Realizability: A Historical Essay

Jaap van Oosten *
Department of Mathematics
P.O.Box 80.010
3508 TA Utrecht
The Netherlands
jvoosten@math.uu.nl

January 20, 2000

Dedicated to Anne S. Troelstra at his 60th Birthday

Introduction

The purpose of this short paper is to sketch the development of a few basic topics in the history of Realizability. The number of topics is quite limited and reflects very much my own personal taste, biases and prejudices.

Realizability has, over the past 60 years, developed into a subject of such dimensions that a comprehensive overview would require a fat book. Maybe someone, some day ought to write such a book. But it will not be easy. Quite apart from the huge amount of literature to cover, there is the task of creating unity where there is none. For Realizability has many faces, each of them turned towards different areas of Logic, Mathematics and Computer Science, and this proliferation shows no signs of diminishing in our days. Like a venomous carcinoma, Realizability stretches out its tentacles to ever more remote fields: Linear Logic, Complexity Theory and Rewrite Theory have already been infected. The theory of Subrecursive Hierarchies too. Everything connected to the λ-calculus is heavily engaged. Proof Theory is suffering. Intuitionism is dead.

Just to name a few! Did you think, that *at least* the realm of classical logic would be safe? Recently, Krivine came up with a Realizability interpretation for ZF set theory!

Confronted with this mess, I have acted like the classical impostor who walked into the hospital claiming to be a surgeon, and is now wielding the knives in the operating theatre: I took the nearest scalpel at hand and cut out everything that wouldn't fit into either one of my two major streams: metamathematics of intuitionistic arithmetical theories, and topos-theoretic developments.

*Research supported by PIONIER – NWO, the Netherlands

Needless to say, there is no question of even starting to list what I omitted. Sometimes to my great regret, although I realize that such hollow apologies just reverberate in the vast emptiness I have created[1].

Therefore, let's get physical and say something concrete about what *is* in this paper. According to me, there are three landmark publications in Realizability. These are:

1) Kleene's original 1945 paper, *On the Interpretation of Intuitionistic Number Theory* ([47])

2) Troelstra's *Metamathematical Investigations* from 1973 ([87])

3) Hyland's *The Effective Topos* from 1981 ([37])

Of these three, both 1) and 3) initiated a whole new strand of research. I have therefore decided that the material I wished to present, naturally divides into two *periods*, viz. 1940–1980 and 1980–2000. This is not to say that suddenly there were, after 1980, no more purely syntactical presentations of Realizabilities (quite on the contrary, thanks to Computer Science syntax is back!), but I do feel that although many of these matters still need and deserve to be investigated (and need all the elegance and expository skills we can muster), no radically new vistas have emerged from this research. Therefore, in my account of the second period I have concentrated on what I regard as more innovative research.

The second item in my list is of a different kind. This monumental work brought together all existing results, many of which were due to its author, and ordered them in such a way that the diligent student could see at once the similarities between them. It charted the territory, and in this way achieved something of conceptual value: the notion that all these systems, interpretations and axiomatizations were manifestations of a pattern that they had in common. What exactly this pattern is, we still don't know. But it is my feeling that the categorical analyses of later years owe a lot to this work.

It made, when it appeared, a 'daunting' impression on some people. And certainly it did so on me when I was his student. But now I experience a sensation of dry, austere beauty in its relentless pursuit of order. And let us not forget it set new standards of presentation and notation. For although Kleene's first paper is a gem of readability, regrettably Kleene later adopted a style of writing which was so cluttered with notation that it takes a strong man to fight himself through.

I have therefore decided to dedicate this paper to Anne Troelstra, my mentor who has contributed so much to the subject matter, in gratitude.

Acknowledgements. I am grateful to Joan Moschovakis, who had a careful look at a preliminary version and drew my attention to a few embarrassing mistakes. I'd also like to thank Lars Birkedal, Martin Hyland, Pino Rosolini and Dana Scott for discussions and for providing last-minute bibliographical and background information.

[1]Let the disappointed reader be solaced by the availability of an *excellent* proof-theoretical survey on Realizability: [88]

1 The first 40 years: 1940-1980

1.1 The origin of Realizability

In his overview paper: "Realizability: a retrospective survey" ([54]), Stephen Cole Kleene recounts how his idea for numerical realizability developed. He wished to give some precise meaning to the intuition that there should be a connection between Intuitionism and the theory of recursive functions (both theories stressing the importance of extracting information *effectively*). He started to think about this in 1940[2].

In order to appreciate the originality of his thinking, one should recall that the formal system of intuitionistic arithmetic **HA** did not exist at the time [Well, ... There is a system closely resembling **HA** in Gödel's paper [26]. Kleene appears to have been at least initially unaware of this, for although his 1945 paper gives the reference, the retrospective survey stresses that "Heyting Arithmetic [...] does not occur as a subsystem readily separated out from Heyting's full system of intuitionistic mathematics", and quotes Kleene's own formalism, which later appeared in [48], as the thing he had in mind].

As an example of a precise connection between Intuitionism and the theory of recursive functions, Kleene starts by conjecturing a weak form of Church's Rule: if a closed formula of the form $\forall x \exists y \varphi(x, y)$ is provable in intuitionistic number theory, then there must be a general recursive function F such that for all n, the formula $\varphi(\overline{n}, \overline{F(n)})$ is true. One arrives at this conjecture by unravelling the meaning that such a statement must have for an intuitionist.

Conjecturing this, at a time when Intuitionism was still clouded by Brouwer's mysticism, the formal system in question hardly established, and the content of the conjecture blatantly false for Peano Arithmetic, was imaginative indeed!

But, this was still far away from the actual development of Realizability. Often, one encounters the opinion that Realizability was inspired by the so-called "Brouwer-Heyting-Kolmogorov interpretation" (a mantra which, rather than being an interpretation, is itself in need of one). This was not the case. Kleene starts by quoting Hilbert and Bernays ([35]). They, in their "Grundlagen der Mathematik", explain the "finitist" position in Mathematics. The relevant passage is the one about "existential statements as incomplete communications", which, since it is philosophy, can only be appropriately understood in the original German:

> Ein *Existenzsatz* über Ziffern, also ein Satz von der Form "es gibt eine Ziffer n von der Eigenschaft $\mathcal{A}(n)$" ist finit aufzufassen als ein "Partialurteil", d.h. als eine unvollständige Mitteilung einer genauer bestimmten Aussage, welche entweder in der direkten Angabe einer Ziffer von der Eigenschaft $\mathcal{A}(n)$ oder der Angabe eines Verfahrens zur Gewinnung einer solchen Ziffer besteht [...].[3]

[2] For some biographical details on Kleene and a personal appreciation, see the obituary by his friend Saunders Mac Lane, [61]

[3] An *existential statement* about numbers, i.e. a statement of the form "there exists a

Kleene then asks: "Can we generalize this idea to think of *all*[4] (except, trivially, the simplest) intuitionistic statements as incomplete communications?"[5]

He outlines in which sense every logical sentence is "incomplete" and what would constitute its "completion". For the implication case, Kleene interestingly says that first he tried an inductive clause inspired by "Heyting's 'proof-interpretation' ", but that it "didn't work" and so, "Heyting's proof-interpretation failed to help me to my goal". Since Kleene doesn't reveal what this first try was, we are free to conjecture. It is just conceivable that he tried: a realizer for $A \to B$ is a partial recursive function which sends proofs of A to proofs of B.

Kleene's realizability was, at least conceptually, a major advance. Its achievement is not so much a philosophical explanation of the intuitionistic connectives. Troelstra ([87], p.188) says: "it cannot be said to make the intended meaning of the logical operators more precise. As a "philosophical reduction" of the interpretation of the logical operators it is also moderately successful; e.g. negative formulae are essentially interpreted by themselves." In fact, Kleene admits this explicitly in his 1945 paper[6]. On the other hand, by providing an interpretation which can be read and checked by the classical mathematician, he did put forward an interpretation of the intuitionistic connectives in terms of the classical ones (this, in contrast to the so-called BHK or "proof"-interpretation, which interprets the intuitionistic connectives in terms of themselves).

More importantly, realizability, as it is designed to handle "information" about formulas rather than proofs, already hints at the role Intuitionism would come to play in theoretical Computer Science some 40 years later: it foreshadows the view of intuitionistic formulas as *datatypes*, and intuitionistic logic as the logic of *information*.

