Equipartition of interval partitions and an application to number theory

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ABSTRACT. We give wider application and simpler proofs of results describing the rate at which the digits of one number theoretic expansion determine those of another. The proofs are based on general measure-theoretic covering arguments and not on the dynamics of specific maps.

1. Introduction

Let $x \in [0,1)$, and suppose we are given the first n digits of some number theoretic expansion of x (what such an expansion is will be defined in a moment). Then these n digits determine $m(n,x)$ digits of some other expansion of x. In this paper we address the question: what can one say about the limit

$$\lim_{n\to\infty} \frac{m(n,x)}{n}$$

for almost all $x \in [0,1)$?

In 1964, G. Lochs [L] was the first to address this question when comparing the decimal and the continued fraction expansions. Let $x = .d_1 d_2 \cdots$ be the decimal expansion of x (which is generated by iterating the map $Sx = 10x \pmod 1$), and

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} = [0; a_1, a_2, \cdots] \tag{1.1}$$

its regular continued fraction (RCF) expansion (generated by the map $Tx = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$). Let $y = .d_1 d_2 \cdots d_n$ be the rational number determined by the first n decimal digits of x, and let $z = y + 10^{-n}$. Then, $[y,z]$ is the decimal cylinder of order n containing x, which we also denote by $B_n(x)$. Now let $y = [0; b_1, b_2, \cdots b_l]$ and $z = [0; c_1, c_2, \cdots c_k]$ be the RCF expansion of y and z. Let

$$m(n,x) = \max\{ i \le \max(l,k) : \text{ for all } j \le i,\ b_j = c_j \}.$$

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In other words, $m(n,x)$ is the largest integer such that $B_n(x) \subset C_{m(n,x)}(x)$, where $C_j(x)$ denotes the continued fraction cylinder of order j containing x. Lochs [L] proved the following theorem:

THEOREM 1. *Let λ denote Lebesgue measure on $[0,1)$. Then for a.e. $x \in [0,1)$*

$$\lim_{n\to\infty} \frac{m(n,x)}{n} = \frac{6\log 2 \log 10}{\pi^2}.$$

Lochs' proof was based on the intricate arithmetic properties of the RCF map, and on a result by Lévy [B] which states that

$$\lim_{n\to\infty} \frac{Q_n}{n} = \frac{\pi^2}{12\log 2} \text{ a.e.}(\lambda),$$

where $\frac{P_n}{Q_n} = [0; a_1, a_2, \cdots, a_n]$.

In 1999, Bosma Dajani and Kraaikamp [BDK] generalized theorem 1 to a wider class of transformations on $[0,1)$, by noticing that Lochs's theorem is concerned with the way the decimal cylinders fit in the CF cylinders, and that the limit is in fact the ratio of the entropies of the maps under consideration. Their proof was based on Shannon-McMillan-Breiman Theorem [B], and the dynamics of the underlying transformations as reflected in the way the partitions are refined under iterations of the corresponding maps.

A surjective map $T : [0,1) \to [0,1)$ is called a *number theoretic fibered map* (NTFM) if it satisfies the following conditions:

(a) there exists a finite or countable partition of intervals $\mathcal{P} = \{P_i; i \in D\}$ such that T restricted to each atom of $\mathcal{P}$ (cylinder set of order 0) is monotone, continuous and injective,
(b) T is ergodic with respect to Lebesgue measure λ, and there exists a T invariant probability measure μ equivalent to λ with bounded density. (Both $\frac{d\mu}{d\lambda}$ and $\frac{d\lambda}{d\mu}$ are bounded).

Iterations of T generate expansions of points $x \in [0,1)$ with digits in D. We refer to the resulting expansion as the *T-expansion of x*.

Almost all known expansions on $[0,1)$ are generated by a NTFM. Among them are the n-adic expansions ($Tx = nx \pmod 1$, where n is a positive integer), β expansions ($Tx = \beta x \pmod 1$, where $\beta > 1$ is a real number), continued fraction expansions ($Tx = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$), Lüroth series ($Tx = \lfloor \frac{1}{x} \rfloor \left(\lfloor \frac{1}{x} \rfloor + 1 \right) x - \lfloor \frac{1}{x} \rfloor$), and all f-expansions [R].

