Travelling waves solutions to the K-P-P equation at the critical wave speed: continuing Simon Harris' probabilistic analysis

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Abstract

Recently Harris (1999), using probabilistic arguments alone, has given new proofs of the (known) existence, asymptotics and uniqueness of travelling wave solutions to the K-P-P equation. This paper is a sequel to Kyprianou (2000b) which provides alternative probabilistic arguments for supercritical wave speeds. We complete our probabilistic analysis here for the more difficult case of critical wave speeds. The analysis is centered around the study of additive and multiplicative martingales and the construction of size-biased measures on a space of non-homogenous marked trees generated by a truncated branching Brownian motion. As part of our results, we also obtain a martinale convergence theorem for the derivative of the additive martingale. Some of the main ideas are inspired by the techniques found in Kyprianou and Biggins (2000) and Lyons (1997). The value of these new probabilistic proofs is their generic nature which in principle can be generalized to study other types of spatial branching diffusions and associated travelling waves.

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1 Introduction

A branching Brownian motion is constructed as follows. An initial ancestor begins its existence at the origin of one-dimensional Euclidean space and time. This individual is immortal and moves according to an independent copy of standard Brownian motion B. The initial ancestor produces a random number of offspring, X, at times which form a Poisson process, n, with rate $\beta > 0$. We shall assume that X has distribution $(p_k : k \geq 0)$ such that $m := \sum_{k \geq 0} k p_k < \infty$. Starting from their point of creation on the path of their parent, each of these children moves and reproduces according to an independent copy of the triple (B, n, X). Let Z_t be the point process describing the number and positions of individuals alive at time t, $\{\Xi_k(t) : k = 1, ..., Z_t(\mathbb{R})\}$. In this text we shall use the Ulam-Harris labelling notation such that an individual u is identified by its line of decent from the initial ancestor. That is, if $u = (i_1, ..., i_{n-1}, i_n)$ then she is the i_nth child of the i_{n-1}th child of ....of the i_1th child of the initial ancestor. Thus uv refers to the individual who, from u's perspective, has line of descent expressed as v.

A natural martingale that arises in branching Brownian motion is of the form

$$W_t(\lambda) := \sum_{u \in N_t} e^{-\lambda(\Xi_u(t) + c_\lambda t)}$$

for t and λ positive, where N_t is the set of individuals alive at time t and $c_\lambda = \lambda/2 + \beta m/\lambda$. See Chauvin (1991), Kingman (1975), Biggins (1977) and Neveu (1988) for further details. Recently there has been a lot of interest in the construction of so called size-biased probability measures on branching trees. These size-biased measures have been used in conjunction with a fundamental measure theoretic result by Lyons *et al.* (1995) and others to show necessary and sufficient conditions for the convergence of additive martingales similar to $W_t(\lambda)$ within the context of a variety of different branching processes. For further references, see Lyons (1997), Kurtz *et al.* (1997), Olofsson (1998) and Athreya (1999).

In the case of branching Brownian motion, it is known that $W(\lambda) := \lim_{t \uparrow \infty} W_t(\lambda)$ exists almost surely and is non-negative. Further, $W(\lambda) \equiv 0$ for $\lambda \geq \sqrt{2\beta m}$ and when $\lambda \in [0, \sqrt{2\beta m})$, $W(\lambda)$ is also the L^1 martingale limit. In this paper we shall use once again the method of size-biased measures by applying it to a study of the limit of the negative derivative of $W_t(\lambda)$

with respect to λ. More specifically,

$$\begin{aligned} \partial W_t(\lambda) &: = -\frac{\partial}{\partial t} W_t(\lambda) \\ &= \sum_{u \in N_t} \left(\Xi_u(t) + \lambda t\right) e^{-\lambda(\Xi_u(t) + c_\lambda t)} \end{aligned}$$

where t and λ are both positive. This martingale we shall refer to from now as the *derivative martingale*.

Interest in the limit of the derivative martingale is stimulated by its intimate connection with travelling wave solutions to the Kolmogorov-Petrovski-Piscounov equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u(f(u) - 1), \tag{1}$$

where $f(s) = E\left(s^X\right)$, taking solutions $u : \mathbb{R} \times \mathbb{R}^+ \to [0,1]$. This reaction-diffusion equation has been studied by many authors both probabilistically and analytically[see references Kolmogorov *et al.*(1937), Fisher (1937), Skorohod (1964) , McKean (1975), Bramson (1978, 1983), Neveu (1988), Uchiyama (1978), Aronson and Wienburger (1975) Karpelevich *et al.* (1993) and Kelbert and Suhov (1995) to name but a few]. Of particular interest however is the recent exposition of Harris (1999) which, using probabilistic arguments *alone*, gives an excellent derivation of the well known existence, uniqueness and asymptotics of travelling wave solutions to (1). By a travelling wave solution it is meant a twice continuously differentiable, monotone increasing function $\Phi_c : \mathbb{R} \to [0,1]$ such that $\Phi_c(-\infty) = 0 = 1 - \Phi_c(\infty)$ with $u(x,t) = \Phi_c(x - ct)$ a solution to (1); $c \geq 0$ is the wave speed. A short calculation shows that Φ_c also solves the ordinary differential equation

$$\frac{1}{2}\Phi_c'' + c\Phi_c' + \Phi_c\left(f(\Phi_c) - 1\right) = 0. \tag{2}$$

We shall now give a brief account of the connection between these travelling waves and the limits of $W_t(\lambda)$ and $\partial W_t(\lambda)$. For further information, one should consult Neveu (1988), McKean (1975), Chauvin (1993) and for a complete account, Harris (1999).