But the scope of realizability is wider than just "interpreting the logic". Realizability also provides models for theories which are classically *inconsistent*, models therefore whose internal logic is strictly non-classical (important examples are: Brouwer's theory of Choice Sequences; parts of (suitably formalized) recursive analysis; set-theoretic interpretations of the polymorphic λ-calculus; Synthetic Domain Theory). It is in some of these models, that the statement "Realizability is equivalent to truth" can be given a precise meaning. And for the intuitionist, (an abstract form of) realizability *does* represent the intuitionistic connectives faithfully, as follows from [91].

---

number n with property $\mathcal{A}(n)$" is finitistically taken as a "partial judgement", that is, as an incomplete rendering of a more precisely determined proposition, which consists in either giving directly a number n with the property $\mathcal{A}(n)$, or a procedure by which such a number can be found [...]

[4] my italics

[5] It is, however, fair to say that Hilbert and Bernays did not limit their treatment of the finitist position to existential statements; they had a lot more to say, and also included negations and $\forall\exists$-statements in their account

[6] "The analysis which leads to this truth definition is not to be regarded as more than a partial analysis of the intuitionistic meaning of the statements [...]" (§2)

1.2 Formalized Realizability and q-Realizability

The definition of Realizability involves only first-order properties of indices of partial recursive functions; hence, as was immediately noticed by Kleene, can be formalized in **HA** itself. This is already in [47]; the details are in [66]. One has a translation:

$$\varphi \mapsto \exists x(x \text{ realizes } \varphi)$$

which Nelson observed to be *idempotent* up to provable equivalence in **HA**. One has the theorem:

$$\mathbf{HA} \vdash \varphi \Rightarrow \text{ for some number } n,\ \mathbf{HA} \vdash \overline{n} \text{ realizes } \varphi$$

Now in [47], there is also a modification of the realizability definition, a notion he later ([48]) called $\vdash$-realizability: in the clauses for disjunction and existential quantification, at suitable places a $\vdash$ is inserted, giving for example:

$$n \vdash\text{-realizes } \exists y\varphi(y) \text{ if } (n)_1 \vdash\text{-realizes } \varphi(\overline{(n)_0}) \text{ and } \vdash \varphi(\overline{(n)_0})$$

One has the theorem:

$$\mathbf{HA} \vdash \varphi \Leftrightarrow \text{ for some } n,\ n \vdash\text{-realizes } \varphi$$

Now let's get back to the weak form of Church's Rule, mentioned in 1.1: if $\mathbf{HA} \vdash \forall x \exists y \varphi(x,y)$ then some n $\vdash$-realizes this formula, which means that n is the index of a total recursive function F, and for $G = \Lambda x.(F(x))_0$ we have that for all m, $\mathbf{HA} \vdash \varphi(\overline{m}, \overline{G(m)})$, hence this formula is true. This conclusion is already in [47].

Now from this weak Church's Rule one immediately obtains the Existence Property for **HA**: if $\mathbf{HA} \vdash \exists x\varphi(x)$ for closed $\exists x\varphi(x)$, then for some n, $\mathbf{HA} \vdash \varphi(\overline{n})$, and as a consequence the Disjunction Property: $\mathbf{HA} \vdash \varphi \vee \psi$ for closed φ, ψ, then either $\mathbf{HA} \vdash \varphi$ or $\mathbf{HA} \vdash \psi$. These conclusions are *not* explicitly in [47], contrary to what Kleene later said[7].

However the strongest proof-theoretic results are obtained if one combines the idea of $\vdash$-realizability with formalized realizability. One defines a *formula* $x\,\mathbf{q}\,\varphi$ (x **q**-realizes φ), simply replacing (in the definition of $\vdash$-realizability) occurrences of "$\vdash \psi$" by ψ itself. Then one can prove that the recursive function from the weak Church's Rule is in fact *provably recursive*. But this version (and the even stronger 'Extended Church's Rule') appears first in Troelstra's [86] (also in [87]), although there is a **q**-version for "analysis" in [53]. These **q**-realizabilities generally suffered from the defect that they were not closed under equivalence. Grayson ([28]) is usually credited with a definition which doesn't have this defect.

[7] The first proof of the Existence and Disjunction properties for **HA** was given by Harrop in [31]. In [51], Kleene says Harrop "rediscovered" these results, and in a footnote he details: "[the Existence property] appears explicitly in [47] p. 115 lines 8-7 from below, or [48] p. 509 lines 15-11 from below, taking $n = 0$. [the Disjunction property] is included in [this]". (Reference numbers changed) These references are also given in [89], vol. I,p.175-6. However, it is simply not there. Kleene was not above drawing obvious inferences, so one can safely assume that the existence property had not occurred to him at the time

1.3 The Logic of Realizability

Kleene's original conjecture that realizability might mirror intuitionistic reasoning faithfully, was disproved: Rose ([74]) and later Ceitin, gave examples of propositional formulas that are realizable (even "absolutely": there is a number n which realizes every substitution instance of the formula, where one substitutes **HA**-sentences for the propositional variables), but not provable in the intuitionistic calculus[8]. The "predicate logic of realizability" is quite complicated, and was investigated by the Russian Plisko in a series of papers. Of course, there are several ways to define what it means for a formula in predicate logic to be "realizable". An interesting theorem ([69]) of his concerns what he calls "absolutely realizable predicate formulas". Consider a purely relational formula $\varphi = \varphi[P_1, \ldots, P_k]$ with all predicate symbols shown, P_i being n_i-ary. Let $F_i : \mathbb{N}^{n_i} \to \mathcal{P}(\mathbb{N})$ be a k-tuple of functions. We can now define the notion n realizes φ, relative to $(F_1, \ldots, F_k)$, by letting the variables run over $\mathbb{N}$, and putting

$$n \text{ realizes } P_i(m_1, \ldots, m_{n_i}) \text{ if and only if } n \in F_i(m_1, \ldots, m_{n_i})$$

Say that a sentence φ of purely relational predicate logic is *absolutely realizable* if there is a number n such that for all k-tuples $(F_1, \ldots, F_k)$, n realizes φ relative to $(F_1, \ldots, F_k)$. The theorem is, that the logic of absolutely realizable predicate formulas is Π^1_1-complete.

However, the logic of realizability can be viewed in a different light. Making use of formalized realizability, one can consider the collection of (say, propositional) formulas φ such that every arithmetical substitution instance (again, by substituting **HA**-sentences for the propositional variables) is provably realized in **HA**. This notion can be formalized in second-order intuitionistic arithmetic **HAS**[9]. Gavrilenko ([25]) has the interesting theorem: suppose φ is a propositional formula with the property that **HAS** proves that every arithmetical substitution instance of it is realizable. Then φ is a theorem of intuitionistic propositional logic[10]. Anticipating further developments, I mention here the following theorem of my own ([91]): let $\mathbf{HA}^+$ be an expansion of **HA** by new constants **k** and **s**, a partial binary function (or ternary relation which is single-valued) and axioms saying that this structure is a partial combinatory algebra. One can define realizability with respect to this. Suppose that φ is a purely relational predicate formula all of whose arithmetical substitution instances are realizable in this abstract sense, provably in $\mathbf{HA}^+$. Then φ is provable in the intuitionistic predicate calculus.

[8] Ceitin's example is: $[\neg(p_1 \wedge p_2) \wedge (\neg p_1 \to q_1 \vee q_2) \wedge (\neg p_2 \to q_1 \vee q_2)] \to [(\neg p_1 \to q_1) \vee (\neg p_1 \to q_2) \vee (\neg p_2 \to q_1) \vee (\neg p_2 \to q_2)]$

[9] One needs second-order, since it involves a truth definition for Gödel numbers of formulas

[10] Regrettably, recently Albert Visser and the author discovered that Gavrilenko's proof contains a gap. Nevertheless we remain convinced that his theorem is true, and that the proof can be patched

1.4 Axiomatization of Realizability

As we have seen, the logic of Realizability is too complicated to axiomatize. Quite different is the situation for formalized realizability. The formulas (x realizes A) all have a syntactic property: they are (up to equivalence) *almost negative*, that is: built from Σ^0_1-formulas using only $\wedge$, $\rightarrow$ and $\forall$. Conversely, if A is an almost negative formula, there is a "partial term" t_A (an expression of arithmetic expressing a –possibly non-terminating – computation), containing the same free variables as A, such that the equivalence

$$A \leftrightarrow t_A\!\downarrow \wedge\, t_A \text{ realizes } A$$

is provable in **HA** ("$t_A\!\downarrow$" means that the computation t_A represents, terminates). This was observed by Kleene in [50].

