1,k} \le 1, \tag{E}$$

(F) $$\|\hat{S}_{k,m}\|_{1,k\leftarrow 1-\alpha,k} \lesssim (\beta_k^{-1} m^{-\gamma})^{\alpha} \qquad \textit{(smoothing property).}$$

Furthermore, since the spaces $\mathcal{V}_k$ are generally non-nested we need an explicit prolongation

$$I_{k-1}^k : \mathcal{V}_{k-1} \to \mathcal{V}_k.$$

The **Cascade iteration** now reads as follows:

- Compute the exact solution u_0 of (2.3) on level 0. Put $u_0^* = u_0$.
- For $k = 1, \dots, j$: On level k, apply $m_k^{(j)}$ iterations of the smoother to the equations (2.3) using $I_{k-1}^k u_{k-1}^*$ as starting value. Denote the result as u_k^*.

Below, we analyze $u_j - u_j^*$ in the $\|\,\|_{1,j}$-norm.

Lemma 3.1. *Let $\hat{I}_{k-1}^k : \mathcal{V}_{k-1} \to \mathcal{V}_k$ be defined by*

(3.1) $$a_k(\hat{I}_{k-1}^k w_{k-1}, v_k) = a_k(w_{k-1}, v_k) \qquad (w_{k-1} \in \mathcal{V}_{k-1},\, v_k \in \mathcal{V}_k).$$

Assume (D), (E), (F),

(3.2) $$\sum_{i=0}^{j-1} (m_{j-i}^{(j)})^{-\gamma\alpha} \lesssim 1,$$

(3.3) $$\|I_{k-1}^k - \hat{I}_{k-1}^k\|_{1-\alpha,k\leftarrow 1,k-1} \lesssim \beta_k^{\alpha},$$

and that for some $s \in [0, 1-\alpha]$ the exact solutions of (2.3) satisfy

(3.4) $$\|u_k - I_{k-1}^k u_{k-1}\|_{1-\alpha,k} \lesssim \beta_k^{2\alpha} \|f\|_{(\mathcal{H}^{1-\alpha})'} + \beta_k^{1+\alpha-s} \|f\|_{(\mathcal{H}^s)'} \qquad (f \in (\mathcal{H}^s)').$$

Then

(3.5)
$$\|u_j - u_j^*\|_{1,j} \lesssim \sum_{i=0}^{j-1} \left(\beta_{j-i}^{\alpha} (m_{j-i}^{(j)})^{-\gamma\alpha} \|f\|_{(\mathcal{H}^{1-\alpha})'} + \beta_{j-i}^{1-s} (m_{j-i}^{(j)})^{-\gamma\alpha} \|f\|_{(\mathcal{H}^s)'} \right) \qquad (f \in (\mathcal{H}^s)').$$

Proof. For $1 \le k \le j$, there holds

$$u_k - u_k^* = S_{k,m_k^{(j)}} (u_k - I_{k-1}^k u_{k-1}^*),$$

and so by (D),

$$\|u_k - u_k^*\|_{1,k} \le \|\hat{S}_{k,m_k^{(j)}} (u_k - I_{k-1}^k u_{k-1}^*)\|_{1,k}.$$

Following [BD99], we write

$$u_k - I_{k-1}^k u_{k-1}^* = u_k - I_{k-1}^k u_{k-1} + \hat{I}_{k-1}^k (u_{k-1} - u_{k-1}^*) + (I_{k-1}^k - \hat{I}_{k-1}^k)(u_{k-1} - u_{k-1}^*).$$

From

$$\|\hat{S}_{k,m_k^{(j)}} (u_k - I_{k-1}^k u_{k-1})\|_{1,k} \lesssim \beta_k^{\alpha} (m_k^{(j)})^{-\gamma\alpha} \|f\|_{(\mathcal{H}^{1-\alpha})'} + \beta_k^{1-s} (m_k^{(j)})^{-\gamma\alpha} \|f\|_{(\mathcal{H}^s)'}$$

by (F) and (3.4);

$$\|\hat{S}_{k,m_k^{(j)}} \hat{I}_{k-1}^k (u_{k-1} - u_{k-1}^*)\|_{1,k} \leq \|u_{k-1} - u_{k-1}^*\|_{1,k}$$

by (E) and (3.1); and

$$\|\hat{S}_{k,m_k^{(j)}} (I_{k-1}^k - \hat{I}_{k-1}^k)(u_{k-1} - u_{k-1}^*)\|_{1,k} \lesssim (m_k^{(j)})^{-\gamma\alpha} \|u_{k-1} - u_{k-1}^*\|_{1,k}$$

by (F) and (3.3), we conclude that there exists a constant $\hat{c}$ such that

$$\begin{aligned} \|u_k - u_k^*\|_{1,k} \quad \leq \quad & \hat{c}\beta_k^\alpha (m_k^{(j)})^{-\gamma\alpha} \|f\|_{(\mathcal{H}^{1-\alpha})'} + \hat{c}\beta_k^{1-s} (m_k^{(j)})^{-\gamma\alpha} \|f\|_{(\mathcal{H}^s)'} \\ & + (1 + \hat{c}(m_k^{(j)})^{-\gamma\alpha}) \|u_{k-1} - u_{k-1}^*\|_{1,k}. \end{aligned} \tag{3.6}$$

Assumption (3.2) shows that

$$\| \prod_{i=0}^{j-1} (1 + \hat{c}(m_{j-i}^{(j)})^{-\gamma\alpha}) \leq e^{\hat{c} \sum_{i=0}^{j-1} (m_{j-i}^{(j)})^{-\gamma\alpha}} \lesssim 1,$$

and so (3.5) follows from a recursive application of (3.6). □

Theorem 3.2. *(cf. (*[BD96, BD99]*)) Assume (D), (E), (F), (3.3) and (3.4). Let*

$$\beta_k \eqsim \rho^{-rk} \quad \text{and} \quad \dim \mathcal{V}_k \eqsim \rho^{dk} \tag{3.7}$$

where one may think of $\rho > 1$ as the mesh refinement factor*, $2r > 0$ as the* order of the equation*, and d as the* space dimension*. Assume that the computational work involved in performing the prolongation I_{k-1}^k and m iterations of the smoother on level k is proportional to* $\dim \mathcal{V}_k$ *and* $m \dim \mathcal{V}_k$ *respectively.*

(a). *Let $r < d\gamma$. Choose $m_{j-i}^{(j)} \eqsim \tilde{m} c^i$ for some $c \in (\rho^{r/\gamma}, \rho^d)$. Then the approximate solution u_j^* from the Cascade iteration satisfies*

$$\|u_j - u_j^*\|_{1,j} \lesssim (\beta_j \tilde{m}^{-\gamma})^\alpha \|f\|_{(\mathcal{H}^{1-\alpha})'} + (\beta_j \tilde{m}^{-\gamma})^\alpha \|f\|_{(\mathcal{H}^s)'} \times \left\{ \begin{array}{ll} \beta_j^{1-s-\alpha} & \text{if } c > \frac{r(1-s)}{\gamma\alpha} \\ j\beta_j^{1-s-\alpha} & \text{if } c = \frac{r(1-s)}{\gamma\alpha} \\ (\frac{\rho^{r/\gamma}}{c})^{\gamma\alpha j} & \text{if } c < \frac{r(1-s)}{\gamma\alpha} \end{array} \right\},$$

requiring a number of arithmetic operations that is proportional to $\tilde{m}$ $\dim \mathcal{V}_j$ (optimal complexity)*.*

(b). *Let $r = d\gamma$. Choose $m_{j-i}^{(j)} \eqsim \tilde{m} \rho^{di} j^{1/(\gamma\alpha)}$. Then*

$$\|u_j - u_j^*\|_{1,j} \lesssim \left\{ \begin{array}{ll} (\beta_j \tilde{m}^{-\gamma})^\alpha \|f\|_{(\mathcal{H}^{1-\alpha})'} & \text{if } s = 1 - \alpha \\ (\beta_j \tilde{m}^{-\gamma})^\alpha (\|f\|_{(\mathcal{H}^{1-\alpha})'} + \frac{1}{j} \|f\|_{(\mathcal{H}^s)'}) & \text{if } s < 1 - \alpha \end{array} \right\},$$

whereas the required number of arithmetic operations is proportional to $\tilde{m}$ $\dim \mathcal{V}_j (1 + \log(\dim \mathcal{V}_j))^{1+d/(r\alpha)}$ (suboptimal complexity)*.*

Proof. Apart from a straightforward counting of the number of operations, the proof follows by applying Lemma 3.1, where the appearing geometrical sums should be estimated in an obvious way. Note that (3.2) follows from the fact that $i \mapsto m_{j-i}^{(j)}$ is an exponentially increasing function. □

4. Preconditioned conjugate gradients as a smoother

Since with the Cascade iteration on lower levels many smoothing iterations are applied, it pays off to accelerate a basic iterative method by conjugate gradients (CG), in which case the basic iterative method is viewed as a preconditioner. As we will see, in common situations the resulting iteration satisfies the smoothing property (F) with $\gamma = 1$, instead of $\gamma = \frac{1}{2}$ that one would get without this acceleration. As shown in Theorem 3.2, having a sufficiently large value of γ is essential for getting an optimal method.

We assume that for each k, some scalar product $(\!(\ ,\ )\!)_k$ on $\mathcal{V}_k$ is given, which, as explained at the end of this section, in applications will incorporate the choice of the preconditioner. We put $|\!|\!|\ |\!|\!|_k = (\!(\ ,\ )\!)_k^{\frac{1}{2}}$. By Riesz' representation theorem, there exists a linear operator $\tilde{A}_k : \mathcal{V}_k \to \mathcal{V}_k$ and an $\tilde{f}_k \in \mathcal{V}_k$ such that

$$(4.1) \qquad a_k(w_k, v_k) = (\!(\tilde{A}_k w_k, v_k)\!)_k \quad (w_k, v_k \in \mathcal{V}_k) \quad \text{and} \quad f_k(v_k) = (\!(\tilde{f}_k, v_k)\!)_k \quad (v_k \in \mathcal{V}_k),$$

and so (2.3) is equivalent to

$$(4.2) \qquad \tilde{A}_k u_k = \tilde{f}_k.$$

Since $\tilde{A}_k$ is SPD with respect to $(\!(\ ,\ )\!)_k$, we can apply CG to (4.2) as our smoothing iteration, and in the following we will verify the assumptions (D), (F) and (F).

Using $\|\cdot\|_{1,k} = |\!|\!|\tilde{A}_k^{\frac{1}{2}} \cdot |\!|\!|_k$, it is well-known that the error $S_{k,m}(v_k)$ after m CG-iterations starting with v_k satisfies

$$(4.3) \qquad \|S_{k,m}(v_k)\|_{1,k} = \min_{p \in P_m, p(0)=1} \|p(\tilde{A}_k) v_k\|_{1,k}.$$

Following [Sha96], cf. also [Hac85, Exercise 6.6.8(i)], for $\Lambda > 0$ we define

$$\phi_{\Lambda,m}(x) = (-1)^{m+1}(2m+1)^{-1}\sqrt{\tfrac{\Lambda}{x}}\, T_{2m+1}(\sqrt{\tfrac{x}{\Lambda}}\,),$$

where T_{2m+1} is the Chebychev polynomial of order $2m+1$. The polynomial $\phi_{\Lambda,m}$ minimizes $\max_{x \in [0,\Lambda]} |\sqrt{x} p(x)|$ over $\{p \in P_m : p(0) = 1\}$. There holds

$$(4.4) \qquad \max_{x \in [0,\Lambda]} |\sqrt{x}\phi_{\Lambda,m}(x)| = (2m+1)^{-1}\sqrt{\Lambda}$$

and

$$(4.5) \qquad \max_{x \in [0,\Lambda]} |\phi_{\Lambda,m}(x)| = 1.$$

Defining $\tilde{S}_{j,m} = \phi_{\rho(\tilde{A}_k),m}(\tilde{A}_k)$, assumptions (D) and (E) follow from (4.3) and (4.5) respectively. From (4.4) and

$$\rho(\tilde{A}_k) = \sup_{0 \neq v_k \in \mathcal{V}_k} \frac{a_k(v_k, v_k)}{|\!|\!|v_k|\!|\!|_k^2} \leq \beta_k^{-2} \sup_{0 \neq v_k \in \mathcal{V}_k} \frac{\|v_k\|_{\mathcal{H}^0}^2}{|\!|\!|v_k|\!|\!|_k^2},$$

we find that

$$(4.6) \qquad \|\tilde{S}_{k,m} v_k\|_{1,k} \leq (2m+1)^{-1} \rho(\tilde{A}_k)^{\frac{1}{2}} |\!|\!|v_k|\!|\!|_k \leq (2m+1)^{-1} c_k \beta_k^{-1} \|v_k\|_{\mathcal{H}^0},$$

where

$$c_k := \sup_{0 \neq v_k \in \mathcal{V}_k} \frac{\|v_k\|_{\mathcal{H}^0}}{|||v_k|||_k} \sup_{0 \neq v_k \in \mathcal{V}_k} \frac{|||v_k|||_k}{\|v_k\|_{\mathcal{H}^0}}. \tag{4.7}$$

By applying interpolation to (E) and (4.6), we conclude that a sufficient condition for the CG-iteration to satisfy (F) with $\gamma = 1$, is that $(\!(\ ,\ )\!)_k$ is selected such that

$$c_k \lesssim 1. \tag{4.8}$$

Now we come to the selection of $(\!(\ ,\ )\!)_k$ and the discussion of the implementation of above CG-iteration.

Let $\{\varphi_{k,i} : i \in K_k\}$ be a basis of $\mathcal{V}_k$, and let

$$\Phi_k : \mathbb{R}^{\dim \mathcal{V}_k} \to \mathcal{V}_k : \mathbf{v}_k \mapsto \sum_{i \in K_k} \mathbf{v}_{k,i} \varphi_{k,i}$$

denote the corresponding bijection between the coordinates of an element and the element in $\mathcal{V}_k$ itself. Define the *mass*- and *stiffness*-matrices $\mathbf{M}_k$, $\mathbf{A}_k \in \mathbb{R}^{\dim \mathcal{V}_k \times \dim \mathcal{V}_k}$ and the vector $\mathbf{f}_k \in \mathbb{R}^{\dim \mathcal{V}_k}$ by

$$<\mathbf{M}_k \mathbf{w}_k, \mathbf{v}_k> = (\Phi_k \mathbf{w}_k, \Phi_k \mathbf{v}_k)_{\mathcal{H}^0}, <\mathbf{A}_k \mathbf{w}_k, \mathbf{v}_k> = a_k(\Phi_k \mathbf{w}_k, \Phi_k \mathbf{v}_k) \text{ and } <\mathbf{f}_k, \mathbf{v}_k> = f_k(\Phi_k \mathbf{v}_k),$$

where $<,>$ denotes an Euclidean scalar product.

