

# Singular Reduction for Proper Actions

R. Cushman<sup>1</sup> and J. Śniatycki<sup>2</sup>

## 1 Introduction

The goal of this paper is to give a simplified proof of the singular reduction theorem for a proper Hamiltonian action of a Lie group  $G$  on a connected smooth symplectic manifold  $(P, \omega)$  which has a coadjoint equivariant momentum mapping. Our proof does not use the assumption that the coadjoint orbit is locally closed, any local normal form, or the shifting trick. These were essential ingredients of the existing proof [4].

The space  $P/G$  of orbits of the symmetry group  $G$  is a topological quotient space of the original phase space  $P$ . We show that  $G$ -invariant smooth functions on  $P$  define a differential structure on  $P/G$  in the sense of Sikorski [12]. Singular reduction gives rise to a partition of  $P/G$  by symplectic manifolds.

We begin by describing regular reduction [9], [10]. Let

$$\Phi : G \times P \rightarrow P : (g, p) \mapsto \Phi_g(p) = g \cdot p$$

be a proper Hamiltonian action of a Lie group  $G$  on a connected symplectic manifold  $(P, \omega)$  with coadjoint equivariant momentum mapping  $J : P \rightarrow \mathcal{G}^*$ . If the action is free, the momentum map  $J$  is a submersion [3], see note A. Thus every  $\alpha \in \mathcal{G}^*$  is a regular value of  $J$ . The level set  $J^{-1}(\alpha)$  is a presymplectic submanifold of  $P$ . Let  $G_\alpha$  be the isotropy group of the coadjoint action at  $\alpha$ . Then the space  $J^{-1}(\alpha)/G_\alpha$  of  $G_\alpha$ -orbits on  $J^{-1}(\alpha)$  is a symplectic manifold [10].

Let  $\mathcal{O}_\alpha$  be the coadjoint orbit through  $\alpha$ . On the one hand, if  $\mathcal{O}_\alpha$  is not locally closed, then  $J^{-1}(\mathcal{O}_\alpha)$  is an immersed submanifold of  $P$ , see note

---

<sup>1</sup>Mathematics Institute, University of Utrecht, Utrecht, the Netherlands

<sup>2</sup>Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada

<sup>3</sup>preprint # 1133 University of Utrecht, 10 February 1999.

B. Moreover, there is a natural bijection between the immersed submanifold  $J^{-1}(\mathcal{O}_\alpha)/G$  of the smooth orbit space  $P/G$  and  $J^{-1}(\alpha)/G_\alpha$ , see notes C and D. On the other hand, if  $\mathcal{O}_\alpha$  is locally closed then  $J^{-1}(\mathcal{O}_\alpha)$  is a submanifold of  $P$ , see note E, and the space  $J^{-1}(\mathcal{O}_\alpha)/G$  of  $G$ -orbits on  $J^{-1}(\mathcal{O}_\alpha)$  is a symplectic manifold, which is naturally diffeomorphic to  $J^{-1}(\alpha)/G_\alpha$ .

If the momentum map  $J : P \rightarrow \mathcal{G}^*$  is not coadjoint equivariant, then it is equivariant with respect to an affine coadjoint action on  $\mathcal{G}^*$ , which is defined as follows. For each  $p \in P$  the map

$$\tilde{\sigma}_p : G \rightarrow \mathcal{G}^* : g \rightarrow J(\Phi_g(p)) - \text{Ad}_{g^{-1}}^t J(p)$$

does not depend on the choice of the point  $p$ . Thus it defines a mapping  $\sigma : G \rightarrow \mathcal{G}^*$  which is a  $\mathcal{G}^*$ -cocycle, that is, for every  $g, h \in G$

$$\sigma(gh) = \sigma(g) + \text{Ad}_{g^{-1}}^t \sigma(h).$$

Let

$$A : G \times \mathcal{G}^* \rightarrow \mathcal{G}^* : (g, h) \rightarrow \text{Ad}_{g^{-1}}^t h + \sigma(g). \quad (1)$$

Then  $A$  is an action of  $G$  on  $\mathcal{G}^*$  called the *affine coadjoint action*. The momentum mapping  $J$  is equivariant with respect to the action  $A$ , that is, for every  $(g, p) \in G \times P$

$$J(\Phi_g(p)) = A_g(J(p)).$$

The regular reduction theorem also holds when the momentum mapping is equivariant with respect to the affine coadjoint action [8].

All the results for free proper actions quoted above do not require the hypothesis that  $G$  is connected.

We now turn to discussing singular reduction. Since the Hamiltonian  $G$ -action  $\Phi$  on  $(P, \omega)$  is not necessarily free, the isotropy groups

$$G_p = \{g \in G \mid \Phi(g, p) = p\}$$

at the point  $p \in P$  play an essential role. Because the  $G$ -action is proper,  $G_p$  is a compact subgroup of  $G$  for each  $p \in P$ . Let  $K$  be a compact subgroup of  $G$ . The set of points of  $P$  of *symmetry type*  $K$  is

$$P_K = \{p \in P \mid G_p = K\}$$

and the set of points of  $P$  of orbit type  $K$  is

$$P_{(K)} = \{p \in P \mid G_p \text{ is conjugate to } K\}.$$

For nonfree proper actions, reduction of the zero level set of a coadjoint equivariant momentum map has been extensively studied [3], [14], [5] and [15]. Their results can be extended to cover reduction of a nonzero level set using the “shifting trick”, [1], [7]. In the above cited papers the authors show that for each compact subgroup  $K$  of  $G$ , each connected component of  $P_{(K)} \cap J^{-1}(0)$  is a presymplectic submanifold of  $P$  and that each connected component of the space  $(P_{(K)} \cap J^{-1}(0))/G$  of  $G$ -orbits on  $P_{(K)} \cap J^{-1}(0)$  is a symplectic manifold. The stratification of  $J^{-1}(0)/G$  by the symplectic manifolds  $(P_{(K)} \cap J^{-1}(0))/G$  was studied in [6], [14] and [4].

In this paper we give a simple proof that for each  $\alpha \in \mathcal{G}^*$  each connected component of  $P_{(K)} \cap J^{-1}(\alpha)$  is a presymplectic submanifold of  $P$  and its space of  $G_\alpha$ -orbits is a symplectic manifold. Our simplification is obtained by a detailed investigation of  $J^{-1}(\mathcal{O}_\alpha) \cap P_K$ , which reduces our argument to the case of a free action.

## 2 Symmetry type

**Theorem 1.** Let  $K$  be a compact subgroup of  $G$ ,  $M$  a connected component of  $P_K$ , and  $\iota_M : M \rightarrow P$  the inclusion map. Then

- i)  $M$  is a submanifold of  $P$  and  $\omega_M = \iota_M^* \omega$  is a symplectic form on  $M$ .
- ii) For each  $G$ -invariant function  $f$  on  $P$ , the flow  $\varphi_t$  of the Hamiltonian vector field  $X_f$  associated to  $f$  preserves  $M$ .
- iii) When  $f$  is a  $G$ -invariant function on  $P$ , the restriction to  $(M, \omega)$  of the Hamiltonian vector field  $X_f$  is a Hamiltonian vector field on  $(M, \omega_M)$  associated to the restriction of  $f$  to  $M$ .