Exploiting the idempotency of the formalized realizability translation, one can then prove that formalized realizability is axiomatized by the scheme:

$$\forall x(A(x) \rightarrow \exists y B(x,y)) \rightarrow \exists e \forall x(A(x) \rightarrow \\ \exists y(T(e,x,y) \wedge B(x,U(y))))$$

where $A(x)$ must be an almost negative formula. This scheme is called ECT_0. The exact formulation of the axiomatization is:

i) $\mathbf{HA} + \mathrm{ECT}_0 \vdash \varphi \leftrightarrow \exists x(x \text{ realizes } \varphi)$

ii) $\mathbf{HA} \vdash \exists x(x \text{ realizes } \varphi) \Leftrightarrow \mathbf{HA} + \mathrm{ECT}_0 \vdash \varphi$

The same axiomatization holds true if **HA** is augemented with Markov's Principle MP: $\forall x(A(x) \vee \neg A(x)) \rightarrow (\neg\neg\exists x A(x) \rightarrow \exists x A(x))$. These axiomatization results were obtained, independently, by Dragalin ([19]) and Troelstra ([86]; see also [87] for a thorough exposition).

Let us look at a minor application. Obviously, Markov's Principle is an example of a predicate logical scheme which is intuitionistically underivable. But one can prove that the following scheme:

$$\forall x(A(x) \vee \neg A(x)) \wedge (\forall x A(x) \rightarrow \exists y B) \rightarrow \exists y(\forall x A(x) \rightarrow B)$$

is derivable in $\mathbf{HA} + \mathrm{MP} + \mathrm{ECT}_0$. So one sees that the introduction of realizability influences the predicate logic, at least if MP is assumed[11].

Another application is, that the scheme IP of Independence of Premisses: $(\neg A \rightarrow \exists y B) \rightarrow \exists y(\neg A \rightarrow B)$ (y not free in B) is not derivable in **HA**, since it is easily shown to be inconsistent with ECT_0 ([87]).

1.5 Extensions and Generalizations of Realizability

The first realizability definition based on a general notion of *combinatory algebra* appears in [81]. Feferman, in [21], sets out to code what he calls "explicit

[11] It is, to my knowledge, still an open problem whether the predicate logic of $\mathbf{HA} + \mathrm{ECT}_0$ properly extends intuitionistic predicate logic

mathematics" in a language for *partial* combinatory algebras (the system was later called **APP** by Troelstra and Van Dalen). The combinator axioms for a partial combinatory algebra[12]:

$$\begin{array}{ll} (\mathbf{k}) & \mathbf{k}xy = x \\ (\mathbf{s}) & \mathbf{s}xyz \simeq xz(yz) \end{array}$$

mirror the two schemes which axiomatize intuitionistic purely implicational logic: $A \to (B \to A)$ and $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$. In the axiom $(\mathbf{s})$, $\simeq$ means: one side is defined iff the other is, in which case equality holds. However, as observed by several people (e.g., [1]), with this convention the $(\mathbf{s})$-axiom is slightly stronger than needed. It is enough to assume that if $xz(yz)$ is defined, then so is $\mathbf{s}xyz$, and $\mathbf{s}xyz = xz(yz)$ (this weakening also occurs in the $\leq$-pca's of [93], and recent work of John Longley).

Of course, the natural numbers with partial recursive application form a partial combinatory algebra. Another example is the set of functions $\mathbb{N} \to \mathbb{N}$. Every function α codes a partial continuous operation (with open domain): $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$[13]. This partial combinatory algebra was at the basis of Kleene's *function realizability* ([52],[55],[53]). This was an interpretation of "intuitionistic analysis" (a theory which treats numerical functions as well as natural numbers; the functions often being seen as reals). Function realizability vindicates Brouwer's opinion[14] that every well-defined function on the reals must be continuous. A $\mathbf{q}$-variant of function realizability establishes for this system the following rule: if an existential statement $\exists\alpha A(\alpha)$ can be proved (α a variable for reals), then $A(r)$ can be established for some *recursive* real r.

At this point it is worthwile to mention an older version for function realizability, which appeared in [46]. This version used relative computability with total functions as oracles. The notion is formulated as "e realizes$^{\Phi}\varphi$" where Φ is a string of functions. Using a Gödel numbering for Turing machines with oracles, let φ^{Φ}_{e} the partial function coded by e using oracles Φ. The clause for $\forall\alpha\psi$ reads: e realizes$^{\Phi}\forall\alpha\psi$ iff for all functions α, $\varphi^{\alpha,\Phi}_{e}$ realizes$^{\alpha,\Phi}\psi(\alpha)$. So if $\forall\alpha\exists\beta\psi$ is realized (relative to oracles Φ), β is obtained recursively in α, Φ. One says a closed formula is realizable, if some number realizes it w.r.t. all oracles. Later, Kleene dismissed this version because the later notion is "equivalent" ([55]). However, I think that this version is closely connected to recently investigated notions of "relative realizability" (see section 2.5, where also an explanation of the 'equivalence' is given).

A different type of generalization is Kreisel's *Modified Realizability*; originally conceived for the system $\mathbf{HA}^{\omega}$. $\mathbf{HA}^{\omega}$ is "Gödel's T with predicate logic". One builds a type structure from one basic type o and type constructors $\times$ and $\Rightarrow$; one has variables of each type, typed combinators for pairing and projections, $\mathbf{k}$ and $\mathbf{s}$ of each appropriate type, and combinators for primitive recursion. For any formula A, a formula "x realizes A" can be defined in a completely

[12] as is well known, partial combinatory algebras are models of **APP**, and vice versa
[13] for details see, e.g., [87]
[14] he called it a "theorem"

straightforward way: the type of the variable x is determined by the logical form of A. So if the type of realizers of A is σ, and the type of realizers of B is τ, the type of realizers of $A \to B$ is $(\sigma \Rightarrow \tau)$. This "typed realizability", defined by Kreisel in 1959 ([56])[15], predates the slogan "formulae as types" (Howard, [36]) by 10 years! Of course, it came to be used in the late seventies to interpret versions of Martin-Löf's type theory (e.g.,[17] and the thesis [84]), and analogous versions for systems based on PCF have been studied by John Longley. Troelstra found an axiomatization for modified realizability for $\mathbf{HA}^\omega$ ([87]).

But, it is the untyped "collapse" of this realizability, that most people know as 'modified realizability'. The structure of Hereditary Recursive Operations ([87]) is a typed structure which models $\mathbf{HA}^\omega$ and is itself definable in **HA**. Using that **HA** is a subsystem of $\mathbf{HA}^\omega$, one can construct out of Kreisel's definition a new notion of ralizability for **HA**. Each formula gets *two* sets of realizers, the *actual* realizers being a subset of the *potential* ones[16]. Features of HRO-modified realizability for **HA** are that it validates the scheme IP (see the last paragraph of 1.4) and refutes Markov's Principle. By a **q**-version of this realizability one can obtain an IP-rule for **HA** (I believe this was first noticed in [90]). Beeson ([4]) applies modified realizability to show that although, in formalizations of elementary recursion theory, the Myhill-Shepherdson and Kreisel-Lacombe-Shoenfield theorems seem to require Markov's Principle, they don't conversely imply it, for these theorems hold under modified realizability.

The idea of actual and potential realizers can of course be applied to different partial combinatory algebras, and was so, by Kleene ("special realizability" in [55]) and Joan Moschovakis ([65]). Moschovakis shows the consistency of Kleene and Vesley's "Basic System" of intuitionistic analysis together with the scheme $(\neg A \to \exists\alpha B(\alpha)) \to \exists\alpha(\neg A \to B(\alpha))$ and the scheme $\exists\alpha A(\alpha) \to \exists\alpha(\mathrm{GR}(\alpha) \wedge A(\alpha))$ for *closed* $\exists\alpha A(\alpha)$ (the formula $\mathrm{GR}(\alpha)$ expresses that α is recursive). She uses the partial combinatory algebra of functions together with its subalgebra of recursive functions; her work is definitely closely related to recent work of Birkedal et al ([3]; see also section 2.5). In general, as shown e.g. in [94], modified realizability interpretations are intimately connected with what the author of these lines has called "Kripke models of realizability" ([90]); see next section.

Recently, modified realizability has enjoyed renewed interest, mainly by the efforts of Thomas Streicher, Martin Hyland and Luke Ong ([82],[41]; see also [94]).

For an extension of formalized Kleene-realizability to second-order arithmetic **HAS**, see [87]. Troelstra shows that the following principle of second-order arithmetic is valid under his extension:

$$\mathrm{UP} \quad \forall X \exists n A(X,n) \to \exists n \forall X A(X,n)$$

[15] in a footnote!