It is known (by both probabilistic and analytical methods) that under these circumstances, travelling waves exist if and only if the speed $c \geq \underline{c} := \sqrt{2\beta m}$. We can parameterize wave speeds such that $0 \leq c \in \{c_\lambda : \lambda \geq 0\}$, in which case the critical wave speed, $\underline{c}$, occurs when $\lambda = \sqrt{2\beta m} =: \underline{\lambda}$. When

$c_\lambda > \underline{c}$ (supercriticality) such that $\lambda \in [0, \underline{\lambda})$ there exists a unique (modulo an additive constant in the argument) travelling wave which can be expressed as the exponentially rescaled Laplace transform of $W(\lambda)$. That is to say

$$\Phi_{c_\lambda}(x) = E\left(\exp\{-e^{-\lambda x} W(\lambda)\}\right).$$

At criticality, when $\lambda = \underline{\lambda}$ and $W(\underline{\lambda}) \equiv 0$, there is again a unique travelling wave (modulo an additive constant in the argument) which this time is the exponentially rescaled Laplace transform of $\lim_{t\uparrow\infty} \partial W_t(\underline{\lambda})$ thus implying that the limit exists and has all its mass in $(0, \infty)$. Thus

$$\Phi_{\underline{c}}(x) = E\left(\exp\left\{-e^{-\underline{\lambda} x} \partial W(\underline{\lambda})\right\}\right).$$

In dealing with travelling waves at the critical wave speed, Harris (1999) proves existence and uniqueness using the connection between a Seneta-Heyde norming result, multiplicative martingales and the known asymptotics of $1 - \Phi_{\underline{c}}(x)$. As a consequence of his proofs he simultaneously recovers the existence of the almost sure limit of $\partial W_t(\underline{\lambda})$ to a strictly positive variable as t tends to infinity. Harris also conjectures that the proof of existence and uniqueness should be possible by first establishing directly the almost sure convergence of $\partial W_t(\underline{\lambda})$ to a strictly positive limit. By stopping the martingale $\partial W_t(\underline{\lambda}) + x W_t(\underline{\lambda})$ appropriately, he is able to show directly the almost sure convergence of $\partial W_t(\underline{\lambda})$ to a non-negative limit. However, his method does not establish that the limit has strictly positive mass. Hence in this paper we shall expose once again how probabilistic arguments lead to an analysis of critical travelling waves to the K-P-P equation by following the path conjectured by Harris.

The reason for pursuing probabilistic proofs of the existence uniqueness and asymptotics of these travelling waves goes deeper than pure aesthetics. The method we present here is inspired by techniques used in Kyprianou and Biggins (2000) which studies a discrete time analogue of the ODE (2). It is anticipated that the probabilistic view will also shed more light on the problem of understanding the asymptotic behaviour of the position of right most particle in a spatial branching diffusions. Bramson (1978, 1983) has already treated branching Brownian motion in this respect, but the rightmost particle issue remains unresolved for the branching random walk and other types of spatial branching diffusions.

In the next section we shall discuss the close connection between Brownian motion and Bessel-3 processes which form a fundamental part of the

analysis presented in later sections. In section 3 we introduce a new positive martingale, $\{xV_t^x(\lambda)\}_{t\geq 0}$, that approximates $\{\partial W_t(\lambda) + xW_t(\lambda)\}_{t\geq 0}$. This new martingale now becomes the focus of our study as its limit can be made to look like the limit of $\partial W_t(\lambda)$ as t tends to infinity when $\lambda \geq \underline{\lambda}$. In section 4 we construct a non-homogenous branching diffusion by truncating ancestral paths that contribute to large negative summands in $\partial W_t(\lambda)$ as t tends to infinity. This procedure is inspired by a similar construction demonstrated for the branching random walk in Kyprianou and Biggins (2000). The truncation is done in a way that, when $\lambda \geq \underline{\lambda}$, the new martingale, $xV_\lambda^x(t)$, and $\partial W_t(\lambda)$ have the same limit on the part of the event space where the non-homogenous and homogenous trees are the same. By applying a method of size-biasing, we are able in section 5 to investigate the positivity of the limit of $V_\lambda^x(t)$ for the cases $\lambda \geq \underline{\lambda}$. By arbritrarily relaxing the severity of the truncation (which depends on the parameter x), we infer the existence and location of mass of the limit of $\partial W_t(\lambda)$ for $\lambda \geq \underline{\lambda}$. In section 6 we apply our results to establishing the existence, asymptotics and uniqueness of travelling wave solutions at the critical wave speed. Essentially once we are in a possession of the limit of $\partial W_t(\underline{\lambda})$, the existence follows easily. We use a stopping line argument as in kyprianou (2000b) to establish the asymptotic

$$1 - \phi_{\underline{c}}(x) \sim k \times xe^{-\underline{\lambda}x}$$

for some constant $k > 0$. This tail behaviour of the travelling wave induces uniqueness, again with little effort.

As in its companion Kyprianou (2000b), this paper contains more proofs than Theorems and Lemmas. Whilst the *proofs* we offer are new, we have only presented new *results* in the context of a Lemma or Theorem.

2 Brownian motion and Bessel-3 processes

Let us begin by recalling some results that show the intimate relationship between Brownian motion and Bessel-3 processes that will be of use later in this paper. Our main references are McKean (1963), Williams (1974), Pitman (1975) and Bertoin (1993).

Suppose that with respect to the law $\mathbb{P}$, $B = \{B_t\}_{t\geq 0}$ is a standard Brownian motion started at 0 and $\mathbb{P}_t$ is its restriction to $\sigma(B_s : s \leq t)$. Define $\rho^\lambda(B) = \inf\{t > 0 : B_t + \lambda t = 0\}$, the first time that B meets the space-time line passing through the origin with negative gradient λ. Now let $\mathbb{Q}^{(x,\lambda)}$ be

the law of a Bessel-3 process on $(-x,\infty)$ started from 0 with negative drift λ. It is known that

$$\frac{d\mathbb{Q}_t^{(x,0)}}{d\mathbb{P}_t}(B) = \frac{x+B_t}{x} I(\rho^0(x+B) > t).$$