**Proof.**

- i. The proof of i) can be found in [7], [5].
- ii. Since  $f$  is  $G$ -invariant,  $g \cdot \varphi_t(p) = \varphi_t(g \cdot p)$  for all  $g \in G$ , and  $p \in P$ . Hence if  $g \in G_p$ , then  $g \in G_{\varphi_t(p)}$ . Since  $\varphi_t$  is a local diffeomorphism, we find that, if

$g \in G_{\varphi_t(p)}$ , then  $g \in G_{\varphi_t^{-1}(\varphi_t(p))} = G_p$ . Hence  $G_{\varphi_t(p)} = G_p$  and  $\varphi_t(p) \in P_K$  for all  $p \in M$ . Since  $\varphi_t(p)$  and  $p$  are in the same connected component of  $P_K$ , it follows that  $\varphi_t(p) \in M$  for all  $p \in M$ . This proves ii.

iii. Since  $M$  is a symplectic submanifold of  $P$  for each  $p \in M$ , the symplectic annihilator  $T_p^\omega M$  of  $T_p M$ , defined by

$$T_p^\omega M = \{u \in T_p P \mid \omega(p)(u, v) = 0 \quad \forall v \in T_p M\}, \quad (2)$$

is a symplectic subspace of  $T_p P$  complementary to  $T_p M$ , that is,

$$T_p P = T_p M \oplus T_p^\omega M. \quad (3)$$

The Hamiltonian flow  $\varphi_t$  of a  $G$ -invariant function  $f$  is generated by  $X_f$ , which satisfies the equation  $X_f \lrcorner \omega = df$ . Since  $\varphi_t$  preserves  $M$ ,  $X_f$  is tangent to  $M$ . Hence for every  $u \in T_p^\omega M$ ,

$$\langle df(p) | u \rangle = \omega(p)(X_f(p), u) = 0.$$

Therefore for every  $v \in T_p M$ ,  $(X_f \lrcorner \omega)v = \langle df | v \rangle$ , which implies that  $X_f \lrcorner \omega_M = df|_M$ . This proves iii.  $\square$

Given  $p \in M$ , for each  $k \in K$ ,  $\Phi(k, p) = \Phi_k(p) = p$ . Hence the tangent at  $p$  of  $\Phi_k$  defines an action  $\Psi$  of  $K$  on  $T_p P$ , which fixes every  $u \in T_p M$ . The normaliser of  $K$  in  $G$  is

$$N^K = \{g \in G \mid gKg^{-1} = K\}.$$

For every  $p \in P$ ,

$$\begin{aligned} G_{g \cdot p} &= \{g' \in G \mid g'g \cdot p = g \cdot p\} = \{g' \in G \mid g^{-1}g'g \cdot p = p\} \\ &= \{g' \in G \mid g^{-1}g'g \in G_p\} = \{g' \in G \mid g' \in gG_p g^{-1}\} = gG_p g^{-1}. \end{aligned}$$

Hence  $g \in G$  preserves  $P_K$  if and only if  $g \in N^K$ . Let  $N_M$  be the subgroup of  $N^K$  preserving the component  $M \subseteq P_K$ , that is,

$$N_M = \{g \in N^K \mid g \cdot p \in M \quad \forall p \in M\}.$$

Note that  $K$  is a normal subgroup of  $N_M$ . The subgroup  $N_M$  contains the connected component of the identity of  $N^K$  and is a closed subgroup of  $G$ . Let  $\mathcal{N}$  be the Lie algebra of  $N_M$ . For each  $\xi \in \mathcal{N}$  and each  $p \in M$ , we have

$$\exp(t\xi) \cdot p = \Phi(\exp(t\xi), p) = \Phi_p(\exp(t\xi)) \in M.$$

Hence  $X^\xi(p) = T_e \Phi_p(\xi) \in T_p M$ . For each  $k \in K$ , there exists  $k' \in K$  such that  $k \cdot \exp(t\xi) = \exp(t\xi) \cdot k'$ . Hence

$$\begin{aligned} \Phi_k(\Phi_p(\exp(t\xi))) &= \Phi_k(\exp(t\xi) \cdot p) = \Phi(k, \exp(t\xi) \cdot p) = \Phi(k \exp(t\xi), p) \\ &= \Phi(\exp(t\xi)k', p) = \Phi(\exp(t\xi), k' \cdot p) = \Phi(\exp(t\xi), p) \\ &= \Phi_p(\exp(t\xi)). \end{aligned}$$

Therefore

$$T_p \Phi_k(X^\xi(p)) = X^\xi(p) \quad \forall k \in K, \xi \in \mathcal{N}, \text{ and } p \in M. \quad (4)$$

The quotient group  $G_M = N_M/K$  is a Lie group which acts on  $M$  by

$$\Phi_M : G_M \times M \rightarrow M : ([g], p) \mapsto \Phi(g, p), \quad (5)$$

where  $[g] \in G_M$  is the coset containing  $g \in N_M$ .

**Theorem 2.** The action  $\Phi_M$  of  $G_M$  on  $M$  is free and proper.

**Proof.** The action  $\Phi_M$  is free by the construction of  $G_M$ . To prove properness we argue as follows. Suppose that the sequence  $\{p_n\}$  of points in  $M$  converges to  $p \in P_K$  and let  $\{[g_n]\}$  be a sequence of elements of  $G_M$  such that  $\Phi_M([g_n], p_n) \rightarrow p' \in M$ . Then  $\Phi(g_n, p_n) = \Phi_M([g_n], p_n) \rightarrow p'$ . By properness of the action of  $G$  on  $P$ , there is a subsequence  $\{g_{n_m}\}$  in  $N_M$  converging to  $g \in G$  such that  $\Phi(g, p) = p'$ . Since  $N_M$  is closed, the limit  $g$  lies in  $N_M$  and  $p \in M$ . Hence, the subsequence  $\{[g_{n_m}]\}$  converges to  $[g] \in G_M$  and  $\Phi_M([g], p) = p'$ . Thus the action  $\Phi_M$  is proper.  $\square$

Let  $\overline{P} = P/G$  be the space of  $G$ -orbits on  $P$  and let  $\pi : P \rightarrow \overline{P}$  be the orbit map.

**Corollary.** The space  $\overline{M} = M/G_M$  of  $G_M$ -orbits on  $M$  is a connected manifold. The space  $\pi(M)$  has the structure of a smooth manifold induced by the natural bijection  $\mu : \pi(M) \rightarrow \overline{M}$ .

**Proof.** Since the action of  $G_M$  on  $M$  is free and proper,  $\overline{M} = M/G_M$  is a smooth manifold. Let  $\pi_M : M \rightarrow \overline{M}$  be the  $G_M$ -orbit map. Since  $M$  is connected and  $\pi_M$  is continuous, it follows that  $\overline{M}$  is connected.  $\square$

For each  $p \in M$ ,  $\pi(p) = G \cdot p$  is the orbit of  $G$  through  $p$ . The intersection of  $G \cdot p$  with  $M$  is the unique  $G_M$ -orbit  $\pi_M(p) = G_M \cdot p$  through  $p$ . In other words,

$$\pi(p) \cap M = G \cdot p \cap M = G_M \cdot p = \pi_M(p).$$

Consequently, the map

$$\mu : \pi(M) \rightarrow \overline{M} : \pi(p) \rightarrow \pi_M(p),$$

is bijective. Moreover,  $\mu$  induces a manifold structure on  $\pi(M)$ .  $\square$

Recall that  $\mathcal{N}$  is the Lie algebra of  $N_M$ . For each  $\xi \in \mathcal{N}$ , the vector field  $X^\xi$  is tangent to  $M$ . For each  $p \in M$ , let

$$\mathcal{M}(p) = \left\{ \xi \in \mathcal{G} \mid X^\xi(p) \in T_p^\omega M \right\}, \quad (6)$$

where  $T_p^\omega M$  (2) is the symplectic annihilator of  $T_p M$ .

**Lemma 1.** For each  $p \in M$

$$\mathcal{N} + \mathcal{M}(p) = \mathcal{G}.$$

**Proof.** Since  $p$  is fixed by  $K$ , the action  $\Phi|(K \times P)$  of  $K$  on  $P$  induces a  $K$ -action  $\Psi$  on  $T_p P$ . The tangent space  $T_p M$  consists of vectors  $v \in T_p P$  which are invariant under this induced action. In other words,

$$T_p M = \left\{ v \in T_p P \mid \Psi_k(v) = T_p \Phi_k(v) = v \quad \forall k \in K \right\}.$$

Moreover, for every  $\xi \in \mathcal{N}$  we have  $X^\xi(p) \in T_p M$ .