In [BDK], the authors call a NTFM T r-regular if, loosely speaking (i) the lengths of two adjacent T cylinders of order n (contained in the same T cylinder of order $n-1$) differ by at most a constant factor L, and (ii) for any interval I of $[0,1)$, if D_m is the smallest T cylinder of order m containing I, then for a.e. $x \in I$, D_m has a refinement of order $m+r$ contained in I with the property that it or an adjacent cylinder contains x. They show that the continued fraction map is 3-regular, and exhibit many examples with this property. They prove

THEOREM 2 (BDK). *Let S and T be NTFM's on $[0,1)$, and suppose T is r-regular. Then for λ a.e. x,*

$$\lim_{n\to\infty}\frac{m(n,x)}{n}=\frac{h(S)}{h(T)},$$

where $h(S)$ and $h(T)$ denote the entropies of the maps S and T respectively, and $m(n,x)$ is the order of the smallest T cylinder containing the S cylinder of x of order n.

Although n-adic expansions are not r-regular they were able to extend their result to the case when $Sx = mx \pmod 1$ and $Tx = nx \pmod 1$ indicating that r-regularity may not be needed. In this paper we show that Lochs' theorem is true for any two sequences of interval partitions on $[0,1)$ satisfying the conclusion of Shannon-McMillan-Breiman theorem. As a corollary we prove a conjecture in [BDK] that Loch's theorem is true for all pairs of number theoretic fibered maps

We end this section with few definitions that will be used in the arguments to follow.

DEFINITION 1. *By an* interval partition, *we mean a finite or countable partition of $[0,1)$ into subintervals. If P is an interval partition and $x\in[0,1)$, we let $P(x)$ denote the interval of P containing x.*

Let $\mathcal{P}=\{P_n\}_{n=1}^{\infty}$ be a sequence of interval partitions. Let λ denote a Borel measure on $[0,1)$.

DEFINITION 2. *Let $c\geq 0$. We say that $\mathcal{P}$* has entropy c a.e. with respect to λ *if*

$$-\frac{\log\lambda\left(P_n\left(x\right)\right)}{n}\to c \text{ a.e.}(\lambda).$$

We say that $\mathcal{P}$ has entropy c in measure with respect to λ *if*

$$-\frac{\log\lambda\left(P_n\left(x\right)\right)}{n}\to c \text{ in measure } (\lambda)$$

Note that we do not assume that each P_n is refined by P_{n+1}.

Suppose that $\mathcal{P}=\{P_n\}_{n=1}^{\infty}$ and $\mathcal{Q}=\{Q_n\}_{n=1}^{\infty}$ are sequences of interval partitions. For each $n\in\mathbb{N}$ and $x\in[0,1)$, define

$$m_{\mathcal{P},\mathcal{Q}}\left(n,x\right)=\sup\left\{m\mid P_n\left(x\right)\subset Q_m\left(x\right)\right\}.$$

2. Almost everywhere convergence

THEOREM 3. *Let $\mathcal{P}=\{P_n\}_{n=1}^{\infty}$ and $\mathcal{Q}=\{Q_n\}_{n=1}^{\infty}$ be sequences of interval partitions and λ a Borel measure on $[0,1)$. Suppose that for some constants $c>0$ and $d>0$, $\mathcal{P}$ has entropy$c$ a.e with respect to λ and $\mathcal{Q}$ has entropy d a.e. with respect to λ. Then*

$$\frac{m_{\mathcal{P},\mathcal{Q}}\left(n,x\right)}{n}\to\frac{c}{d}\text{ a.e. }(\lambda)$$

PROOF. First we show that

$$\limsup_{n\to\infty}\frac{m_{\mathcal{P},\mathcal{Q}}\left(n,x\right)}{n}\leq\frac{c}{d}\text{ a.e.}$$