Having the basis on $\mathcal{V}_k$ fixed, there is clearly an one-to-one correspondence between SPD matrices $\mathbf{W}_k \in \mathbb{R}^{\dim \mathcal{V}_k \times \dim \mathcal{V}_k}$ and scalar products $(\!(\ ,\ )\!)_k$ on $\mathcal{V}_k$ via the relation

$$<\mathbf{W}_k \cdot, \cdot> = (\!(\Phi_k \cdot, \Phi_k \cdot)\!)_k.$$

Using this correspondence, c_k defined in (4.7) can be rewritten as

$$c_k = \kappa(\mathbf{W}_k^{-1} \mathbf{M}_k),$$

and so (4.8) means that $\mathbf{W}_k$ should be spectrally equivalent to some multiple of $\mathbf{M}_k$, uniformly in k.

From the definitions of $\tilde{A}_k$ and $\tilde{f}_k$ in (4.1) it easily follows that

$$\tilde{A}_k = \Phi_k \mathbf{W}_k^{-1} \mathbf{A}_k \Phi_k^{-1} \quad \text{and} \quad \tilde{f}_k = \Phi_k \mathbf{W}_k^{-1} \mathbf{f}_k.$$

We conclude that the result $\hat{u}_k$ of the application of m CG-iterations to (4.2) using the scalar product $(\!(\ ,\ )\!)_k$ and initial value $\hat{u}_k^0$, satisfies $\hat{u}_k = \Phi_k \hat{\mathbf{u}}_k$, where $\hat{\mathbf{u}}_k$ is the result of the application, with initial value $\Phi_k^{-1} \hat{u}_k^0$, of m CG-iterations to $\mathbf{W}_k^{-1} \mathbf{A}_k \mathbf{u}_k = \mathbf{W}_k^{-1} \mathbf{f}_k$ using the scalar product $<\mathbf{W}_k \cdot, \cdot>$, or equivalently, m *preconditioned* CG-iterations to $\mathbf{A}_k \mathbf{u}_k = \mathbf{f}_k$ using the Euclidean scalar product $<\cdot, \cdot>$ and preconditioner $\mathbf{W}_k$.

5. Error estimates for nonconforming discretizations

Making use of the assumptions from §2 throughout this section, we will derive sufficient conditions for (3.3) and (3.4). Our analysis will follow the lines from [Bre99] quite closely, in particular Lemma 3.4, Theorem 3.5, Lemma 3.7 and Theorem 3.8 from that paper. Differences are that we end up with simpler constructions of the discrete right-hand sides f_k, and that within the same framework we derive a reduced set of conditions for the fully regular case $\alpha = 1$. Furthermore we allow that $\mathcal{V} \hookrightarrow \mathcal{H}^{1-\alpha}$ is not dense (cf. Remark 2.4).

We start by considering $f \in (\mathcal{H}^{1-\alpha})'$, i.e. $s = 1 - \alpha$, and

$$f_k = f \circ I_k \quad \text{for some} \quad I_k : \mathcal{V}_k \to \mathcal{H}^{1-\alpha}. \tag{5.1}$$

Since for applying duality arguments we will consider solutions corresponding to different right-hand sides simultaneously, in this section we will use the notations $u^{(f)}$ and $u_k^{(f)}$ to denote the solutions of $a(u, v) = f(v)$ $(v \in \mathcal{V})$ ((2.1)) and $a_k(u_k, v_k) = f(I_k v_k)$ $(v_k \in \mathcal{V}_k)$ ((2.3)) respectively.

Remark 5.1. In [Bre99] it is assumed that I_k maps into $\mathcal{V}$ instead of only into $\mathcal{H}^{1-\alpha}$. The present approach will give rise to a reduced set of conditions for the case $\alpha = 1$.

Theorem 5.2. *(a). Assume*

$$\text{(G)} \qquad \inf_{v_k \in \mathcal{V}_k} \|u - v_k\|_{a_k} \lesssim \beta_k^\alpha \|u\|_{\mathcal{H}^{1+\alpha}} \quad (u \in \mathcal{H}^{1+\alpha}) \quad \textit{(approximation)},$$

$$\text{(H)} \quad |a_k(u^{(f)}, v_k) - f(I_k v_k)| \lesssim \beta_k^\alpha \|f\|_{(\mathcal{H}^{1-\alpha})'} \|v_k\|_{1,k} \, (f \in (\mathcal{H}^{1-\alpha})',\, v_k \in \mathcal{V}_k) \quad \textit{(consistency)}.$$

Then

$$\|u^{(f)} - u_k^{(f)}\|_{a_k} \lesssim \beta_k^\alpha \|f\|_{(\mathcal{H}^{1-\alpha})'} \qquad (f \in (\mathcal{H}^{1-\alpha})'), \tag{5.2}$$

i.e., with $f_k = f \circ I_k$, the error estimate (2.4) is valid for $s = 1 - \alpha$.

(b). If, in addition,

$$|a_k(u^{(f)} - u_k^{(f)}, u_k^{(g)})| \lesssim \beta_k^{2\alpha} \|f\|_{(\mathcal{H}^{1-\alpha})'} \|g\|_{(\mathcal{H}^{1-\alpha})'} \quad (f, g \in (\mathcal{H}^{1-\alpha})'), \tag{5.3}$$

then

$$\|u^{(f)} - I_k u_k^{(f)}\|_{\mathcal{H}^{1-\alpha}} \lesssim \beta_k^{2\alpha} \|f\|_{(\mathcal{H}^{1-\alpha})'} \qquad (f \in (\mathcal{H}^{1-\alpha})') \tag{5.4}$$

Proof. (a). Given $f \in (\mathcal{H}^{1-\alpha})'$, let $\tilde{u}_k \in \mathcal{V}_k$ satisfy $a_k(\tilde{u}_k, v_k) = a_k(u^{(f)}, v_k)$ $(v_k \in \mathcal{V}_k)$. Then

$$\|u^{(f)} - \tilde{u}_k\|_{a_k} = \inf_{v_k \in \mathcal{V}_k} \|u^{(f)} - v_k\|_{a_k}.$$

On the other hand, there holds

$$\|\tilde{u}_k - u_k^{(f)}\|_{1,k} = \sup_{0 \neq v_k \in \mathcal{V}_k} \frac{|a_k(\tilde{u}_k - u_k^{(f)}, v_k)|}{\|v_k\|_{1,k}} = \sup_{0 \neq v_k \in \mathcal{V}_k} \frac{|a_k(u^{(f)}, v_k) - f(I_k v_k)|}{\|v_k\|_{1,k}}.$$

The proof of (5.2) follows from (G), $\|u^{(f)}\|_{\mathcal{H}^{1+\alpha}} \lesssim \|f\|_{(\mathcal{H}^{1-\alpha})'}$ (i.e. (B)) and (H).

(b). Let $f \in (\mathcal{H}^{1-\alpha})'$. There holds $\|u^{(f)} - I_k u_k^{(f)}\|_{\mathcal{H}^{1-\alpha}} = \sup_{0 \neq g \in (\mathcal{H}^{1-\alpha})'} \frac{|g(u^{(f)} - I_k u_k^{(f)})|}{\|g\|_{(\mathcal{H}^{1-\alpha})'}}$. For arbitrary $g \in (\mathcal{H}^{1-\alpha})'$, we write

$$\begin{aligned}
&|g(u^{(f)} - I_k u_k^{(f)})| = |a(u^{(g)}, u^{(f)}) - a_k(u_k^{(g)}, u_k^{(f)})| \\
&= \quad |a_k(u^{(g)} - u_k^{(g)}, u_k^{(f)}) + a_k(u^{(g)} - u_k^{(g)}, u^{(f)} - u_k^{(f)}) + a_k(u_k^{(g)}, u^{(f)} - u_k^{(f)})| \\
&\lesssim \quad \beta_k^{2\alpha} \|g\|_{(\mathcal{H}^{1-\alpha})'} \|f\|_{(\mathcal{H}^{1-\alpha})'}
\end{aligned}$$

by (5.3) and (5.2), which completes the proof. □

We discuss the non-standard condition (5.3). We first consider a special case:

Proposition 5.3. *If for all k, I_k is the trivial injection (necessary is $\mathcal{V}_k \subset \mathcal{H}^{1-\alpha}$), then (5.3) follows from (5.2) and*

(I) $\qquad |a_k(u^{(f)}, v_k) - f(v_k)| \lesssim \beta_k^\alpha \|f\|_{(\mathcal{H}^{1-\alpha})'} \|v_k\|_{a_k} \qquad (f \in (\mathcal{H}^{1-\alpha})',\ v_k \in \mathcal{V}_k + \mathcal{V}),$

which is a slightly stronger assumption than (H) for this I_k.

Proof. Given $f, g \in (\mathcal{H}^{1-\alpha})'$, by (I) and (5.2) we have

$$\begin{aligned}
a_k(u^{(f)} - u_k^{(f)}, u_k^{(g)}) &= |a_k(u^{(f)}, u_k^{(g)} - u^{(g)}) + a(u^{(f)}, u^{(g)})) - a_k(u_k^{(f)}, u_k^{(g)}| \\
&= |a_k(u^{(f)}, u_k^{(g)} - u^{(g)}) - f(u_k^{(g)} - u^{(g)})| \\
&\lesssim \beta_k^\alpha \|f\|_{(\mathcal{H}^{1-\alpha})'} \|u_k^{(g)} - u^{(g)}\|_{a_k} \lesssim \beta_k^{2\alpha} \|f\|_{(\mathcal{H}^{1-\alpha})'} \|g\|_{(\mathcal{H}^{1-\alpha})'},
\end{aligned}$$

which is (5.3). □

The following proposition shows that in the general case (5.3) can be deduced from an extra consistency assumption.

Proposition 5.4. *Assume (5.2), and let $\Pi^k : \mathcal{H}^{1+\alpha} \to \mathcal{V}_k$ be such that*

(J) $\qquad \|(I - \Pi^k) u\|_{a_k} \lesssim \beta_k^\alpha \|u\|_{\mathcal{H}^{1+\alpha}} \qquad (u \in \mathcal{H}^{1+\alpha})$

and

(K) $\qquad |a_k(u^{(f)}, \Pi^k v) - f(I_k \Pi^k v)| \lesssim \beta_k^{2\alpha} \|f\|_{(\mathcal{H}^{1-\alpha})'} \|v\|_{\mathcal{H}^{1+\alpha}} \qquad (f \in (\mathcal{H}^{1-\alpha})', v \in \mathcal{H}^{1+\alpha}).$

Then (5.3) is valid.

Proof. The proof follows from the assumptions and (B) by writing for $f, g \in (\mathcal{H}^{1-\alpha})'$,

$$\begin{aligned}
a_k(u^{(f)} - u_k^{(f)}, u_k^{(g)}) = a_k(u^{(f)} - u_k^{(f)}, \Pi^k u^{(g)}) + a_k(u^{(f)} - u_k^{(f)}, u_k^{(g)} - \Pi^k u^{(g)}) = \\
a_k(u^{(f)}, \Pi^k u^{(g)}) - f(I_k \Pi^k u^{(g)}) + a_k(u^{(f)} - u_k^{(f)}, u_k^{(g)} - u^{(g)} + (I - \Pi^k) u^{(g)}).
\end{aligned}$$

□

It will turn out that (5.4) can be used to prove (3.4):

Proposition 5.5. *(a). Assume (5.4) and*

$$\|v_k\|_{1-\alpha,k} \eqsim \|I_k v_k\|_{\mathcal{H}^{1-\alpha}} \qquad (v_k \in \mathcal{V}_k). \tag{5.5}$$

Let $\Pi^k : \mathcal{H}^{1+\alpha} \to \mathcal{V}_k$ *be a mapping such that*

$$\|I - I_k \Pi^k\|_{\mathcal{H}^{1-\alpha} \leftarrow \mathcal{H}^{1+\alpha}} \leq \beta_k^{2\alpha}. \tag{L}$$

Then

$$\|\Pi^k u^{(f)} - u_k^{(f)}\|_{1-\alpha,k} \lesssim \beta_k^{2\alpha} \|f\|_{(\mathcal{H}^{1-\alpha})'} \qquad (f \in (\mathcal{H}^{1-\alpha})'). \tag{5.6}$$

(b). If, in addition, the prolongation I_{k-1}^k *satisfies*

$$\|(\Pi^k - I_{k-1}^k \Pi^{k-1}) u\|_{1-\alpha,k} \lesssim \beta_k^{2\alpha} \|u\|_{\mathcal{H}^{1+\alpha}} \qquad (u \in \mathcal{H}^{1+\alpha}) \tag{M}$$

and

$$\|I_{k-1}^k\|_{1-\alpha,k \leftarrow 1-\alpha,k-1} \lesssim 1, \tag{N}$$

then

$$\|u_k^{(f)} - I_{k-1}^k u_{k-1}^{(f)}\|_{1-\alpha,k} \lesssim \beta_k^{2\alpha} \|f\|_{(\mathcal{H}^{1-\alpha})'} \qquad (f \in (\mathcal{H}^{1-\alpha})'), \tag{5.7}$$

i.e., with $f_k = f \circ I_k$*, (3.4) is valid for* $s = 1 - \alpha$*.*

Proof. (a). By (5.5), (L), (B) and (5.4), for $f \in (\mathcal{H}^{1-\alpha})'$ it holds that

$$\begin{aligned} \|\Pi^k u^{(f)} - u_k^{(f)}\|_{1-\alpha,k} &\eqsim \|I_k(\Pi^k u^{(f)} - u_k^{(f)})\|_{\mathcal{H}^{1-\alpha}} \\ &\leq \|(I_k \Pi^k - I) u^{(f)}\|_{\mathcal{H}^{1-\alpha}} + \|u^{(f)} - I_k u_k^{(f)}\|_{\mathcal{H}^{1-\alpha}} \lesssim \beta_k^{2\alpha} \|f\|_{(\mathcal{H}^{1-\alpha})'}. \end{aligned}$$

(b). By writing

$$u_k^{(f)} - I_{k-1}^k u_{k-1}^{(f)} = u_k^{(f)} - \Pi^k u^{(f)} + (\Pi^k - I_{k-1}^k \Pi^{k-1}) u^{(f)} + I_{k-1}^k (\Pi^{k-1} u^{(f)} - u_{k-1}^{(f)}),$$

the proof follows from (5.6), (M), (N), (B) and (C). □

In a special case (5.7) follows already from (5.4) and a condition slightly stronger than (N):

Proposition 5.6. *Let* $\alpha = 1$ *and* I_k *be the trivial injection, and suppose that* $I_{k-1}^k : \mathcal{V}_{k-1} \to \mathcal{V}_k$ *has an extension to a projector* $\tilde{I}_{k-1}^k : \mathcal{V}_{k-1} + \mathcal{V}_k \to \mathcal{V}_k$*. Then (5.4) and*

$$\|\tilde{I}_{k-1}^k v_k\|_{0,k} \lesssim \|v_k\|_{\mathcal{H}^0} \qquad (v_k \in \mathcal{V}_{k-1} + \mathcal{V}_k). \tag{O}$$

imply (5.7), i.e. (3.4) with $f_k = f|_{\mathcal{V}_k}$ *and* $s = 1 - \alpha = 0$*.*

Proof. For $f \in (\mathcal{H}^0)'$, we have

$$\|u_k^{(f)} - I_{k-1}^k u_{k-1}^{(f)}\|_{0,k} = \|\hat{I}_{k-1}^k (u_k^{(f)} - u_{k-1}^{(f)})\|_{0,k} \lesssim \|u_k^{(f)} - u_{k-1}^{(f)}\|_{\mathcal{H}^0}$$

by (O), and

$$\|u_k^{(f)} - u_{k-1}^{(f)}\|_{\mathcal{H}^0} \leq \|u_k^{(f)} - u^{(f)}\|_{\mathcal{H}^0} + \|u^{(f)} - u_{k-1}^{(f)}\|_{\mathcal{H}^0} \lesssim \beta_k^{2\alpha} \|f\|_{(\mathcal{H}^0)'}$$

by (5.4) and (C). □

For $\alpha < 1$, estimate (5.5) allows us to switch between discrete and continuous fractional norms. It can be verified using the following proposition.