Since  $\omega$  is  $G$ -invariant and  $T_p M$  is  $\Psi_k$ -invariant, it follows that  $T_p^\omega M$  is also  $\Psi_k$ -invariant. For every  $u \in T_p M$ , the average of  $u$  over  $K$  is

$$\bar{u} = \int_K \Psi_k(u) dk = \int_K T_p \Phi_k(u) dk, \quad (7)$$

where  $dk$  denotes Haar measure of  $K$  normalised so that  $\text{vol } K = 1$ . Since  $\bar{u}$  is  $\Psi_k$ -invariant, it belongs to  $T_p M$ . If  $u \in T_p^\omega M$ , then  $\bar{u} \in T_p^\omega M$  because  $T_p^\omega M$  is  $\Psi_k$ -invariant. Hence if  $u \in T_p^\omega M$ , it follows that  $\bar{u} \in T_p M \cap T_p^\omega M = \{0\}$ . Thus

$$T_p^\omega M = \left\{ u \in T_p P \mid \bar{u} = 0 \right\}. \quad (8)$$

For each  $\xi \in \mathcal{G}$ , let

$$\bar{\xi} = \int_K T_e L_k(\xi) dk,$$

where  $L_k : G \rightarrow G : g \rightarrow kg$  is left translation by  $k$ . Since the map

$$\mathcal{G} \rightarrow T_p P : \xi \rightarrow X^\xi(p)$$

is equivariant, that is,  $X^{T_e L_k \xi}(p) = T_p \Phi_k \xi$ , and has kernel  $\mathcal{K}$ , it follows that

$$\mathcal{M}(p) = \{\xi \in \mathcal{G} \mid \bar{\xi} \in \mathcal{K}\}.$$

For every  $\xi \in \mathcal{G}$ , we have  $\xi = \bar{\xi} + (\xi - \bar{\xi})$ , where  $\overline{(\xi - \bar{\xi})} = 0$ . Since  $T_e L_k \bar{\xi} = \bar{\xi}$  for all  $k \in K$ , it follows that  $T_p \Phi_k(X^{\bar{\xi}}(p)) = X^{\bar{\xi}}(p)$  for  $k \in K$ . So  $X^{\bar{\xi}}(p) \in T_p M$ , that is  $\bar{\xi} \in \mathcal{N}$ . Moreover  $(\xi - \bar{\xi}) = 0 \in \mathcal{K}$ , which implies that  $\xi - \bar{\xi} \in \mathcal{M}(p)$ . Hence  $\mathcal{G} = \mathcal{N} + \mathcal{M}(p)$ .  $\square$

### 3 Reduction

In this section we prove

**Theorem 3.** The action of  $G_M$  on  $(M, \omega_M)$  has a momentum map  $J_M : M \rightarrow \mathcal{G}_M^*$ , which is equivariant with respect to an affine coadjoint action

$$A : G_M \times \mathcal{G}_M^* \rightarrow \mathcal{G}_M^* : ([g], \phi) \mapsto A_{[g]}\phi.$$

For every  $G$ -coadjoint orbit  $\mathcal{O}_\alpha \subseteq \mathcal{G}^*$  with  $J^{-1}(\mathcal{O}_\alpha) \cap M \neq \emptyset$ , there exists an orbit  $\mathcal{O}_M$  of the action  $A$  such that

$$J^{-1}(\mathcal{O}_\alpha) \cap M = J_M^{-1}(\mathcal{O}_M).$$

**Proof:** Let  $\kappa : \mathcal{K} \rightarrow \mathcal{G}$ ,  $\mu : \mathcal{K} \rightarrow \mathcal{N}$ , and  $\nu : \mathcal{N} \rightarrow \mathcal{G}$  be inclusion mappings and  $\lambda : \mathcal{N} \rightarrow \mathcal{G}_M$  be the natural projection map. Their transposes are the mappings  $\kappa^* : \mathcal{G}^* \rightarrow \mathcal{K}^*$ ,  $\mu^* : \mathcal{N}^* \rightarrow \mathcal{K}^*$ ,  $\nu^* : \mathcal{G}^* \rightarrow \mathcal{N}^*$ , and  $\lambda^* : \mathcal{G}_M^* \rightarrow \mathcal{N}^*$ , respectively. Let  $J|_M : M \rightarrow \mathcal{G}^*$  be the restriction of  $J$  to  $M$ .

To prove the theorem we need the following two lemmas.

**Lemma 2.**  $\kappa^* \circ J|_M : M \rightarrow \mathcal{K}^*$  is constant.

**Proof.** For every  $\xi \in \mathcal{G}$ , we have  $X^\xi \lrcorner \omega = dJ_\xi$ . Moreover,  $\xi \in \mathcal{K}$  implies that  $X^\xi(p) = 0$  for all  $p \in M$ . Hence  $d(\kappa^* \circ J|_M) = \kappa^* \circ dJ|_M = 0$ , and  $\kappa^* \circ J|_M$  is constant on  $M$ .  $\square$

Since  $\mu^* : \mathcal{N}^* \rightarrow \mathcal{K}^*$  is onto and  $\kappa^* \circ J|_M : M \rightarrow \mathcal{K}^*$  is constant, there exists a constant map  $j_M : M \rightarrow \mathcal{N}^*$  such that

$$\mu^* \circ j_M = \kappa^* \circ J|_M.$$

**Lemma 3.** There exists a unique map  $J_M : M \rightarrow \mathcal{G}_M^*$  such that

$$\lambda^* \circ J_M = \nu^* \circ J|_M - j_M. \quad (9)$$

**Proof.** We have

$$\mu^* \circ (\nu^* \circ J|_M - j_M) = \kappa^* \circ J|_M - \kappa^* J|_M = 0.$$

The existence of a unique lift  $J_M : M \rightarrow \mathcal{G}_M^*$  of  $(\nu^* \circ J|_M - j_M) : M \rightarrow \mathcal{N}^*$  follows from the exactness of the sequence

$$0 \longrightarrow \mathcal{G}_M^* \xrightarrow{\lambda^*} \mathcal{N}^* \xrightarrow{\mu^*} \mathcal{K}^* \longrightarrow 0. \quad \square \quad (10)$$

Continuing with the proof of the first assertion in theorem 3, we now show that the map  $J_M : M \rightarrow \mathcal{G}_M^*$  is a momentum map for the action of  $G_M$  on  $M$ .

For each  $\xi \in \mathcal{N} \subseteq \mathcal{G}$ , the action of the one parameter subgroup  $\exp t\lambda(\xi)$  of  $G_M$  on  $M$  coincides with the action of the subgroup  $\exp t\xi$  of  $G$ . This latter action is generated by the Hamiltonian vector field  $X^\xi$  of  $J_\xi$  restricted to  $M$ . Hence

$$\begin{aligned} X^\xi \lrcorner \omega_M &= d\langle J|_M | \nu(\xi) \rangle = d\langle \nu^* \circ J|_M | \xi \rangle \\ &= d\langle \lambda^* \circ J_M + j_M | \xi \rangle = \langle d(\lambda^* \circ J_M) | \xi \rangle + \langle dj_M | \xi \rangle \\ &= d\langle J_M | \lambda(\xi) \rangle. \end{aligned}$$

Thus  $X^\xi$  is the Hamiltonian vector field of  $\langle J_M | \lambda(\xi) \rangle$ . Hence  $J_M$  is a momentum map for the action  $G_M$  on  $M$ .

We note that the momentum map  $J_M : M \rightarrow \mathcal{G}_M^*$  need not be coadjoint equivariant. However, there exists a  $\mathcal{G}_M^*$ -cocycle  $\sigma : G_M \rightarrow \mathcal{G}_M^*$  such that the map

$$A : G_M \times \mathcal{G}_M^* \rightarrow \mathcal{G}_M^* : ([g], \phi) \mapsto A_M([g], \phi) = \text{Ad}_{[g]}^t \phi + \sigma([g])$$

is an action of  $G_M$  on  $\mathcal{G}_M^*$  and

$$J_M([g] \cdot p) = A_{[g]}(J_M(p)).$$

This completes the proof of the first assertion in theorem 3.

