Singular Reduction for Proper Actions

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1 Introduction

The goal of this paper is to give a simplified proof of the singular reduction theorem for a proper Hamiltonian action of a Lie group G on a connected smooth symplectic manifold (P, ω) which has a coadjoint equivariant momentum mapping. Our proof does not use the assumption that the coadjoint orbit is locally closed, any local normal form, or the shifting trick. These were essential ingredients of the existing proof [4].

The space P/G of orbits of the symmetry group G is a topological quotient space of the original phase space P. We show that G-invariant smooth functions on P define a differential structure on P/G in the sense of Sikorski [12]. Singular reduction gives rise to a partition of P/G by symplectic manifolds.

We begin by describing regular reduction [9], [10]. Let

$$\Phi : G \times P \to P : (g, p) \mapsto \Phi_g(p) = g \cdot p$$

be a proper Hamiltonian action of a Lie group G on a connected symplectic manifold (P, ω) with coadjoint equivariant momentum mapping $J : P \to \mathcal{G}^*$. If the action is free, the momentum map J is a submersion [3], see note A. Thus every $\alpha \in \mathcal{G}^*$ is a regular value of J. The level set $J^{-1}(\alpha)$ is a presymplectic submanifold of P. Let G_α be the isotropy group of the coadjoint action at α. Then the space $J^{-1}(\alpha)/G_\alpha$ of G_α-orbits on $J^{-1}(\alpha)$ is a symplectic manifold [10].

Let $\mathcal{O}_\alpha$ be the coadjoint orbit through α. On the one hand, if $\mathcal{O}_\alpha$ is not locally closed, then $J^{-1}(\mathcal{O}_\alpha)$ is an immersed submanifold of P, see note

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B. Moreover, there is a natural bijection between the immersed submanifold $J^{-1}(\mathcal{O}_\alpha)/G$ of the smooth orbit space P/G and $J^{-1}(\alpha)/G_\alpha$, see notes C and D. On the other hand, if $\mathcal{O}_\alpha$ is locally closed then $J^{-1}(\mathcal{O}_\alpha)$ is a submanifold of P, see note E, and the space $J^{-1}(\mathcal{O}_\alpha)/G$ of G-orbits on $J^{-1}(\mathcal{O}_\alpha)$ is a symplectic manifold, which is naturally diffeomorphic to $J^{-1}(\alpha)/G_\alpha$.

If the momentum map $J : P \to \mathcal{G}^*$ is not coadjoint equivariant, then it is equivariant with respect to an affine coadjoint action on $\mathcal{G}^*$, which is defined as follows. For each $p \in P$ the map

$$\widetilde{\sigma}_p : G \to \mathcal{G}^* : g \to J(\Phi_g(p)) - \mathrm{Ad}^t_{g^{-1}} J(p)$$

does not depend on the choice of the point p. Thus it defines a mapping $\sigma : G \to \mathcal{G}^*$ which is a $\mathcal{G}^*$-cocycle, that is, for every $g, h \in G$

$$\sigma(gh) = \sigma(g) + \mathrm{Ad}^t_{g^{-1}} \sigma(h).$$

Let

$$A : G \times \mathcal{G}^* \to \mathcal{G}^* : (g, h) \to \mathrm{Ad}^t_{g^{-1}} \alpha + \sigma(g). \tag{1}$$

Then A is an action of G on $\mathcal{G}^*$ called the *affine coadjoint action*. The momentum mapping J is equivariant with respect to the action A, that is, for every $(g, p) \in G \times P$

$$J(\Phi_g(p)) = A_g(J(p)).$$

The regular reduction theorem also holds when the momentum mapping is equivariant with respect to the affine coadjoint action [8].

All the results for free proper actions quoted above do not require the hypothesis that G is connected.

We now turn to discussing singular reduction. Since the Hamiltonian G-action Φ on (P, ω) is not necessarily free, the isotropy groups

$$G_p = \left\{ g \in G \,\middle|\, \Phi(g, p) = p \right\}$$

at the point $p \in P$ play an essential role. Because the G-action is proper, G_p is a compact subgroup of G for each $p \in P$. Let K be a compact subgroup of G. The set of points of P of *symmetry type* K is

$$P_K = \left\{ p \in P \,\middle|\, G_p = K \right\}$$

and the set of points of P of *orbit type* K is

$$P_{(K)} = \left\{ p \in P \middle| G_p \text{ is conjugate to } K \right\}.$$

For nonfree proper actions, reduction of the zero level set of a coadjoint equivariant momentum map has been extensively studied [3], [14], [5] and [15]. Their results can be extended to cover reduction of a nonzero level set using the "shifting trick", [1], [7]. In the above cited papers the authors show that for each compact subgroup K of G, each connected component of $P_{(K)} \cap J^{-1}(0)$ is a presymplectic submanifold of P and that each connected component of the space $(P_{(K)} \cap J^{-1}(0))/G$ of G-orbits on $P_{(K)} \cap J^{-1}(0)$ is a symplectic manifold. The stratification of $J^{-1}(0)/G$ by the symplectic manifolds $(P_{(K)} \cap J^{-1}(0))/G$ was studied in [6], [14] and [4].

In this paper we give a simple proof that for each $\alpha \in \mathcal{G}^*$ each connected component of $P_{(K)} \cap J^{-1}(\alpha)$ is a presymplectic submanifold of P and its space of G_α-orbits is a symplectic manifold. Our simplification is obtained by a detailed investigation of $J^{-1}(\mathcal{O}_\alpha) \cap P_K$, which reduces our argument to the case of a free action.

2 Symmetry type

Theorem 1. Let K be a compact subgroup of G, M a connected component of P_K, and $\iota_M : M \to P$ the inclusion map. Then

i) M is a submanifold of P and $\omega_M = \iota_M^* \omega$ is a symplectic form on M.

ii) For each G-invariant function f on P, the flow φ_t of the Hamiltonian vector field X_f associated to f preserves M.

iii) When f is a G-invariant function on P, the restriction to (M, ω) of the Hamiltonian vector field X_f is a Hamiltonian vector field on (M, ω_M) associated to the restriction of f to M.