Write

$$\Lambda_t^{(x,\lambda)}(B) = \frac{(x+B_t+\lambda t)}{x} e^{-\lambda\left(B_t+\frac{1}{2}\lambda t\right)} I(\rho^\lambda(x+B) > t)$$

and note that this expression has the multiplicative decomposition

$$\Lambda_t^{(x,\lambda)}(B) = \Lambda_s^{(x,\lambda)}(B) \times \tilde{\Lambda}_{t-s}^{(x+B_s+\lambda s,\lambda)}\left(\tilde{B}\right) \tag{3}$$

where $s \in (0,t)$ and $\tilde{\Lambda}$ and $\tilde{B}$ are an independent copies of Λ and B respectively. For any $\mathcal{A} \subseteq \mathcal{B}(-x,\infty)$

$$\begin{aligned}
\int I(B_t &\in \mathcal{A}) \Lambda_t^{(x,\lambda)}(B)\, d\mathbb{P}_t \\
&= E\left[I\left(\rho^\lambda(x+B) > t, B_t \in \mathcal{A}\right) \frac{x+B_t+\lambda t}{x} e^{-\lambda\left(B_t+\frac{1}{2}\lambda t\right)}\right] \\
&= E\left[I\left(\rho^0(x+B) > t, B_t - \lambda t \in \mathcal{A}\right) \frac{x+B_t}{x}\right] \\
&= \int I(B_t - \lambda t \in \mathcal{A}) d\mathbb{Q}_t^{(x,0)} \qquad (4)
\end{aligned}$$

where the second equality follows by the Cameron-Martin-Girsanov Theorem. In particular choosing $\mathcal{A} = (-x,\infty)$ we have $\int \Lambda_t^{(x,\lambda)}(B)\, d\mathbb{P}_t = 1$ for all $x,t > 0$. Therefore, in view of the decomposition (3), $\left\{\Lambda_t^{(x,\lambda)}(B)\right\}_{t>0}$ is a $\mathbb{P}$-martingale. Further, from the computation in (4) one sees that

$$\frac{d\mathbb{Q}_t^{(x,\lambda)}}{d\mathbb{P}_t}(B) = \Lambda_t^{(x,\lambda)}(B)$$

for all $t > 0$. The work of Bertoin (1993) or McKean (1963), for example, shows that $\mathbb{Q}^{(x,0)}$ is also the law of a Brownian motion started at x and conditioned not to enter the half-line $(-\infty,0]$. The calculation (4) shows that $\mathbb{Q}^{(x,\lambda)}$ is thus the law of a Brownian motion started at x and conditioned not to meet the space-time barrier

$$\Gamma^{(x,\lambda)} := \left\{(y,t) \in \mathbb{R} \times \mathbb{R}^+ : y + \lambda t + x = 0\right\}$$

3 Branching Brownian motion with a barrier

Suppose for each $t > 0$ in the branching Brownian motion we define a subset of N_t, say $\tilde{N}_t$, consisting of all individuals whose ancestral paths on the branching tree have not met the space-time barrier $\Gamma^{(x,\lambda)}$ by time t. Define

$$V_t^x(\lambda) = \sum_{u \in \tilde{N}_t} \frac{(x + \Xi_u(t) + \lambda t)}{x} e^{-\lambda(\Xi_u(t) + c_\lambda t)}.$$

Given that the summands associated with each individual u looks like a version of $\Lambda_t^{(x,\lambda)}(B) \times e^{-\beta m t}$ under $\mathbb{P}$ and that $\tilde{N}_t$ grows at a rate on average that should be close to the original growth rate $e^{\beta m t}$, we could intuitively reason that $EV_t^x(\lambda) = 1$ for all $x, t > 0$. The following Lemma builds upon this idea.

Lemma 1 *The sequence $\{V_t^x(\lambda)\}_{t>0}$ is a mean 1 martingale with respect to $\{\mathcal{F}_t\}_{t>0}$ (the sigma algebra describing the evolution of the branching Brownian motion up to time t).*

Before proceeding to the proof we should make some remarks about why this is martingale is of particular interest to the problem at hand. Let us temporarily assume that $\lambda \geq \underline{\lambda}$. Note that $V_t^x(\lambda)$ is always positive and therefore has a limit almost surely. Further, we can identify $xV_t^x(\lambda)$ as contributing to the positive part of the martingale $\partial W_t(\lambda) + xW_t(\lambda)$ and thus serves as an approximation to it. In fact, on the event (which we shall call $\gamma^{(x,\lambda)}$) that the branching Brownian motion remains entirely to the right of $\Gamma^{(x,\lambda)}$, the truncated process and the original process are the same and thus so are the limits of $xV_t^x(\lambda)$ and $\partial W_t(\lambda) + xW_t(\lambda)$. Since for $\lambda \geq \underline{\lambda}$ the martingale limit $W(\lambda) \equiv 0$ then on $\gamma^{(x,\lambda)}$ we have

$$\lim_{t \uparrow \infty} xV_t^x(\lambda) = \lim_{t \uparrow \infty} \partial W_t(\lambda).$$

Denoting by L_t the position of the left most particle in the branching Brownian motion, since we know $W(\underline{\lambda}) \equiv 0$ it follows that $\lim_{t\uparrow\infty} L_t + \underline{\lambda} t = \infty$ and hence $\inf_{t \geq 0}\{L_t + \lambda t\} > -\infty$ almost surely. Consequently

$$P\left(\gamma^{(x,\lambda)}\right) = P\left(\inf_{t \geq 0}\{L_t + \lambda t\} > -x\right) \uparrow 1 \text{ as } x \uparrow \infty. \tag{5}$$

Thus we have established an almost sure limit for the derivative martingale exists almost surely and has all mass in $[0, \infty)$.

Let us return then to the proof of the Lemma.

Proof. We break the proof into two parts. The first shows that $EV_t^x(\lambda) = 1$ for all $x, t > 0$ and the second uses this result to get the martingale property.