Fix $\varepsilon > 0$. Let $x \in [0,1)$ be a point at which the convergence conditions of the hypotheses are met. Fix $\eta > 0$ so that $\frac{c+\eta}{c-\frac{c}{d}\eta} < 1+\varepsilon$. Choose N so that for all $n \geq N$

$$\lambda\left(P_n\left(x\right)\right) > 2^{-n(c+\eta)}$$

and

$$\lambda\left(Q_n\left(x\right)\right) < 2^{-n(d-\eta)}.$$

Fix n so that $\min\left\{n, \frac{c}{d}n\right\} \geq N$, and let m' denote any integer greater than $(1+\varepsilon)\frac{c}{d}n$. By the choice of η,

$$\lambda\left(P_n\left(x\right)\right) > \lambda\left(Q_{m'}\left(x\right)\right)$$

so that $P_n(x)$ is not contained in $Q_{m'}(x)$. Therefore

$$m_{\mathcal{P},\mathcal{Q}}\left(n,x\right) \leq \left(1+\varepsilon\right)\frac{c}{d}n$$

and so

$$\limsup_{n\to\infty}\frac{m_{\mathcal{P},\mathcal{Q}}\left(n,x\right)}{n} \leq \left(1+\varepsilon\right)\frac{c}{d} \text{ a.e.}$$

Since $\varepsilon > 0$ was arbitrary, we have the desired result.

Now we show that

$$\liminf_{n\to\infty}\frac{m_{\mathcal{P},\mathcal{Q}}\left(n,x\right)}{n} \geq \frac{c}{d} \text{ a.e.}$$

Fix $\varepsilon \in (0,1)$. Choose $\eta > 0$ so that $\zeta =: \varepsilon c - \eta\left(1 + (1-\varepsilon)\frac{c}{d}\right) > 0$. For each $n \in \mathbb{N}$ let $\bar{m}(n) = \left\lceil (1-\varepsilon)\frac{c}{d}n \right\rceil$. For brevity, for each $n \in \mathbb{N}$ we call an element of P_n (respectively Q_n) $(n,\eta)-$good if

$$\lambda\left(P_n\left(x\right)\right) < 2^{-n(c-\eta)}$$

(respectively

$$\lambda\left(Q_n\left(x\right)\right) > 2^{-n(d+\eta)}).$$

For each $n \in \mathbb{N}$, let

$$D_n\left(\eta\right) = \left\{ x \mid \begin{array}{c} P_n\left(x\right) \text{ is } \left(n,\eta\right)-\text{good and } Q_{\bar{m}(n)}\left(x\right) \text{ is } \left(\bar{m}\left(n\right),\eta\right)-\text{good} \\ \text{and } P_n\left(x\right) \nsubseteq Q_{\bar{m}(n)}\left(x\right) \end{array} \right\}.$$

If $x \in D_n(\eta)$, then $P_n(x)$ contains an endpoint of the $(\bar{m}(n),\eta)-$good interval $Q_{\bar{m}(n)}(x)$. By the definition of $D_n(\eta)$ and $\bar{m}(n)$,

$$\frac{\lambda\left(P_n\left(x\right)\right)}{\lambda\left(Q_{\bar{m}(n)}\left(x\right)\right)} < 2^{-n\zeta}.$$

Since no more than one atom of P_n can contain a particular endpoint of an atom of $Q_{\bar{m}(n)}$, we see that $\lambda\left(D_n\left(\eta\right)\right) < 2 \cdot 2^{-n\zeta}$ and so

$$\sum_{n=1}^{\infty}\lambda\left(D_n\left(\eta\right)\right) < \infty,$$

which implies that

$$\lambda\left\{x \mid x \in D_n\left(\eta\right) \text{ i.o.}\right\} = 0.$$

Since $\bar{m}(n)$ goes to infinity as n does, we have shown that for almost every $x \in [0,1)$, there exists $N \in \mathbb{N}$, so that for all $n \geq N$, $P_n(x)$ is (n,η) −good and $\left(Q_{\bar{m}(n)}(x)\right)$ is $(\bar{m}(n),\eta)$ −good and $x \notin D_n(\eta)$. In other words, for almost every $x \in [0,1)$, there exists $N \in \mathbb{N}$, so that for all $n \geq N$, $P_n(x)$ is (n,η) −good and $\left(Q_{\bar{m}(n)}(x)\right)$ is $(\bar{m}(n),\eta)$ −good and $P_n(x) \subset \left(Q_{\bar{m}(n)}(x)\right)$. Thus, for almost every $x \in [0,1)$, there exists $N \in \mathbb{N}$, so that for all $n \geq N$, $m_{\mathcal{P},\mathcal{Q}}(n,x) \geq \bar{m}(n)$, so that

$$\frac{m_{\mathcal{P},\mathcal{Q}}(n,x)}{n} \geq (1-\varepsilon).$$

This proves that

$$\liminf_{n\to\infty} \frac{m_{\mathcal{P},\mathcal{Q}}(n,x)}{n} \geq 1-\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have established the theorem. □