Proof.
i. The proof of i) can be found in [7], [5].

ii. Since f is G-invariant, $g \cdot \varphi_t(p) = \varphi_t(g \cdot p)$ for all $g \in G$, and $p \in P$. Hence if $g \in G_p$, then $g \in G_{\varphi_t(p)}$. Since φ_t is a local difeomorphism, we find that, if

$g \in G_{\varphi_t(p)}$, then $g \in G_{\varphi_t^{-1}(\varphi_t(p))} = G_p$. Hence $G_{\varphi_t(p)} = G_p$ and $\varphi_t(p) \in P_K$ for all $p \in M$. Since $\varphi_t(p)$ and p are in the same connected component of P_K, it follows that $\varphi_t(p) \in M$ for all $p \in M$. This proves ii.

iii. Since M is a symplectic submanifold of P for each $p \in M$, the symplectic annihilator $T_p^\omega M$ of T_pM, defined by

$$T_p^\omega M = \Big\{ u \in T_pP \,\Big|\, \omega(p)(u,v) = 0 \quad \forall\, v \in T_pM \Big\}, \tag{2}$$

is a symplectic subspace of T_pP complementary to T_pM, that is,

$$T_pP = T_pM \oplus T_p^\omega M. \tag{3}$$

The Hamiltonian flow φ_t of a G-invariant function f is generated by X_f, which satisfies the equation $X_f \lrcorner\, \omega = df$. Since φ_t preserves M, X_f is tangent to M. Hence for every $u \in T_p^\omega M$,

$$\langle df(p) | u \rangle = \omega(p)(X_f(p), u) = 0.$$

Therefore for every $v \in T_pM$, $(X_f \lrcorner\, \omega)v = \langle df | v \rangle$, which implies that $X_f \lrcorner\, \omega_M = df\big|_M$. This proves iii. $\square$

Given $p \in M$, for each $k \in K$, $\Phi(k,p) = \Phi_k(p) = p$. Hence the tangent at p of Φ_k defines an action Ψ of K on T_pP, which fixes every $u \in T_pM$. The normaliser of K in G is

$$N^K = \{ g \in G \mid gKg^{-1} = K \}.$$

For every $p \in P$,

$$\begin{aligned} G_{g\cdot p} &= \{ g' \in G \mid g'g \cdot p = g \cdot p \} = \{ g' \in G \mid g^{-1}g'g \cdot p = p \} \\ &= \{ g' \in G \mid g^{-1}g'g \in G_p \} = \{ g' \in G \mid g' \in gG_pg^{-1} \} = gG_pg^{-1}. \end{aligned}$$

Hence $g \in G$ preserves P_K if and only if $g \in N^K$. Let N_M be the subgroup of N^K preserving the component $M \subseteq P_K$, that is,

$$N_M = \Big\{ g \in N^K \mid g \cdot p \in M \quad \forall\, p \in M \Big\}.$$

Note that K is a normal subgroup of N_M. The subgroup N_M contains the connected component of the identity of N^K and is a closed subgroup of G. Let $\mathcal{N}$ be the Lie algebra of N_M. For each $\xi \in \mathcal{N}$ and each $p \in M$, we have

$$\exp(t\xi) \cdot p = \Phi(\exp(t\xi), p) = \Phi_p(\exp(t\xi)) \in M.$$

Hence $X^\xi(p) = T_e\Phi_p(\xi) \in T_pM$. For each $k \in K$, there exists $k' \in K$ such that $k \cdot \exp(t\xi) = \exp(t\xi) \cdot k'$. Hence

$$\begin{aligned}
\Phi_k(\Phi_p(\exp(t\xi)) &= \Phi_k(\exp(t\xi) \cdot p) = \Phi(k, \exp(t\xi) \cdot p) = \Phi(k \exp(t\xi), p) \\
&= \Phi(\exp(t\xi)k', p) = \Phi(\exp(t\xi), k' \cdot p) = \Phi(\exp(t\xi), p) \\
&= \Phi_p(\exp(t\xi)).
\end{aligned}$$

Therefore

$$T_p\Phi_k(X^\xi(p)) = X^\xi(p) \quad \forall\, k \in K,\ \xi \in \mathcal{N},\ \text{and } p \in M. \tag{4}$$

The quotient group $G_M = N_M/K$ is a Lie group which acts on M by

$$\Phi_M : G_M \times M \to M : ([g], p) \mapsto \Phi(g, p), \tag{5}$$

where $[g] \in G_M$ is the coset containing $g \in N_M$.

Theorem 2. The action Φ_M of G_M on M is free and proper.

Proof. The action Φ_M is free by the construction of G_M. To prove properness we argue as follows. Suppose that the sequence $\{p_n\}$ of points in M converges to $p \in P_K$ and let $\{[g_n]\}$ be a sequence of elements of G_M such that $\Phi_M([g_n], p_n) \to p' \in M$. Then $\Phi(g_n, p_n) = \Phi_M([g_n], p_n) \to p'$. By properness of the action of G on P, there is a subsequence $\{g_{n_m}\}$ in N_M converging to $g \in G$ such that $\Phi(g, p) = p'$. Since N_M is closed, the limit g lies in N_M and $p \in M$. Hence, the subsequence $\{[g_{n_m}]\}$ converges to $[g] \in G_M$ and $\Phi_M([g], p) = p'$. Thus the action Φ_M is proper. □

Let $\overline{P} = P/G$ be the space of G-orbits on P and let $\pi : P \to \overline{P}$ be the orbit map.

Corollary. The space $\overline{M} = M/G_M$ of G_M-orbits on M is a connected manifold. The space $\pi(M)$ has the structure of a smooth manifold induced by the natural bijection $\mu : \pi(M) \to \overline{M}$.

Proof. Since the action of G_M on M is free and proper, $\overline{M} = M/G_M$ is a smooth manifold. Let $\pi_M : M \to \overline{M}$ be the G_M-orbit map. Since M is connected and π_M is continuous, it follows that $\overline{M}$ is connected. □

For each $p \in M$, $\pi(p) = G \cdot p$ is the orbit of G through p. The intersection of $G \cdot p$ with M is the unique G_M-orbit $\pi_M(p) = G_M \cdot p$ through p. In other words,

$$\pi(p) \cap M = G \cdot p \cap M = G_M \cdot p = \pi_M(p).$$

Consequently, the map

$$\mu : \pi(M) \to \overline{M} : \pi(p) \to \pi_M(p),$$

is bijective. Moreover, μ induces a manifold structure on $\pi(M)$. $\square$

Recall that $\mathcal{N}$ is the Lie algebra of N_M. For each $\xi \in \mathcal{N}$, the vector field X^ξ is tangent to M. For each $p \in M$, let

$$\mathcal{M}(p) = \Big\{ \xi \in \mathcal{G} \mid X^\xi(p) \in T_p^\omega M \Big\}, \tag{6}$$

where $T_p^\omega M$ (2) is the symplectic annihilator of T_pM.

Lemma 1. For each $p \in M$

$$\mathcal{N} + \mathcal{M}(p) = \mathcal{G}.$$

Proof. Since p is fixed by K, the action $\Phi|(K \times P)$ of K on P induces a K-action Ψ on T_pP. The tangent space T_pM consists of vectors $v \in T_pP$ which are invariant under this induced action. In other words,

$$T_pM = \Big\{ v \in T_pP \mid \Psi_k(v) = T_p\Phi_k(v) \ = \ v \quad \forall\, k \in K \Big\}.$$

Moreover, for every $\xi \in \mathcal{N}$ we have $X^\xi(p) \in T_pM$.