[16] This modified realizability is also reminiscent of Kolmogorov's interpretation of intuitionism by "problems"; see, e.g.,[62]

The initials UP stand for Uniformity Principle. This principle received much attention in connection with the Effective Topos: see sections 2.1 and 2.2. Saying that every function from sets of numbers to numbers must be constant, it is very non-classical; however, it can be shown that **HAS** + UP has no non-classical first-order consequences ([90]).

1.6 Kripke Models of Realizability

This, of course, is a prelude to a general topos-theoretic account of realizability. But topos theory was slow to catch up with realizability, and long after the logical significance of toposes had been grasped, it was not yet clear what toposes could do for realizability.

A Kripke model of realizability is a Kripke model of the theory **APP**, that is: a system of partial combinatory algebras $(A_p)_{p\in P}$ indexed by some partially ordered set P, together with maps $A_p \to A_q$ for $p \leq q$, satisfying the usual conditions. As a simple example, take the partial order $\{0 < 1\}$, let A_1 the pca of function realizability and A_0 its sub-pca of recursive functions. One can also take: A_1 the graph model $\mathcal{P}(\omega)$ and A_0 its subalgebra on the r.e. subsets of $\mathbb{N}$. See section 2.5 for more about this.

In general, if $(A_p)_{p\in P}$ is a Kripke model of realizability, to any formula φ a P-indexed system $([\![\varphi]\!]_p)_{p\in P}$ of sets of realizers is assigned (which is a subset of $(A_p)_{p\in P}$ in the sense of Kripke models).

The first example I know of such a Kripke model of realizability, is the unpublished paper [16]. De Jongh wished to establish the theorem that a formula A is provable in intuitionistic predicate calculus if and only if each of its arithmetical substitutions is provable in **HA**. He succeeded partially: the full theorem was first proved by Leivant in his thesis (and Leivant used proof theory). In [91] I was able to revive De Jongh's original realizability method to prove the full theorem.

Another example occurs in [27]. The models of De Jongh and Goodman are strikingly similar: in both cases, A_p is the set of indices of functions partial recursive in some set $X_p \subseteq \mathbb{N}$, with $X_p \subseteq X_q$ for $p \leq q$. However, Goodman, whose aim was to interpret a version of $\mathbf{HA}^\omega$ with decidable equality at all types, also brings the $\neg\neg$-translation into the picture, so strictly speaking his model transcends the definition of a Kripke model of realizability, and might rather be called a (generalized) Beth model of realizability.

Much work on combinations of realizability with Kripke forcing was done by Jim Lipton ([58],[59]).

1.7 Extensional Realizability

"Extensional realizability" defines not just realizers, but simultaneously an equivalence relation on them; the idea is that a realizer for an implication $A \to B$ should send equivalent realizers for A to equivalent realizers for B. The origin is, of course, again Kreisel's modified realizability; just as HRO is a model for $\mathbf{HA}^{\omega}$ which is definable in $\mathbf{HA}$, we have the models HEO of 'hereditarily effective operations' and $\mathrm{HRO}_{\mathrm{E}}$, the extensional collapse of HRO (see [87]). HEO in combination with modified realizability is already considered in Troelstra ([87]), but the first extensional realizability for $\mathbf{HA}^{\omega}$, in combination with Kripke forcing, was used by Beeson ([5]), who extended Goodman's theorem to the statement that $\mathbf{E}-\mathbf{HA}^{\omega}+\mathrm{AC}$ is conservative over $\mathbf{HA}$.

The first time a definition for extensional realizability appeared in print that was suitable for first-order arithmetic, was in Pitts' thesis ([68]), although Pitts claims the idea came from Robin Gandy.

Extensional realizability was used by Beeson ([6] and [7]) in connection with Martin-Löf's Type Theory, and by Diller, Troelstra and Renardel ([18],[70]). Martin Hyland studied extensional realizability from a topos-theoretic point of view, and noted its salient higher-order logical properties in [43] (see also the next chapter).

In [93], two versions of extensional realizability for $\mathbf{HA}$, analogous to HEO and $\mathrm{HRO}_{\mathrm{E}}$, are compared and found non-equivalent. It is shown that the HEO-version is not idempotent, but nevertheless an axiomatization for this realizability is obtained over a conservative extension of $\mathbf{HA}$. The usual Troelstra-type results are obtained: a $\mathbf{q}$-version is defined, and an "Extensional Church's Rule" for $\mathbf{HA}$ is derived.

2 The period 1980-2000

Around 1970, Lawvere and Tierney had generalized Grothendieck's notion of "topos" to the definition of *elementary topos*; in subsequent work they (and also others, like Michael Barr and Peter Freyd) had shown that very many results in the theory of Grothendieck topoi can in fact be derived from the axioms for an elementary topos. An impressive account of elementary topos theory (I mean 'theory of elementary toposes'; the theory itself is at places far from 'elementary') of the 70ies, which has served as a standard reference to this day, is Johnstone's [44].

Logicians discovered that toposes generalized semantical ideas that had developed in the 60ies: Cohen forcing for ZF set theory (later, by Solovay[17] reformulated in terms of Boolean-valued models[18]), Kripke and Beth models for intuitionistic predicate logic, and topological models. All these semantics fall, from the point of view of a topos theorist, under the header "localic toposes", or to use a more familiar term for logicians: Heyting-valued semantics.

Denis Higgs ([33],[34]) had proved in 1973 that the category of '$\mathcal{H}$-valued sets' is equivalent to the topos of sheaves over $\mathcal{H}$, for a complete Heyting algebra $\mathcal{H}$. So Kripke semantics, topological semantics etc. have a *natural* extension to higher-order languages[19]. This is important for the development of intuitionistic elementary mathematics: the real numbers are constructed by Dedekind cuts which needs second-order arithmetic (logicians had been describing models for analysis, completely independent of second-order arithmetic).

It seems that no one in the traditional logicians' world of the 70ies was more influential in pushing topos semantics than Dana Scott. Martin Hyland has testified[20] that Scott's coming to Oxford in the mid-70ies meant a "change in ways of doing logic". Much of this can probably be attributed to a different cultural background: most of all, the model theorist Scott advocated the view of realizability (and other 'interpretations') as *models*, to be treated as syntax-free as possible.

Anyway, the reader who wishes to see a representative sample of work from the 70ies on sheaf models, is referred to the "Durham Proceedings" ([22]). All this work concerns *Grothendieck topoi* however, and realizability was markedly absent. In fact, what did one know about non-Grothendieck topoi? Finite sets (not very entertaining); and yes; the Lawvere/Tierney axioms are sufficiently *algebraic* to ensure that a *free topos* exists; but what did one know about it? Finally, there were the toposes arising by the so-called *filter-quotient* construction which had been used to give topos-theoretic proofs of Cohen's independence results.

[17] and, independently, by Scott and Vopěnka; see Scott's Foreword to [8]

[18] It was Scott who first observed that Cohen's forcing over a poset was Kripke forcing combined with the $\neg\neg$-translation

[19] This point is emphasized in Scott's Foreword to [8], where the failure by logicians to spot this fact, is attributed to "the first-order disease"

[20] in his lecture at the Realizability workshop in Trento

2.1 The effective topos

A completely new type of topoi was discovered around 1979 (apparently following some ideas of Scott; independently, there had been work of W.Powell along similar lines) by Martin Hyland, Peter Johnstone and Andy Pitts. The relevant publications are [40], [68] and [37].

It was well-known, and amply demonstrated in Fourman and Scott's paper [23] that Boolean-valued sets generalize to Heyting-valued sets for a complete Heyting algebra. The completeness of the algebra is used for interpretation of the quantifiers. Now in [23], Fourman and Scott had dissected the construction of the topos of $\mathcal{H}$-sets into two, logically meaningful steps. First, one has a model of many-sorted intuitionistic predicate logic without equality. The predicates of sort X (where X is a set) are functions from X into the set of propositions $\mathcal{H}$. Since $\mathcal{H}$ itself exists as a sort, one has in fact second-order propositional logic too. The next step is adding equality as a general $\mathcal{H}$-valued symmetric and transitive (but not necessarily reflexive!) relation, and consider *all* possible such. One obtains a topos, and the validity of a formula φ in the internal logic of this topos is connected to the validity in the underlying model of many-sorted predicate logic of a translation of φ into the "logic of identity and existence" ([78]).