We make a short digression to find an explicit expression for the cocycle  $\lambda^*\sigma$ , which will not be used in the remainder of the proof. Comparing equations (9) and (1) we see that for  $\xi \in \mathcal{N}$ ,

$$\begin{aligned} \langle \sigma([g]) \mid \lambda(\xi) \rangle &= \langle J_M([g] \cdot p) - \text{Ad}_{[g]}^* J_M(p) \mid \lambda(\xi) \rangle \\ &= \langle \text{Ad}_{g^{-1}}^* j_M(p) \mid \xi \rangle - \langle j_M(g \cdot p) \mid \xi \rangle = \langle j_M \mid \text{Ad}_{g^{-1}} \xi - \xi \rangle \\ &= \langle \text{Ad}_{g^{-1}}^* j_M - j_M \mid \xi \rangle. \end{aligned}$$

Hence

$$\lambda^*(\sigma([g])) = \text{Ad}_{g^{-1}}^* j_M - j_M. \quad (11)$$

We now turn to proving the second assertion of theorem 3. If  $p, p' \in J^{-1}(\mathcal{O}_\alpha) \cap M$  then  $J(p') = \text{Ad}_g^* J(p) = J(g \cdot p)$  for some  $g \in N_M$ . Since,  $g \cdot p = [g] \cdot p$ , where  $[g]$  is the coset of  $g$  in  $G_M = N_M/K$ , equation (9) yields

$$\begin{aligned} \lambda^* \circ J_M(p') &= \nu^* \circ J(p') - j_M = \nu^* \circ \text{Ad}_{g^{-1}}^* J(p) - j_M \\ &= \nu^* \circ J(g \cdot p) - j_M = (\lambda^* \circ J_M([g] \cdot p) + j_M) - j_M \\ &= \lambda^* \circ J_M([g] \cdot p) = \lambda^* \circ A_{[g]}(J_M(p)). \end{aligned}$$

Since  $\ker \lambda^* = 0$ , it follows that

$$J_M(p') = A_{[g]}(J_M(p)).$$

This implies that  $J_M(p')$  and  $J_M(p)$  are in the same orbit  $\mathcal{O}_M$  of the affine coadjoint action  $A$  of  $G_M$  on  $\mathcal{G}_M^*$ , that is,

$$J^{-1}(\mathcal{O}_\alpha) \cap M \subseteq J_M^{-1}(\mathcal{O}_M). \quad (12)$$

Conversely, if  $p, p' \in J_M^{-1}(\mathcal{O}_M)$ , then  $J_M(p) = A_{[g]}(J_M(p'))$  where  $g \in N_M$ . Therefore,

$$\nu^* \circ J(p) = \nu^* \circ \text{Ad}_{g^{-1}}^* J(p').$$

But  $\nu^* : \mathcal{G}^* \rightarrow \mathcal{N}^*$  is the transpose of the inclusion mapping  $\nu : \mathcal{N} \rightarrow \mathcal{G}$ . So

$$\ker \nu^* = \mathcal{N}^0 = \{ \alpha \in \mathcal{G}^* \mid \langle \alpha \mid \xi \rangle = 0 \quad \forall \xi \in \mathcal{N} \}.$$

This implies that

$$J(p) - \text{Ad}_{g^{-1}}^* J(p') \in \mathcal{N}^0.$$

Hence for every  $\xi \in \mathcal{N}$ , we have

$$\langle J(p) - \text{Ad}_{g^{-1}}^* J(p') \mid \xi \rangle = 0.$$

On the one hand, differentiating this equation in a direction  $u$  tangent to  $J_M^{-1}(\mathcal{O}_M)$  at  $p$ , we get

$$\langle T_p \left( J(p) - \text{Ad}_{g^{-1}}^* J(p') \right) (u) \mid \xi \rangle = 0 \quad (13)$$

for every  $u \in T_p J_M^{-1}(\mathcal{O}_M)$  and every  $\xi \in \mathcal{N}$ . On the other hand, from (6) we see that  $X^\xi(p) \in T_p^\omega M$  for  $\xi \in \mathcal{M}(p)$ . But

$$\langle T_p J(u) \mid \xi \rangle = dJ^\xi(p)u = \omega_M(p)(X^\xi(p), u) = 0,$$

for all  $u \in T_p J_M^{-1}(\mathcal{O}_M)$  and  $\xi \in \mathcal{M}(p)$ . Therefore

$$\langle T_p \left( J(p) - \text{Ad}_{g^{-1}}^* J(p') \right) (u) \mid \xi \rangle = 0 \quad (14)$$

for every  $u \in T_p J_M^{-1}(\mathcal{O}_M)$  and every  $\xi \in \mathcal{M}(p)$ . Since  $\mathcal{N} + \mathcal{M}(p) = \mathcal{G}$ , equations (13) and (14) imply that  $J(p) - \text{Ad}_{g^{-1}}^* J(p')$  is independent of  $p \in J_M^{-1}(\mathcal{O}_M)$ . Moreover,  $g \in N_M$  implies that  $g \cdot p' = [g] \cdot p' \in J_M^{-1}(\mathcal{O}_M)$ . Hence taking  $p = g \cdot p'$ , we get

$$J(p) - \text{Ad}_{g^{-1}}^* J(p') = J(g \cdot p') - \text{Ad}_{g^{-1}}^* J(p') = 0,$$

because  $J$  is coadjoint equivariant. Thus  $J(p)$  and  $J(p')$  are in the same coadjoint orbit  $\mathcal{O}_\alpha$ . Therefore,

$$J_M^{-1}(\mathcal{O}_M) \subseteq J^{-1}(\mathcal{O}_\alpha) \cap M.$$

Taking into account inclusion (12) we obtain  $J_M^{-1}(\mathcal{O}_M) = J^{-1}(\mathcal{O}_\alpha) \cap M$ . This completes the proof of theorem 3.  $\square$

**Theorem 4.** If  $J^{-1}(\mathcal{O}_\alpha) \cap M \neq \emptyset$ , then its projection  $(M \cap J^{-1}(\mathcal{O}_\alpha)) / N_M$  to  $\overline{M}$  is a symplectic manifold.

**Proof.** Let  $\beta = J_M(p)$  for some  $p \in M \cap J^{-1}(\mathcal{O}_\alpha)$ . Then

$$\begin{aligned} (M \cap J^{-1}(\mathcal{O}_\alpha)) / N_M &= (M \cap J^{-1}(\mathcal{O}_\alpha)) / G_M \\ &= J_M^{-1}(\mathcal{O}_M) / G_M = J_M^{-1}(\beta) / G_{M_\beta}, \end{aligned}$$

where  $G_{M_\beta}$  is the isotropy group of  $\beta$  in  $G_M$ . But  $J_M^{-1}(\beta) / G_{M_\beta}$  is a symplectic manifold using the regular reduction theorem.  $\square$

## 4 Differential spaces

Let  $K$  be a compact subgroup of  $G$ . Our symplectic manifold  $(P, \omega)$  is partitioned by connected components  $M$  of the manifold  $P_K$  of symmetry type  $K$  and by connected components  $L$  of the manifold  $P_{(K)}$  of orbit type  $K$ . The partition of  $P$  by orbit type, given by

$$P = \bigcup_{K \prec G} P_{(K)} = \bigcup_{K \prec G} \coprod_{L \text{ c.c. } P_{(K)}} L,$$

is  $G$ -invariant. Here  $\prec$  means compact subgroup of and c.c. means connected component of. It induces a partition

$$\bar{P} = \bigcup_{K \prec G} \coprod_{L \text{ c.c. } P_{(K)}} \bar{L}$$

of the orbit space  $\bar{P}$ .