Proposition 5.7. *Assume* $\operatorname{Im} I_k \subset \mathcal{H}^1$ *and*

$$\text{(P)} \qquad \|I_k v_k\|_{\mathcal{H}^0} \eqsim \|v_k\|_{0,k} \qquad (v_k \in \mathcal{V}_k),$$

$$\text{(Q)} \qquad \|I_k v_k\|_{\mathcal{H}^1} \lesssim \|v_k\|_{1,k} \qquad (v_k \in \mathcal{V}_k).$$

Let $\Pi^k : \mathcal{H}^1 \to \mathcal{V}_k$ *be some mapping satisfying*

$$\text{(R)} \qquad \|\Pi^k v\|_{1,k} \lesssim \|v\|_{\mathcal{H}^1} \qquad (v \in \mathcal{H}^1),$$

$$\text{(S)} \qquad \|I - I_k \Pi^k\|_{\mathcal{H}^0 \leftarrow \mathcal{H}^1} \lesssim \beta_k.$$

Then $\|I_k v_k\|_{\mathcal{H}^t} \eqsim \|v_k\|_{t,k}$ *(*$t \in [0,1]$*,* $v_k \in \mathcal{V}_k$*), i.e., (5.5) is valid.*

Proof. Using $\|I_k v_k\|_{\mathcal{H}^0} \lesssim \|v_k\|_{0,k}$ and (Q), interpolation shows that $\|I_k v_k\|_{\mathcal{H}^t} \lesssim \|v_k\|_{t,k}$ ($t \in [0,1], v_k \in \mathcal{V}_k$).

The estimate $\|I_k v_k\|_{\mathcal{H}^0} \gtrsim \|v_k\|_{0,k}$ implies that there exists an $F^k : \operatorname{Im} I_k \to \mathcal{V}_k$ with $F^k I_k = Id$ and $\|F^k \cdot\|_{0,k} \lesssim \|\cdot\|_{\mathcal{H}^0}$ on $\operatorname{Im} I_k$. Let $Q^k : \mathcal{H}^0 \to \operatorname{Im} I_k$ denote the $\mathcal{H}^0$-orthogonal projector onto $\operatorname{Im} I_k$. Then for $F^k Q^k : \mathcal{H}^0 \to \mathcal{V}_k$ we have

$$\|F^k Q^k v\|_{0,k} \lesssim \|Q^k v\|_{\mathcal{H}^0} \leq \|v\|_{\mathcal{H}^0} \qquad (v \in \mathcal{H}^0),$$

and

$$\|F^k Q^k v\|_{1,k} \lesssim \beta_k^{-1} \|F^k Q^k (I - I_k \Pi^k) v\|_{0,k} + \|\Pi^k v\|_{1,k} \lesssim \|v\|_{\mathcal{H}^1} \qquad (v \in \mathcal{H}^1)$$

by (2.2), (R) and (S). Interpolation shows that $\|v_k\|_{t,k} = \|F^k Q^k I_k v_k\|_{t,k} \lesssim \|I_k v_k\|_{\mathcal{H}^t}$ ($t \in [0,1]$, $v_k \in \mathcal{V}_k$). ☐

Remark 5.8. From Remark 2.4 we learn that if $\mathcal{V} = \mathcal{H}^1$, and

$$|a(u,v)| \lesssim \|u\|_{\mathcal{H}^{1+\alpha}} \|v\|_{\mathcal{H}^{1-\alpha}} \qquad (u \in \mathcal{H}^{1+\alpha},\ v \in \mathcal{H}^1) \qquad ((2.5)),$$

then (H) and (K) can be rewritten as

$$(5.8) \qquad |a_k(u, v_k) - a(u, I_k v_k)| \lesssim \beta_k^{\alpha} \|u\|_{\mathcal{H}^{1+\alpha}} \|v_k\|_{1,k} \qquad (u \in \mathcal{H}^{1+\alpha},\ v_k \in \mathcal{V}_k),$$

$$(5.9) \qquad |a_k(u, \Pi^k v) - a(u, I_k \Pi^k v)| \lesssim \beta_k^{2\alpha} \|u\|_{\mathcal{H}^{1+\alpha}} \|v\|_{\mathcal{H}^{1+\alpha}} \qquad (u \in \mathcal{H}^{1+\alpha}, v \in \mathcal{H}^{1+\alpha}),$$

where $a(\,,\,)$ here is the extended form on $\mathcal{H}^{1+\alpha} \times \mathcal{H}^{1-\alpha}$.

This reformulation of the consistency assumptions has the following advantage: Suppose there is some $\delta \geq \alpha$ and a Hilbert space that we denote by $\mathcal{H}^{1+\delta}$, such that $\mathcal{H}^{1+\alpha} = [\mathcal{H}^1, \mathcal{H}^{1+\delta}]_{1+\alpha}$, and for which we are able to verify (2.5), (5.8) and (5.9) with α replaced by δ. Then since, assuming (Q) and (R), the estimates (2.5), (5.8) and (5.9) are also valid for α replaced by zero, the method of complex interpolation shows that they are valid for the original α, and so (H) and (K) hold.

We note that it is generally not possible to apply interpolation arguments to (H) and (K) directly because of the regularity limitation.

In our application concerning the Stokes problem, $\mathcal{V}$ is a proper subspace of $\mathcal{H}^1$. For this application we will verify (H), but it can be shown that (5.8) is not valid.

Now we have derived sufficient conditions for (3.4) with $f_k = f \circ I_k$, we come to the verification of (3.3). We start with a lemma.

Lemma 5.9. *Assume (5.4), (5.5) and (5.7). Then for $P_k^{k-1} : \mathcal{V}_k \to \mathcal{V}_{k-1}$ defined by*

$$a_{k-1}(P_k^{k-1} u_k, v_{k-1}) = a_k(u_k, I_{k-1}^k v_{k-1}) \qquad (u_k \in \mathcal{V}_k,\ v_{k-1} \in \mathcal{V}_{k-1}),$$

it holds that

$$\|P_k^{k-1} u_k^{(f)} - u_{k-1}^{(f)}\|_{1-\alpha,k-1} \lesssim \beta_k^{2\alpha} \|f\|_{(\mathcal{H}^{1-\alpha})'} \qquad (f \in (\mathcal{H}^{1-\alpha})'). \tag{5.10}$$

Proof. By (5.5) and duality, we have to prove that

$$|g(I_{k-1}(P_k^{k-1} u_k^{(f)} - u_{k-1}^{(f)}))| \lesssim \beta_k^{2\alpha} \|g\|_{(\mathcal{H}^{1-\alpha})'} \|f\|_{(\mathcal{H}^{1+\alpha})'} \qquad (f, g \in (\mathcal{H}^{1-\alpha})').$$

We write

$$\begin{aligned} g(I_{k-1}(P_k^{k-1} u_k^{(f)} - u_{k-1}^{(f)})) &= a_{k-1}(u_{k-1}^{(g)}, P_k^{k-1} u_k^{(f)} - u_{k-1}^{(f)}) = \\ & a_k(I_{k-1}^k u_{k-1}^{(g)}, u_k^{(f)}) - a_{k-1}(u_{k-1}^{(g)}, u_{k-1}^{(f)}) = f((I_k I_{k-1}^k - I_{k-1}) u_{k-1}^{(g)}), \end{aligned}$$

and

$$(I_k I_{k-1}^k - I_{k-1}) u_{k-1}^{(g)} = I_k(I_{k-1}^k u_{k-1}^{(g)} - u_k^{(g)}) + I_k u_k^{(g)} - u^{(g)} + u^{(g)} - I_{k-1} u_{k-1}^{(g)}.$$

From (5.5), (5.7), (5.4) and (C), we conclude that

$$|f((I_k I_{k-1}^k - I_{k-1}) u_{k-1}^{(g)})| \lesssim \|f\|_{(\mathcal{H}^{1-\alpha})'} \beta_k^{2\alpha} \|g\|_{(\mathcal{H}^{1-\alpha})'},$$

which completes the proof. □

Remark 5.10. In relation to multi-grid convergence theory, we note here that (5.5), (N), (5.7) and (5.10) imply that $\|I - I_{k-1}^k P_k^{k-1}\|_{1-\alpha,k\leftarrow 1+\alpha,k} \lesssim \beta_k^{2\alpha}$, which is the so-called 'approximation property' (cf. [Hac85]). One may consult [Bre99, proof of Lemma 4.2] to verify this statement.

Proposition 5.11. *Assume (5.2), (5.5) and (5.10), then*

$$\|I_{k-1}^k - \hat{I}_{k-1}^k\|_{1-\alpha,k\leftarrow 1,k-1} \lesssim \beta_k^{\alpha},$$

i.e., (3.3) is valid.

Proof. By (5.5) and duality, we have to prove that

$$|g(I_k(I_{k-1}^k - \hat{I}_{k-1}^k) u_{k-1})| \lesssim \beta_k^{\alpha} \|g\|_{(\mathcal{H}^{1-\alpha})'} \|u_{k-1}\|_{1,k-1} \qquad (g \in (\mathcal{H}^{1-\alpha})',\ u_{k-1} \in \mathcal{V}_{k-1}).$$

By the definition of $\hat{I}_{k-1}^k$ given in (3.1), we have

$$\begin{aligned} g(I_k(I_{k-1}^k - \hat{I}_{k-1}^k) u_{k-1}) &= a_k(u_k^{(g)}, (I_{k-1}^k - \hat{I}_{k-1}^k) u_{k-1}) = a_k(u_k^{(g)}, (I_{k-1}^k - I) u_{k-1}) \\ &= a_{k-1}(P_k^{k-1} u_k^{(g)} - u_{k-1}^{(g)}, u_{k-1}) + a_k(u_{k-1}^{(g)} - u_k^{(g)}, u_{k-1}). \end{aligned}$$

By (5.2) and (C), there holds

$$|a_k(u_{k-1}^{(g)} - u_k^{(g)}, u_{k-1})| \lesssim \beta_k^\alpha \|g\|_{(\mathcal{H}^{1-\alpha})'} \|u_{k-1}\|_{1,k-1}.$$

The inverse inequality (2.2) and (5.10) show that

$$\begin{aligned} |a_{k-1}(P_k^{k-1} u_k^{(g)} - u_{k-1}^{(g)}, u_{k-1})| &\lesssim \beta_k^{-\alpha} \|P_k^{k-1} u_k^{(g)} - u_{k-1}^{(g)}\|_{1-\alpha,k-1} \|u_{k-1}\|_{1,k-1} \\ &\lesssim \beta_k^\alpha \|g\|_{(\mathcal{H}^{1-\alpha})'} \|u_{k-1}\|_{1,k-1}, \end{aligned}$$

which completes the proof. □

Finally in this section, we return to the verification of (2.4) and (3.4). So far we assumed that $f_k = f \circ I_k$, where, besides other conditions, I_k satisfied (5.5), i.e., $\|\cdot\|_{1-\alpha,k} \eqsim \|I_k \cdot\|_{\mathcal{H}^{1-\alpha}}$. For $\alpha = 1$, this condition is easily fulfilled, but for $\alpha < 1$ the verification will be difficult without assuming that I_k maps into $\mathcal{H}^1$. On the other hand, to ensure that for $f \in (\mathcal{H}^{1-\alpha})'$, $f_k = f \circ I_k$ is well-defined, it is already sufficient that I_k maps into $\mathcal{H}^{1-\alpha}$.

In view of this observation, in the following we relax the conditions on the construction of f_k. Yet, if $\mathcal{V}_k \not\subset \mathcal{H}^{1-\alpha}$ it will not be possible to take $f_k = f|_{\mathcal{V}_k}$. In that case it may make sense to consider only right-hand sides $f \in (\mathcal{H}^s)'$ for some $s < 1-\alpha$, which then allows for a further simplification of the construction of f_k. However, as noted before, due to the regularity restriction, imposing stronger conditions on f will not lead to qualitatively better error estimates as function of β_k.

Theorem 5.12. *Let $s \in [0, 1-\alpha]$, and let G_k be a mapping from $\mathcal{V}_k$ into $\mathcal{H}^s$. For $f \in (\mathcal{H}^s)'$, let $u_k^{(f)}$ now denote the solution of $a_k(u_k, v_k) = f(G_k v_k)$ $(v_k \in \mathcal{V}_k)$, which is the system (2.3) with $f_k = f \circ G_k$.*

(a). Assume (5.2) and

$$\text{(T)} \qquad \|(I_k - G_k)v_k\|_{\mathcal{H}^s} \lesssim \beta_k^{1-s} \|v_k\|_{1,k} \qquad (v_k \in \mathcal{V}_k).$$

Then

$$\|u^{(f)} - u_k^{(f)}\|_{a_k} \lesssim \beta_k^\alpha \|f\|_{(\mathcal{H}^{1-\alpha})'} + \beta_k^{1-s} \|f\|_{(\mathcal{H}^s)'} \qquad (f \in (\mathcal{H}^s)'),$$

i.e, (2.4) is valid.