Hyland, Johnstone and Pitts discovered a useful generalization of the first step in this construction, calling it 'tripos' for 'topos-representing indexed preordered set'[21]. The 'Theory of triposes' is the subject matter of Andy Pitts' thesis [68], but a major application of the idea is the 'effective topos', discovered by Martin Hyland and described in the classic paper [37]. Let the 'domain of propositions' be the powerset of $\mathbb{N}$. For any set X, the set of predicates on X i.e. the set $\mathcal{P}(\mathbb{N})^X$ is preordered by: $\varphi \leq \psi$ if and only if there is a partial recursive function F such that for each $x \in X$ and each $n \in \varphi(x)$, $F(n)$ is defined and $F(n) \in \psi(x)$. Then $\mathcal{P}(\mathbb{N})^X$ is a Heyting (pre)algebra, and although it is not complete, adjoints to the map $\mathcal{P}(\mathbb{N})^f : \mathcal{P}(\mathbb{N})^Y \to \mathcal{P}(\mathbb{N})^X$ (for functions $f : X \to Y$) exist. One can mimick the the construction of the topos of $\mathcal{H}$-valued sets completely, and one gets the *Effective topos* $\mathcal{E}ff$.

In $\mathcal{E}ff$, the standard truth definition for first-order arithmetic (based on the natural numbers object) is equivalent to Kleene's 1945-realizability. But much more is true: standard second-order arithmetic in $\mathcal{E}ff$ is captured by an informal reading of Troelstra's realizability for **HAS** (as shown in [90]), and standard analysis in $\mathcal{E}ff$ (using the Dedekind reals) turns out to be equivalent to Bishop-style recursive analysis. The finite type structure over the natural numbers is the structure HEO. All these different, hitherto unrelated bits of research fell into their right place.

Even more strikingly, also the *proof-theoretic* results obtained by realizability received a wider significance in the effective topos. The role of the almost negative formulas is explained by the fact that the category of Sets in contained in $\mathcal{E}ff$ as "$\neg\neg$-sheaves" (see the section "Basic facts from the logic of sheaves"

[21] 'Tripos' is also the name of the major Mathematics exam at the University of Cambridge. A typical Cambridge pun, in more than one way

in [37]).

In a little series of never-published, hand-written notes, Robin Grayson ([28],[30],[29]) gave accounts of results obtained by Hyland. He described the construction of toposes for modified and extensional realizability. He explained the topos-theoretic counterpart of **q**-realizability. By *glueing* the toposes Sets and $\mathcal{E}ff$ along the embedding (see [96] for this construction) one gets a topos corresponding to a sort of **q**-realizability. Replacing Sets by the free topos $\mathcal{F}$ and constructing $\mathcal{E}ff$ over $\mathcal{F}$, one obtains versions of existence properties for higher-order intuitionistic arithmetic **HAH** and Church's Rule for **HAH**[22]. Let us sketch the argument for Church's Rule. So $\mathcal{F}$ is the free topos, $\mathcal{E}ff(\mathcal{F})$ the effective topos constructed over it, and $\mathcal{E}$ the glueing of $\mathcal{F}$ to $\mathcal{E}ff(\mathcal{F})$. The satisfaction relation $\mathcal{E} \models \varphi$ can be expressed in $\mathcal{F}$. Now suppose $\mathbf{HAH} \vdash \forall x : N \exists y : N\psi(x,y)$, so $\mathcal{E} \models \forall x \exists y \psi$. By the realizability construction, we have

$$\mathcal{F} \models \exists f : N \forall x : N \exists y : N(T(f,x,y) \wedge \mathcal{E} \models \psi(x, U(y)))$$

Now there is a logical functor $\mathcal{E} \to \mathcal{F}$ (a general feature of the glueing construction), whence

$$\mathcal{F} \models \exists f : N \forall x : N \exists y : N(T(f,x,y) \wedge \psi(x, U(y)))$$

so **HAH** proves the formula, and we are done.[23]

2.2 Modest Sets and Internal Completeness

In his paper [37], Hyland had singled out an interesting subcategory of $\mathcal{E}ff$: the subcategory on what he called 'effective objects'. This category generalizes Eršov's "numerated sets" ([20]): it is equivalent to the category whose objects are pairs (X, μ) with X a set and $\mu : A \to X$ a surjective function from a subset of $\mathbb{N}$ to X; morphisms $(X,\mu) \to (Y,\nu)$ are functions $f : X \to Y$ such that for some partial recursive function F, $F(n)$ is defined for all $n \in \mathrm{dom}(\mu)$ and $F(n) \in \mathrm{dom}(\nu)$ and $f(\mu(n)) = \nu(F(n))$. Abstractly the effective objects are (in $\mathcal{E}ff$) $\neg\neg$-separated quotients of subobjects of N. The concrete representation just given, was later called the *category of modest sets* by Dana Scott ([79]).

Hyland noticed that the effective objects allow an interesting generalization of Troelstra's Uniformity Principle (see section 1.5). Recall that Sets is included in $\mathcal{E}ff$ as $\neg\neg$-sheaves. Now any function from a quotient of a set to an effective object is necessarily constant in $\mathcal{E}ff$; in fact, for an effective object A and a quotient B of a set, the diagonal embedding $A \to A^B$ is an isomorphism.

[22] The existence property for **HAH** was first proved by Lambek and Ph.Scott in 1978, using Friedman-style **q**-realizability. That this was essentially a glueing construction, was realized by Peter Freyd, who appears to have been surprised by the fact that in $\mathcal{F}$ the terminal object is indecomposable and projective, but nevertheless gave an algebraic proof of it. Freyd's proof was again syntacticized by Lambek and Ph.Scott in [57]

[23] By the way, existence properties for **HAS** had first been obtained by Friedman in [24] using **q**-realizability. Note, that Friedman's "set existence property for **HAS**" is *not* automatically subsumed by the existence property for full **HAH**

Around 1985, Moggi and Hyland made an important discovery. This 'Uniformity Principle' meant that a specific internal category in $\mathcal{E}ff$ (basically, the internal full subcategory of separated subquotients of N) was *complete* in a sense, without being a preorder[24].

This meant several things. For example, Scott used it in [79] to show that intuitionistically it may happen that a set A is in bijective correspondence with 2^{2^A}[25]. It could also be used to obtain a set-theoretic interpretation of Girard's second-order λ-calculus F[26].

The precise meaning of 'complete' (this is not expressible in the internal language of the topos) took a while to sort out. A basic observation came from Freyd: take the property that $A \rightarrow A^B$ is an isomorphism for each set B (in fact, just the set 2 will suffice; but note the *set* 2, not the *object* 2 in $\mathcal{E}ff$!) as a *defining* property A can have; call A 'discrete' if it has this property. Eventually, Hyland, Robinson and Rosolini showed that the discrete objects, as a fibration over $\mathcal{E}ff$, are complete, and weakly equivalent to the fibration obtained by 'externalizing' the aforementioned internal category in $\mathcal{E}ff$; from this, it follows that the internal category is 'weakly complete'[27]. This is explained in [42] and [38].

Of course this does *not* mean that the category of modest sets is complete, as [72] and [77] hastened to point out. But it may serve very well for interpretations of theories in, say, system F and related programming languages such as Quest. Such 'PER' models were constructed by Abadi, Cardelli, Longo, Freyd, Hyland, Robinson, Rosolini and many, many others; by now, PER models form a standard tool in the semantics of programming languages.

For historical reasons, quotients of sets are called 'uniform objects'. The notions 'uniform' and 'discrete' can be applied to *maps* as well and give rise to a factorization system on $\mathcal{E}ff$ very much in analogy with the 'monotone-light' factorization system on the category of T_0-topological spaces (see [12]).

Important applications of the completeness of 'pers' come from Synthetic Domain Theory (see section 2.6)

2.3 Realizability as a universal construction

The effective topos has intriguing, not to say mystifying aspects. One way of attacking its mystery is to look for universal properties it may enjoy. Around 1990, two papers appeared with rather similar-looking constructions of $\mathcal{E}ff$: [11] and [73]. The key word here is *completion.*

We have seen that the effective topos is a two-step construction. But there are many ways in which to cover a distance by two steps ...