Since ω is G-invariant and T_pM is Ψ_k-invariant, it follows that $T_p^\omega M$ is also Ψ_k-invariant. For every $u \in T_pM$, the average of u over K is

$$\overline{u} = \int_K \Psi_k(u)\ \mathrm{dk} \ = \ \int_K T_p\Phi_k(u)\ \mathrm{dk}, \tag{7}$$

where dk denotes Haar measure of K normalised so that $\mathrm{vol}\, K = 1$. Since $\overline{u}$ is Ψ_k-invariant, it belongs to T_pM. If $u \in T_p^\omega M$, then $\overline{u} \in T_p^\omega M$ because $T_p^\omega M$ is Ψ_k-invariant. Hence if $u \in T_p^\omega M$, it follows that $\overline{u} \in T_pM \cap T_p^\omega M = \{0\}$. Thus

$$T_p^\omega M = \Big\{ u \in T_pP \mid \overline{u} = 0 \Big\}. \tag{8}$$

For each $\xi \in \mathcal{G}$, let

$$\overline{\xi} = \int_K T_eL_k(\xi)\ \mathrm{dk},$$

where $L_k : G \to G : g \to kg$ is left translation by k. Since the map

$$\mathcal{G} \to T_pP : \xi \to X^\xi(p)$$

is equivariant, that is, $X^{T_e L_k \xi}(p) = T_p\Phi_k \xi$, and has kernel $\mathcal{K}$, it follows that

$$\mathcal{M}(p) = \left\{ \xi \in \mathcal{G} \mid \overline{\xi} \in \mathcal{K} \right\}.$$

For every $\xi \in \mathcal{G}$, we have $\xi = \overline{\xi} + (\xi - \overline{\xi})$, where $\overline{(\xi - \overline{\xi})} = 0$. Since $T_e L_k \overline{\xi} = \overline{\xi}$ for all $k \in K$, it follows that $T_p\Phi_k(X^{\overline{\xi}}(p)) = X^{\overline{\xi}}(p)$ for $k \in K$. So $X^{\overline{\xi}}(p) \in T_pM$, that is $\overline{\xi} \in \mathcal{N}$. Moreover $\overline{(\xi - \overline{\xi})} = 0 \in \mathcal{K}$, which implies that $\xi - \overline{\xi} \in \mathcal{M}(p)$. Hence $\mathcal{G} = \mathcal{N} + \mathcal{M}(p)$. □

3 Reduction

In this section we prove

Theorem 3. The action of G_M on (M, ω_M) has a momentum map $J_M : M \to \mathcal{G}_M^*$, which is equivariant with respect to an affine coadjoint action

$$A : G_M \times \mathcal{G}_M^* \to \mathcal{G}_M^* : ([g], \phi) \mapsto A_{[g]}\phi.$$

For every G-coadjoint orbit $\mathcal{O}_\alpha \subseteq \mathcal{G}^*$ with $J^{-1}(\mathcal{O}_\alpha) \cap M \neq \emptyset$, there exists an orbit $\mathcal{O}_M$ of the action A such that

$$J^{-1}(\mathcal{O}_\alpha) \cap M = J_M^{-1}(\mathcal{O}_M).$$

Proof: Let $\kappa : \mathcal{K} \to \mathcal{G}$, $\mu : \mathcal{K} \to \mathcal{N}$, and $\nu : \mathcal{N} \to \mathcal{G}$ be inclusion mappings and $\lambda : \mathcal{N} \to \mathcal{G}_M$ be the natural projection map. Their transposes are the mappings $\kappa^* : \mathcal{G}^* \to \mathcal{K}^*$, $\mu^* : \mathcal{N}^* \to \mathcal{K}^*$, $\nu^* : \mathcal{G}^* \to \mathcal{N}^*$, and $\lambda^* : \mathcal{G}_M^* \to \mathcal{N}^*$, respectively. Let $J\big|_M : M \to \mathcal{G}^*$ be the restriction of J to M.

To prove the theorem we need the following two lemmas.

Lemma 2. $\kappa^* \circ J\big|_M : M \to \mathcal{K}^*$ is constant.

Proof. For every $\xi \in \mathcal{G}$, we have $X^\xi \lrcorner\, \omega = \mathrm{d}J_\xi$. Moreover, $\xi \in \mathcal{K}$ implies that $X^\xi(p) = 0$ for all $p \in M$. Hence $\mathrm{d}(\kappa^* \circ J\big|_M) = \kappa^* \circ \mathrm{d}J\big|_M = 0$, and $\kappa^* \circ J\big|_M$ is constant on M. □

Since $\mu^* : \mathcal{N}^* \to \mathcal{K}^*$ is onto and $\kappa^* \circ J\big|_M : M \to \mathcal{K}^*$ is constant, there exists a constant map $j_M : M \to \mathcal{N}^*$ such that

$$\mu^* \circ j_M = \kappa^* \circ J\big|_M.$$

Lemma 3. There exists a unique map $J_M : M \to \mathcal{G}_M^*$ such that

$$\lambda^* \circ J_M = \nu^* \circ J\Big|_M - j_M. \tag{9}$$

Proof. We have

$$\mu^* \circ (\nu^* \circ J\Big|_M - j_M) = \kappa^* \circ J\Big|_M - \kappa^* J\Big|_M = 0.$$

The existence of a unique lift $J_M : M \to \mathcal{G}_M^*$ of $(\nu^* \circ J\Big|_M - j_M) : M \to \mathcal{N}^*$ follows from the exactness of the sequence

$$0 \longrightarrow \mathcal{G}_M^* \xrightarrow{\lambda^*} \mathcal{N}^* \xrightarrow{\mu^*} \mathcal{K}^* \longrightarrow 0. \qquad \square \tag{10}$$

Continuing with the proof of the first assertion in theorem 3, we now show that the map $J_M : M \to \mathcal{G}_M^*$ is a momentum map for the action of G_M on M.

For each $\xi \in \mathcal{N} \subseteq \mathcal{G}$, the action of the one parameter subgroup $\exp t\lambda(\xi)$ of G_M on M coincides with the action of the subgroup $\exp t\xi$ of G. This latter action is generated by the Hamiltonian vector field X^ξ of J_ξ restricted to M. Hence

$$\begin{aligned} X^\xi \mathbin{\lrcorner} \omega_M &= \mathrm{d}\Big\langle J\Big|_M \mid \nu(\xi)\Big\rangle \;=\; \mathrm{d}\Big\langle \nu^* \circ J\Big|_M \mid \xi\Big\rangle \\ &= \mathrm{d}\langle \lambda^* \circ J_M + j_M \mid \xi\rangle \;=\; \langle \mathrm{d}(\lambda^* \circ J_M) \mid \xi\rangle + \langle \mathrm{d}j_M \mid \xi\rangle \\ &= \mathrm{d}\langle J_M \mid \lambda(\xi)\rangle\,. \end{aligned}$$

Thus X^ξ is the Hamiltonian vector field of $\langle J_M \mid \lambda(\xi)\rangle$. Hence J_M is a momentum map for the action G_M on M.

We note that the momentum map $J_M : M \to \mathcal{G}_M^*$ need not be coadjoint equivariant. However, there exists a $\mathcal{G}_M^*$-cocycle $\sigma : G_M \to \mathcal{G}_M^*$ such that the map

$$A : G_M \times \mathcal{G}_M^* \to \mathcal{G}_M^* : ([g], \phi) \mapsto A_M([g], \phi) = \mathrm{Ad}^t_{[g]^{-1}}\phi + \sigma([g])$$

is an action of G_M on $\mathcal{G}_M^*$ and

$$J_M([g] \cdot p) = A_{[g]}(J_M(p)).$$

This completes the proof of the first assertion in theorem 3.