(b). In addition, assume (5.5), and

$$\text{(U)} \qquad \|(I_k - G_k)\Pi^k\|_{\mathcal{H}^s \leftarrow \mathcal{H}^{1+\alpha}} \lesssim \beta_k^{1+\alpha-s}$$

for some mapping $\Pi^k : \mathcal{H}^{1+\alpha} \to \mathcal{V}_k$ satisfying

$$\text{(V)} \qquad \|(I - \Pi^k)v\|_{a_k} \lesssim \beta_k^\alpha \|v\|_{\mathcal{H}^{1+\alpha}} \qquad (v \in \mathcal{H}^{1+\alpha}).$$

Then (5.6) gives

$$\text{(5.11)} \qquad \|\Pi^k u^{(f)} - u_k^{(f)}\|_{1-\alpha,k} \lesssim \beta_k^{2\alpha} \|f\|_{(\mathcal{H}^{1-\alpha})'} + \beta_k^{1+\alpha-s} \|f\|_{(\mathcal{H}^s)'} \qquad (f \in (\mathcal{H}^s)'),$$

and (N) and (5.7) give

$$\text{(5.12)} \qquad \|u_k^{(f)} - I_{k-1}^k u_{k-1}^{(f)}\|_{1-\alpha,k} \lesssim \beta_k^{2\alpha} \|f\|_{(\mathcal{H}^{1-\alpha})'} + \beta_k^{1+\alpha-s} \|f\|_{(\mathcal{H}^s)'} \qquad (f \in (\mathcal{H}^s)'),$$

i.e., (3.4) is valid.

Proof. In this proof, let $\hat{u}_k^{(f)}$ denote the solution of $a_k(u_k, v_k) = f(I_k v_k)$ $(v_k \in \mathcal{V}_k)$.

(a). By (T), we have

$$|a_k(\hat{u}_k^{(f)} - u_k^{(f)}, v_k)| = |f((I_k - G_k)v_k)| \lesssim \beta_k^{1-s} \|f\|_{(\mathcal{H}^s)'} \|v_k\|_{1,k} \qquad (v_k \in \mathcal{V}_k),$$

or

$$\|\hat{u}_k^{(f)} - u_k^{(f)}\|_{a_k} \lesssim \beta_k^{1-s} \|f\|_{(\mathcal{H}^s)'}. \tag{5.13}$$

The proof now follows from (5.2).

(b). By (5.6) or (N), (5.7) and (C), it is sufficient to show that

$$\|\hat{u}_k^{(f)} - u_k^{(f)}\|_{1-\alpha,k} \lesssim \beta_k^{1+\alpha-s} \|f\|_{(\mathcal{H}^s)'} \qquad (f \in (\mathcal{H}^s)'),$$

which by (5.5) and duality is equivalent to

$$|g(I_k(\hat{u}_k^{(f)} - u_k^{(f)}))| \lesssim \beta_k^{1+\alpha-s} \|g\|_{(\mathcal{H}^{1-\alpha})'} \|f\|_{(\mathcal{H}^s)'} \qquad (g \in (\mathcal{H}^{1-\alpha})',\ f \in (\mathcal{H}^s)'). \tag{5.14}$$

By writing

$$\begin{aligned} g(I_k(\hat{u}_k^{(f)} - u_k^{(f)})) &= a_k(\hat{u}_k^{(g)}, \hat{u}_k^{(f)} - u_k^{(f)}) \\ &= a_k(\Pi^k \hat{u}^{(g)}, \hat{u}_k^{(f)} - u_k^{(f)}) + a_k(\hat{u}_k^{(g)} - \Pi^k \hat{u}^{(g)}, \hat{u}_k^{(f)} - u_k^{(f)}), \end{aligned}$$

(5.14) follows from

$$|a_k(\Pi^k \hat{u}^{(g)}, \hat{u}_k^{(f)} - u_k^{(f)})| = |f((I_k - G_k)\Pi^k \hat{u}^{(g)})| \lesssim \|f\|_{(\mathcal{H}^s)'} \beta_k^{1+\alpha-s} \|g\|_{(\mathcal{H}^{1-\alpha})'}$$

by (U) and (B), and

$$|a_k(\hat{u}_k^{(g)} - \hat{u}^{(g)} + \hat{u}^{(g)} - \Pi^k \hat{u}^{(g)}, \hat{u}_k^{(f)} - u_k^{(f)})| \lesssim \beta_k^{\alpha} \|g\|_{(\mathcal{H}^{1-\alpha})'} \beta_k^{1-s} \|f\|_{(\mathcal{H}^s)'}$$

by (5.2), (V) and (B), and (5.13). ☐

As we will see in the applications in §6, the conditions of above theorem can be satisfied for G_k mapping only into $\mathcal{H}^s$. Properties of the generally more complicated I_k are still used to obtain relevant estimates, but I_k will not enter the practical computations. In particular for $s = 1 - \alpha$, this approach will give rise to new, cheaper nonconforming discretizations.

6. Applications

6.1. **Nonconforming P_1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain without slits. We consider the Poisson equation with homogeneous Dirichlet boundary conditions, i.e., we take

$$\mathcal{H}^0 = L^2(\Omega), \quad \mathcal{H}^t = H_0^1(\Omega) \cap H^t(\Omega) \ \ (t \geq 1), \quad \mathcal{V} = \mathcal{H}^1$$

and $a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx$. Then (A) holds, and elliptic regularity theory shows that there exists an $\alpha \in (\frac{1}{2}, 1]$, such that for $f \in H^{-1+\alpha}(\Omega)$, the solution u of (2.1) satisfies $\|u\|_{H^{1+\alpha}} \lesssim \|f\|_{H^{-1+\alpha}}$ ((B)).

Let $\tau_0, \tau_1, \ldots$ be a sequence of conforming triangulations of Ω, such that τ_{k+1} is generated from τ_k by refinement, $\sup_{T \in \tau_k} \operatorname{diam}(T) \eqsim 2^{-k}$, and the triangles satisfy a shape regularity condition uniformly over the levels. We define $\overline{E}_k$, $\overline{V}_k$ as the set of all edges and vertices of τ_k, and E_k, V_k as the set of internal edges and vertices of τ_k. For $e \in \overline{E}_k$, m_e will denote the midpoint of e, and $\mathbf{n}_e$ is a unit vector normal to e.

We consider the nonconforming P_1 finite element space, i.e., we take $\mathcal{V}_k = N_k$ where

$$N_k = \{v \in \textstyle\prod_{T\in\tau_k} P_1(T)\colon\ v \text{ is continuous at } m_e \text{ for } e \in E_k, \\ \text{and it vanishes at } m_e \text{ for } e \in \overline{E}_k \backslash E_k\},$$

and define

$$a_k(u_k, v_k) = \sum_{T\in\tau_k} \int_T \nabla u_k \cdot \nabla v_k dx.$$

It holds that $\beta_k \eqsim h_k := 2^{-k}$ and so (C) is valid.

We equip the spaces N_k with nodal bases $\{\eta_{k,e} : e \in E_k\}$, defined by

$$\eta_{k,e}(m_{\tilde{e}}) = \delta_{e,\tilde{e}} \qquad (e, \tilde{e} \in E_k). \tag{6.1}$$

From $(\eta_{k,e}, \eta_{k,\tilde{e}})_{L^2} \eqsim \delta_{e,\tilde{e}} h_k^2$, it follows that the mass matrices are uniformly well-conditioned (diagonal) matrices. The analysis from §4 shows that CG-smoothing with any preconditioning matrices that are uniformly well-conditioned satisfies (D), (E) and (F) with $\gamma = 1$. Since the values of the other parameters appearing in Theorem 3.2 are given by $r = 1$, $\rho = 2$ and $d = 2$, this theorem shows that for any $c \in (2, 4)$ the Cascade iteration with $m_{j-i}^{(j)} \eqsim \tilde{m} c^i$ yields an optimal solver in case (3.3) and (3.4) are valid.

We define the prolongation in the usual way, that is,

$$(I_{k-1}^k v_{k-1})(m_e) = \text{ average}_i \text{ of } v_{k-1}|_{T_i}(m_e) \qquad (e \in E_k),$$

where $\tau_{k-1} \ni T_i \supset e$.

It is clear that I_{k-1}^k extends to a projector $\tilde{I}_{k-1}^k : N_{k-1} + N_k \to N_k$ that is L^2-bounded. This means that if we had confined ourselves to the $\alpha = 1$ case, which corresponds to Ω being convex, then by Theorem 5.2, Proposition 5.3, Proposition 5.6, Lemma 5.9 and Proposition 5.12, it would have been sufficient to verify only (G) and (I) to conclude (3.3) and (3.4) (with $s = 1 - \alpha = 0$ and $f_k = f|_{N_k}$). It is well-known that (G) and (H) (with I_k being the trivial injection) are valid, see for example [BS94, §8.3]. Exactly the same technique that yields (H) shows the slightly stronger (I) as well.

In the general case, we have to verify a larger set of conditions from §5. For convenience of the reader, we give arguments for all these conditions, however, since most of them have been discussed earlier in the literature, our treatment will be concise. The reader who prefers more details is referred to [Bre99] and the references cited there.

Following [Bre99], we define an auxiliary space

$$\tilde{N}_k = \prod_{T\in\tau_k} P_2(T) \cap C(\Omega) \cap H_0^1(\Omega), \tag{6.2}$$

i.e., $\tilde{N}_k$ is the space of continuous piecewise quadratics with respect to τ_k, which are zero on $\partial\Omega$. We define $I_k : N_k \to \tilde{N}_k \subset H_0^1(\Omega)$ by

$$(I_k v_k)(m_e) = v_k(m_e) \ \ (e \in E_k), \quad (I_k v_k)(p) = v_k|_T(p) \ \ (p \in V_k),$$

where $T \in \tau_k$ is some triangle that contains p.

By the L^2-stability of the properly scaled canonical bases of N_k and $\tilde{N}_k$, estimate (P) follows easily.

By the continuity in the midpoints of the edges, if $v_k \in N_k$ is constant on a $T_0 \in \tau_k$ as well as on each of its direct neighbors $T_1, \dots, T_q \in \tau_k$, it is constant on $\cup_{i=0}^q T_i$, and so $(I - I_k)v_k = 0$ on T_0. Using a homogeneity argument one can now conclude that for arbitrary $v_k \in N_k$, there holds $\|(I - I_k)v_k\|_{L^2(T_0)} \lesssim h_k \sqrt{\sum_{i=0}^q |v_k|^2_{H^1(T_i)}}$, and so

$$\|(I - I_k)v_k\|_{L^2} \lesssim h_k \|v_k\|_{1,k} \qquad (v_k \in N_k), \tag{6.3}$$

which using an inverse inequality gives (Q).

The mapping $\Pi^k : H_0^1(\Omega) \to N_k$ that we will use on all places in §5 is defined by

$$(\Pi^k v)(m_e) = \frac{1}{|e|}\int_e v ds \qquad (e \in E_k).$$

Since Π^k locally reproduces linear polynomials, an application of the Bramble-Hilbert lemma, a homogeneity argument and interpolation show that

$$\|(I - \Pi^k)v\|_{L^2} + h_k \|(I - \Pi^k)v\|_{a_k} \lesssim h_k^t \|v\|_{H^t} \qquad (t \in [1,2],\ v \in H_0^1(\Omega) \cap H^t(\Omega)), \tag{6.4}$$

which gives (J), and thus (G), as well as (R) and (V).

Since also $I_k \Pi^k : H_0^1(\Omega) \to H_0^1(\Omega)$ locally reproduces linear polynomials, the same arguments show that

$$\|(I - I_k\Pi^k)v\|_{H^q} \lesssim h_k^{t-q} \|v\|_{H^t} \qquad (q \in [0,1],\ t \in [1,2],\ v \in H_0^1(\Omega) \cap H^t(\Omega)), \tag{6.5}$$

giving (L) and (S).

Similar arguments that gave (6.3) show that

$$\|(I - I_{k-1}^k)v_{k-1}\|_{L^2} \lesssim h_k \|v_{k-1}\|_{1,k-1} \qquad (v_{k-1} \in N_{k-1}), \tag{6.6}$$

which by applying inverse inequalities gives (N).

Since both Π^k and $I_{k-1}^k \Pi^{k-1}$ locally reproduce linear polynomials, the Bramble-Hilbert lemma gives

$$\|(\Pi^k - I_{k-1}^k \Pi^{k-1})v\|_{L^2} \lesssim h_k^2 \|v\|_{H^2} \qquad (v \in H_0^1(\Omega) \cap H^2(\Omega)).$$

Since furthermore by $\|\Pi^k v\|_{1,k} \lesssim \|v\|_{H^1}$ ((R)), and $\|I_{k-1}^k v_{k-1}\|_{1,k} \lesssim \|v_{k-1}\|_{1,k-1}$, which follows from (6.6), there holds $\|(\Pi^k - I_{k-1}^k \Pi^{k-1})v\|_{1,k} \lesssim \|v\|_{H^1}$, interpolation gives (M).

For $u \in H_0^1(\Omega) \cap H^2(\Omega)$, $v \in H_0^1(\Omega)$, integration by parts shows that

$$\left|\int_\Omega \nabla u \cdot \nabla v dx\right| = \left| -\int_\Omega \triangle u \, v dx\right| \lesssim \|u\|_{H^2} \|v\|_{L^2}, \tag{6.7}$$

where $(u,v) \mapsto -\int_\Omega \triangle u \, v dx$ is the unique extension of $a(\ ,\ )$ to a bounded bilinear form on $(H_0^1(\Omega) \cap H^2(\Omega)) \times L^2(\Omega)$. For $u \in H_0^1(\Omega) \cap H^2(\Omega)$, $v_k \in N_k$, integration by parts on each

$T \in \tau_k$ shows that

$$a_k(u, v_k) + \int_\Omega \triangle u\, I_k v_k dx = \int_\Omega \triangle u\,(I_k - I)v_k dx - \sum_{e \in \overline{E}_k} \int_e (\partial_{\mathbf{n}_e} u)[v_k] ds, \tag{6.8}$$

where $[v_k]$ denotes the jump of v_k across e in the direction of $\mathbf{n}_e$. By (6.3), $|\int \triangle u\,(I_k - I)v_k dx| \lesssim h_k \|u\|_{H^2} \|v_k\|_{1,k}$. Since $[v_k]$ is linear on e, and zero in its midpoint, it holds that $\int_e [v_k] ds = 0$. An application of [CR73, Lemma 3] now shows that $|\sum_e \int_e (\partial_{\mathbf{n}_e} u)[v_k] ds| \lesssim h_k \|u\|_{H^2} \|v_k\|_{1,k}$, and so

$$|a_k(u, v_k) + \int_\Omega \triangle u\, I_k v_k dx| \lesssim h_k \|u\|_{H^2} \|v_k\|_{1,k}. \tag{6.9}$$

We now substitute $v_k = \Pi^k v$ in (6.8), where $v \in H_0^1(\Omega) \cap H^2(\Omega)$. Using (6.4) and (6.5) we find that $|\int \triangle u\,(I_k - I)\Pi^k v dx| \lesssim h_k^2 \|u\|_{H^2} \|v\|_{H^2}$. Since $\int_e [\Pi^k v - v] ds = 0$, [CR73, Lemma 3] and (6.4) give

$$|\sum_e \int_e (\partial_{\mathbf{n}_e} u)[\Pi^k v] ds| = |\sum_e \int_e (\partial_{\mathbf{n}_e} u)[\Pi^k v - v] ds| \lesssim h_k \|u\|_{H^2} \|(\Pi^k - I)v\|_{a_k} \lesssim h_k^2 \|u\|_{H^2} \|v\|_{H^2},$$

and so

$$|a_k(u, \Pi^k v) + \int_\Omega \triangle u\, I_k \Pi^k v dx| \lesssim h_k^2 \|u\|_{H^2} \|v\|_{H^2}. \tag{6.10}$$

Because of (6.7), (6.9), (6.10), Remark 5.8 with $\delta = 1$ shows (H) and (K).