[24] contradicting a classical theorem of Peter Freyd

[25] Contradicting Cantor's theorem

[26] Contradicting a well-known result of Reynolds

[27] Basically, the problem resides in the absence of choice in $\mathcal{E}ff$. Call the internal category C. For an arbitrary other, say D, we have the object C^D of diagrams in C of type D, and an object E of pairs (d, c) where d is a diagram, and c a limit for this diagram. The projection: $E \rightarrow C^D$ is an epimorphism in $\mathcal{E}ff$, but there need not be a section of it, which would assign a limit to each diagram

Let us consider two completion processes: given a finite-limit category C one can add coproducts to it; or one can add stable quotients of equivalence relations to it, making it *exact*. The first construction belongs to folklore and results in $\mathrm{Fam}(C)$: objects are families $(C_i)_{i\in I}$ of objects of C indexed by a set I; a morphism $(C_i)_{i\in I} \to (D_j)_{j\in J}$ consists of a function $f : I \to J$ and an I-indexed collection of arrows $(f_i : C_i \to D_{f(i)})_{i\in I}$ of C. The second construction is detailed in [13] and results in the category $(C)_{\mathrm{ex/lex}}$.

Performing the two in succession gives $(\mathrm{Fam}(C))_{\mathrm{ex/lex}}$ which is a topos, the topos $\mathrm{Sets}^{C^{\mathrm{op}}}$[28].

Now suppose one does not add *all* coproducts, just the *recursive* ones. That is, take $\mathrm{Fam}_R(C)$: objects are now families indexed by a subset I of $\mathbb{N}$, and morphisms $(C_i)_{i\in I} \to (D_j)_{j\in J}$ need a *partial recursive* function $I \to J$. The main result of [73] is: $(\mathrm{Fam}_R(\mathrm{Sets}))_{\mathrm{ex/lex}}$ is a topos, *the effective topos.* Note the mirroring in the two cases: for a Grothendieck topos, at least for presheaf toposes, one completes a *small* category with all coproducts indexed by Sets; for $\mathcal{E}\!f\!f$, one completes Sets by coproducts indexed by a small category R!

It follows from the general theory of ex/lex completions that the category $\mathrm{Fam}_R(\mathrm{Sets})$ (into which Sets embeds) is equivalent to the full subcategory of *projective objects* of $\mathcal{E}\!f\!f$; and moreover, that every object of $\mathcal{E}\!f\!f$ is a quotient of a projective object.

On the other hand, the construction of [11] presents $\mathcal{E}\!f\!f$ as $(\mathrm{Asm})_{\mathrm{ex/reg}}$; that is, make Asm exact but preserve the regular structure, where Asm is the category of *assemblies*, the $\neg\neg$-separated objects of the effective topos[29].

It is amusing to note that $(\mathrm{Asm})_{\mathrm{ex/lex}}$ also yields a topos; now not the effective topos, but a topos for *extensional realizability* (see [93])[30].

An interesting result in this area is due to John Longley ([60]). We can construct $\mathcal{E}\!f\!f$ over any partial combinatory algebra A; call it $\mathcal{E}\!f\!f(A)$. How "functorial" is $\mathcal{E}\!f\!f(A)$ in A? Longley defines a 2-category **Pca** of partial combinatory algebras, such that the category $\mathbf{Pca}(A, B)$ is equivalent to the category of exact functors $\mathcal{E}\!f\!f(A) \to \mathcal{E}\!f\!f(B)$ which commute with the inclusions from Sets into these toposes. At first sight, his definition looks like a hack, but: a 1-cell from A to B in **Pca** is nothing but an *internal* partial combinatory algebra in $\mathrm{Asm}(B)$ (assemblies over B; that is: a $\neg\neg$-separated internal pca in $\mathcal{E}\!f\!f(B)$ for which the domain of the application map is $\neg\neg$-closed) with global sections A; a 2-cell between such is an internal 'ordinary' pca-morphism. Viewed in this way, and combined with Pitts' *iteration results* ([68]), the construction becomes a lot more transparent, and its connection to the exact completions business should be obvious.

Recently, a lot of work was devoted to the question of when an exact completion is (locally) cartesian closed: see [75] and [14]. Much of this work was prompted by the appearance of Scott's "New Category" ([80])[31]. This category

[28] For a recent explanation of when (if) $(C)_{\mathrm{ex/lex}}$ is a topos, see [63]

[29] The constructions ex/lex and ex/reg are well explained in [15] and [10]

[30] [63] has an independent, abstract argument that $\mathrm{Asm}_{\mathrm{ex/lex}}$ is a topos

[31] I like "New Category" as a name, better than this category's official name, **Equ**, pronounced 'Eek'. "New Category" is like "New Foundations" and things like that

is 'almost' an exact completion of the category of T_0-topological spaces.

2.4 Axiomatization Revisited

In his seminal paper [37], Hyland had finished with the comment:

> What we lack, above all [...] is any real information analogous to the results obtained in Troelstra ([87]) axiomatizing realizability [...] we have no good information in this area. We can not properly be said to understand realizability until we do.

Wasn't it about time, after 1990 and all these further results on $\mathcal{E}ff$ has appeared, to use them in order to obtain more "information in this area"?

In [92], the construction of a series of theories of higher order arithmetic (2nd,3rd, ... order) is given, which are true in $\mathcal{E}ff$, and realizabilities for these theories which are also true in $\mathcal{E}ff$, and which can be axiomatized over the theories. This is based on the fact that in $\mathcal{E}ff$, realizability can be defined in such a way that in $\mathcal{E}ff$, a sentence is equivalent to its own realizability. The details are worked out for 2nd and 3rd order arithmetic; the axioms characterizing the 2nd order realizability are Uniformity Principle, Extended Church's Thesis and Shanin's Principle which says that for any subset X of N there is a $\neg\neg$-closed subset A of N such that

$$X = \{x \mid \exists y \langle x, y\rangle \in A\}$$

The construction of these theories is motivated by the fact that the relevant arithmetical objects are covered by *definable* projective objects; e.g., Ω^N is covered by $(\Omega_{\neg\neg})^N$; that this is a cover, is the content of Shanin's Principle.

A corollary of the treatment for 3rd order arithmetic is, that from the axioms which characterize its realizability, one can prove a completeness property of the category of modest sets.

Yet, we are a long way from understanding realizability axiomatically. We may ask the following question. For an arbitrary topos $\mathcal{E}$ with natural numbers object, let $\mathcal{E}ff(\mathcal{E})$ be the effective topos constructed over it. The construction $\mathcal{E} \mapsto \mathcal{E}ff(\mathcal{E})$ is *not* idempotent ([68]) up to equivalence, although Pitts shows it gives rise to a monad ("the effective monad") on a certain category of toposes and geometric morphisms. Is there any way of characterizing the algebras for this monad? Is there any reasonable system of meaningful conditions on $\mathcal{E}$ ensuring that $\mathcal{E} \to \mathcal{E}ff(\mathcal{E})$ is an equivalence?

What does $\mathcal{E}ff(\mathcal{F})$ look like? One thing for sure: it is *not* an exact completion!

2.5 Relative Realizability

From around 1997, a group of talented young people around Dana Scott at CMU in Pittsburgh has been working on Realizability: Steve Awodey, Andrej Bauer and Lars Birkedal. In a few recent papers ([3],[2],[9]) they study what they call 'relative realizability'.

Suppose a pca A has a subset $A_\sharp$ which is closed under the application and contains a choice for **k** and **s** for A; in other words, a sub-pca. One can define a tripos on Sets in the following way: predicates on X are functions $X \to \mathcal{P}(A)$, but the order between two such functions has to be realized by an element of $A_\sharp$. Call the resulting topos $\mathcal{E}ff(A_\sharp, A)$.

Usually, $A_\sharp$ consists of 'recursive' or 'recursively enumerable' elements of A; see the examples cited in section 1.6. Part of the motivation for studying this situation is the "study of computable operations and maps on data that is not necessarily computable, such as the space of all real numbers".

$\mathcal{E}ff(A_\sharp, A)$ compares nicely to the toposes $\mathcal{E}ff(A_\sharp)$ and $\mathcal{E}ff(A)$: there is a geometric morphism $\mathcal{E}ff(A_\sharp) \to \mathcal{E}ff(A_\sharp, A)$ which is *local*, and there is a *logical functor* $\mathcal{E}ff(A_\sharp, A) \to \mathcal{E}ff(A)$.

Now I present this here because I believe that this work ties up with research done in the past. I already alluded to this when discussing Joan Moschovakis' work. Also Kleene's first attempt at function realizability should be recalled: if I am not mistaken, it is of the form $(A_\sharp, A)$ where $A = \mathbb{N}^{\mathbb{N}}$ and $A_\sharp$ its sub-pca of total recursive functions. The 'equivalence' Kleene noticed between this and his later function realizability, is then a consequence of a more general fact: the logical functor above.