For each  $\alpha \in \mathcal{G}^*$ , the level set  $J^{-1}(\alpha)$  is invariant under the motion of a Hamiltonian vector field associated to a  $G$ -invariant Hamiltonian. Hence we have a further partition of  $P$

$$P = \bigcup_{K \prec G} \bigcup_{\alpha \in \mathcal{G}^*} \coprod_{M \text{ c.c. } P_K} J^{-1}(\alpha) \cap M$$

by subsets which are invariant under the motion. However,  $J^{-1}(\alpha)$  is not  $G$ -invariant. Since the momentum map  $J : P \rightarrow \mathcal{G}^*$  is  $G$ -equivariant,  $G \cdot J^{-1}(\alpha) = J^{-1}(\mathcal{O}_\alpha)$ , where  $\mathcal{O}_\alpha$  is the coadjoint orbit through  $\alpha$ . Hence a  $G$ -invariant partition of  $P$  is

$$P = \bigcup_{K \prec G} \bigcup_{\mathcal{O}_\alpha \subseteq \mathcal{G}^*} \coprod_{L \text{ c.c. } P_{(K)}} (L \cap J^{-1}(\mathcal{O}_\alpha)).$$

Our aim in this section is to analyze this partition and the induced partition of  $\bar{P} = P/G$  given by

$$\bar{P} = \bigcup_{K \prec G} \bigcup_{\mathcal{O}_\alpha \subseteq \mathcal{G}^*} \coprod_{M \text{ c.c. } P_K} (M \cap J^{-1}(\mathcal{O}_\alpha)) / G_M.$$

Since coadjoint orbits need not be even locally closed [11, p. 512], the differential structure of the partition of  $P$  by the inverse image  $J^{-1}(\mathcal{O}_\alpha)$  of

coadjoint orbits  $\mathcal{O}_\alpha \subseteq \mathcal{G}^*$  cannot be described in terms of Whitney smooth functions. A generalization of Whitney's theory, which is adequate for this purpose, is Sikorski's theory of differential spaces [12], [13].

A *differential structure* on a topological space  $Q$  is a set  $C^\infty(Q)$  of continuous functions on  $Q$  which have the following properties:

- I. The topology of  $Q$  is generated by the functions in  $C^\infty(Q)$ , that is, the collection

$$\{f^{-1}(U) \mid f \in C^\infty(Q) \text{ where } U \text{ is an open subset of } \mathbf{R}\}$$

is a subbasis for the topology of  $Q$ .

- II. For every  $F \in C^\infty(\mathbf{R}^n)$  and every  $f_1, \dots, f_n \in C^\infty(Q)$ ,  $F(f_1, \dots, f_n) \in C^\infty(Q)$ .
- III. If  $f : Q \rightarrow \mathbf{R}$  is a function such that for every  $p \in Q$  there is an open neighbourhood  $U$  of  $p$  in  $Q$  and a function  $f_U \in C^\infty(Q)$  satisfying  $f|_U = f_U|_U$ , then  $f \in C^\infty(Q)$ .

A topological space  $Q$  endowed with a differential structure  $C^\infty(Q)$  is called a *differential space* [12, sec. 6]. An element of  $C^\infty(Q)$  is called a *smooth function*. Thus  $C^\infty(Q)$  is the set of smooth functions on  $Q$ . From property II it follows that  $C^\infty(Q)$  is a commutative ring under addition and pointwise multiplication.

**Example 1.** If  $Q$  is a smooth manifold, then the collection of smooth functions on  $P$ , defined in terms of the manifold structure of  $P$ , is a differential structure on  $P$ .

**Proof.** To verify property I we need only show that given a  $q \in Q$  and an open neighborhood  $U$  of  $q$  in  $Q$ , then there is a smooth function  $f$  on  $Q$  such that  $f^{-1}(0, 1)$  is an open neighborhood of  $q$  contained in  $U$ . We may assume that  $U$  is contained in the domain of a chart  $(V, \varphi)$  of  $Q$  and that the closure  $\overline{U}$  of  $U$  is compact. Then there is a nonnegative smooth function  $g$  on  $\mathbf{R}^n$  whose support is contained in  $\varphi(\overline{U})$  and whose range is contained in  $[0, \frac{1}{2}]$ . The smooth function  $f = g \circ \varphi$  has the desired property.

Property II is obvious.

To prove III suppose that  $f : Q \rightarrow \mathbf{R}$  has the property that for each  $q \in Q$  there is an open neighborhood  $U = U_q$  of  $q$  in  $Q$  and a smooth function  $F_U$  such that  $f|_U = F_U|_U$ . Shrinking  $U$  if necessary, we may assume that  $U$  is contained in a chart domain  $V_q$  containing  $q$  of the manifold  $Q$ . Since  $\{V_q\}_{q \in Q}$  cover  $Q$  and  $f|_{V_q}$  is smooth, it follows that  $f$  is smooth.  $\square$

**Example 2.** Let  $\Phi : G \times P \rightarrow P$  be a proper action of a Lie group  $G$  on a smooth manifold  $P$ . Let  $\overline{P} = P/G$  be the space of  $G$ -orbits on  $P$  (which is not necessarily a smooth manifold) and let  $\pi : P \rightarrow \overline{P}$  be the orbit map. We say that the function  $\overline{f} : \overline{P} \rightarrow \mathbf{R}$  is smooth if there is a smooth  $G$ -invariant function on  $P$  such that  $\overline{f} \circ \pi = f$ . In other words, if  $f : P \rightarrow \mathbf{R}$  is a smooth  $G$ -invariant function on  $P$ , then the induced function  $\overline{f} : \overline{P} \rightarrow \mathbf{R}$  is smooth. Hence the set  $C^\infty(\overline{P})$  of smooth functions on  $\overline{P}$  is induced from the space  $C^\infty(P)^G$  of smooth  $G$ -invariant functions on  $P$ . The pair  $(C^\infty(\overline{P}), \overline{P})$  is a differential space.

**Proof.** Property I. It suffices to show that given  $\overline{p} \in \overline{P}$  and an open neighborhood  $\overline{U}$  of  $\overline{p}$  in  $\overline{P}$ , there is a smooth function  $\overline{f}$  on  $\overline{P}$  such that  $\overline{f}^{-1}(0,1)$  is an open neighborhood of  $\overline{p}$  contained in  $\overline{U}$ . Let  $p \in \pi^{-1}(\overline{p})$  and let  $S_p$  be a slice to the  $G$ -action on  $P$  at  $p$ . Then  $V = S_p \cap \pi^{-1}(\overline{U})$  is an open neighborhood of  $p$  in  $S_p$ . There is a smooth  $G_p$ -invariant nonnegative function  $\tilde{f}$  on  $S_p$  whose support is a compact subset contained in  $V$  which contains  $p$  and whose range is contained in  $[0, \frac{1}{2})$ . Define the function  $f$  by  $f(\Phi_g(v)) = \tilde{f}(v)$  for every  $g \in G$  and every  $v \in V$ . Then  $f$  is a smooth  $G$ -invariant function on  $P$  with support contained in  $G \cdot V$  and whose range is contained in  $[0, \frac{1}{2}]$ . Thus  $f$  induces a smooth function  $\overline{f}$  on  $\overline{P}$  such that  $\overline{f}^{-1}(0,1)$  is an open subset of  $\overline{U}$  containing  $\overline{p}$ .

Property II follows immediately from the fact that property II holds for  $C^\infty(P)^G$ .

We now prove property III. Let  $\overline{f} : \overline{P} \rightarrow \mathbf{R}$  be a function such that for each  $\overline{p} \in \overline{P}$  there is an open neighborhood  $\overline{U}$  of  $\overline{p}$  in  $\overline{P}$  and a smooth function  $\overline{f}_{\overline{U}}$  on  $\overline{P}$  so that  $\overline{f}|_{\overline{U}} = \overline{f}_{\overline{U}}|_{\overline{U}}$ . Now  $\pi^*\overline{f} : P \rightarrow \mathbf{R}$  is  $G$ -invariant and

$$\pi^*\overline{f}|_{\pi^{-1}(\overline{U})} = \pi^*\overline{f}_{\overline{U}}|_{\pi^{-1}(\overline{U})}.$$

But  $\pi^*\overline{f}_{\overline{U}} \in C^\infty(P)^G$ . Hence  $\pi^*\overline{f} \in C^\infty(P)^G$ , which implies that  $\overline{f} \in C^\infty(\overline{P})$ .  $\square$

Suppose that  $(C^\infty(Q), Q)$  is a differential space and that  $M$  is a *subset* of  $Q$ . Then we can define a differential structure on  $M$  as follows. We say that a function  $f : M \rightarrow \mathbf{R}$  is *smooth* on  $M$  if for every  $m \in M$  there is an open neighborhood  $U$  of  $m$  in  $Q$  and a function  $f_U \in C^\infty(Q)$  such that  $f|(M \cap U) = f_U|(M \cap U)$ .