From Theorem 5.2, Propositions 5.4, 5.5 and 5.7, Lemma 5.9 and Proposition 5.11, we now conclude (3.3) and (3.4) with $s = 1 - \alpha$ and $f_k = f \circ I_k$.

Finally, we will simplify the construction of f_k. Since $1 - \alpha < \frac{1}{2}$, it is known, see [Osw94], that $N_k \subset H^{1-\alpha}(\Omega)$, and furthermore that the following inverse inequality is valid:

$$\|v_k\|_{H^{1-\alpha}} \lesssim h_k^{\alpha - 1} \|v_k\|_{L^2} \qquad (v_k \in N_k). \tag{6.11}$$

This means that with G_k being the trivial injection, (T) and (U) with $s = 1 - \alpha$ follow from (6.3) and (6.5) respectively. Theorem 5.13 now shows that (3.4) with $s = 1 - \alpha$ is also valid when $f_k = f|_{N_k}$. From Theorem 3.2 we conclude the optimality of the resulting Cascade iteration, that is,

$$\|u_j - u_j^*\|_{1,j} \lesssim h_k^\alpha \|f\|_{H^{-1+\alpha}} \qquad (f \in H^{-1+\alpha}(\Omega))$$

taking $\mathcal{O}(\dim \mathcal{V}_j)$ operations.

Moreover, we note that by (5.11) we have the optimal error estimate

$$\|\Pi^k u - u_k\|_{1-\alpha,k} \lesssim h_k^{2\alpha} \|f\|_{H^{-1+\alpha}} \qquad (f \in H^{-1+\alpha}(\Omega)),$$

which seems new for $f_k = f|_{N_k}$ and $\alpha \in (\frac{1}{2}, 1)$.

6.2. **Morley element.** Let $\Omega \subset I\!R^2$ be a bounded polygonal domain without slits. We consider the biharmonic equation with homogeneous Dirichlet boundary conditions, i.e., we take

$$\mathcal{H}^0 = L^2(\Omega), \quad \mathcal{H}^t = H_0^2(\Omega) \cap H^{2t}(\Omega) \ (t \geq 1), \quad \mathcal{V} = \mathcal{H}^1,$$

and $a(u,v) = \int_\Omega \sum_{i,j=1}^2 \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} dx$. Then (A) holds, and elliptic regularity theory shows that there exists an $\alpha \in (\frac{1}{4}, \frac{1}{2}]$, where $\alpha = \frac{1}{2}$ corresponds to the case of Ω being convex, such that for $f \in H^{-2+2\alpha}(\Omega)$, the solution u of (2.1) satisfies $\|u\|_{H^{2+2\alpha}} \lesssim \|f\|_{H^{-2+2\alpha}}$ ((B)).

Let $\tau_0, \tau_1, \ldots$ be a sequence of conforming triangulations of Ω, such that τ_{k+1} is generated from τ_k by refinement, $\sup_{T \in \tau_k} \operatorname{diam}(T) \eqsim 2^{-k}$, and the triangles satisfy a shape regularity condition uniformly over the levels. We define $\overline{E}_k$, $\overline{V}_k$, E_k, V_k, m_e, $\mathbf{n}_e$ as in §6.1.

We consider the Morley finite element space, i.e., we take $\mathcal{V}_k = M_k$ where

$$\begin{aligned} M_k = \{v \in \textstyle\prod_{T \in \tau_k} P_2(T) \colon & \ v \text{ is continuous at } p \in V_k, \text{ and it vanishes at } p \in \overline{V}_k \backslash V_k; \\ & \partial_{\mathbf{n}_e} v \text{ is continuous at } m_e \text{ for } e \in E_k, \text{ and it vanishes at } m_e \text{ for } e \in \overline{E}_k \backslash E_k\}. \end{aligned}$$

and define

$$a_k(u_k, v_k) = \sum_{T \in \tau_k} \int_T \sum_{i,j=1}^2 \frac{\partial^2 u_k}{\partial x_i \partial x_j} \frac{\partial^2 v_k}{\partial x_i \partial x_j} dx.$$

With $h_k := 2^{-k}$, it holds that $\beta_k \eqsim h_k^2$ and so (C) is valid.

We equip M_k with the properly scaled canonical basis

$$\{\zeta_{k,e} : e \in E_k\} \cup \{\theta_{k,p} : p \in V_k\} \tag{6.12}$$

defined by

$$\begin{array}{rclcrclc} \zeta_{k,e}(p) & = & 0 & (p \in V_k), & \theta_{k,e}(\tilde{p}) & = & \delta_{p,\tilde{p}} & (\tilde{p} \in V_k), \\ \partial_{\mathbf{n}_{\tilde{e}}} \zeta_{k,e}(m_{\tilde{e}}) & = & |e|^{-1} \delta_{e,\tilde{e}} & (\tilde{e} \in E_k), & \partial_{\mathbf{n}_e} \theta_{k,p}(m_e) & = & 0 & (e \in E_k), \end{array}$$

The resulting mass-matrix is uniformly well-conditioned as function of k, and so the analysis from §4 shows that CG-smoothing without, or with any preconditioners that are uniformly well-conditioned satisfies (D), (E) and (F) with $\gamma = 1$. Since the values of the other parameters appearing in Theorem 3.2 are given by $r = 2$, $\rho = 2$ and $d = 2$, this theorem shows that with $m_{j-i}^{(j)} \eqsim \tilde{m} 4^i j^{1/\alpha}$, the Cascade iteration yields a *sub*optimal solver in case (3.3) and (3.4) are valid. Later, in §6.4 we will return to this point, where we will introduce an even more powerful smoother resulting in an optimal Cascade iteration.

We take the prolongation I_{k-1}^k commonly used in connection with the Morley finite element space defined by

$$\begin{aligned} (I_{k-1}^k v_{k-1})(p) &= \text{average}_i \text{ of } v_{k-1}|_{T_i}(p) & (p \in V_k), \\ \partial_{\mathbf{n}_e}(I_{k-1}^k v_{k-1})(m_e) &= \text{average}_i \text{ of } \partial_{\mathbf{n}_e}(v_{k-1}|_{T_i})(m_e) & (e \in E_k), \end{aligned} \tag{6.13}$$

where $\tau_{k-1} \ni T_i \ni p$ or $\tau_{k-1} \ni T_i \supset e$.

Let $\tilde{M}_k$ be the so-called Hsieh-Clough-Tocher macro finite element space corresponding to τ_k (see e.g. [Cia91]), where the degrees of freedom corresponding to boundary points are set to zero. Following [Bre99], we define $I_k : M_k \to \tilde{M}_k \subset H_0^2(\Omega)$ by

$$\begin{array}{rcll} (I_k v_k)(p) &=& v_k(p) & (p \in V_k), \\ \partial_{\mathbf{n}_e}(I_k v_k)(m_e) &=& \partial_{\mathbf{n}_e} v_k(m_e) & (e \in E_k), \\ \partial_{x_j}(I_k v_k)(p) &=& \text{average}_i \text{ of } \partial_{x_j}(v_k|_{T_i})(p) & (p \in V_k,\ j \in \{1,2\}), \end{array}$$

where $\tau_k \ni T_i \ni p$. By the L^2-stability of the properly scaled canonical bases of M_k and $\tilde{M}_k$, estimate (P) follows easily. The mapping $\Pi^k : H_0^2(\Omega) \to M_k$ that will be used on all places in §5 is defined by

$$(\Pi^k v)(p) = v(p) \quad (p \in V_k), \quad \partial_{\mathbf{n}_e}(\Pi^k v)(m_e) = \frac{1}{|e|}\int_e \partial_{\mathbf{n}_e} v\, ds \quad (e \in E_k).$$

Analogously as in §6.1, the following estimates can be shown. For details we refer to [Bre99] and the references cited there.

$$\|(I - I_k)v_k\|_{L^2} \lesssim h_k^2 \|v_k\|_{1,k} \qquad (v_k \in M_k), \tag{6.14}$$

$$\|(I - \Pi^k)v\|_{L^2} + h_k^2 \|(I - \Pi^k)v\|_{a_k} \lesssim h_k^t \|v\|_{H^t} \qquad (t \in [2,3],\ v \in H_0^2(\Omega) \cap H^t(\Omega)), \tag{6.15}$$

$$\|(I - I_k\Pi^k)v\|_{H^q} \lesssim h_k^{t-q} \|v\|_{H^t} \qquad (q \in [0,2],\ t \in [2,3],\ v \in H_0^2(\Omega) \cap H^t(\Omega)), \tag{6.16}$$

$$\|(I - I_{k-1}^k)v_{k-1}\|_{L^2} \lesssim h_k^2 \|v_{k-1}\|_{1,k-1} \qquad (v_{k-1} \in M_{k-1}), \tag{6.17}$$

$$\|(\Pi^k - I_{k-1}^k \Pi^{k-1})v\|_{L^2} \lesssim h_k^3 \|v\|_{H^3} \qquad (v \in H_0^2(\Omega) \cap H^3(\Omega)). \tag{6.18}$$

Similarly as in §6.1, these estimates imply (G), (J), (L), (M), (N), (Q), (S) and (V).

For $u \in H_0^2(\Omega) \cap H^3(\Omega)$, $v \in H_0^2(\Omega)$, integration by parts shows that

$$|a(u,v)| = |-\int_\Omega \nabla(\triangle u) \cdot \nabla v\, dx| \lesssim \|u\|_{H^3} \|v\|_{H^1}, \tag{6.19}$$

where $(u,v) \mapsto -\int_\Omega \nabla(\triangle u) \cdot \nabla v\, dx$ is the unique extension of $a(\ ,\ )$ to a bounded bilinear form on $(H_0^2(\Omega) \cap H^3(\Omega)) \times H_0^1(\Omega)$. For $u \in H_0^2(\Omega) \cap H^3(\Omega)$, $v_k \in M_k$, integration by parts on each $T \in tau_k$ shows that

$$a_k(u, v_k) + \int_\Omega \nabla(\triangle u) \cdot \nabla(I_k v_k)\, dx = \tag{6.20}$$

$$\sum_{T \in \tau_k} \int_T \nabla(\triangle u) \cdot \nabla((I_k - I)v_k)\, dx - \sum_{j=1}^{2} \sum_{e \in \overline{E}_k} \int_e (\partial_{\mathbf{n}_e} \partial_{x_j} u)[\partial_{x_j} v_k]\, ds,$$

where $[\partial_{x_j} v_k]$ denotes the jump of $\partial_{x_j} v_k$ across e in the direction of $\mathbf{n}_e$. By an inverse inequality on $M_k + \tilde{M}_k$ and (6.14), there holds

$$|\sum_{T \in \tau_k} \int_T \nabla(\triangle u) \cdot \nabla((I_k - I)v_k)\, dx| \lesssim \|u\|_{H^3} h_k^{-1} \|(I - I_k)v_k\|_{L^2} \lesssim h_k \|u\|_{H^3} \|v_k\|_{1,k}.$$

Using the continuity of $v_k \in M_k$ in the vertices, it is easily verified that in m_e for $e \in E_k$ not only the normal, but also the tangential derivative of v_k is continuous, and that it vanishes in m_e for $e \in \overline{E}_k \backslash E_k$. Since furthermore $[\partial_{x_j} v_k]$ is linear, $\int_e [\partial_{x_j} v_k] ds = 0$, which by [CR73, Lemma 3] gives $|\sum_j \sum_e \int_e (\partial_{\mathbf{n}_e} \partial_{x_j} u)[\partial_{x_j} v_k] ds| \lesssim h_k \|u\|_{H^3} \|v_k\|_{1,k}$, and so

$$|a_k(u, v_k) + \int_\Omega \nabla(\triangle u) \cdot \nabla(I_k v_k) dx| \lesssim h_k \|u\|_{H^3} \|v_k\|_{1,k}. \tag{6.21}$$

Analogously as in §6.1, by substituting $v_k = \Pi^k v$ in (6.20), where $v \in H_0^2(\Omega) \cap H^3(\Omega)$,

$$|a_k(u, \Pi^k v) + \int_\Omega \nabla(\triangle u) \cdot \nabla(I_k \Pi^k v) dx| \lesssim h_k^2 \|u\|_{H^3} \|v\|_{H^3} \tag{6.22}$$

follows using $\|(I_k - I)\Pi^k)v\|_{L^2} \lesssim h_k^3 \|v\|_{H^3}$ by (6.15) and (6.16). Because of (6.19), (6.21) and (6.22), Remark 5.8 with $\delta = \frac{1}{2}$ shows (H) and (K).

From Theorem 5.2, Propositions 5.4, 5.5 and 5.7, Lemma 5.9 and Proposition 5.11, we now conclude (3.3) and (3.4) with $s = 1 - \alpha$ and $f_k = f \circ I_k$.

Finally, we will simplify the construction of f_k. We start by considering in Theorem 5.13 the $s = 1 - \alpha$ case, i.e., $f \in H^{2\alpha - 2}(\Omega)$. Let

$$\tilde{N}_k = \prod_{T \in \tau_k} P_2(T) \cap C(\Omega) \cap H_0^1(\Omega),$$

which is the same space as in (6.2). Since $\alpha \in (\frac{1}{4}, \frac{1}{2}]$ and thus $2 - 2\alpha \in [1, \frac{3}{2})$, it is known, see [Osw94], that $\tilde{N}_k \subset H_0^{2-2\alpha}(\Omega)$, and furthermore that the following inverse inequality is valid:

$$\|\tilde{v}_k\|_{H^{2-2\alpha}} \lesssim h_k^{2\alpha - 2} \|\tilde{v}_k\|_{L^2} \qquad (\tilde{v}_k \in \tilde{N}_k). \tag{6.23}$$

We define $G_k : M_k \to \tilde{N}_k$ by

$$(G_k v_k)(p) = v_k(p) \quad (p \in V_k), \quad (G_k v_k)(m_e) = \text{ average}_i \text{ of } v_k|_{T_i}(m_e) \quad (e \in E_k),$$

where $\tau_k \ni T_i \supset e$.