We make a short digression to find an explicit expression for the cocycle $\lambda^*\sigma$, which will not be used in the remainder of the proof. Comparing equations (9) and (1) we see that for $\xi \in \mathcal{N}$,

$$\begin{aligned}
\langle \sigma([g]) \mid \lambda(\xi) \rangle &= \left\langle J_M([g]\cdot p) - \mathrm{Ad}^*_{[g]^{-1}} J_M(p) \mid \lambda(\xi) \right\rangle \\
&= \left\langle \mathrm{Ad}^*_{g^{-1}} j_M(p) \mid \xi \right\rangle - \langle j_M(g\cdot p) \mid \xi \rangle \;=\; \langle j_M \mid \mathrm{Ad}_{g^{-1}}\xi - \xi \rangle \\
&= \left\langle \mathrm{Ad}^*_{g^{-1}} j_M - j_M \mid \xi \right\rangle .
\end{aligned}$$

Hence

$$\lambda^*(\sigma([g])) = \mathrm{Ad}^*_{g^{-1}} j_M - j_M. \tag{11}$$

We now turn to proving the second assertion of theorem 3. If $p, p' \in J^{-1}(\mathcal{O}_\alpha) \cap M$ then $J(p') = \mathrm{Ad}^*_{g^{-1}} J(p) = J(g\cdot p)$ for some $g \in N_M$. Since, $g\cdot p = [g]\cdot p$, where $[g]$ is the coset of g in $G_M = N_M/K$, equation (9) yields

$$\begin{aligned}
\lambda^* \circ J_M(p') &= \nu^* \circ J(p') - j_M \;=\; \nu^* \circ \mathrm{Ad}^*_{g^{-1}} J(p) - j_M \\
&= \nu^* \circ J(g\cdot p) - j_M \;=\; (\lambda^* \circ J_M([g]\cdot p) + j_M) - j_M \\
&= \lambda^* \circ J_M([g]\cdot p) \;=\; \lambda^* \circ A_{[g]}(J_M(p)).
\end{aligned}$$

Since $\ker \lambda^* = 0$, it follows that

$$J_M(p') = A_{[g]}(J_M(p)).$$

This implies that $J_M(p')$ and $J_M(p)$ are in the same orbit $\mathcal{O}_M$ of the affine coadjoint action A of G_M on $\mathcal{G}^*_M$, that is,

$$J^{-1}(\mathcal{O}_\alpha) \cap M \subseteq J_M^{-1}(\mathcal{O}_M). \tag{12}$$

Conversely, if $p, p' \in J_M^{-1}(\mathcal{O}_M)$, then $J_M(p) = A_{[g]}(J_M(p'))$ where $g \in N_M$. Therefore,

$$\nu^* \circ J(p) = \nu^* \circ \mathrm{Ad}^*_{g^{-1}} J(p').$$

But $\nu^* : \mathcal{G}^* \to \mathcal{N}^*$ is the transpose of the inclusion mapping $\nu : \mathcal{N} \to \mathcal{G}$. So

$$\ker \nu^* = \mathcal{N}^0 = \left\{ \alpha \in \mathcal{G}^* \mid \langle \alpha \mid \xi \rangle = 0 \quad \forall\, \xi \in \mathcal{N} \right\}.$$

This implies that

$$J(p) - \mathrm{Ad}^*_{g^{-1}} J(p') \in \mathcal{N}^0.$$

Hence for every $\xi \in \mathcal{N}$, we have

$$\left\langle J(p) - \mathrm{Ad}^*_{g^{-1}} J(p') \mid \xi \right\rangle = 0.$$

On the one hand, differentiating this equation in a direction u tangent to $J_M^{-1}(\mathcal{O}_M)$ at p, we get

$$\left\langle T_p \left(J(p) - Ad^*_{g^{-1}} J(p') \right) (u) \mid \xi \right\rangle = 0 \tag{13}$$

for every $u \in T_p J_M^{-1}(\mathcal{O}_M)$ and every $\xi \in \mathcal{N}$. On the other hand, from (6) we see that $X^\xi(p) \in T_p^\omega M$ for $\xi \in \mathcal{M}(p)$. But

$$\langle T_p J(u) \mid \xi \rangle = \mathrm{d}J^\xi(p)u \;=\; \omega_M(p)(X^\xi(p), u) \;=\; 0,$$

for all $u \in T_p J_M^{-1}(\mathcal{O}_M)$ and $\xi \in \mathcal{M}(p)$. Therefore

$$\left\langle T_p \left(J(p) - Ad^*_{g^{-1}} J(p') \right) (u) \mid \xi \right\rangle = 0 \tag{14}$$

for every $u \in T_p J_M^{-1}(\mathcal{O}_M)$ and every $\xi \in \mathcal{M}(p)$. Since $\mathcal{N} + \mathcal{M}(p) = \mathcal{G}$, equations (13) and (14) imply that $J(p) - Ad^*_{g^{-1}} J(p')$ is independent of $p \in J_M^{-1}(\mathcal{O}_M)$. Moreover, $g \in N_M$ implies that $g \cdot p' = [g] \cdot p' \in J_M^{-1}(\mathcal{O}_M)$. Hence taking $p = g \cdot p'$, we get

$$J(p) - \mathrm{Ad}^*_{g^{-1}} J(p') \;=\; J(g \cdot p') - \mathrm{Ad}^*_{g^{-1}} J(p') \;=\; 0,$$

because J is coadjoint equivariant. Thus $J(p)$ and $J(p')$ are in the same coadjoint orbit $\mathcal{O}_\alpha$. Therefore,

$$J_M^{-1}(\mathcal{O}_M) \subseteq J^{-1}(\mathcal{O}_\alpha) \cap M.$$

Taking into account inclusion (12) we obtain $J_M^{-1}(\mathcal{O}_M) = J^{-1}(\mathcal{O}_\alpha) \cap M$. This completes the proof of theorem 3. □

Theorem 4. If $J^{-1}(\mathcal{O}_\alpha) \cap M \neq \emptyset$, then its projection $\left(M \cap J^{-1}(\mathcal{O}_\alpha)\right)/N_M$ to $\overline{M}$ is a symplectic manifold.