Let $\tilde{\eta}_t$ be the intensity measure of $\tilde{Z}_t$. Watanabe (1964) has studied the branching Brownian motion with an absorbing barrier. From his work it can easily be deduced that for $y + \lambda t + x \geq 0$,

$$\tilde{\eta}_t(dy) = e^{\beta m t} \mathbb{P}_t\left(B_t \in dy, \rho^\lambda(x + B) > t\right).$$

Thus,

$$\begin{aligned} EV_t^x(\lambda) &= \int \frac{(x + y + \lambda t)}{x} e^{-\lambda(z + c_\lambda t)} \tilde{\eta}_t(dy) \\ &= \int \frac{(x + y + \lambda t)}{x} e^{-\lambda(y + c_\lambda t)} I(\rho^\lambda(x + B) > t) e^{\beta m t} d\mathbb{P}_t(B_t \in dy) \\ &= \int \Lambda_t^{(x,\lambda)}(B)\, d\mathbb{P}_t \\ &= \int d\mathbb{Q}_t^{(x,\lambda)} \\ &= 1 \end{aligned} \tag{6}$$

where the last equality follows since the first integral is taken over

$$\{y \in \mathbb{R} : x + y + \lambda t \geq 0\}$$

which is tantamount to taking the final integral over all possible paths up to time t of a Bessel-3 process on $(-x, \infty)$ started at 0 with negative drift λ.

Since any line of descent in branching Brownian motion is itself a Brownian motion, we shall adopt the notation of the previous section and for $u \in N_t$ let

$$\Lambda_t^{(x,\lambda)}(\Xi_u) = \frac{(x + \Xi_u(t) + \lambda t)}{x} e^{-\lambda\left(\Xi_u(t) + \frac{1}{2}\lambda t\right)} I\left(\rho^\lambda(x + \Xi_u) > t\right)$$

so that

$$V_t^x(\lambda) = e^{-\beta m t} \sum_{u \in \tilde{N}_t} \Lambda_t^{(x,\lambda)}(\Xi_u)$$

For each $s > 0$, let $\theta_{u,s}$ be the shift operator which renders the individual $u \in \tilde{N}_s$ the initial ancestor. When $0 < s < t$, $u \in \tilde{N}_s$ and $uv \in \tilde{N}_t$ we have the decomposition

$$\Lambda_t^{(x,\lambda)}(\Xi_{uv}) = \Lambda_s^{(x,\lambda)}(\Xi_u) \times \widetilde{\Lambda}_{t-s}^{(x+\Xi_u(s)+\lambda s,\lambda)}(\theta_{u,s} \circ \Xi_v)$$

where $\tilde{\Lambda}$ is an independent copy of Λ. Hence

$$\begin{aligned} & E(V_t^x(\lambda) | \mathcal{F}_s) \\ = \; & E\left(e^{-\beta m s} \sum_{u \in \tilde{N}_s} \Lambda_s^{(x,\lambda)}(\Xi_u) e^{-(t-s)} \sum_{v \in \theta_{u,s} \circ \tilde{N}_{t-s}} \tilde{\Lambda}_{t-s}^{(x+\Xi_u(s)+\lambda s,\lambda)}(\theta_{u,s} \circ \Xi_v) \middle| \mathcal{F}_s \right) \\ = \; & e^{-\beta m s} \sum_{u \in \tilde{N}_s} \Lambda_s^{(x,\lambda)}(\Xi_u) E\left(V_{t-s}^{(x+\Xi_u(s)+\lambda s)}(\lambda, u) \middle| \mathcal{F}_s \right) \end{aligned} \tag{7}$$

where $V_{t-s}^{(\cdot)}(\lambda, u)$ are independent copies of $V_{t-s}^{(\cdot)}(\lambda)$. Since $EV_t^x(\lambda) = 1$ for all $x, t > 0$ the proof is complete. □

4 Change of measure on non-homogenous branching diffusions

Continuing with the same notation as in the previous section we define a non-homogenous branching diffusion as follows. In branching Brownian motion, any individual whose path meets the barrier $\Gamma^{(x,\lambda)}$ is immediately killed. This process we shall refer to as *truncated branching Brownian motion.* In the truncated process, the number alive at time t is described precisely by $\tilde{N}_t$. Suppose that $\mathcal{T}^{(x,\lambda)}$ is the space of marked trees generated by the branching process truncated at $\Gamma^{(x,\lambda)}$. Let $\mu^{(x,\lambda)}$ be the probability measure on $\mathcal{T}^{(x,\lambda)}$ that corresponds to the truncated branching Brownian motion as defined in the previous section. We use $\mu_t^{(x,\lambda)}$ for its restriction to $\mathcal{F}_t$.

For any $\tau \in \mathcal{T}^{(x,\lambda)}$, by following the spatial path of any ancestral line of decent from the origin of space and time we identify a process $\xi = (\xi_t : t \geq 0)$ which we shall refer to as a *spine.* Essentially any spine is a Brownian motion that is killed on meeting the barrier $\Gamma^{(x,\lambda)}$. Now let $\widetilde{\mathcal{T}}^{(x,\lambda)}$ be the space of marked trees in $\mathcal{T}^{(x,\lambda)}$ but with distinguished spines. If ξ is a spine that has survived the truncation by time t, then there are n_t points of fission before time t where $n = (\nu_i : i = 1, ..., n_t)$ is a Poisson process of rate β.