**Proposition 1.** The set  $C^\infty(M)$  of smooth functions on  $M$  is a differential structure.

**Proof.**

Property I. Let  $m \in M$  and suppose that  $U$  is an open neighborhood of  $m$  in  $Q$ . Then there is an open set  $V$  in  $\mathbf{R}$  and a smooth function  $f$  on  $Q$  such that  $f^{-1}(V)$  is an open neighborhood  $\mathcal{U}$  of  $m$  in  $Q$  contained in  $U$ . Now  $f|M$  is a smooth function on  $M$  because for every  $p \in M \subseteq Q$  there is an open neighborhood  $\tilde{U}$  of  $p$  in  $Q$  and a smooth function  $\tilde{f}$  on  $Q$  such that  $\tilde{f}|_{\tilde{U}} = f|_{\tilde{U}}$ . But  $f|M = f|_{\tilde{U}}|_{M \cap \tilde{U}}$ . Hence  $f|(M \cap \tilde{U}) = \tilde{f}|_{\tilde{U}}|(M \cap \tilde{U})$ . Moreover,  $(f|M)^{-1}(V) = \mathcal{U} \cap M$ , which is an open neighborhood of  $m$  in the induced topology on  $M$  contained in  $M \cap U$ .

Property II follows immediately from the fact that property II holds for  $C^\infty(Q)$ .

Property III follows from the definition of smooth function on  $M$ . □

We say that the differential structure on  $M$  given in proposition 1 is *inherited from* the differential structure on  $Q$ . *Locally* a smooth function on  $M$  in the differential structure on  $M$  inherited from the differential structure on  $Q$  is a restriction to  $M$  of a smooth function on  $Q$ .

**Proposition 2.** If  $M$  is a closed subset of a smooth paracompact manifold  $Q$  then every smooth function on  $M$  extends to a smooth function on  $Q$ .

**Proof.** Let  $f \in C^\infty(M)$  and  $\{U_p \mid p \in M\}$  the covering of  $M$  by opens sets in  $Q$  such that for each  $p \in M$ ,  $U_p \ni p$  and there exists  $f_{U_p} \in C^\infty(Q)$  satisfying  $f_{U_p}|_{U_p \cap M} = f|_{U_p \cap M}$ . Since  $M$  is closed in  $Q$ , its complement  $M'$  is open in  $Q$  and the family  $\{U_p \mid p \in M\} \cup M'$  is an open covering of  $Q$ . Let  $\{\varphi_\alpha\}$  be a partition of unity subordinate to this covering. Each  $\varphi_\alpha \in C^\infty(Q)$  has support in some  $U_{p_\alpha}$  or in  $M'$ . Moreover  $\sum_\alpha \varphi_\alpha = 1$ . Let  $g = \sum_\alpha \varphi_\alpha f_{U_{p_\alpha}}$ , where the sum is taken over  $\alpha$  such that the support of  $\varphi_\alpha$  has nonempty intersection with  $M$ . Clearly,  $g \in C^\infty(Q)$ . Since  $M' \cap M = \emptyset$ , it follows that  $g|M = f$ . □

If  $M$  is not closed in  $Q$ , and  $\{p_n\}$  is a sequence of points in  $M$  converging to  $p \notin M$ , then we can construct a smooth function  $f$  on  $M$  such that  $f(p_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Hence  $f$  cannot be a restriction to  $M$  of a function on  $C^\infty(Q)$ .

**Corollary.** If  $M$  is a closed submanifold of  $Q$ , then the differential structure on  $M$  induced by restriction is the space of smooth functions using the manifold structure of  $Q$ .

**Proof.** This is a special case of proposition 2 because a submanifold  $M$  of  $Q$  is a closed subset of  $Q$ . Moreover the differential structure induced by restriction is just the set of smooth functions using the manifold structure of  $M$ .  $\square$

## 5 Reduced Poisson algebra

As before, let  $(P, \omega)$  be a symplectic manifold with  $\Phi : G \times P \rightarrow P$  a proper Hamiltonian action of a Lie group  $G$  on  $P$ . Let  $\overline{P} = P/G$  the space of  $G$ -orbits with orbit map  $\pi : P \rightarrow \overline{P}$ .

In example 2 we have shown that the space  $C^\infty(\overline{P})$  of all functions on  $\overline{P}$  which pull back under the  $G$ -orbit map  $\pi$  to a smooth  $G$ -invariant function on  $P$  is a differential structure on  $\overline{P}$ . In fact,

**Proposition 3.**  $(C^\infty(\overline{P}), \{, \}_{\overline{P}}, \cdot)$  is a Poisson algebra.

**Proof.** Here  $\cdot$  is pointwise multiplication of smooth functions and  $\{, \}_{\overline{P}}$  is a Poisson bracket on  $C^\infty(\overline{P})$ , which is defined as follows. Let  $\overline{f}, \overline{h} \in C^\infty(\overline{P})$ . At each  $p \in P$  let

$$\{\overline{f}, \overline{h}\}_{\overline{P}}(\pi(p)) = \{f, h\}(p),$$

where  $\pi^*\overline{f} = f$ ,  $\pi^*\overline{h} = h$  with  $f, h \in C^\infty(P)^G$ . Moreover,  $\{, \}$  is the usual Poisson bracket on the space of smooth functions on the symplectic manifold  $(P, \omega)$ . To see that the Poisson bracket  $\{, \}_{\overline{P}}$  is well defined, suppose that  $\tilde{f}$  is another smooth  $G$ -invariant function on  $P$  which induces the function  $\overline{f}$  on  $\overline{P}$ . Then

$$0 = \pi^*\overline{f} - \pi^*\tilde{f} = f - \tilde{f}$$

on  $P$ , since  $\pi$  is surjective. Hence  $\{f, h\} = \{\tilde{f}, h\}$ , which implies that  $\{\overline{f}, \overline{h}\}_{\overline{P}}$  does not depend on the choice of representative of  $\overline{f}$ . Since  $\{, \}_{\overline{P}}$  is skew

symmetric, the same argument shows that  $\{\bar{f}, \bar{h}\}_{\bar{P}}$  does not depend on the choice of representative of  $\bar{h}$  either. Hence  $\{, \}_{\bar{P}}$  is well defined.

From the fact that  $(C^\infty(P)^G, \{, \}, \cdot)$  is a Poisson algebra [5, p. 347 (5.1)], it follows that  $(C^\infty(\bar{P}), \{, \}_{\bar{P}}, \cdot)$  is a Poisson algebra.  $\square$

Let  $\mathcal{O}_\alpha \subseteq \mathcal{G}^*$  be a  $G$ -coadjoint orbit and let  $M$  be a connected component of the manifold  $P_K$  of points of  $P$  of symmetry type  $K$ . Furthermore, suppose that  $P_{M,\alpha} = M \cap J^{-1}(\mathcal{O}_\alpha)$  is nonempty. From theorem 3 we know that  $P_{M,\alpha} = J^{-1}(\mathcal{O}_M)$  for some  $G_M$ -affine coadjoint orbit  $\mathcal{O}_M$ . Thus it follows from note B, that  $P_{M,\alpha}$  is an immersed submanifold of the symplectic manifold  $(M, \omega_M)$ , (see theorem 1). In addition,  $P_{M,\alpha}$  is invariant under the flow of every  $G_M$ -invariant Hamiltonian vector field on  $(M, \omega_M)$  [5, p. 343 (4.10)]. Consider the space  $C^\infty(P_{M,\alpha})^{G_M}$  of smooth  $G_M$ -invariant functions on  $P_{M,\alpha}$  inherited from the space  $C^\infty(M)^{G_M}$  defined by the manifold structure of  $M$ . Then  $C^\infty(P_{M,\alpha})^{G_M}$  is a differential structure on  $P_{M,\alpha}$ .