3. Application

Let S and T be number theoretic fibered maps on $[0,1)$ with invariant probability measures μ_1 and μ_2 equivalent to Lebesgue measure and with generating partitions (cylinders of order 0) P and Q respectively. Denote by P_n and Q_n the interval partitions of $[0,1)$ into cylinder sets of order n, and as above, denote by $P_n(x)$ the element of P_n containing x (similarly for $Q_n(x)$), and set

$$m(n,x) = \sup\{m \mid P_n(x) \subset Q_m(x)\}.$$

Suppose that $h(S) > 0$ and $h(T) > 0$.

THEOREM 4. *Let S and T be as above and let λ denote Lebesgue measure on* $[0,1)$. *Then*

$$\lim_{n\to\infty} \frac{m(n,x)}{n} = \frac{h(S)}{h(T)} \quad a.e.\,(\lambda)$$

PROOF. Since (S,μ_1) and (T,μ_2) are ergodic, by Shannon-McMillan-Breiman theorem,

$$-\frac{\log \mu_1(P_n(x))}{n} \to h(S) a.e.\,(\mu_1))$$

and

$$-\frac{\log \mu_2(Q_n(x))}{n} \to h(T) a.e.\,(\mu_2),$$

where $h(S) > 0$ and $h(T) > 0$. Since μ_1 and μ_2 are equivalent to λ and have bounded densities with respect to λ, it follows that in the above limits one can replace μ_1 and μ_2 by λ. By theorem 3, the result follows. □

4. Convergence in measure

In general, a sequence of interval partitions $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ may have entropy in measure with respect to a measure λ, but no entropy a.e. (λ). However, given a pair of such sequences of partitions, and with one additional hypothesis, we obtain a convergence in measure result analogous to the previous theorem.

THEOREM 5. *Let $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ and $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ be sequences of interval partitions and λ a Borel measure on $[0,1)$. Suppose that for some constants $c > 0$ and $d > 0$, $\mathcal{P}$* has entropy c in measure *and $\mathcal{Q}$* has entropy d in measure. *Suppose in addition that for each n, Q_n is refined by Q_{n+1}. Then*

$$\frac{m_{\mathcal{P},\mathcal{Q}}(n,x)}{n} \to \frac{c}{d} \text{ in measure.}$$

PROOF. Fix $\varepsilon \in (0,1)$. Choose $\eta > 0$, so that $\eta < \varepsilon$ and $\zeta =: (c-\eta) - (1-\varepsilon)\left(c + \frac{c}{d}\eta\right) > 0$. Choose N so large that for all $n \geq N$,

$$\lambda\left\{x \mid \left|-\frac{\log \lambda(P_n(x))}{n} - c\right| < \eta\right\} > 1 - \eta$$

and

$$\lambda\left\{x \mid \left|-\frac{\log \lambda(Q_n(x))}{n} - d\right| < \eta\right\} > 1 - \eta.$$

We refer to a partition element $P_n(x)$ as (n,η)*−good* if it satisfies

$$\left|-\frac{\log \lambda(P_n(x))}{n} - c\right| < \eta$$

and make a similar definition for $Q_n(x)$. Choose $n > N$ so that $m(n) =: \left\lceil (1-\varepsilon)\frac{c}{d}n \right\rceil > N$, and $2^{-n\zeta} < \varepsilon$. Each (n,η)−good $P_n(x)$ and $(m(n),\eta)$−good $Q_m(y)$ satisfy

$$\lambda(P_n(x)) < 2^{-n(c-\eta)}$$

and

$$\lambda(Q_m(y)) > 2^{-m(n)(d+\eta)}$$

so that

$$\frac{\lambda(P_n(x))}{\lambda(Q_m(y))} < \frac{2^{-n(c-\eta)}}{2^{-m(n)(d+\eta)}} \leq 2^{-n\zeta} < \varepsilon.$$