Further at the i-th point of fission, there are $X_i \geq 0$ new trees growing $\left\{\tau_j \in \mathcal{T}^{\left(\xi_{\nu_i}+\lambda\nu_i+x,\lambda\right)} : 0 \leq j \leq X_i\right\}$ where X_i is an independent copy of X for each i. We construct the (non-probability) measure $\widetilde{\mu}_t^{(x,\lambda)}$ $(t \geq 0)$ so that for $(\tau, \xi) \in \widetilde{\mathcal{T}}^{(x,\lambda)}$

$$d\widetilde{\mu}_t^{(x,\lambda)}(\tau,\xi) = I(\rho^\lambda(\xi+x) > t) d\mathbb{P}_t(\xi)\, d\mathbb{L}_t^{(\beta)}(n) \prod_{i=1}^{n_t} p_{X_i} \prod_{j=1}^{X_i} d\mu_{t-\nu_i}^{\left(\xi_{\nu_i}+\lambda\nu_i+x,\lambda\right)}(\tau_j)$$

where $\mathbb{L}_t^{(\beta)}$ is the law of a Poisson process with rate β up to time t and empty products are understood to be equal to 1. This new measure is a decomposition of μ_t over $\widetilde{\mathcal{T}}^{(x,\lambda)}$ so that

$$d\mu_t^{(x,\lambda)}(\tau) = \sum_{u \in \tilde{N}_t} I(\Xi_u(t) = \xi_t)\, d\widetilde{\mu}_t^{(x,\lambda)}(\tau,\xi) \tag{8}$$

for all $t \geq 0$. Consider also the bivariate probability measure $\widetilde{\pi}_t$ $(t \geq 0)$ on $\widetilde{\mathcal{T}}^{(x,\lambda)}$ where

$$\begin{aligned}
d\widetilde{\pi}_t^{(x,\lambda)}(\tau,\xi) &= \Lambda^{(x,\lambda)}(\xi) d\widetilde{\mu}_t^{(x,\lambda)}(\tau,\xi) \\
&= \frac{x+\xi_t+\lambda t}{x} \times e^{-\lambda\left(\xi_t+\frac{1}{2}\lambda t\right)} \times e^{-\beta m t}(m+1)^{n_t} \\
&\qquad \times \prod_{i=1}^{n_t} \frac{X_i+1}{m+1} \times \frac{1}{X_i+1} \times d\widetilde{\mu}_t^{(x,\lambda)}(\tau,\xi) \\
&= d\mathbb{Q}_t^{(x,\lambda)}(\xi) \times d\mathbb{L}_t^{(\beta(m+1))}(n) \\
&\qquad \times \prod_{i=1}^{n_t} \frac{X_i+1}{m+1} p_{X_i} \times \frac{1}{X_i+1} \times \prod_{j=1}^{X_i} d\mu_{t-\nu_i}^{\left(\xi_{\nu_i}+\lambda\nu_i+x,\lambda\right)}(\tau_j)
\end{aligned} \tag{9}$$

In the second equality, we have used the fact that if $\mathbb{L}_t^{((m+1)\beta)}$ is the law of a Poisson process up to time t with rate $(m+1)\beta$ then

$$\frac{d\mathbb{L}_t^{((m+1)\beta)}}{d\mathbb{L}_t^{(\beta)}}(n) = e^{-\beta m t}(m+1)^{n_t}.$$

Equality (8) can be used to confirm that $\widetilde{\pi}_t$ has total mass equal to 1. Using (8) again, we can marginalize the measure $\widetilde{\pi}_t^{(x,\lambda)}$ to $\mathcal{T}^{(x,\lambda)}$ so that we have

probability measure $\pi_t^{(x,\lambda)}$ $(t \geq 0)$ which satisfies

$$\frac{d\pi_t^{(x,\lambda)}}{d\mu_t^{(x,\lambda)}} = V_t^x(\lambda) \tag{10}$$

for all $t \geq 0$.

The definition of $\widetilde{\pi}$ with its decomposition in (9) corresponds to a branching Brownian motion truncated at $\Gamma^{(x,\lambda)}$ having distinguished spine ξ such that:

(i) the process $(\xi_t + \lambda t : t \geq 0)$ moves according to a Bessel-3 process on $(-x, \infty)$ started at 0 (that is to say the spine moves away from the barrier $\Gamma^{(x,\lambda)}$ as a Bessel-3 processes and therefore never meets it),

(ii) the points of fission along the spine form a Poisson process with accelerated rate $(m+1)\beta$,

(iii) the distribution of offspring numbers at each point of fission on the spine has tilted measure $(\widetilde{p}_k = (k+1)p_k/(m+1) : k \geq 0)$, and finally

(iv) the spine is chosen randomly so that at each fission point, the next individual to represent the spine is chosen with uniform probability from the current representative and its offspring.

5 Convergence of $\partial W_t(\lambda)$ for $\lambda \geq \underline{\lambda}$

Pursuing the same line as Lyons (1997), Kurtz *et al.* (1995), Olofsson (1998) and Athreya (1999) we define for any $x > 0$, $V^x(\lambda) = \limsup_{t\uparrow\infty} V_t^x(\lambda)$ (which is also equal to $\lim_{t\uparrow\infty} V_t^x(\lambda)$ μ-a.s.) and check whether this limit has any μ-mass away from zero by using the following fundamental result from measure theory [see Durrett (1991) or Athreya (1999)]

$$V^x(\lambda) = \infty \quad \pi\text{-a.s} \implies V^x(\lambda) = 0 \quad \mu\text{-a.s.}, \tag{11}$$

$$V^x(\lambda) < \infty \quad \pi\text{-a.s} \implies \int V^x(\lambda)\, d\mu = 1. \tag{12}$$

In view of the discussion following Lemma 1, it then follows that the limit of ∂W_t exists and has all its mass concentrated at zero or in $(0, \infty)$ accordingly as $V^x(\lambda)$ does (for any $x > 0$).

Theorem 1 *For $\lambda \geq \underline{\lambda}$, $\partial W(\lambda) := \lim_{t\uparrow\infty} \partial W_t(\lambda)$ exists almost surely and is valued on $[0,\infty)$. Further, if $\lambda = \underline{\lambda}$ and $E(X(\log X)^{2+\delta}) < \infty$ for some $\delta > 0$ then its limit is strictly positive with probability one and if $\lambda > \underline{\lambda}$ or $E(X(\log X)^{2-\delta}) = \infty$ for some $\delta > 0$ the limit is identically zero with probability one.*

Before proceeding to the proof we need to note a few things about the behaviour of Bessel-3 processes and also the asymptotics of the sequence $(X_n : n \geq 1)$ of independent copies of X representing the numbers of offspring of ξ at each point of fission.