Let  $\bar{P}_{M,\alpha}$  be the space of  $G_M$ -orbits on  $P_{M,\alpha}$ . Since a smooth function on  $\bar{P}_{M,\alpha}$  is induced by a smooth  $G_M$ -invariant function on  $P_{M,\alpha}$ , we see that  $C^\infty(\bar{P}_{M,\alpha})$  is a differential structure on  $\bar{P}_{M,\alpha}$ . Define a Poisson bracket  $\{, \}_{\bar{P}_{M,\alpha}}$  on  $C^\infty(\bar{P}_{M,\alpha})$  as follows, compare with [2]. For each  $\bar{f}, \bar{h} \in C^\infty(\bar{P}_{M,\alpha})$  and each  $p \in P_{M,\alpha}$  let

$$\{\bar{f}, \bar{h}\}_{\bar{P}_{M,\alpha}}(\rho(p)) = \{f, h\}_M(p), \quad (15)$$

where  $\{, \}_M$  is the usual Poisson bracket on  $(M, \omega_M)$ ,  $f, h \in C^\infty(P_{M,\alpha})^{G_M}$  such that  $\rho^* \bar{f} = f, \rho^* \bar{h} = h$ , and  $\rho$  is the restriction of the  $G_M$ -orbit map  $\pi$  to  $P_{M,\alpha}$ . Note that the right hand side of (15) is well defined, because locally every smooth  $G_M$ -invariant function on  $P_{M,\alpha}$  is the restriction of a smooth  $G_M$ -invariant function on  $M$ .

**Proposition 4.** The Poisson bracket  $\{, \}_{\bar{P}_{M,\alpha}}$  is well defined.

**Proof.** Suppose that  $\tilde{f} \in C^\infty(P_{M,\alpha})^{G_M}$  induces the function  $\bar{f}$  on  $\bar{P}_{M,\alpha}$ . Then

$$\tilde{f} - f = \rho^* \bar{f} - \rho^* \bar{f} = 0$$

on  $\bar{P}_{M,\alpha}$ . Hence for each  $p \in P_{M,\alpha}$ , we have

$$\{\tilde{f} - f, h\}_M(p) = -d(\tilde{f} - f)(p)X_h(p) = -L_{X_h}0 = 0,$$

where the last equality follows from the fact the  $P_{M,\alpha}$  is invariant under the flow of every  $G_M$ -invariant Hamiltonian vector field on  $M$ . Hence  $\{\bar{f}, \bar{h}\}_{\bar{P}_{M,\alpha}}$  does not depend on the choice of representative of  $\bar{f}$ . Since  $\{, \}_{\bar{P}_{M,\alpha}}$  is skew symmetric, it does not depend on the choice of representative of  $\bar{h}$  either. Hence the Poisson bracket  $\{, \}_{\bar{P}_{M,\alpha}}$  on  $C^\infty(\bar{P}_{M,\alpha})$  is well defined.  $\square$

From the fact that  $(C^\infty(M)^{G_M}, \{, \}_M, \cdot)$  is a Poisson subalgebra of  $(C^\infty(M), \{, \}_M, \cdot)$  and that  $P_{M,\alpha}$  is  $G_M$ -invariant, it follows that  $(C^\infty(P_{M,\alpha})^{G_M}, \{, \}_M, \cdot)$  is a Poisson algebra. Hence  $(C^\infty(\bar{P}_{M,\alpha}), \{, \}_{\bar{P}_{M,\alpha}}, \cdot)$  is a Poisson algebra. In fact

**Theorem 5.**  $(C^\infty(\bar{P}_{M,\alpha}), \{, \}_{\bar{P}_{M,\alpha}}, \cdot)$  is a nondegenerate Poisson algebra, that is, every smooth function on  $\bar{P}_{M,\alpha}$  which is a Casimir, is locally constant.

**Proof.** Suppose that  $\bar{f} \in C^\infty(\bar{P}_{M,\alpha})$  is a Casimir. Then for each  $p \in P_{M,\alpha}$

$$0 = \{\bar{f}, \bar{h}\}_{\bar{P}_{M,\alpha}}(\rho(p)),$$

for every  $\bar{h} \in C^\infty(\bar{P}_{M,\alpha})$ . From the definition of the Poisson bracket  $\{, \}_{\bar{P}_{M,\alpha}}$  it follows that

$$0 = \{f, h\}_M(p) = -df(p)X_h(p), \quad (16)$$

for every  $p \in P_{M,\alpha}$  and every  $h \in C^\infty(P_{M,\alpha})^{G_M}$ . Here  $\rho^*\bar{f} = f$  and  $\rho^*\bar{h} = h$ . But  $P_{M,\alpha} = J_M^{-1}(\mathcal{O}_M) = G_M \cdot J_M^{-1}(\beta)$  by theorem 3. Since the  $G_M$ -action on  $M$  (and hence on  $P_{M,\alpha}$ ) is free,  $\beta$  is a regular value of the  $G_M$  momentum mapping  $J_M$ , (see note A). Consequently,

$$T_p J_M^{-1}(\beta) = \text{span}\{X_h(p) \mid h \in C^\infty(P_{M,\alpha})^{G_M}\},$$

see [5, p. 343–4 (4.12)]. Hence equation (16) reads  $0 = df(p)v_p$  for every  $v_p \in T_p J_M^{-1}(\beta)$ . Thus  $f$  is locally constant on  $P_{M,\alpha}$ , since it is  $G_M$ -invariant. This implies that  $\bar{f}$  is locally constant on  $J_M^{-1}(\mathcal{O}_M)$  and hence on  $\bar{P}_{M,\alpha}$ .  $\square$

**Corollary.** If the coadjoint orbit  $\mathcal{O}_\alpha \subseteq \mathcal{G}^*$  is locally closed, then  $\bar{P}_{M,\alpha}$  is a smooth symplectic manifold and the differential structure  $C^\infty(\bar{P}_{M,\alpha})$  on  $\bar{P}_{M,\alpha}$  coincides with that formed by the smooth functions coming from its manifold structure.

**Proof.** From theorem 3 and the hypothesis it follows that  $J_M^{-1}(\mathcal{O}_M) = J^{-1}(\mathcal{O}_\alpha) \cap M = P_{M,\alpha}$  is locally closed and hence is a submanifold of  $M$ ,

see note E. The assertions of the corollary now follow from corollary 2 of proposition 2 and regular reduction.  $\square$

Suppose that we define a relation  $\sim$  on  $\overline{P}$  by saying that  $\overline{p}_1 \sim \overline{p}_2$  if they can be joined by a piecewise smooth curve in  $\overline{P}$  each of whose pieces is an integral curve of a Hamiltonian vector field of a smooth function on  $\overline{P}$ . Clearly  $\sim$  is an equivalence relation. The following theorem [14] holds.

**Theorem 6.** The partition

$$\overline{P} = \bigcup_{K \prec G} \bigcup_{\mathcal{O}_\alpha \subseteq \mathcal{G}^*} \prod_{M \text{ c.c. } P_K} (M \cap J^{-1}(\mathcal{O}_\alpha)) / G_M \quad (17)$$

is the same as the partition of  $\overline{P}$  by the equivalence classes of  $\sim$ .

In other words, the Poisson algebra  $(C^\infty(\overline{P}), \{, \}_{\overline{P}}, \cdot)$  determines the partition (17).

## 6 Notes

None of the proofs in this section are original. They are included only for convenience of the reader.

All the arguments below hold if  $J$  is not coadjoint equivariant but only equivariant under an affine coadjoint action.