Proof. Let $\beta = J_M(p)$ for some $p \in M \cap J^{-1}(\mathcal{O}_\alpha)$. Then

$$\begin{aligned} \left(M \cap J^{-1}(\mathcal{O}_\alpha)\right)/N_M &= \left(M \cap J^{-1}(\mathcal{O}_\alpha)\right)/G_M \\ &= J_M^{-1}(\mathcal{O}_M)/G_M \;=\; J_M^{-1}(\beta)/G_{M_\beta}, \end{aligned}$$

where G_{M_β} is the isotropy group of β in G_M. But $J_M^{-1}(\beta)/G_{M_\beta}$ is a symplectic manifold using the regular reduction theorem. □

4 Differential spaces

Let K be a compact subgroup of G. Our symplectic manifold (P, ω) is partioned by connected components M of the manifold P_K of symmetry type K and by connected components L of the manifold $P_{(K)}$ of orbit type K. The partition of P by orbit type, given by

$$P = \bigcup_{K \prec G} P_{(K)} = \bigcup_{K \prec G} \coprod_{L \text{ c.c. } P_{(K)}} L,$$

is G-invariant. Here $\prec$ means compact subgroup of and c.c. means connected component of. It induces a partition

$$\overline{P} = \bigcup_{K \prec G} \coprod_{L \text{ c.c. } P_{(K)}} \overline{L}$$

of the orbit space $\overline{P}$.

For each $\alpha \in \mathcal{G}^*$, the level set $J^{-1}(\alpha)$ is invariant under the motion of a Hamiltonian vector field associated to a G-invariant Hamiltonian. Hence we have a further partition of P

$$P = \bigcup_{K \prec G} \bigcup_{\alpha \in \mathcal{G}^*} \coprod_{M \text{ c.c. } P_K} J^{-1}(\alpha) \cap M$$

by subsets which are invariant under the motion. However, $J^{-1}(\alpha)$ is not G-invariant. Since the momentum map $J : P \to \mathcal{G}^*$ is G-equivariant, $G \cdot J^{-1}(\alpha) = J^{-1}(\mathcal{O}_\alpha)$, where $\mathcal{O}_\alpha$ is the coadjoint orbit through α. Hence a G-invariant partition of P is

$$P = \bigcup_{K \prec G} \bigcup_{\mathcal{O}_\alpha \subseteq \mathcal{G}^*} \coprod_{L \text{ c.c. } P_{(K)}} \left(L \cap J^{-1}(\mathcal{O}_\alpha) \right).$$

Our aim in this section is to analyze this partition and the induced partition of $\overline{P} = P/G$ given by

$$\overline{P} = \bigcup_{K \prec G} \bigcup_{\mathcal{O}_\alpha \subseteq \mathcal{G}^*} \coprod_{M \text{ c.c. } P_K} \left(M \cap J^{-1}(\mathcal{O}_\alpha) \right) / G_M.$$

Since coadjoint orbits need not be even locally closed [11, p. 512], the differential structure of the partition of P by the inverse image $J^{-1}(\mathcal{O}_\alpha)$ of coadjoint orbits $\mathcal{O}_\alpha \subseteq \mathcal{G}^*$ cannot be described in terms of Whitney smooth functions. A generalization of Whitney's theory, which is adequate for this purpose, is Sikorski's theory of differential spaces [12], [13].

A *differential structure* on a topological space Q is a set $C^\infty(Q)$ of continuous functions on Q which have the following properties:

I. The topology of Q is generated by the functions in $C^\infty(Q)$, that is, the collection

$$\left\{ f^{-1}(U) \big| f \in C^\infty(Q) \, \text{where } U \text{ is an open subset of } \mathbf{R} \right\}$$

is a subbasis for the topology of Q.

II. For every $F \in C^\infty(\mathbf{R}^n)$ and every $f_1, \ldots, f_n \in C^\infty(Q)$, $F(f_1, \ldots, f_n) \in C^\infty(Q)$.

III. If $f : Q \to \mathbf{R}$ is a function such that for every $p \in Q$ there is an open neighbourhood U of p in Q and a function $f_U \in C^\infty(Q)$ satisfying $f\big|_U = f_U \mid U$, then $f \in C^\infty(Q)$.

A topological space Q endowed with a differential structure $C^\infty(Q)$ is called a *differential space* [12, sec. 6]. An element of $C^\infty(Q)$ is called a *smooth* function. Thus $C^\infty(Q)$ is the set of smooth functions on Q. From property II it follows that $C^\infty(Q)$ is a commutative ring under addition and pointwise multiplication.

Example 1. If Q is a smooth manifold, then the collection of smooth functions on P, defined in terms of the manifold structure of P, is a differential structure on P.

Proof. To verify property I we need only show that given a $q \in Q$ and an open neighborhood U of q in Q, then there is a smooth function f on Q such that $f^{-1}(0,1)$ is an open neighborhood of q contained in U. We may assume that U is contained in the domain of a chart (V, φ) of Q and that the closure $\overline{U}$ of U is compact. Then there is a nonnegative smooth function g on $\mathbf{R}^n$ whose support is contained in $\varphi(\overline{U})$ and whose range is contained in $[0, \frac{1}{2}]$. The smooth function $f = g \circ \varphi$ has the desired property.

Property II is obvious.

To prove III suppose that $f : Q \to \mathbf{R}$ has the property that for each $q \in Q$ there is an open neighborhood $U = U_q$ of q in Q and a smooth function F_U such that $f|U = F_U|U$. Shrinking U if necessary, we may assume that U is contained in a chart domain V_q containing q of the manifold Q. Since $\{V_q\}_{q \in Q}$ cover Q and $f|V_q$ is smooth, it follows that f is smooth. $\square$

Example 2. Let $\Phi : G \times P \to P$ be a proper action of a Lie group G on a smooth manifold P. Let $\overline{P} = P/G$ be the space of G-orbits on P (which is not necessarily a smooth manifold) and let $\pi : P \to \overline{P}$ be the orbit map. We say that the function $\overline{f} : \overline{P} \to \mathbf{R}$ is smooth if there is a smooth G-invariant function on P such that $\overline{f} \circ \pi = f$. In other words, if $f : P \to \mathbf{R}$ is a smooth G-invariant function on P, then the induced function $\overline{f} : \overline{P} \to \mathbf{R}$ is smooth. Hence the set $C^\infty(\overline{P})$ of smooth functions on $\overline{P}$ is induced from the space $C^\infty(P)^G$ of smooth G-invariant functions on P. The pair $(C^\infty(\overline{P}), \overline{P})$ is a differential space.

Proof. Property I. It suffices to show that given $\overline{p} \in \overline{P}$ and an open neighborhood $\overline{U}$ of $\overline{p}$ in $\overline{P}$, there is a smooth function $\overline{f}$ on $\overline{P}$ such that $\overline{f}^{-1}(0,1)$ is an open neighborhood of $\overline{p}$ contained in $\overline{U}$. Let $p \in \pi^{-1}(\overline{p})$ and let S_p be a slice to the G-action on P at p. Then $V = S_p \cap \pi^{-1}(\overline{U})$ is an open neighborhood of p in S_p. There is a smooth G_p-invariant nonnegative function $\widetilde{f}$ on S_p whose support is a compact subset contained in V which contains p and whose range is contained in $[0, \frac{1}{2})$. Define the function f by $f(\Phi_g(v)) = \widetilde{f}(v)$ for every $g \in G$ and every $v \in V$. Then f is a smooth G-invariant function on P with support contained in $G \cdot V$ and whose range is contained in $[0, \frac{1}{2}]$. Thus f induces a smooth function $\overline{f}$ on $\overline{P}$ such that $\overline{f}^{-1}(0,1)$ is an open subset of $\overline{U}$ containing $\overline{p}$.