Using the local reproduction of linear polynomials, similar arguments that were used for (6.3) show that

$$\|(I - G_k)v_k\|_{L^2} \lesssim h_k^2 \|v_k\|_{1,k} \qquad (v_k \in M_k). \tag{6.24}$$

Since G_k even locally reproduces quadratic polynomials, the Bramble-Hilbert lemma gives

$$\|(I - G_k)\Pi^k v\|_{L^2} \lesssim h_k^3 \|v\|_{H^3} \qquad (v \in H_0^2(\Omega) \cap H^3(\Omega)). \tag{6.25}$$

From (6.23), (6.24) and (6.25), and concerning I_k, the estimates (6.14) and (6.16) and the inverse inequality $\| \|_{H^{2-2\alpha}} \lesssim h_k^{2\alpha-2} \| \|_{L^2}$ on $\tilde{M}_k \supset \operatorname{Im} I_k$, we conclude that

$$\|(I_k - G_k)v_k\|_{H^{2-2\alpha}} \lesssim h_k^{2\alpha} \|v_k\|_{1,k} \qquad (v_k \in M_k),$$

$$\|(I_k - G_k)\Pi^k v\|_{H^{2-2\alpha}} \lesssim h_k^{4\alpha} \|v\|_{H^{2+2\alpha}} \qquad (v \in H_0^2(\Omega) \cap H^{2+2\alpha}(\Omega)),$$

which are the conditions (T) and (U) respectively for $s = 1 - \alpha$.

Theorem 5.13 now shows that (3.4) with $s = 1-\alpha$ is also valid when $f_k = f \circ G_k$. From Theorem 3.2 we conclude the suboptimality of the resulting Cascade iteration, that is,

$$\|u_j - u_j^*\|_{1,j} \lesssim h_k^{2\alpha}\|f\|_{H^{-2+2\alpha}} \qquad (f \in H^{-2+2\alpha}(\Omega)),$$

taking $\mathcal{O}(\mathrm{dim} M_j(1+\log(\mathrm{dim} M_j))^{1+1/\alpha})$ operations.

By imposing stronger conditions on f, the construction of f_k can be simplified further. For $s < \frac{1}{4}$, there holds $M_k \subset H^{2s}(\Omega)$ and the following inverse inequality is valid:

$$\|v_k\|_{H^{2s}} \lesssim h^{-2s}\|v_k\|_{L^2} \qquad (v_k \in M_k).$$

From (6.14) and (6.16), we conclude

$$\|(I_k - I)v_k\|_{H^{2s}} \lesssim h_k^{2-2s}\|v_k\|_{1,k} \qquad (v_k \in M_k)$$

and

$$\|(I_k - I)\Pi^k v\|_{H^{2s}} \lesssim h_k^{2+2\alpha-2s}\|v\|_{H^{2+2\alpha}} \qquad (v \in H_0^2(\Omega) \cap H^{2+2\alpha}(\Omega)),$$

which are (T) and (U) with G_k replaced by the trivial injection. Theorems 5.13 and 3.2 now show that for such $s \in [0, \frac{1}{4})$ and $f_k = f|_{M_k}$, there holds

$$\|u_j - u_j^*\|_{1,j} \lesssim h_k^{2\alpha}(\|f\|_{H^{-2+2\alpha}} + \frac{1}{j}\|f\|_{H^{-2s}}) \qquad (f \in H^{-2s}(\Omega)),$$

requiring $\mathcal{O}(\mathrm{dim} M_j(1+\log(\mathrm{dim} M_j))^{1+1/\alpha})$ operations.

It is interesting to compare error estimates for the following constructions of f_k:

(a). $f_k = f|_{M_k}$, which is the standard Morley method;
(b). $f_k = f \circ I_k$, which was analyzed first by Brenner in [Bre99];
(c). $f_k = f \circ \hat{G}_k$, where $\hat{G}_k : M_k \to \hat{N}_k := \prod_{T \in \tau_k} P_1(T) \cap C(\Omega) \cap H_0^1(\Omega)$ is the *linear* interpolator. This modified Morley method was proposed by Arnold and Brezzi in [AB85];
(d). $f_k = f \circ G_k$, introduced in this paper.

As for $\tilde{N}_k$, there holds $\hat{N}_k \subset H_0^{2-2\alpha}(\Omega)$, and so also (c) is well-defined for any $f \in H^{-2+2\alpha}(\Omega)$. Furthermore the following inverse inequality is valid:

$$\|\hat{v}_k\|_{H^{2-2\alpha}} \lesssim h^{2\alpha-2}\|\hat{v}_k\|_{L^2} \qquad (\hat{v}_k \in \hat{N}_k).$$

By this inverse inequality and the local reproduction by $\hat{G}_k$ of linear polynomials, there holds $\|(I_k - \hat{G}_k)v_k\|_{H^{2-2\alpha}} \lesssim h_k^{2\alpha}\|v_k\|_{1,k}$ $(v_k \in M_k)$, which is (T) with $s = 1-\alpha$ and G_k replaced by $\hat{G}_k$. Theorem 5.13(a) or (5.2) show that

$$\|u - u_k\|_{a_k} \lesssim h_k^{2\alpha}\|f\|_{H^{-2+2\alpha}} \qquad (f \in H^{-2+2\alpha}(\Omega)),$$

for (b), (c) and (d); and for any $s \in [0, \frac{1}{4})$,

$$\|u - u_k\|_{a_k} \lesssim h_k^{2\alpha}\|f\|_{H^{-2+2\alpha}} + h_k^{2-2s}\|f\|_{H^{-2s}} \qquad (f \in H^{-2s}(\Omega)),$$

for (a). So with respect to the energy-norm, (b), (c) and (d) give optimal results in terms of smoothness of f that is required, where (c) is the cheapest of these.

The estimate (5.11) shows that the errors in the $\| \, \|_{1-\alpha,k}$-norm satisfy

$$\|\Pi^k u - u_k\|_{1-\alpha,k} \lesssim h_k^{4\alpha}\|f\|_{H^{-2+2\alpha}} \qquad (f \in H^{-2+2\alpha}(\Omega)),$$

for (b) and (d), and

$$\|\Pi^k u - u_k\|_{1-\alpha,k} \lesssim h_k^{4\alpha}\|f\|_{H^{-2+2\alpha}} + h_k^{2+2\alpha-2s}\|f\|_{H^{-2s}} \qquad (f \in H^{-2s}(\Omega),\ s \in [0, \frac{1}{4})),$$

for (a).

Since $\hat{G}_k \Pi^k$ does not reproduce quadratics, (U) with G_k replaced by $\hat{G}_k$ is not valid for any $s \in [0, 1-\alpha]$. Instead, for $s \in [0, 1-\alpha]$, there holds

$$\|(I_k - \hat{G}_k)\Pi^k v\|_{H^{2s}} \lesssim h_k^{2-2s}\|v\|_{H^2} \leq h_k^{2-2s}\|v\|_{H^{2+2\alpha}} \qquad (v \in H_0^2(\Omega) \cap H^{2+2\alpha}(\Omega)),$$

from which, using a minor modification of Theorem 5.13(b), we infer that for (c) there holds

$$\|\Pi^k u - u_k\|_{1-\alpha,k} \lesssim h_k^{4\alpha}\|f\|_{H^{-2+2\alpha}} + h_k^{2-2s}\|f\|_{H^{-2s}} \qquad (f \in H^{-2s}(\Omega),\ s \in [0, 1-\alpha]).$$

This bound is of order $h_k^{2\alpha}$ only if $s \leq 1 - 2\alpha$, which means that (d) requires smoother f than (b) and (d) to give a bound of the same quality. Since $1 - 2\alpha = \frac{1}{4}$ for $\alpha = \frac{3}{8}$, on basis of these bounds, (c) should be preferred to (a) for $\alpha \leq \frac{3}{8}$, but for $\frac{3}{8} < \alpha \leq \frac{1}{2}$, the situation is even reversed.

We conclude that at least with respect to the $\| \, \|_{1-\alpha,k}$-norm, the best method is (d). In our situation of having nested triangulations, we may even replace G_k by the quadratic interpolator with respect to the coarse mesh τ_{k-1}, which has the advantage that with respect to the canonical bases on M_k and $\tilde{N}_{k-1}$, this mapping is represented by a (non-square) diagonal matrix. Numerical experiments should indicate whether or not this modification has a quantitatively adverse effect on the resulting discretization error.

6.3. **A nonconforming finite element discretization of the Stokes equation.** On some bounded convex polygonal domain $\Omega \subset \mathbb{R}^2$, we consider the stationary Stokes equations written in variational form: For $\mathbf{f} \in H^{-1}(\Omega)^2$, find $\mathbf{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q\,dx = 0\}$, such that

$$\begin{array}{rcll} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) & = & \mathbf{f}(\mathbf{v}) & (\mathbf{v} \in H_0^1(\Omega)^2) \\ b(\mathbf{u},q) & = & 0 & (q \in L_0^2(\Omega)), \end{array} \tag{6.26}$$

where $a(\mathbf{u},\mathbf{v}) = \int_\Omega \sum_{j=1}^2 \nabla \mathbf{u}_j \cdot \nabla \mathbf{v}_j \, dx$ and $b(\mathbf{v},q) = -\int_\Omega q \operatorname{div} \mathbf{v} dx$. It is known that $\mathbf{f} \in (L^2(\Omega)')^2$ implies $\mathbf{u} \in H^2(\Omega)^2$ and $p \in H^1(\Omega)$ with

$$\|\mathbf{u}\|_{(H^2)^2} + \|p\|_{H^1} \lesssim \|\mathbf{f}\|_{((L^2)')^2}. \tag{6.27}$$

Let $\tau_0, \tau_1, \ldots$ be a sequence of conforming triangulations of Ω, such that τ_{k+1} is generated from τ_k by refinement, $\sup_{T \in \tau_k} \operatorname{diam}(T) \eqsim 2^{-k}$, and the triangles satisfy a shape regularity condition uniformly over the levels. We define $\overline{E}_k$, $\overline{V}_k$, E_k, V_k, m_e, $\mathbf{n}_e$ as in §6.1.

For $\mathbf{f} \in (L^2(\Omega)')^2$, and with $Q_k = \prod_{T\in\tau_k} P_0(T) \cap L^2_0(\Omega)$ and N_k being the nonconforming P_1 finite element space with respect to τ_k from §6.3, we consider the following discretization: Find $\mathbf{u}_k \in N_k^2$ and $p_k \in Q_k$ such that

$$
\begin{array}{rcll}
a_k(\mathbf{u}_k, \mathbf{v}_k) + b_k(\mathbf{v}_k, p_k) & = & \mathbf{f}(\mathbf{v}_k) & (\mathbf{v}_k \in N_k^2) \\
b_k(\mathbf{u}_k, q_k) & = & 0 & (q_k \in Q_k),
\end{array}
\tag{6.28}
$$

where $a_k(\mathbf{u}_k, \mathbf{v}_k) = \sum_{T\in\tau_k} \int_T \sum_{j=1}^2 \nabla(\mathbf{u}_k)_j \cdot \nabla(\mathbf{v}_k)_j \, dx$ and $b_k(\mathbf{v}_k, q) = -\sum_{T\in\tau_k} \int_T q \operatorname{div} \mathbf{v}_k dx$.

With $\mathbf{Z}$ being the closed subspace of $H^1_0(\Omega)^2$ defined by

$$\mathbf{Z} = \{\mathbf{v} \in H^1_0(\Omega)^2 : b(\mathbf{v}, q) = 0 \ (q \in L^2_0(\Omega))\},$$

the velocity component $\mathbf{u}$ of the solution of (6.26) can be characterized as the solution of the following elliptic problem: Find $\mathbf{u} \in \mathbf{Z}$ such that

$$a(\mathbf{u}, \mathbf{v}) = \mathbf{f}(\mathbf{v}) \qquad (\mathbf{v} \in \mathbf{Z}). \tag{6.29}$$

Analogously, with

$$\mathbf{Z}_k = \{\mathbf{v}_k \in N_k^2 : b_k(\mathbf{v}_k, q_k) = 0 \ (q_k \in Q_k)\},$$

the velocity component $\mathbf{u}_k$ of the solution of (6.28) is the unique solution of the problem of finding $\mathbf{u}_k \in \mathbf{Z}_k$ such that

$$a_k(\mathbf{u}_k, \mathbf{v}_k) = \mathbf{f}(\mathbf{v}_k) \qquad (\mathbf{v} \in \mathbf{Z}_k). \tag{6.30}$$

Particular for the pair (N_k^2, Q_k) is that a local basis of $\mathbf{Z}_k$ is available, and so that instead of solving the saddle-point problem (6.28), solving (6.30) is a feasible method to approximate $\mathbf{u}$. We will consider the Cascade iteration.

Above problem fits in our general framework with $a(\,,\,)$ and $a_k(\,,\,)$ as above, $\mathbf{f}_k = \mathbf{f}|_{\mathbf{Z}_k}$ assuming $\mathbf{f} \in (L^2(\Omega)')^2$, and

$$\mathcal{H}^0 = L^2(\Omega)^2,\ \mathcal{H}^1 = H^1_0(\Omega)^2,\ \mathcal{H}^2 = H^1_0(\Omega)^2 \cap H^2(\Omega)^2,\ \mathcal{V} = \mathbf{Z},\ \mathcal{V}_k = \mathbf{Z}_k.$$

Clearly (A) holds and (6.27) shows (B) with $\alpha = 1$. Note that $\mathbf{Z}_{k-1} \not\subset \mathbf{Z}_k \not\subset \mathbf{Z}$. There holds $\beta_k \eqsim h_k := 2^{-k}$ and so (C) is valid.

Remark 6.1. In view of Remark 2.4, we note that here we encounter a case where $\mathbf{Z} \hookrightarrow \mathcal{H}^0 = L^2(\Omega)^2$ is not dense, which means that the mapping $\mathbf{f} \mapsto \mathbf{u}$, even restricted to $(L^2(\Omega)')^2$, is not injective. Moreover, since $\mathbf{Z}_k \not\subset \mathbf{Z}$, an $\mathbf{f} \in (L^2(\Omega)')^2$ that yields a zero solution $\mathbf{u}$, may give a nonzero discrete solution $\mathbf{u}_k$, from which we infer that $\mathbf{u}_k$ is not a function of $\mathbf{u}$.