Also the work of Thomas Streicher ([83]) deserves mention: he did the same thing and also noted the logical functor. This logical functor, by the way, is a *filter quotient situation.*

Note that the motivation of letting computable things act on non-computable data, is reminiscent of Kleene's setup for higher-type recursive functionals ([49] and later papers).

Finally I'd like to add that the relative situation can profitably be studied from the point of view of the topos $\text{Sets}^{\to}$, but this is current work.

2.6 Non-classical Theories

A useful feature of $\mathcal{E}ff$ and related topoi is that in them one often finds models for inherently non-classical theories, theories which have no classical models (sometimes not even models in Grothendieck topoi).

Here I just point at a few interesting topics that deserve further research.

Synthetic Domain Theory aims for a suitable category of objects which carry a *natural* domain structure, such that between these objects *any* map is automatically continuous. Suggested by Dana Scott. Scott's student Rosolini ([76]) was the first who made real progress in setting up the theory; later work was done by, among others, Hyland ([39]), Phoa ([67]), Taylor ([85]), and Streicher/Reus ([71]). In [95], the force of a truly axiomatic and rigorously internal approach is advocated.

Algebraic Set Theory. In their elegant little book ([45]), Joyal and Moerdijk present a novel way of looking at set theory. They point to a model in $\mathcal{E}ff$, which needs to be further investigated.

Intuitionistic Nonstandard Arithmetic. Also for this, there are interesting models in $\mathcal{E}ff$, as pointed out in ([64]). This must also definitely be studied more closely.

References

[1] P.H.G. Aczel. A note on interpreting intuitionistic higher-order logic, 1980. Handwritten note.

[2] S. Awodey and L. Birkedal. Elementary axioms for local maps of toposes. Manuscript, submitted for 1999 Category Theory Conference in Coimbra, 1999.

[3] S. Awodey, L. Birkedal, and D.S. Scott. Local realizability toposes and a modal logic for computability. Presented at *Tutorial Workshop on Realizability Semantics, FLoC'99*, Trento, Italy 1999., 1999.

[4] M.J. Beeson. The nonderivability in intuitionistic formal systems of theorems on the continuity of effective operations. *Journal of Symbolic Logic*, 41:321–346, 1975.

[5] M.J. Beeson. Goodman's theorem and beyond. *Pacific Journal of Mathematics*, 84:1–28, 1979.

[6] M.J. Beeson. Recursive models for constructive set theories. *Annals of Pure and Applied Logic*, 23:127–178, 1982.

[7] M.J. Beeson. *Foundations of constructive mathematics.* Springer-Verlag, 1985.

[8] J.L. Bell. *Boolean-valued Models and Independence Proofs in Set Theory.* Clarendon Press, Oxford, 1977.

[9] L. Birkedal. Developing theories of types and computability via realizability. PhD Thesis, December 1999.

[10] A. Carboni. Some free constructions in realizability and proof theory. *Journal of Pure and Applied Algebra*, 103:117–148, 1995.

[11] A. Carboni, P.J. Freyd, and A. Scedrov. A categorical approach to realizability and polymorphic types. In M. Main, A. Melton, M. Mislove, and D.Schmidt, editors, *Mathematical Foundations of Programming Language Semantics*, volume 298 of *Lectures Notes in Computer Science*, pages 23–42, New Orleans, 1988. Springer-Verlag.

[12] A. Carboni, G. Janelidze, G. M. Kelly, and R. Paré. On localization and stabilization for factorization systems. *Appl. Categ. Structures*, 5(1):1–58, 1997.

[13] A. Carboni and R. Celia Magno. The free exact category on a left exact one. *Journal of Australian Mathematical Society*, 33(A):295–301, 1982.

[14] A. Carboni and G. Rosolini. Locally cartesian closed exact completions, 1998. to appear.

[15] A. Carboni and E.M. Vitale. Regular and exact completions. *Journal of Pure and Applied Algebra*, 125:79–117, 1998.

[16] D.H.J. de Jongh. The maximality of the intuitionistic predicate calculus with respect to Heyting's Arithmetic, 1969. typed manuscript from University of Wisconsin, Madison.

[17] J. Diller. Modified realization and the formulae–as–types notion. In J. P. Seldin and J. R. Hindley, editors, *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 491–501. Academic Press, New York, 1980.

[18] J. Diller and A.S.Troelstra. Realizability and intuitionistic logic. *Synthese*, 60:253–282, 1984.

[19] A.G. Dragalin. Transfinite completions of constructive arithmetical calculus (Russian). *Doklady*, 189:458–460, 1969. Translation *SM* 10, pp. 1417–1420.

[20] Yu.L. Eršov. Theorie der Numerierungen. *Zeitschrift für Math. Log.*, 19(4):289–388, 1973.

[21] S. Feferman. A language and axioms for explicit mathematics. In J.N. Crossley, editor, *Algebra and Logic*, pages 87–139. Springer-Verlag, 1975.

[22] M.P. Fourman, C.J. Mulvey, and D.S. Scott, editors. *Applications of Sheaves*. Springer (LNM 753), 1979.

[23] M.P. Fourman and D.S. Scott. Sheaves and logic. In M.P. Fourman, C.J. Mulvey, and D.S. Scott, editors, *Applications of Sheaves*, pages 302–401. Springer-Verlag, 1979.

[24] H.M. Friedman. On the derivability of instantiation properties. *Journal of Symbolic Logic*, 42:506–514, 1977.

[25] Yu. V. Gavrilenko. Recursive realizability from the intuitionistic point of view (Russian). *Doklady*, 256:18–22, 1981. Translation *SM* 23, pp. 9–14.

[26] K. Gödel. Zur intuitionistischen arithmetik und zahlentheorie. *Ergebnisse eines mathematisches Kolloquiums*, 4:34–38, 1932.

[27] N.D. Goodman. Relativized realizability in intuitionistic arithmetic of all finite types. *Journal of Symbolic Logic*, 43:23–44, 1978.

[28] R.J. Grayson. Derived rules obtained by a model-theoretic approach to realisability, 1981. Handwritten notes from Münster University.

[29] R.J. Grayson. Modified realisability toposes, 1981. Handwritten notes from Münster University.

[30] R.J. Grayson. Note on extensional realizability, 1981. Handwritten notes from Münster University.

[31] R. Harrop. On disjunctions and existential statements in intuitionistic systems of logic. *Mathematische Annalen*, 132:347–361, 1956.

[32] A. Heyting, editor. *Constructivity in Mathematics*. North-Holland Publishing Company, 1959.

[33] D. Higgs. A category approach to boolean-valued set theory. Technical report, University of Waterloo, 1973.

[34] D. Higgs. Injectivity in the topos of complete heyting algebra valued sets. *Canadian Journal of Mathematics*, 36:550–568, 1984.

[35] D. Hilbert and P. Bernays. *Grundlagen der Mathematik I*. Springer Verlag, 1934.

[36] W.A. Howard. To H.B. Curry: The formulae-as-types notion of construction. In J. Hindley and J. Seldin, editors, *Essays on Combinatory Logic, Lambda Calculus, and Formalism*. Academic Press, 1969.

[37] J.M.E. Hyland. The effective topos. In A.S. Troelstra and D. Van Dalen, editors, *The L.E.J. Brouwer Centenary Symposium*, pages 165–216. North Holland Publishing Company, 1982.

[38] J.M.E. Hyland. A small complete category. *Journal of Pure and Applied Logic*, 40:135–165, 1988.

[39] J.M.E. Hyland. First steps in synthetic domain theory. In A. Carboni, M.C. Pedicchio, and G. Rosolini, editors, *Category Theory '90*, volume 1144 of *Lectures Notes in Mathematics*, pages 131–156, Como, 1992. Springer-Verlag.

[40] J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. Tripos theory. *Math. Proc. Camb. Phil. Soc.*, 88:205–232, 1980.

[41] J.M.E. Hyland and C.-H. L. Ong. Modified realizability toposes and strong normalization proofs. In J.F. Groote and M. Bezem, editors, *Typed Lambda Calculi and Applications*, volume 664 of *Lecture Notes in Computer Science*, pages 179–194. Springer-Verlag, 1993.

[42] J.M.E. Hyland, E.P. Robinson, and G. Rosolini. The discrete objects in the effective topos. *Proceedings of the London Mathematical Society*, 60:1–60, 1990.

[43] J.M.E.Hyland. Realizability toposes. Talk in Amsterdam (notes by Troelstra), 1982.

[44] P.T. Johnstone. *Topos Theory.* Number 10 in LMS Mathematical Monographs. Academic Press, London, 1977.

[45] A. Joyal and I. Moerdijk. *Algebraic Set Theory*, volume 220 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1995.