Property II follows immediately from the fact that property II holds for $C^\infty(P)^G$.

We now prove property III. Let $\overline{f} : \overline{P} \to \mathbf{R}$ be a function such that for each $\overline{p} \in \overline{P}$ there is an open neighborhood $\overline{U}$ of $\overline{p}$ in $\overline{P}$ and a smooth function $\overline{f}_{\overline{U}}$ on $\overline{P}$ so that $\overline{f}|\overline{U} = \overline{f}_{\overline{U}}|\overline{U}$. Now $\pi^*\overline{f} : P \to \mathbf{R}$ is G-invariant and

$$\pi^*\overline{f}|\pi^{-1}(\overline{U}) = \pi^*\overline{f}_{\overline{U}}|\pi^{-1}(\overline{U}).$$

But $\pi^*\overline{f}_{\overline{U}} \in C^\infty(P)^G$. Hence $\pi^*\overline{f} \in C^\infty(P)^G$, which implies that $\overline{f} \in C^\infty(\overline{P})$. $\square$

Suppose that $(C^\infty(Q), Q)$ is a differential space and that M is a *subset* of Q. Then we can define a differential structure on M as follows. We say that a function $f : M \to \mathbf{R}$ is *smooth* on M if for every $m \in M$ there is an open neighborhood U of m in Q and a function $f_U \in C^\infty(Q)$ such that $f|(M \cap U) = f_U|(M \cap U)$.

Proposition 1. The set $C^\infty(M)$ of smooth functions on M is a differential structure.

Proof.
Property I. Let $m \in M$ and suppose that U is an open neighborhood of m in Q. Then there is an open set V in $\mathbf{R}$ and a smooth function f on Q such that $f^{-1}(V)$ is an open neighborhood $\mathcal{U}$ of m in Q contained in U. Now $f|M$ is a smooth function on M because for every $p \in M \subseteq Q$ there is an open neighborhood $\widetilde{U}$ of p in Q and a smooth function $\widetilde{f}$ on Q such that $\widetilde{f}_{\widetilde{U}} = f|\widetilde{U}$. But $f|M = f|\widetilde{U}$ on $M \cap \widetilde{U}$. Hence $f|(M \cap \widetilde{U}) = \widetilde{f}_{\widetilde{U}}|(M \cap \widetilde{U})$. Moreover, $(f|M)^{-1}(V) = \mathcal{U} \cap M$, which is an open neighborhood of m in the induced topology on M contained in $M \cap U$.

Property II follows immediately from the fact that property II holds for $C^\infty(Q)$.

Property III follows from the definition of smooth function on M. $\square$

We say that the differential structure on M given in proposition 1 is *inherited from* the differential structure on Q. *Locally* a smooth function on M in the differential structure on M inherited from the differential structure on Q is a restriction to M of a smooth function on Q.

Proposition 2. If M is a closed subset of a smooth paracompact manifold Q then every smooth function on M extends to a smooth function on Q.

Proof. Let $f \in C^\infty(M)$ and $\{U_p \mid p \in M\}$ the covering of M by opens sets in Q such that for each $p \in M$, $U_p \ni p$ and there exists $f_{U_p} \in C^\infty(Q)$ satisfying $f_{U_p} \mid U_p \cap M = f \mid U_p \cap M$. Since M is closed in Q, its complement M' is open in Q and the family $\{U_p \mid p \in M\} \cup M'$ is an open covering of Q. Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to this covering. Each $\varphi_\alpha \in C^\infty(Q)$ has support in some U_{p_α} or in M'. Moreover $\sum_\alpha \varphi_\alpha = 1$. Let $g = \sum_\alpha \varphi_\alpha f_{U_{p_\alpha}}$, where the sum is taken over α such that the support of φ_α has nonempty intersection with M. Clearly, $g \in C^\infty(M)$. Since $M' \cap M = \emptyset$, it follows that $g|M = f$. $\square$

If M is not closed in Q, and $\{p_n\}$ is a sequence of points in M converging to $p \notin M$, then we can construct a smooth function f on M such that $f(p_n) \to \infty$ as $n \to \infty$. Hence f cannot be a restriction to M of a function on $C^\infty(Q)$.

Corollary. If M is a closed submanifold of Q, then the differential structure on M induced by restriction is the space of smooth functions using the manifold structure of Q.

Proof. This is a special case of proposition 2 because a submanifold M of Q is a closed subset of Q. Moreover the differential structure induced by restriction is just the set of smooth functions using the manifold structure of M. □

5 Reduced Poisson algebra

As before, let (P, ω) be a symplectic manifold with $\Phi : G \times P \to P$ a proper Hamiltonian action of a Lie group G on P. Let $\overline{P} = P/G$ the space of G-orbits with orbit map $\pi : P \to \overline{P}$.

In example 2 we have shown that the space $C^\infty(\overline{P})$ of all functions on $\overline{P}$ which pull back under the G-orbit map π to a smooth G-invariant function on P is a differential structure on $\overline{P}$. In fact,

Proposition 3. $(C^\infty(\overline{P}),\ \{\, ,\, \}_{\overline{P}}, \cdot)$ is a Poisson algebra.

Proof. Here $\cdot$ is pointwise multiplication of smooth functions and $\{\, ,\, \}_{\overline{P}}$ is a Poisson bracket on $C^\infty(\overline{P})$, which is defined as follows. Let $\overline{f}, \overline{h} \in C^\infty(\overline{P})$. At each $p \in P$ let

$$\{\overline{f}, \overline{h}\}_{\overline{P}}(\pi(p)) = \{f, h\}(p),$$

where $\pi^*\overline{f} = f$, $\pi^*\overline{h} = h$ with $f, h \in C^\infty(P)^G$. Moreover, $\{\, ,\, \}$ is the usual Poisson bracket on the space of smooth functions on the symplectic manifold (P, ω). To see that the Poisson bracket $\{\, ,\, \}_{\overline{P}}$ is well defined, suppose that $\tilde{f}$ is another smooth G-invariant function on P which induces the function $\overline{f}$ on $\overline{P}$. Then

$$0 = \pi^*\overline{f} - \pi^*\overline{f} \ = f - \tilde{f}$$

on P, since π is surjective. Hence $\{f, h\} = \{\tilde{f}, h\}$, which implies that $\{\overline{f}, \overline{h}\}_{\overline{P}}$ does not depend on the choice of representative of $\overline{f}$. Since $\{\, ,\, \}_{\overline{P}}$ is skew symmetric, the same argument shows that $\{\overline{f}, \overline{h}\}_{\overline{P}}$ does not depend on the choice of representative of $\overline{h}$ either. Hence $\{\,,\,\}_{\overline{P}}$ is well defined.