We postpone the discussion of the smoother, and first verify estimates (3.3) and (3.4) using the theory from §5.

It is known that the sequence of pairs $(N_k^2, Q_k)_k$ is Ladyšenskaja-Babuška-Brezzi (LBB) stable, i.e.,

$$\gamma_k := \inf_{0\neq q_k\in Q_k} \sup_{0\neq \mathbf{v}_k\in N_k^2} \frac{|b_k(\mathbf{v}_k, q_k)|}{\|\mathbf{v}_k\|_{a_k} \|q_k\|_{L^2}} \gtrsim 1,$$

and obviously also that

$$\Gamma_k := \sup_{0 \neq q \in L_0^2(\Omega), 0 \neq \mathbf{v}_k \in H_0^1(\Omega)^2 + N_k^2} \frac{|b_k(\mathbf{v}_k, q)|}{\|\mathbf{v}_k\|_{a_k} \|q\|_{L^2}} \lesssim 1.$$

The general theory of mixed methods (see e.g. [BS94, §10]) shows that for $\mathbf{v} \in \mathbf{Z}$,

$$\inf_{\mathbf{v}_k \in \mathbf{Z}_k} \|\mathbf{v} - \mathbf{v}_k\|_{1,k} \leq (1 + \frac{\Gamma_k}{\gamma_k}) \inf_{\mathbf{v}_k \in N_k^2} \|\mathbf{v} - \mathbf{v}_k\|_{a_k},$$

and so (G) follows from

$$\inf_{\mathbf{v}_k \in N_k^2} \|\mathbf{v} - \mathbf{v}_k\|_{a_k} \lesssim h_k \|\mathbf{v}\|_{(H^1)^2} \qquad (\mathbf{v} \in H_0^1(\Omega)^2).$$

For $\mathbf{f} \in (L^2(\Omega)')^2$, let $(\mathbf{u}, p) \in (H_0^1(\Omega)^2 \cap H^2(\Omega)^2) \times (L_0^2(\Omega) \cap H^1(\Omega))$ denote the corresponding solution of (6.26). Integration by parts and a density argument shows that for all $\mathbf{v} \in L^2(\Omega)^2$,

$$\int_\Omega (-\triangle \mathbf{u} + \nabla p) \cdot \mathbf{v} dx = \mathbf{f}(\mathbf{v}),$$

and so for $\mathbf{v}_k \in H_0^1(\Omega)^2 + N_k^2$, integration by parts on each $T \in \tau_k$ gives

$$a_k(\mathbf{u}, \mathbf{v}_k) - \mathbf{f}(\mathbf{v}_k) = -b_k(\mathbf{v}_k, p) + \sum_{e \in \overline{E}_k} \int_e \sum_j (\partial_{\mathbf{n}_e} \mathbf{u}_j)[(\mathbf{v}_k)_j] - p[\mathbf{v}_k \cdot \mathbf{n}_e] ds, \tag{6.31}$$

where $[w]$ denotes the jump of w across e in the direction of $\mathbf{n}_e$. Since both $[(\mathbf{v}_k)_j]$ and $[\mathbf{v}_k \cdot \mathbf{n}_e]$ are linear on e and zero in its midpoint, an application of [CR73, Lemma 3] shows that

$$|\sum_{e \in \overline{E}_k} \int_e \sum_j (\partial_{\mathbf{n}_e} \mathbf{u}_j)[(\mathbf{v}_k)_j] - p[\mathbf{v}_k \cdot \mathbf{n}_e] ds| \lesssim h_k (\|\mathbf{u}\|_{(H^2)^2} + \|p\|_{H^1}) \|\mathbf{v}_k\|_{a_k}. \tag{6.32}$$

If $\mathbf{v}_k \in \mathbf{Z} + \mathbf{Z}_k$, then from $Q_k \subset L_0^2(\Omega)$ we have

$$|b_k(\mathbf{v}_k, p)| = \inf_{q_k \in Q_k} |b_k(\mathbf{v}_k, p - q_k)| \leq \Gamma_k \|\mathbf{v}_k\|_{a_k} \inf_{q_k \in Q_k} \|p - q_k\|_{L^2}. \tag{6.33}$$

From (6.31), (6.32), (6.33),

$$\inf_{q_k \in Q_k} \|q - q_k\|_{L^2} \lesssim h_k \|q\|_{H^1} \qquad (q \in L_0^2(\Omega) \cap H^1(\Omega))$$

and (6.27), we conclude (I).

Before introducing the prolongation, from [FM90] we recall that $\mathbf{curl}_k$, defined by $(\mathbf{curl}_k\, v)|_T = \mathbf{curl}\, v|_T$ $(T \in \tau_k)$ with $\mathbf{curl} = [\frac{\partial}{\partial x_2} \; -\frac{\partial}{\partial x_1}]^T$, is a bijection between the Morley finite element space M_k and $\mathbf{Z}_k$. Since starting from here until the end of this paper, we will consider simultaneously the Stokes equations discretized on $\mathbf{Z}_k$ and the biharmonic equation discretized on M_k, to avoid confusion we will use the following notations:

Definition 6.2. With $a_k^{(\mathrm{St})}(\,,\,)$, $I_{k-1}^{k\ (\mathrm{St})}$, $S_{k,m}^{(\mathrm{St})}$, $\hat{S}_{k,m}^{(\mathrm{St})}$, $\|\ \|_{t,k}^{(\mathrm{St})}$ and $a_k^{(\mathrm{bih})}(\,,\,)$, $I_{k-1}^{k\ (\mathrm{bih})}$, $S_{k,m}^{(\mathrm{bih})}$, $\hat{S}_{k,m}^{(St}$), $\|\ \|_{t,k}^{(\mathrm{bih})}$, we mean the bilinear form, the prolongation, the operators related to the smoother and the norms corresponding to the discretized Stokes equations on $\mathbf{Z}_k$ and the discretized Biharmonic equation on M_k respectively.

We now define

$$I_{k-1}^{k\ (\mathrm{St})} = \mathbf{curl}_k \circ I_{k-1}^{k\ (\mathrm{bih})} \circ \mathbf{curl}_{k-1}^{-1} : \mathbf{Z}_{k-1} \to \mathbf{Z}_k.$$

Remark 6.3. If, for all k, M_k and $\mathbf{Z}_k$ are equipped with bases such that $\mathbf{curl}_k$ is a bijection between both sets of basisfunctions, then obviously the matrix representations of $I_{k-1}^{k\ (\mathrm{St})}$ and $I_{k-1}^{k\ (\mathrm{bih})}$ are equal, up to permutations.

The canonical extension of $I_{k-1}^{k\ (\mathrm{bih})}$ to an operator $\tilde{I}_{k-1}^{k\ (\mathrm{bih})} : M_{k-1} + M_k \to M_k$ is a projector, and so $I_{k-1}^{k\ (\mathrm{St})}$ extends to a projector $\tilde{I}_{k-1}^{k\ (\mathrm{St})} = \mathbf{curl}_k \circ \tilde{I}_{k-1}^{k\ (\mathrm{bih})} \circ \mathbf{curl}_k^{-1} : \mathbf{Z}_{k-1} + \mathbf{Z}_k \to \mathbf{Z}_k$. The arguments that yield (6.17) for $I_{k-1}^{k\ (\mathrm{bih})}$, show the same result for its extension, i.e.,

$$\|(I - \tilde{I}_{k-1}^{k\ (\mathrm{bih})})v_k\|_{L^2} \lesssim h_k^2 \sqrt{\sum_{T \in \tau_k} |v_k|^2_{H^2(T)}} \qquad (v_k \in M_{k-1} + M_k).$$

By writing $\tilde{I}_{k-1}^{k\ (\mathrm{bih})} = I - (I - \tilde{I}_{k-1}^{k\ (\mathrm{bih})})$, and applying inverse inequalities, it follows that

$$\sqrt{\sum_{T \in \tau_k} |\tilde{I}_{k-1}^{k\ (\mathrm{bih})} v_k|^2_{H^1(T)}} \lesssim \sqrt{\sum_{T \in \tau_k} |v_k|^2_{H^1(T)}} \qquad (v_k \in M_{k-1} + M_k). \tag{6.34}$$

Since $\|\mathbf{curl}\, v_k\|_{(L^2)^2} = \sqrt{\sum_{T \in \tau_k} |v_k|^2_{H^1(T)}}$, (6.34) is equivalent to $\|\tilde{I}_{k-1}^{k\ (\mathrm{St})} \mathbf{v}_k\|_{0,k}^{(\mathrm{St})} \lesssim \|\mathbf{v}_k\|_{(L^2)^2}$ ($\mathbf{v}_k \in \mathbf{Z}_{k-1} + \mathbf{Z}_k$), which is (O).

From Theorem 5.2, Proposition 5.3, Proposition 5.6, Lemma 5.9 and Proposition 5.11, we conclude (3.3) and (3.4) with $s = 1 - \alpha = 0$ and $\mathbf{f}_k = \mathbf{f}|_{\mathbf{Z}_k}$.

We equip $\mathbf{Z}_k$ with the standard basis defined by applying $\mathbf{curl}_k$ to the basis of M_k from (6.12). This basis on $\mathbf{Z}_k$ is given by

$$\{\phi_{k,e} := |e|^{-1}\eta_{k,e}\mathbf{t}_e : e \in E_k\} \cup \{\xi_{k,p} := \sum_{i=1}^{\ell} |e_i|^{-1}\eta_{k,e_i}\mathbf{n}_{e_i,p} : p \in V_k\}, \tag{6.35}$$

where $\eta_{k,e}$ is the nodal basisfunction of N_k defined in (6.1), $\mathbf{t}_e = [(\mathbf{n}_e)_2\ -(\mathbf{n}_e)_1]^T$ is a unit vector tangential to e, the edges $e_i, \dots e_\ell$ are all edges in E_k that contain p, and $\mathbf{n}_{e_i,p}$ is the unit vector normal to e_i pointing in the counterclockwise direction with respect to p, see Figure 1.

A difficulty with defining a suitable smoother for the system (6.30) is that, although properly scaled this basis on $\mathbf{Z}_k$ is not uniformly well-conditioned as function of k with

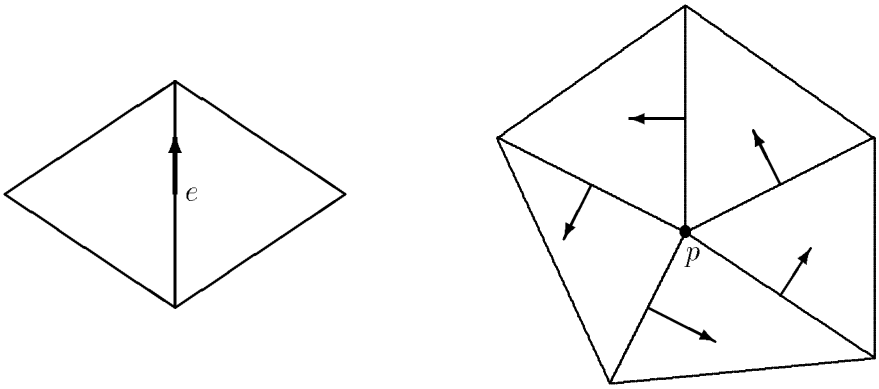

FIGURE 1. Basis functions $\phi_{k,e}$ and $\xi_{k,p}$ of the space $\mathbf{Z}_k$

respect to $\| \, \|_{(L^2)^2}$. In view of results obtained in §4, this means that e.g. unpreconditioned CG not necessarily satisfies (D), (E), (F).

An alternative approach to define a smoother, first followed on [Bre90], is based on the relation

$$a_k^{(\mathrm{bih})}(u_k, v_k) = a_k^{(\mathrm{St})}(\mathbf{curl}_k u_k, \mathbf{curl}_k v_k) \qquad (u_k, v_k \in M_k). \tag{6.36}$$

This relation and the one between the bases (6.12) and (6.35) of M_k and $\mathbf{Z}_k$ show that the stiffness matrices corresponding to both problems are equal, and so any smoothing iteration developed for one problem has its direct counterpart for the other, where the practical realizations are equal.

In §6.2 in the biharmonic framework, we considered CG-iteration without preconditioning, or with preconditioning matrices that are uniformly well-conditioned. Let $(S_{k,m}^{(\mathrm{bih})}, \hat{S}_{k,m}^{(\mathrm{bih})})_k$ denote the corresponding sequence of pairs of operators as meant at the beginning of §4, where $\hat{S}_{k,m}^{(\mathrm{bih})}$ is the linear operator defined using the Chebychev polynomial. We showed that this sequence satisfies (D), (E) and

$$\|\hat{S}_{k,m}^{(\mathrm{bih})} v_k\|_{1,k}^{(\mathrm{bih})} \lesssim (h_k^{-2} m^{-1})^t \|v_k\|_{t,k}^{(\mathrm{bih})} \qquad (t \in [0,1], v_k \in M_k), \tag{6.37}$$

which is (F) with $\gamma = 1$.

Since (6.36) is equivalent to $\| \cdot \|_{1,k}^{(\mathrm{bih})} = \|\mathbf{curl}_k \cdot \|_{1,k}^{(\mathrm{St})}$, we directly conclude that the corresponding operators $S_{k,m}^{(\mathrm{St})} = \mathbf{curl}_k \circ S_{k,m}^{(\mathrm{bih})} \circ \mathbf{curl}_k^{-1}$ and $\hat{S}_{k,m}^{(\mathrm{St})} = \mathbf{curl}_k \circ \hat{S}_{k,m}^{(\mathrm{bih})} \circ \mathbf{curl}_k^{-1}$ in the Stokes framework satisfy (D) and (E). Using [PB87, Proposition 8.1], in [Bre90, Proposition 3] it was shown that

$$\|v_k\|_{\frac{1}{2},k}^{(\mathrm{bih})} \lesssim \|\mathbf{curl}_k v_k\|_{0,k}^{(\mathrm{St})} \qquad (v_k \in M_k). \tag{6.38}$$

From (6.37) with $t = \frac{1}{2}$ and (6.38), we conclude that

$$\|\hat{S}_{k,m}^{(\mathrm{St})} \mathbf{v}_k\|_{1,k}^{(\mathrm{St})} \lesssim h_k^{-1} m^{-\frac{1}{2}} \|\mathbf{v}_k\|_{0,k}^{(\mathrm{St})} \qquad (\mathbf{v}_k \in \mathbf{Z}_k), \tag{6.39}$$

which is (F) with $\gamma = \frac{1}{2}$. Since the values of the other parameters appearing in Theorem 3.2 are given by $r = 1$, $\rho = 2$ and $d = 2$, from this theorem we may only conclude that for such a CG-smoothing and with $m_{j-i}^{(j)} \eqsim \tilde{m}4^i j^2$, the Cascade iteration for solving the discretized Stokes equations yields a *sub*optimal solver.