Firstly, from Theorem 3.2 of Shiga and Watanabe (1973) it is easy to show that $\mathbb{Q}^{(x,0)}$-almost surely, the path of a Bessel-3 process *eventually* (in the sense of the Borel-Cantelli Lemma) becomes bounded by the curves $t^{\frac{1}{2}+\epsilon}$ and $t^{\frac{1}{2}-\epsilon}$ for any $\epsilon > 0$. Secondly a simple calculation shows that when $q > 0$ $E(X(\log X)^q)$ is (in)finite if and only if

$$\sum_{n\geq 1} P_{\tilde{\pi}}(\log X > cn^{\frac{1}{q}})$$

is (in)finite for any $c > 0$. Consequently, by the Borel-Cantelli Lemma, $\limsup_{n\uparrow\infty} n^{-\frac{1}{q}} \log X_n$ is (infinite) zero according whether the given moment is (in)finite.

Proof. As previously mentioned, the discussion following Lemma 1 deals with the first part of this Theorem. For the second part, we shall first consider the case that $\lambda > \underline{\lambda}$. By construction we have the lower bounds

$$V_t^x(\lambda) \geq \frac{(x + \xi_t + \lambda t)}{x} e^{-\lambda(\xi_t + c_\lambda t)}. \tag{13}$$

Recall that under $\widetilde{\pi}$, $\{x + \xi_t + \lambda t\}_{t>0}$ is a Bessel-3 process on $(0,\infty)$ started at zero. As this process eventually grows no faster than $t^{\frac{1}{2}+\epsilon}$ for any $\epsilon > 0$ and since for $\lambda > \underline{\lambda}$, $c_\lambda < \lambda$ it becomes clear from (13) that

$$\limsup_{t\uparrow\infty} V_t^x(\lambda) = \infty \ \widetilde{\pi}\text{-a.s.}$$

and hence $V^x(\lambda)$ and $\partial W(\lambda)$ are identically zero μ-almost surely.

Now suppose that $\lambda \geq \underline{\lambda}$ (so that $c_\lambda \leq \lambda$) and $E(X(\log X)^{2-\delta}) = \infty$ for some $\delta > 0$. Without loss of generality we can assume that δ is small. We have the lower bound

$$V^x_{\nu_k}(\lambda) \geq X_k \frac{(x+\xi_{\nu_k}+\lambda\nu_k)}{x} e^{-\lambda\left(\xi_{\nu_k}+\lambda\nu_k\right)}. \tag{14}$$

By the law of large numbers, $\nu_k \sim [\beta(m+1)]^{-1} k$. We deduce that $(x+\xi_{\nu_k}+\lambda\nu_k)$ eventually grows no faster than $O(k^{\frac{1}{2}+\epsilon})$ for any $\epsilon > 0$. Hence by choosing ϵ sufficiently large, (14) shows that with the given moment condition, $\limsup_{t\uparrow\infty} V^x_t(\lambda) = \infty$ $\widetilde{\pi}$-a.s.. Thus the result follows as before.

Finally suppose that $\lambda = \underline{\lambda}$ (so that $c_\lambda = \underline{c} = \underline{\lambda}$) and $E(X(\log X)^{2+\delta}) < \infty$ for some $\delta > 0$. Again without loss of generality we may assume that δ is small. Let $\mathcal{G}$ be the sigma-algebra generated by the diffusion of the spine ξ, the Poisson process n representing the times of reproduction and $(X_k : k \geq 1)$. We have

$$E_{\widetilde{\pi}}\left(V^x_t(\underline{\lambda})\middle|\,\mathcal{G}\right) = \sum_{i=1}^{n_t} X_i \frac{(x+\xi_{\nu_i}+\underline{\lambda}\nu_i)}{x} e^{-\underline{\lambda}\left(\xi_{\nu_i}+\underline{\lambda}\nu_i\right)} + \frac{(x+\xi_t+\underline{\lambda}t)}{x} e^{-\underline{\lambda}(\xi_t+\underline{\lambda}t)}$$

Taking advantage of the fact that a Bessel-3 process eventually grows no slower than $t^{\frac{1}{2}-\epsilon}$ for any $\epsilon > 0$, the law of large numbers applied to n and the sequence $(\nu_i : i \geq 1)$ plus the given moment condition implies that

$$\lim_{t\uparrow\infty} E_{\widetilde{\pi}}\left(V^x_t(\underline{\lambda})\middle|\,\mathcal{G}\right) < \infty \ \widetilde{\pi}\text{-a.s.}$$

Fatou's Lemma implies that $\liminf_{t\uparrow\infty} V^x_t(\underline{\lambda})$ is also finite π-almost surely. The Radon-Nikodym derivative (10) tells us that $V^x_t(\underline{\lambda})^{-1}$ is π-martingale and therefore has a limit π-almost surely, which by the previous statement must be almost surely finite. This result together with the remarks following Lemma 1 shows that $\partial W(\underline{\lambda})$ has at least some of its mass in $(0,\infty)$. Harris (1999) shows through simple probabilistic means that if the derivative martingale limit exists, then its mass is either all concentrated on $(0,\infty)$ or all at zero with probability one. This concludes the proof. □

Remark 1 *In principle finer moment conditions may be established for the conclusions of this Theorem to hold. In the above Theorem, the moment conditions appear as a consequence of working in the proof with the eventual upper and lower bounds $t^{\frac{1}{2}\pm\epsilon}$ for the paths of Bessel-3 processes. Referring back to Theorem 3.2 of Shiga and Watanabe and also the Law of the Iterated Logarithm for Bessel processes, it is possible to establish a tighter space time envelope that eventually contains sample paths of Bessel-3 processes. An improvment on the given moment conditions above should thus follow.*

6 Existence, asymptotics and uniqueness at criticality

We are now ready to show existence and uniqueness of travelling wave solutions to the K-P-P equation at the critical wave speed. With the exception of the subsection on asymptotics, most of the arguments used below are not new. We simply re-employ existing knowledge in a slightly different context; namely that we are now in possession of the strictly positive limit $\partial W(\underline{\lambda})$. To see the (probabilistic) origin of the arguments used below, one should consult Lalley and Sellke (1987), Neveu (1988), Chauvin (1991), Biggins and Kyprianou (1997), Harris (1999), Kyprianou (1998, 1999).