From the fact that $(C^\infty(P)^G, \{\,,\,\}, \cdot)$ is a Poisson algebra [5, p. 347 (5.1)], it follows that $(C^\infty(\overline{P}),\ \{\,,\,\}_{\overline{P}}, \cdot)$ is a Poisson algebra. $\square$

Let $\mathcal{O}_\alpha \subseteq \mathcal{G}^*$ be a G-coadjoint orbit and let M be a connected component of the manifold P_K of points of P of symmetry type K. Furthermore, suppose that $P_{M,\alpha} = M \cap J^{-1}(\mathcal{O}_\alpha)$ is nonempty. From theorem 3 we know that $P_{M,\alpha} = J^{-1}(\mathcal{O}_M)$ for some G_M-affine coadjoint orbit $\mathcal{O}_M$. Thus it follows from note B, that $P_{M,\alpha}$ is an immersed submanifold of the symplectic manifold (M, ω_M), (see theorem 1). In addition, $P_{M,\alpha}$ is invariant under the flow of every G_M-invariant Hamiltonian vector field on (M, ω_M) [5, p. 343 (4.10)]. Consider the space $C^\infty(P_{M,\alpha})^{G_M}$ of smooth G_M-invariant functions on $P_{M,\alpha}$ inherited from the space $C^\infty(M)^{G_M}$ defined by the manifold structure of M. Then $C^\infty(P_{M,\alpha})^{G_M}$ is a differential structure on $P_{M,\alpha}$.

Let $\overline{P}_{M,\alpha}$ be the space of G_M-orbits on $P_{M,\alpha}$. Since a smooth function on $\overline{P}_{M,\alpha}$ is induced by a smooth G_M-invariant function on $P_{M,\alpha}$, we see that $C^\infty(\overline{P}_{M,\alpha})$ is a differential structure on $\overline{P}_{M,\alpha}$. Define a Poisson bracket $\{\,,\,\}_{\overline{P}_{M,\alpha}}$ on $C^\infty(\overline{P}_{M,\alpha})$ as follows, compare with [2]. For each $\overline{f}, \overline{h} \in C^\infty(\overline{P}_{M,\alpha})$ and each $p \in P_{M,\alpha}$ let

$$\{\overline{f}, \overline{h}\}_{\overline{P}_{M,\alpha}}(\rho(p)) = \{f, h\}_M(p), \tag{15}$$

where $\{\,,\,\}_M$ is the usual Poisson bracket on (M, ω_M), $f, h \in C^\infty(P_{M,\alpha})^{G_M}$ such that $\rho^*\overline{f} = f, \rho^*\overline{h} = h$, and ρ is the restriction of the G_M-orbit map π to $P_{M,\alpha}$. Note that the right hand side of (15) is well defined, because locally every smooth G_M-invariant function on $P_{M,\alpha}$ is the restriction of a smooth G_M-invariant function on M.

Proposition 4. The Poisson bracket $\{\,,\,\}_{\overline{P}_{M,\alpha}}$ is well defined.

Proof. Suppose that $\widetilde{f} \in C^\infty(P_{M,\alpha})^{G_M}$ induces the function $\overline{f}$ on $\overline{P}_{M,\alpha}$. Then

$$\widetilde{f} - f = \rho^*\overline{f} - \rho^*\overline{f} = 0$$

on $\overline{P}_{M,\alpha}$. Hence for each $p \in P_{M,\alpha}$, we have

$$\{\widetilde{f} - f, h\}_M(p) = -\,\mathrm{d}(\widetilde{f} - f)(p)X_h(p) \;=\; -L_{X_h}0 \;=\; 0,$$

where the last equality follows from the fact the $P_{M,\alpha}$ is invariant under the flow of every G_M-invariant Hamiltonian vector field on M. Hence $\{\overline{f},\overline{h}\}_{\overline{P}_{M,\alpha}}$ does not depend on the choice of representative of $\overline{f}$. Since $\{\,,\,\}_{\overline{P}_{M,\alpha}}$ is skew symmetric, it does not depend on the choice of representative of $\overline{h}$ either. Hence the Poisson bracket $\{\,,\,\}_{\overline{P}_{M,\alpha}}$ on $C^\infty(\overline{P}_{M,\alpha})$ is well defined. $\square$

From the fact that $(C^\infty(M)^{G_M},\{\,,\,\}_M,\cdot)$ is a Poisson subalgebra of $(C^\infty(M)\,,$ $\{\,,\,\}_M,\cdot)$ and that $P_{M,\alpha}$ is G_M-invariant, it follows that $(C^\infty(P_{M,\alpha})^{G_M},\{\,,\,\}_M,$ $\cdot)$ is a Poisson algebra. Hence $(C^\infty(\overline{P}_{M,\alpha}),\{\,,\,\}_{\overline{P}_{M,\alpha}},\cdot)$ is a Poisson algebra. In fact

Theorem 5. $(C^\infty(\overline{P}_{M,\alpha}),\{\,,\,\}_{\overline{P}_{M,\alpha}},\cdot)$ is a nondegenerate Poisson algebra, that is, every smooth function on $\overline{P}_{M,\alpha}$ which is a Casimir, is locally constant.

Proof. Suppose that $\overline{f}\in C^\infty(\overline{P}_{M,\alpha})$ is a Casimir. Then for each $p\in P_{M,\alpha}$

$$0=\{\overline{f},\overline{h}\}_{\overline{P}_{M,\alpha}}(\rho(p)),$$

for every $\overline{h}\in C^\infty(\overline{P}_{M,\alpha})$. From the definition of the Poisson bracket $\{\,,\,\}_{\overline{P}_{M,\alpha}}$ it follows that

$$0=\ \{f,h\}_M(p)\ =\ -\,\mathrm{d}f(p)X_h(p), \tag{16}$$

for every $p\in P_{M,\alpha}$ and every $h\in C^\infty(P_{M,\alpha})^{G_M}$. Here $\rho^*\overline{f}=f$ and $\rho^*\overline{h}=h$. But $P_{M,\alpha}=J_M^{-1}(\mathcal{O}_M)=G_M\cdot J_M^{-1}(\beta)$ by theorem 3. Since the G_M-action on M (and hence on $P_{M,\alpha}$) is free, β is a regular value of the G_M momentum mapping J_M, (see note A). Consequently,

$$T_pJ_M^{-1}(\beta)=\mathrm{span}\Big\{X_h(p)\big|\,h\in C^\infty(P_{M,\alpha})^{G_M}\Big\},$$

see [5, p. 343–4 (4.12)]. Hence equation (16) reads $0=\mathrm{d}f(p)v_p$ for every $v_p\in T_pJ_M^{-1}(\beta)$. Thus f is locally constant on $P_{M,\alpha}$, since it is G_M-invariant. This implies that $\overline{f}$ is locally constant on $J_M^{-1}(\mathcal{O}_M)$ and hence on $\overline{P}_{M,\alpha}$. $\square$

Corollary. If the coadjoint orbit $\mathcal{O}_\alpha\subseteq\mathcal{G}^*$ is locally closed, then $\overline{P}_{M,\alpha}$ is a smooth symplectic manifold and the differential structure $C^\infty(\overline{P}_{M,\alpha})$ on $\overline{P}_{M,\alpha}$ coincides with that formed by the smooth functions coming from its manifold structure.