**A. Proof.** Let  $p \in P$  and suppose that  $\text{im} T_p J$  is a proper subspace of  $\mathcal{G}^*$ . Then there is a nonzero  $\xi \in \mathcal{G}$  such that  $\xi \in (\text{im} T_p J)^\circ$ , that is,  $(\text{im} T_p J)(\xi) = 0$ . Hence for every  $v_p \in T_p P$ , we have

$$\omega(p)(X^\xi(p), v_p) = \text{d}J_\xi(p)v_p = ((T_p J)v_p)\xi = 0.$$

Since  $\omega(p)$  is nondegenerate, this implies  $X^\xi(p) = 0$ . Hence  $\exp t\xi \cdot p = p$  for every  $t \in \mathbf{R}$ . Thus for a nonzero  $t \in \mathbf{R}$ , we see that  $\exp t\xi$  is a nonidentity element of the isotropy group  $G_p$ . But this contradicts the hypothesis that the  $G$ -action is free. Thus  $J$  is a surjective submersion. Hence every element of  $\mathcal{G}$  is a regular value of  $J$ .  $\square$

**B. Proof.** Since  $\alpha$  is a regular value of  $J$ , the  $\alpha$ -level set  $J^{-1}(\alpha)$  is a submanifold of  $P$ . Because  $J$  is coadjoint equivariant, the induced  $G_\alpha$ -action on  $J^{-1}(\alpha)$  is defined and is proper. Let  $p \in J^{-1}(\alpha)$ . Then there is a slice  $S_p$  to

the  $G_\alpha$  action at  $p$ . Let  $\mathcal{L}$  be a complementary subspace in  $\mathcal{G}_\alpha$  of  $\mathcal{G}_p$  and let  $L = \exp \mathcal{L}$ . From the definition of a slice, it follows that the map

$$\Psi : L \times S_p \rightarrow J^{-1}(\alpha) : (\ell, s_p) \rightarrow \ell \cdot s_p$$

with  $\Psi(e, p) = p$  is a local diffeomorphism. Because  $J^{-1}(\mathcal{O}_\alpha) = G \cdot J^{-1}(\alpha)$ , the map

$$\lambda : G \times (L \times S_p) \rightarrow J^{-1}(\mathcal{O}_\alpha) : (g, (\ell, s_p)) \rightarrow g \cdot \Psi(\ell, s_p)$$

with  $\lambda(e, (e, p)) = p$  is a local diffeomorphism and hence is a local parametrization of  $J^{-1}(\mathcal{O}_\alpha)$  at  $p$ . Consequently,  $J^{-1}(\mathcal{O}_\alpha)$  is an immersed submanifold of  $P$ .  $\square$

**C. Proof.** Let  $i : J^{-1}(\alpha) \rightarrow J^{-1}(\mathcal{O}_\alpha)$  be the inclusion and let  $\tilde{\pi} : J^{-1}(\mathcal{O}_\alpha) \rightarrow J^{-1}(\mathcal{O}_\alpha)/G$  and  $\tilde{\pi}_\alpha : J^{-1}(\alpha) \rightarrow J^{-1}(\alpha)/G_\alpha$  be the orbit maps. Since  $\tilde{\pi} \circ i$  is  $G_\alpha$ -invariant, it induces a map  $\sigma : J^{-1}(\alpha)/G_\alpha \rightarrow J^{-1}(\mathcal{O}_\alpha)/G$ . The map  $\sigma$  is injective, for if  $p, p' \in J^{-1}(\alpha)$  with  $\tilde{\pi}(i(p)) = \tilde{\pi}(i(p'))$ , then there is a  $g \in G$  such that  $g \cdot i(p) = i(p')$ . Since

$$\alpha = J(i(p')) = \text{Ad}_{g^{-1}}^t J(i(p)) = \text{Ad}_{g^{-1}}^t \alpha,$$

it follows that  $g \in G_\alpha$ . Thus every fiber of  $\tilde{\pi} \circ i$  is a single  $G_\alpha$ -orbit, which shows that  $\sigma$  is injective. To show that  $\sigma$  is surjective, it suffices to show that  $\tilde{\pi} \circ i$  is surjective. Suppose that  $p \in J^{-1}(\mathcal{O}_\alpha)$ . Consider the orbit  $G \cdot p \in J^{-1}(\mathcal{O}_\alpha)/G$ . Since  $J^{-1}(\mathcal{O}_\alpha) = G \cdot J^{-1}(\alpha)$ , there is a  $p' \in J^{-1}(\alpha)$  and a  $g \in G$  such that  $p = g \cdot p'$ . Hence  $\tilde{\pi}(i(p')) = G \cdot p' = G \cdot p$ . Thus  $\sigma$  is surjective. Consequently, there is a bijective mapping between  $J^{-1}(\alpha)$  and  $J^{-1}(\mathcal{O}_\alpha)$ .  $\square$

**D. Proof.** Because the  $G$ -orbit map  $\pi : P \rightarrow P/G$  is a submersion and  $\lambda : U \subseteq G \times (L \times S_p) \rightarrow J^{-1}(\mathcal{O}_\alpha)$  is a local parametrization of  $J^{-1}(\mathcal{O}_\alpha)$  at  $p$  (see B), it follows that  $\pi \circ \lambda$  restricted to  $\{e\} \times (L \times S_p)$  is a local parametrization of  $J^{-1}(\mathcal{O}_\alpha)/G$  at  $\pi(p)$ . Consequently,  $J^{-1}(\mathcal{O}_\alpha)/G$  is an immersed submanifold of  $P/G$ .  $\square$

**E. Proof.** Since  $\mathcal{O}_\alpha$  is locally closed and  $J$  is continuous, it follows that  $J^{-1}(\mathcal{O}_\alpha)$  is locally closed and thus locally compact. Hence the local parametrization  $\lambda$  is a homeomorphism using the topology on  $J^{-1}(\mathcal{O}_\alpha)$  induced from  $P$ . Consequently,  $J^{-1}(\mathcal{O}_\alpha)$  is a submanifold of  $P$ .  $\square$

## References

- [1] J. Arms, *Symmetry and solution set singularities in Hamiltonian field theories*, Acta Phys. Polonica, **B17** (1986) 499–523.
- [2] J. Arms, R. Cushman and M. Gotay, *A universal reduction procedure for Hamiltonian group actions*, In: “The geometry of Hamiltonian systems”, ed. T. Ratiu, p. 31–51, Birkhäuser, Boston, 1991.
- [3] J. Arms, J. Marsden and V. Moncrief, *Symmetry and bifurcation of momentum mappings*, Comm. Math. Phys., **78** (1981) 455–478.
- [4] L. Bates and E. Lerman, *Proper group actions and symplectic stratified spaces*, Pac. J. Math. **181** (1997) 201–229.
- [5] R. Cushman and L. Bates, “Global aspects of classical integrable systems”, Birkhäuser, Basel, 1997.
- [6] R. Cushman and R. Sjamaar, *On singular reduction of Hamiltonian systems*, In: “Symplectic geometry and mathematical physics”, ed. P. Donato et al., p. 114–128, Birkhäuser, Boston, 1991.
- [7] V. Guillemin and S. Sternberg, “Symplectic techniques in physics”, Cambridge University Press, Cambridge, 1984.
- [8] P. Liebermann and C. Marle, “Symplectic geometry and analytical mechanics”, D. Reidel, Dordrecht, 1987.
- [9] K. Meyer, *Symmetries and integrals in mechanics*, In: “Dynamical systems”, ed. M. Piexoto, p. 259–272, Academic Press, New York, 1973.
- [10] J. Marsden and A. Weinstein, *Reduction of symplectic manifolds*, Rep. Math. Phys. **5** (1974), 121–130.
- [11] L. Pukanski, *Unitary representations of solvable groups*, Ann. Sci. Ecole Norm. Sup. **4** (1971) 457–608.
- [12] R. Sikorski, *Abstract covariant derivative*, Colloq. Math. **18** (1967) 151–272.
- [13] R. Sikorski, “Wstep do geometrii różniczkowej”, PWN, Warszawa, 1972. MR 57 7400.
- [14] R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Ann. of Math. **134** (1991) 375–422.

- [15] J. Śniatycki, G. Schwarz and L. Bates: *Yang-Mills and Dirac Fields in a Bag, Constraints and Reduction*, Comm. Math. Phys. **168** (1995) 441-453.
- [16] H. Whitney, *Analytic extensions of differentiable functions defined on closed sets*, Trans. Amer. Math. Soc. **36** (1934) 63-89.