6.1 Existence

It is very easy (if not a little messy) to decompose $\partial W_t(\underline{\lambda})$ into contributions derived from the population at time $0 < s < t$ as was done in (7). We have

$$\begin{aligned} \partial W_t(\underline{\lambda}) &= \sum_{u \in N_s} e^{-\underline{\lambda}(\Xi_u(s)+\underline{\lambda}s)} \partial W_{t-s}(\underline{\lambda}, u) \\ &\quad + \sum_{u \in N_s} (\Xi_u(s) + \underline{\lambda}s)\, e^{-\underline{\lambda}(\Xi_u(s)+\underline{\lambda}s)} W_{t-s}(\underline{\lambda}, u), \end{aligned} \tag{15}$$

where, for each $u \in N_s$, $\partial W_{t-s}(\underline{\lambda}, u)$ and $W_{t-s}(\underline{\lambda}, u)$ are independent copies of $\partial W_{t-s}(\underline{\lambda})$ and $W_{t-s}(\underline{\lambda})$ respectively. Using obvious notation, as t tends to infinity we thus recover the smoothing transform

$$\partial W(\underline{\lambda}) = \sum_{u \in N_s} e^{-\underline{\lambda}(\Xi_u(s)+\underline{\lambda}s)} \partial W(\underline{\lambda}, u) \tag{16}$$

which holds for all $s > 0$. The identity (16) can be also expressed in terms of Laplace transforms. That is to say, if $\Phi(x) := E\left(\exp\left\{-e^{-\underline{\lambda}x}\partial W(\underline{\lambda})\right\}\right)$, then Φ satisfies the functional equation

$$\Phi(x) = E\left[\prod_{u \in N_s} \Phi(x + \Xi_u(s) + \underline{\lambda}s)\right]. \tag{17}$$

for all $s > 0$. Theorem 8 of Kyprianou (1999) concludes that this Φ solves the functional equation (17) if and only if is also solves the ODE (2). Hence we have shown existence.

6.2 Asymptotics

For uniqueness we need to use the known asymptotic behaviour of solutions to (17); namely that a critical travelling wave solution $\Phi_{\underline{\lambda}}$ satisfies

$$1 - \Phi_{\underline{c}}(x) \sim k \times x e^{-\underline{\lambda} x} \text{ as } x \to \infty, \tag{18}$$

where k is a positive constant. Harris (1999) gives a probabilistic derivation of this asymptotic using martingales built from the travelling waves via the Feynman-Kac formula. Analytical methods are also available to establish this result. See Uchyiama (1978) and Bramson (1983). We shall offer here yet another probabilistic derivation of the required asymptotics using additive and multiplicative martingales alone. The advantage of the technique we present is that it is not specific to this type of spatial branching process. In principle, analogues of the above asymptotic could be achieved using this method for other types of spatial branching process.

For branching Brownian motion, we can also consider the barrier $\Gamma^{(-z,\underline{\lambda})}$ where $z > 0$. By arresting lines of decent the first time they hit this barrier we produce a random collection of individuals, namely the *stopping line* C_z. For further information on stopping lines, their rigorous definition and properties, one should consult Chauvin (1991) or Neveu (1988). What is important to note for our purposes is that $\{C_z\}_{z\geq 0}$ is a sequence of dissecting stopping lines tending to infinity on which the branching property holds and whose cardinality, $\{|C_z|\}_{z\geq 0}$ forms a continuous time branching process (z plays the role of time). [Note that by a sequence of dissecting stopping lines tending to infinity we mean any line of decent from the initial ancestor will hit $\Gamma^{(-z,\underline{\lambda})}$ with probability one for all $z > 0$ and $\inf |u| : u \in C_z$ tends to infinity as z tends to infinity]. Let $\{\sigma_u(z) : u \in C_z\}$ be the times at which individuals meet the barrier $\Gamma^{(-z,\underline{\lambda})}$. From the afore mentioned references, it is known that

$$\prod_{u\in C_z} \Phi_{\underline{c}}(x + \Xi_u(\sigma_u(z)) + \underline{\lambda}\sigma_u(z)) = \Phi_{\underline{c}}(x+z)^{|C_z|} \qquad (x > 0)$$

is a martingale with a limit almost surely and in L^1.

Suppose we turn our attention to the branching Brownian motion truncated at $\Gamma^{(x,\underline{\lambda})}$. Define $\tilde{C}_z$ analogously as the set of individuals in the truncated process that hit the barrier $\Gamma^{(-z,\underline{\lambda})}$ for $z > 0$. Recall that on the event $\gamma^{(x,\underline{\lambda})}$ the branching Brownian motion and its truncated version are the same. Thus

we can say that on $\gamma^{(x,\underline{\lambda})}$,

$$\lim_{z\uparrow\infty} -|\tilde{C}_z| \log \Phi_{\underline{c}}(x+z)$$

exists almost surely in $[0,\infty)$ and is non-trivial in the sense that it has mass in $(0,\infty)$ with positive probability for all $x>0$. Now consider the process

$$\begin{aligned} V^x_{\tilde{C}_z} &:= \sum_{u\in\tilde{C}_z} \frac{(x+\Xi_u(\sigma_u(z))+\underline{\lambda}\sigma_u(z))}{x} e^{-\underline{\lambda}(\Xi_u(\sigma_u(z))+\underline{\lambda}\sigma_u(z))} \\ &= \frac{(x+z)}{x} e^{-\underline{\lambda}z}|\tilde{C}_z|. \end{aligned}$$

Using ideas similar to those in Kyprianou (2000b) we can prove the following Lemma. From it will follow immediately the predicted asymptotics of $\Phi_{\underline{c}}$.