[46] S. C. Kleene. Realizability. In *Summaries of Talks presented at the Summer Institute for Symbolic Logic*, pages 100–104. Institute for Defense Analyses, Communications Research Division, Princeton, 1957. Also in *[32]*, pp. 285–289. Errata in *[55]*, page 192.

[47] S.C. Kleene. On the interpretation of intuitionistic number theory. *Journal of Symbolic Logic*, 10:109–124, 1945.

[48] S.C. Kleene. *Introduction to metamathematics.* North-Holland Publishing Company, 1952. Co-publisher: Wolters–Noordhoff; 8th revised ed.1980.

[49] S.C. Kleene. Recursive functionals and quantifiers of finite types I. *Trans. Amer. Math. Soc*, 91, 1959.

[50] S.C. Kleene. Realizability and Shanin's algorithm for the constructive deciphering of mathematical sentences. *Logique et Analyse, Nouvelle Série*, 3:154–165, 1960.

[51] S.C. Kleene. Disjunction and existence under implication in elementary intuitionistic formalisms. *Journal of Symbolic Logic*, 27:11–18, 1962. Addenda in *JSL* **28** (1963), pp. 154–156.

[52] S.C. Kleene. Logical calculus and realizability. *Acta Philosophica Fennica*, 18:71–80, 1965.

[53] S.C. Kleene. *Formalized Recursive Functionals and Formalized Relizability*, volume 89 of *Memoirs of the American Mathematical Society.* American Mathematical Society, 1969.

[54] S.C. Kleene. Realizability: a retrospective survey. In A.R.D. Mathias and H. Rogers, editors, *Cambridge Summer School in Mathematical Logic*, volume 337 of *Lecture Notes in Mathematics*, pages 95–112. Springer-Verlag, 1973.

[55] S.C. Kleene and R.E. Vesley. *The Foundations of Intuitionistic Mathematics, especially in relation to recursive functions*. North-Holland Publishing Company, 1965.

[56] G. Kreisel. Interpretation of analysis by means of functionals of finite type. In A. Heyting, editor, *Constructivity in Mathematics*, pages 101–128. North-Holland, 1959.

[57] J. Lambek and P. J. Scott. *Introduction to Higher Order Categorical Logic*. Cambridge University Press, Cambridge, 1986.

[58] J. Lipton. Constructive Kripke semantics and realizability. In Y.N. Moschovakis, editor, *Logic from Computer Science*. Springer, 1990. Also as a Technical Report, from Cornell University, nr.90–71.

[59] J. Lipton and M.J. O'Donnell. Some intuitions behind realizability semantics for constructive logic: Tableaux and Läuchli countermodels. *Annals of Pure and Applied Logic*, 81:187–239, 1996.

[60] J. Longley. *Realizability Toposes and Language Semantics*. PhD thesis, Edinburgh University, 1995.

[61] S. Mac Lane. Stephen Cole Kleene – a reminiscence. *Annals of Pure and Applied Logic*, 81:3–7, 1996.

[62] Yu.T. Medvedev. Finite problems (Russian). *Doklady*, 142:1015–1018, 1962. Translation *SM* 3, pp. 227–230.

[63] M. Menni. A characterization of the left exact categories whose exact completions are toposes. Manuscript, submitted to Proceedings of 1999 meeting in Category theory in Coimbra, November 1999.

[64] I. Moerdijk. A model for intuitionistic nonstandard arithmetic. *Annals of Pure and Applied Logic*, 73:37–51, 1995.

[65] J.R. Moschovakis. Can there be no nonrecursive functions? *Journal of Symbolic Logic*, 36:309–315, 1971.

[66] D. Nelson. Recursive functions and intuitionistic number theory. *Transactions of the American Mathematical Society*, 61:307–368,556, 1947.

[67] W. Phoa. Effective domains and intrinsic structure. In J. Mitchell, editor, *Proceedings of the 5th Annual IEEE Symposium on Logic in Computer Science*, pages 366–377, Philadelphia, Pennsylvania, 1990. IEEE Computer Society Press.

[68] A.M. Pitts. *The Theory of Triposes*. PhD thesis, Cambridge University, 1981.

[69] V.E. Plisko. Absolute realizability of predicate formulas (Russian). *Izv. Akad. Nauk.*, 47:315–334, 1983. Translation *Math. Izv.* 22, pp. 291–308.

[70] G.R. Renardel de Lavalette. Extended bar induction in applicative theories. *Annals of Pure and Applied Logic*, 50:139–189, 1990.

[71] B. Reus and T. Streicher. General synthetic domain theory—a logical approach. *Mathematical Structures in Computer Science*, 9:177–223, 1999.

[72] E.P. Robinson. How complete is PER? In A.R. Meyer, editor, *Proceedings of the 4th Annual IEEE Symposium on Logic in Computer Science*, pages 106–111, Asilomar, 1989. IEEE Computer Society Press.

[73] E.P. Robinson and G. Rosolini. Colimit completions and the effective topos. *Journal of Symbolic Logic*, 55:678–699, 1990.

[74] G.F. Rose. Propositional calculus and realizibility. *Transactions of the American Mathematical Society*, 75:1–19, 1953.

[75] J. Rosický. Cartesian closed exact completions. Available from the Hypatia Electronic Library: `http://hypatia.dcs.qmw.ac.uk`, 1997.

[76] G. Rosolini. *Continuity and Effectiveness in Topoi.* PhD thesis, University of Oxford, 1986.

[77] G. Rosolini. About modest sets. *International Journal of Foundations of Computer Science*, 1:341–353, 1990.

[78] D.S. Scott. Identity and existence in intuitionistic logic. In M. Fourman, C.J. Mulvey, and D.S. Scott, editors, *Applications of Sheaves*, pages 660–696, Berlin, 1979. Springer (LNM 753).

[79] D.S. Scott. Church's thesis and a unification of types, 1986. Lecture at the Conference on Church's Thesis: Fifty Years Later.

[80] D.S. Scott. A new category? Domains, spaces and equivalence relations. Manuscript, 1996.

[81] J. Staples. Combinator realizability of constructive finite type analysis. In A.R.D. Mathias and H. Rogers, editors, *Cambridge Summer School in Mathematical Logic*, pages 253–273. Springer, 1973.

[82] T. Streicher. Investigations into intensional type theory. Habilitationsschrift, Universität München, 1994.

[83] T. Streicher. A topos for computable analysis, 1997. Note of a talk presented at PSSL, May 1997 in Utrecht, Holland. Available electronially at `www.mathematik.uni-darmstadt.de/ streicher`.

[84] M.D.G. Swaen. *Weak and Strong Sum-Elimination in Intuitionistic Type Theory.* PhD thesis, Universiteit van Amsterdam, 1989.

[85] P. Taylor. The fixed point property in synthetic domain theory. In *6th Symp. on Logic in Computer Science*, pages 152–160. IEEE Computer Society Press, 1991.

[86] A.S. Troelstra. Notions of realizability for intuitionistic arithmetic and intuitionistic arithmetic in all finite types. In J.E. Fenstad, editor, *The Second Scandinavian Logic Symposium*, pages 369–405. North-Holland, 1971.

[87] A.S. Troelstra, editor. *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis.* Springer, 1973. With contributions by A.S. Troelstra, C.A. Smoryński, J.I. Zucker and W.A. Howard.

[88] A.S. Troelstra. Realizability. In S.R. Buss, editor, *Handbook of Proof Theory*, pages 407–473. North-Holland, 1998.

[89] A.S. Troelstra and D. van Dalen. *Constructivism in Mathematics.* North-Holland, 1988. 2 volumes.

[90] J. van Oosten. *Exercises in Realizability.* PhD thesis, Universiteit van Amsterdam, 1991.

[91] J. van Oosten. A semantical proof of De Jongh's theorem. *Archive for Mathematical Logic*, pages 105–114, 1991.

[92] J. van Oosten. Axiomatizing higher-order Kleene realizability. *Annals of Pure and Applied Logic*, 70:87–111, 1994.

[93] J. van Oosten. Extensional realizability. *Annals of Pure and Applied Logic*, 84:317–349, 1997.

[94] J. van Oosten. The modified realizability topos. *Journal of Pure and Applied Algebra*, 116:273–289, 1997.

[95] J. van Oosten and A.K. Simpson. Axioms and (counter)examples in synthetic domain theory. Technical Report 1080, Department of Mathematics, Utrecht University, 1998.

[96] G. Wraith. Artin glueing. *Journal of Pure and Applied Algebra*, 4:345–348, 1974.