Consider the set $B_n = \left\{x \in [0,1) \mid P_n(x) \nsubseteq Q_{m(n)}(x)\right\}$.We have

$$\begin{aligned} B_n \subset\ & \{x \mid P_n(x) \text{ is not } (n,\eta) - \text{good.}\} \\ & \cup \{x \mid P_n(x) \text{ is disjoint from all } (m(n),\eta) - \text{good } Q_{m(n)}(y)\ .\} \\ & \cup \left\{x \mid \begin{array}{c} P_n(x) \text{ is } (n,\eta) - \text{good, } P_n(x) \text{ meets some } (m(n),\eta) - \text{good} \\ Q_{m(n)}(y)\text{, but } P_n(x) \text{ is not contained in } Q_{m(n)}(x)\,. \end{array}\right\}. \end{aligned}$$

Each of the first two of these sets has measure less than η. If x is in the third set, then $P_n(x)$ contains an endpoint of some $(m(n),\eta)$−good $Q_{m(n)}(y)$. No other element of the partition P_n could contain that endpoint. Thus there can be at most two such P_n elements for each (m,η)−good Q_m element. Because of the ratio of the lengths of (n,η)−good P_n and (m,η)−good Q_m elements, we see that the measure of the third set cannot exceed 2ε. Thus $\lambda(B_n) < 2\varepsilon + 2\eta < 4\varepsilon$.

If $x \in [0,1]\backslash B_n$, then $m_{\mathcal{P},\mathcal{Q}}(n,x) \geq m(n)$, so

$$\frac{m_{\mathcal{P},\mathcal{Q}}(n,x)}{n} \geq \frac{\left\lceil (1-\varepsilon)\frac{c}{d}n \right\rceil}{n}.$$

Since ε was arbitrary, we have shown that for all $\delta > 0$ there is an N so that for all $n > N$,

$$\lambda \left\{ x \mid \frac{m_{\mathcal{P},\mathcal{Q}}(n,x)}{n} > 1 - \delta \right\} \geq 1 - \delta.$$

On the other hand, given $\varepsilon > 0$, suppose $\eta > 0$ is chosen so that $(1+\varepsilon)\left(c - \frac{c}{d}\eta\right) - (c+\eta) > 0$ and N is chosen as before. Let $n \geq N$ and let m' denote any integer greater than $(1+\varepsilon)\frac{c}{d}n$. Then each (n,η) −good element of P_n has measure greater than the measure of each (m',η) −good element of $Q_{m'}$ Therefore

$$\begin{aligned} C_n =: &\{x \mid P_n(x) \subset Q_{m'}(x)\} \\ &\subset \{x \mid P_n(x) \text{ is not } (n,\eta) - \text{good}\} \\ &\quad \cup \{x \mid Q_{m'}(x) \text{ is not } (m',\eta) - \text{good}\} \end{aligned}$$

so that $\lambda\{x \mid P_n(x) \subset Q_{m'}(x)\} \leq 2\eta$. But for all $m > m'$, $Q_{m'}$ is refined by Q_m, so that $Q_m(x) \subset Q_{m'}(x)$. Thus if $x \in [0,1)\backslash C_n$, and $m > m'$, $P_n(x) \nsubseteq Q_m(x)$. In other words, for each $x \in [0,1]\backslash C_n$, $m_{\mathcal{P},\mathcal{Q}}(n,x) < m'$ so that

$$\frac{m_{\mathcal{P},\mathcal{Q}}(n,x)}{n} \leq \frac{(1+\varepsilon)\frac{c}{d}n}{n}.$$

Since ε was arbitrary, we have shown that for all $\delta > 0$ there is an N so that for all $n > N$,

$$\lambda \left\{ x \mid \frac{m_{\mathcal{P},\mathcal{Q}}(n,x)}{n} < (1+\delta)\frac{c}{d} \right\} \geq 1 - \delta,$$

which completes the proof. □

Remark 1. *The above results (and hence the application) can be extended to sequences of interval partitions $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ and $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ of entropy c and d, where exactly one of the numbers c and d is zero. If $c = 0$ and $d \neq 0$, we can conclude that*

$$\lim_{n\to\infty} \frac{m_{\mathcal{P},\mathcal{Q}}(n,x)}{n} = 0.$$

In the reverse case we conclude

$$\lim_{n\to\infty} \frac{m_{\mathcal{P},\mathcal{Q}}(n,x)}{n} = \infty.$$

The proofs are easy adaptations of the ones given.

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