Lemma 2 *Let $\left\{\mathcal{F}_{\tilde{C}_z} : z\geq 0\right\}$ be the natural filtration describing everything in the truncated branching tree up to the barrier $\Gamma^{(-z,\underline{\lambda})}$. If $E(X(\log X)^{2+\delta})<\infty$ for some $\delta>0$ then sequence $V^x_{\tilde{C}_z}$ is an $\mathcal{F}_{\tilde{C}_z}$ martingale that converges almost surely and in mean to $V^x(\underline{\lambda})$ (defined in the previous section).*

Proof. For $t>0$, let $A_t\left(\tilde{C}_z\right)=\left\{u\in\tilde{N}_t : v\notin\tilde{C}_z\ \forall v\leq u\right\}$ and define

$$V^x_{t\wedge\tilde{C}_z} = \sum_{u\in A_t\left(\tilde{C}_z\right)} \frac{(x+\Xi_u(t)+\underline{\lambda}t)}{x} e^{-\underline{\lambda}(\Xi_u(t)+\underline{\lambda}t)} + \frac{(x+z)}{x} e^{-\underline{\lambda}z}|\tilde{C}_{z,t}|,$$

where $\tilde{C}_{z,t}=\left\{u\in\tilde{C}_z : \sigma_u(z)\leq t\right\}$. By decomposing members of $\tilde{N}_t$ in accordance with their ancestors (if at all) in $\tilde{C}_z$, it is an easy calculation, similar to (7), to show that

$$E\left(V^x_t(\underline{\lambda})\middle|\mathcal{F}_{\tilde{C}_z}\right)=V^x_{t\wedge\tilde{C}_z}. \tag{19}$$

Since $\tilde{C}_z$ is a dissecting stopping line then $\lim_{t\uparrow\infty}\left|A_t\left(\tilde{C}_z\right)\right|=0$. From the proof of Theorem 1, $V^x_t(\underline{\lambda})$ converges in mean to $V^x(\underline{\lambda})$ as t tends to infinity we thus have

$$\lim_{t\uparrow\infty} E\left(V^x_t(\underline{\lambda})\middle|\mathcal{F}_{\tilde{C}_z}\right)=E\left(V^x(\underline{\lambda})\middle|\mathcal{F}_{\tilde{C}_z}\right)=V^x_{\tilde{C}_z}.$$

The tower property of conditional expectation thus shows that $V^x_{\tilde{C}_z}$ is a martingale. Taking the limit in (19) with respect to z instead gives us similarly

$E\left(V_t^x\left(\underline{\lambda}\right)\middle|\mathcal{F}_\infty\right) = V_t^x\left(\underline{\lambda}\right)$ for all $t > 0$ where $\mathcal{F}_\infty = \sigma\left(\bigcup_{z\geq 0}\mathcal{F}_{\tilde{C}_z}\right)$. This implies that $V_{\underline{\lambda}}^x$ is $\mathcal{F}_\infty$-measurable and hence $V_{\tilde{C}_z}^x$ has limit $V^{\overline{x}}\left(\underline{\lambda}\right)$ as z tends to infinity. $\square$

Since we now know that both $-|\tilde{C}_z|\log\Phi_{\underline{\lambda}}\left(x+z\right)$ and $x^{-1}(x+z)e^{-\underline{\lambda}z}|\tilde{C}_z|$ have limits as z tends to infinity with mass in $(0,\infty)$ on $\gamma^{(x,\underline{\lambda})}$ and that $\Phi_{\underline{\lambda}}\left(\infty\right) = 1$, we thus conclude that

$$k_x = \lim_{z\uparrow\infty}\frac{-\log\left[\Phi_{\underline{c}}\left(x+z\right)\right]}{(x+z)e^{-\underline{\lambda}z}} = \lim_{z\uparrow\infty}\frac{1-\Phi_{\underline{c}}\left(z\right)}{ze^{-\underline{\lambda}z}}$$

for some positive constant k_x depending on x. This result is true for all $x > 0$ and hence the asymptotic (18) is proved.

6.3 Uniqueness

Solutions to (17) can be used to construct L^1-convergent multiplicative martingales of the form

$$M_t\left(x\right) := \prod_{u\in N_t}\Phi_{\underline{c}}\left(x+\Xi_u\left(t\right)+\underline{\lambda}t\right) \text{ for } t > 0.$$

Further, using the asymptotic behaviour of $\Phi_{\underline{c}}$, since $L_t + \underline{\lambda}t \to \infty$ a.s., we have that

$$\begin{aligned}
-\log M_t\left(x\right) &\sim \sum_{u\in N_t} -\log\Phi_{\underline{c}}\left(x+\Xi_u\left(t\right)+\underline{\lambda}t\right)\\
&\sim \sum_{u\in N_t}\left[1-\Phi_{\underline{c}}\left(x+\Xi_u\left(t\right)+\underline{\lambda}t\right)\right]\\
&\sim C\sum_{u\in N_t}\left(x+\Xi_u\left(t\right)+\underline{\lambda}t\right)e^{-\underline{\lambda}(x+\Xi_u(t)+\underline{\lambda}t)}\\
&= Ce^{-\underline{\lambda}x}\left(\partial W_t\left(\underline{\lambda}\right)+xW_t\left(\underline{\lambda}\right)\right)
\end{aligned}$$

as t tends to infinity. Thus any solution to (2) at criticality must satisfy

$$\Phi_{\underline{c}}\left(x\right) = E\left[\lim_{t\uparrow\infty}M_t\left(x\right)\right] = E\left(\exp\left\{-Ce^{-\underline{\lambda}x}\partial W\left(\underline{\lambda}\right)\right\}\right)$$

and therefore uniqueness (modulo an additive constant in the argument) follows.

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