Remark 6.4. We give some comments on multi-grid convergence theory.

Instead of CG, let us consider a linear iteration as a smoother. The above analysis shows that the smoothing property in the biharmonic framework implies this property in the Stokes framework, with a γ that is halved. Here with smoothing property, we mean the condition on the smoother as imposed in the theory from [Hac85].

A different condition on the smoother is imposed in the convergence theory from [BPX91]. It turns out that validity of this condition generally does *not* carry over when switching from the biharmonic to the Stokes framework. Indeed, using [Ste98, Remark 2.9], it can be checked that e.g. damped Richardson iteration with symmetric preconditioning matrices that have uniformly bounded condition numbers satisfies this condition in the biharmonic framework, as it satisfies the smoothing property from [Hac85], but that it not satisfies the condition from [BPX91] in the Stokes framework.

Nevertheless, the theory from [BPX91] may still be used to analyze the multi-grid method applied to the discretized Stokes equations. Indeed, because the error amplification operator of the multi-grid method is linear, the analysis of this operator as a whole can be carried out in the biharmonic framework. So, in particular, one can still prove that the variable V-cycle, which is covered by the theory from [BPX91], yields uniformly bounded condition numbers.

Yet, there is one point where one has to pay for the fact that in the Stokes framework these simple iterations do not satisfy the condition imposed on a smoother in [BPX91]. Since the biharmonic operator is not fully regular, it is not possible to show that the so-called mildly variable V-cycle (see [Ste98]) gives uniformly bounded condition numbers. On the other hand, damped Richardson iteration with preconditioners of the type discussed below does satisfy the condition from [BPX91] in the Stokes framework, and so for this smoother it can be shown that the mildly variable V-cycle gives uniformly bounded condition numbers.

To construct better smoothers in order to obtain a Cascade iteration that is optimal, we study the conditioning of the basis (6.35) of $\mathbf{Z}_k$. Using that $\{\eta_{k,e} : e \in E_k\}$ is an $L^2(\Omega)$-orthogonal basis of N_k, for vectors $\mathbf{c} = (\mathbf{c}_e)_{e\in E_k}$ and $\mathbf{d} = (\mathbf{d}_p)_{p\in V_k}$, we infer that

(6.40)
$$\|\sum_{e\in E_k} \mathbf{c}_e\phi_{k,e} + \sum_{p\in V_k} \mathbf{d}_p\xi_{k,p}\|^2_{(L^2)^2} = \sum_{e\in E_k} |\mathbf{c}_e|^2|e|^{-2}\|\eta_{k,e}\|^2_{L^2} + \sum_{e\in E_k} |\mathbf{d}_{p_e} - \mathbf{d}_{\tilde{p}_e}|^2|e|^{-2}\|\eta_{k,e}\|^2_{L^2},$$

where $p_e, \tilde{p}_e$ denote both vertices of τ_k on $e \in E_k$, and $\mathbf{d}_p := 0$ when $p \in \overline{V}_k \backslash V_k$. Furthermore, there holds

$$\sum_{e\in E_k} |\mathbf{d}_{p_e} - \mathbf{d}_{\tilde{p}_e}|^2|e|^{-2}\|\eta_{k,e}\|^2_{L^2} \eqsim \int_\Omega |\nabla d^I|^2 dx, \tag{6.41}$$

where d^I is the function in the conforming P_1 finite element space $\hat{N}_k = \prod_{T\in\tau_k} P_1(T) \cap C(\Omega) \cap H^1_0(\Omega)$ defined by $d^I(p) = \mathbf{d}_p$.

Defining $\hat{\mathbf{A}}_k \in I\!R^{\dim\hat{N}_k \times \dim\hat{N}_k}$ by $<\hat{\mathbf{A}}_k\mathbf{d}, \tilde{\mathbf{d}}>= \int_\Omega \nabla d^I \cdot \nabla \tilde{d}^I dx$, for $\mathbf{M}_k \in I\!R^{\dim\mathbf{Z}_k\times\dim\mathbf{Z}_k}$ being the mass-matrix corresponding to (6.35), from (6.40) and (6.41) we conclude that

$$\mathbf{M}_k \eqsim \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \hat{\mathbf{A}}_k \end{bmatrix}.$$

Simple conforming multi-grid preconditioners $\hat{\mathbf{W}}_k$ for $\hat{\mathbf{A}}_k$ are available that take $\sim \dim\hat{N}_k$ operations, and for which $\kappa(\hat{\mathbf{W}}_k^{-1}\hat{\mathbf{A}}_k) \lesssim 1$. We now consider CG-smoothing applied to (6.30) with respect to the basis (6.35), with preconditioning matrices $\mathbf{W}_k$ that satisfy $\mathbf{W}_k \eqsim \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \hat{\mathbf{W}}_k \end{bmatrix}$. Let $(S^{(\mathrm{St})}_{k,m}, \hat{S}^{(\mathrm{St})}_{k,m})_k$ denote the corresponding sequence of pairs of operators as meant at the beginning of §4, where $\hat{S}^{(\mathrm{St})}_{k,m}$ is the linear operator defined using the Chebychev polynomial. Since $\kappa(\mathbf{W}_k^{-1}\mathbf{M}_k) \lesssim 1$, the analysis from §4 shows (D), (E) and

$$\|\hat{S}^{(\mathrm{St})}_{k,m}\mathbf{v}_k\|^{(\mathrm{St})}_{1,k} \lesssim h_k^{-1} m^{-1} \|\mathbf{v}_k\|^{(\mathrm{St})}_{0,k} \qquad (\mathbf{v}_k \in \mathbf{Z}_k), \tag{6.42}$$

which is (F) with $\gamma = 1$. Since $r = 1$, $\rho = 2$ and $d = 2$, from Theorem 3.2 we conclude that for any $c \in (2,4)$ and $m^{(j)}_{j-i} \eqsim \tilde{m}c^i$, the Cascade iteration with CG-smoothing and such a preconditioner applied to the discretized Stokes equations yields an optimal solver.

6.4. **The new smoother analyzed in the biharmonic framework.** We have seen that, with respect to bases (6.12) and (6.35) on M_k and $\mathbf{Z}_k$, the stiffness matrices corresponding to the discretization of the biharmonic equation on M_k and that of the Stokes equations on $\mathbf{Z}_k$ are equal. In the previous subsection we used this fact to analyze the application of CG-smoothing without, or with simple preconditioning matrices developed in the biharmonic framework to the Stokes equations. Using $\|\cdot\|^{(\mathrm{bih})}_{\frac{1}{2},k} \lesssim \|\mathbf{curl}\cdot\|^{(\mathrm{St})}_{0,k}$ ((6.38)), it appeared that such a CG-iteration satisfies the smoothing property (F) in the Stokes framework with $\gamma = \frac{1}{2}$. In addition, in the Stokes framework we developed new preconditioners, involving a multi-grid call on a second order scalar problem, and we showed that CG-smoothing with such preconditioners satisfies (F) with $\gamma = 1$.

In this subsection, we analyze CG-smoothing with these new preconditioners in the biharmonic framework. With $(S^{(\mathrm{St})}_{k,m}, \hat{S}^{(\mathrm{St})}_{k,m})_k$ denoting the sequence of pairs of operators corresponding to this preconditioned CG-iteration in the Stokes framework, we define $S^{(\mathrm{bih})}_{k,m} = \mathbf{curl}_k^{-1} \circ S^{(\mathrm{St})}_{k,m} \circ \mathbf{curl}_k$ and $\hat{S}^{(\mathrm{bih})}_{k,m} = \mathbf{curl}_k^{-1} \circ \hat{S}^{(\mathrm{St})}_{k,m} \circ \mathbf{curl}_k$. Obviously, again (6.36) shows (D) and (E) in the biharmonic framework. Condition (F) will follow from the following proposition.

Proposition 6.5. *With* $\|\mathbf{z}_k\|_{-t,k} := \sup_{0\neq\mathbf{w}_k\in\mathbf{Z}_k} \frac{|(\mathbf{z}_k,\mathbf{w}_k)_{(L^2)^2}|}{\|\mathbf{w}_k\|^{(\mathrm{St})}_{1,k}}$ *($t \in [0,1]$, $\mathbf{z}_k \in \mathbf{Z}_k$), there holds*

$$\|\mathbf{curl}_k v_k\|^{(\mathrm{St})}_{-1,k} \lesssim \|v_k\|^{(\mathrm{bih})}_{0,k} \qquad (v_k \in M_k). \tag{6.43}$$

Proof. With rot_k defined by $(\mathrm{rot}_k\,\mathbf{w})_T = \mathrm{rot}\,\mathbf{w}|_T$ ($T \in \tau_k$) where $\mathrm{rot} = [-\frac{\partial}{\partial x_2}\ \frac{\partial}{\partial x_1}]$, integration by parts show that for $v_k \in M_k$, $\mathbf{w}_k \in \mathbf{Z}_k$, there holds

$$(\mathbf{curl}_k\, v_k, \mathbf{w}_k)_{(L^2)^2} = (v_k, \mathrm{rot}_k\,\mathbf{w}_k)_{L^2} + \sum_{e\in\overline{E}_k} \int_e v_k\, [\mathbf{w}_k \times \mathbf{n}_e] ds,$$

where $[\mathbf{w}_k \times \mathbf{n}_e]$ denotes the jump of $\mathbf{w}_k \times \mathbf{n}_e$ across e in the direction of $\mathbf{n}_e$. Since $[\mathbf{w}_k \times \mathbf{n}_e]$ is linear on e and zero in its midpoint, it holds that $\int_e [\mathbf{w}_k \times \mathbf{n}_e] ds = 0$. An application of [CR73, Lemma 3] and an inverse inequality now show that

$$|\sum_{e\in\overline{E}_k} \int_e v_k[\mathbf{w}_k \times \mathbf{n}_e] ds| \lesssim h_k \sqrt{\sum_{T\in\tau_k} |v_k|^2_{H^1(T)}}\|\mathbf{w}_k\|^{(\mathrm{St})}_{1,k} \lesssim \|v\|^{(\mathrm{bih})}_{0,k}\|\mathbf{w}_k\|^{(\mathrm{St})}_{1,k}.$$

Since $|(v_k, \mathrm{rot}_k\,\mathbf{w}_k)_{L^2}| \leq \sqrt{2}\|v_k\|^{(\mathrm{bih})}_{0,k}\|\mathbf{w}_k\|^{(\mathrm{St})}_{1,k}$, we conclude that $|(\mathbf{curl}_k\, v_k, \mathbf{w}_k)_{(L^2)^2}| \lesssim \|\mathbf{w}_k\|^{(\mathrm{St})}_{1,k}$, which completes the proof. ☐

Note that by interpolation, (6.36) and (6.38) imply that

$$\|v_k\|^{(\mathrm{bih})}_{\frac{t+1}{2},k} \lesssim \|\mathbf{curl}_k v_k\|^{(\mathrm{St})}_{t,k} \qquad (t \in [0, 1],\ v_k \in M_k),$$

and (6.36) and (6.43) imply that

$$\|\mathbf{curl}_k v_k\|^{(\mathrm{St})}_{t,k} \lesssim \|v_k\|^{(\mathrm{bih})}_{\frac{t+1}{2},k} \qquad (t \in [-1, 1],\ v_k \in M_k). \tag{6.44}$$

From (6.42), (6.36), and (6.44) with $t = 0$, for $\alpha \leq \frac{1}{2}$ we obtain that

$$\|\hat{S}^{(\mathrm{bih})}_{k,m} v_k\|^{(\mathrm{bih})}_{1,k} \lesssim (h_k^{-2}m^{-2})^{\alpha}\|v_k\|^{(\mathrm{bih})}_{1-\alpha,k} \qquad (v_k \in M_k),$$

which is (F) with $\gamma = 2$. Since $r = 2$, $\rho = 2$ and $d = 2$, from Theorem 3.2 we conclude that for any $c \in (2, 4)$ and $m^{(j)}_{j-i} \eqsim \tilde{m}c^i$, the Cascade iteration with CG-smoothing and this new preconditioner applied to the discretized biharmonic equation yields an optimal solver.

References

[AB85] D.N. Arnold and F. Brezzi. Mixed and nonconforming finite element methods: Implementation, postprocessing and error estimates. *M^2AN*, 19:7–32, 1985.

[BD96] F.A. Bornemann and P. Deuflhard. The cascadic multigrid method for elliptic problems. *Numer. Math.*, 75:135–152, 1996.

[BD99] D. Braess and W. Dahmen. A Cascade algorithm for the Stokes equations. *Numer. Math.*, 82:179–191, 1999.

[BPX91] J.H. Bramble, J.E. Pasciak, and J. Xu. The analysis of multigrid algorithms with nonnested spaces or inherited quadratic forms. *Math. Comp.*, 56(193):1–34, 1991.

[Bre90] S.C. Brenner. A nonconforming multigrid method for the stationary Stokes equations. *Math. Comp.*, 55(192):411–437, 1990.

[Bre99] S.C. Brenner. Convergence of nonconforming multigrid methods without full elliptic regularity. *Math. Comp.*, 68(225):25–53, 1999.

[BS94] S.C. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag, New York, 1994.

[Cia91] P.G. Ciarlet. Basic error estimates for elliptic problems. In P.G. Ciarlet and J.L Lions, editors, *Finite Element Methods (Part 1)*, volume II of *Handbook of Numerical Analysis*, pages 17–352. North-Holland, Amsterdam, 1991.

[CR73] M. Crouzeix and P. A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. *RAIRO Anal. Numér.*, 3:33–76, 1973.

[FM90] R.S. Falk and M.E. Morley. Equivalence of finite element methods for problems in elasticity. *SIAM J. Numer. Anal.*, 27(6):1486–1505, 1990.

[Hac85] W. Hackbusch. *Multi-Grid Methods and Applications.* Springer-Verlag, Berlin, 1985.

[Osw94] P. Oswald. *Multilevel finite element approximation: Theory and applications.* B.G. Teubner, Stuttgart, 1994.

[PB87] P Peisker and D. Braess. A conjugate gradient method and a multigrid algorithm for Morley's finite element approximation of the biharmonic equation. *Numer. Math.*, 50:567–586, 1987.

[Sha96] V.V. Shaidurov. Some estimates of the rate of convergence for the cascadic conjugate gradient method. *Comp. Math. Applic.*, 31:161–171, 1996.

[Ste98] R.P. Stevenson. An analysis of nonconforming multi-grid methods, leading to an improved method for the Morley element. Technical Report 9823, University of Nijmegen, 1998. Submitted.

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