Proof. From theorem 3 and the hypothesis it follows that $J_M^{-1}(\mathcal{O}_M)=J^{-1}(\mathcal{O}_\alpha)\cap M=P_{M,\alpha}$ is locally closed and hence is a submanifold of M, see note E. The assertions of the corollary now follow from corollary 2 of proposition 2 and regular reduction. $\square$

Suppose that we define a relation $\sim$ on $\overline{P}$ by saying that $\overline{p}_1 \sim \overline{p}_2$ if they can be joined by a piecewise smooth curve in $\overline{P}$ each of whose pieces is an integral curve of a Hamiltonian vector field of a smooth function on $\overline{P}$. Clearly $\sim$ is an equivalence relation. The following theorem [14] holds.

Theorem 6. The partition

$$\overline{P} = \bigcup_{K \prec G} \bigcup_{\mathcal{O}_\alpha \subseteq \mathcal{G}^*} \coprod_{M \text{ c.c. } P_K} \left(M \cap J^{-1}(\mathcal{O}_\alpha) \right) / G_M \tag{17}$$

is the same as the partition of $\overline{P}$ by the equivalence classes of $\sim$.

In other words, the Poisson algebra $(C^\infty(\overline{P}), \{\ ,\ \}_{\overline{P}}, \cdot)$ determines the partition (17).

6 Notes

None of the proofs in this section are original. They are included only for convenience of the reader.

All the arguments below hold if J is not coadjoint equivariant but only equivariant under an affine coadjoint action.

A. **Proof**. Let $p \in P$ and suppose that $\mathrm{im} T_p J$ is a proper subspace of $\mathcal{G}^*$. Then there is a nonzero $\xi \in \mathcal{G}$ such that $\xi \in (\mathrm{im} T_p J)^\circ$, that is, $(\mathrm{im} T_p J)(\xi) = 0$. Hence for every $v_p \in T_p P$, we have

$$\omega(p)(X^\xi(p), v_p) = \mathrm{d}J_\xi(p) v_p \ = \ ((T_p J) v_p)\xi \ = \ 0.$$

Since $\omega(p)$ is nondegenerate, this implies $X^\xi(p) = 0$. Hence $\exp t\xi \cdot p = p$ for every $t \in \mathbf{R}$. Thus for a nonzero $t \in \mathbf{R}$, we see that $\exp t\xi$ is a nonidentity element of the isotropy group G_p. But this contradicts the hypothesis that the G-action is free. Thus J is a surjective submersion. Hence every element of $\mathcal{G}$ is a regular value of J. $\square$

B. **Proof**. Since α is a regular value of J, the α-level set $J^{-1}(\alpha)$ is a submanifold of P. Because J is coadjoint equivaraint, the induced G_α-action on $J^{-1}(\alpha)$ is defined and is proper. Let $p \in J^{-1}(\alpha)$. Then there is a slice S_p to the G_α action at p. Let $\mathcal{L}$ be a complementary subspace in $\mathcal{G}_\alpha$ of $\mathcal{G}_p$ and let $L = \exp \mathcal{L}$. From the definition of a slice, it follows that the map

$$\Psi : L \times S_p \to J^{-1}(\alpha) : (\ell, s_p) \to \ell \cdot s_p$$

with $\Psi(e,p) = p$ is a local diffeomorphism. Because $J^{-1}(\mathcal{O}_\alpha) = G \cdot J^{-1}(\alpha)$, the map

$$\lambda : G \times (L \times S_p) \to J^{-1}(\mathcal{O}_\alpha) : (g, (\ell, s_p)) \to g \cdot \Psi(\ell, s_p)$$

with $\lambda(e,(e,p)) = p$ is a local diffeomorphism and hence is a local parametrization of $J^{-1}(\mathcal{O}_\alpha)$ at p. Consequently, $J^{-1}(\mathcal{O}_\alpha)$ is an immersed submanifold of P. $\square$

C. **Proof**. Let $i : J^{-1}(\alpha) \to J^{-1}(\mathcal{O}_\alpha)$ be the inclusion and let $\widetilde{\pi} : J^{-1}(\mathcal{O}_\alpha) \to J^{-1}(\mathcal{O}_\alpha)/G$ and $\widetilde{\pi}_\alpha : J^{-1}(\alpha) \to J^{-1}(\alpha)/G_\alpha$ be the orbit maps. Since $\widetilde{\pi} \circ i$ is G_α-invariant, it induces a map $\sigma : J^{-1}(\alpha)/G_\alpha \to J^{-1}(\mathcal{O}_\alpha)/G$. The map σ is injective, for if $p, p' \in J^{-1}(\alpha)$ with $\widetilde{\pi}(i(p)) = \widetilde{\pi}(i(p'))$, then there is a $g \in G$ such that $g \cdot i(p) = i(p')$. Since

$$\alpha = J(i(p')) \;=\; \mathrm{Ad}^t_{g^{-1}} J(i(p)) \;=\; \mathrm{Ad}^t_{g^{-1}} \alpha,$$

it follows that $g \in G_\alpha$. Thus every fiber of $\widetilde{\pi} \circ i$ is a single G_α-orbit, which shows that σ is injective. To show that σ is surjective, it suffices to show that $\widetilde{\pi} \circ i$ is surjective. Suppose that $p \in J^{-1}(\mathcal{O}_\alpha)$. Consider the orbit $G \cdot p \in J^{-1}(\mathcal{O}_\alpha)/G$. Since $J^{-1}(\mathcal{O}_\alpha) = G \cdot J^{-1}(\alpha)$, there is a $p' \in J^{-1}(\alpha)$ and a $g \in G$ such that $p = g \cdot p'$. Hence $\widetilde{\pi}(i(p')) = G \cdot p' = G \cdot p$. Thus σ is surjective. Consequently, there is a bijective mapping between $J^{-1}(\alpha)$ and $J^{-1}(\mathcal{O}_\alpha)$. $\square$

D. **Proof**. Because the G-orbit map $\pi : P \to P/G$ is a submersion and $\lambda : U \subseteq G \times (L \times S_p) \to J^{-1}(\mathcal{O}_\alpha)$ is a local parmetrization of $J^{-1}(\mathcal{O}_\alpha)$ at p (see B), it follows that $\pi \circ \lambda$ restricted to $\{e\} \times (L \times S_p)$ is a local parmetrization of $J^{-1}(\mathcal{O}_\alpha)/G$ at $\pi(p)$. Consequently, $J^{-1}(\mathcal{O}_\alpha)/G$ is an immersed submanifold of P/G. $\square$

E. **Proof**. Since $\mathcal{O}_\alpha$ is locally closed and J is continuous, it follows that $J^{-1}(\mathcal{O}_\alpha)$ is locally closed and thus locally compact. Hence the local parmetrization λ is a homeomorphism using the topology on $J^{-1}(\mathcal{O}_\alpha)$ induced from P. Consequently, $J^{-1}(\mathcal{O}_\alpha)$ is a submanifold of P. $\square$

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