Martingale convergence and the functional equation in the multi-type branching random walk

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Abstract

A generalization of Biggins' Martingale Convergence Theorem is proved for the multi-type branching random walk. The proof appeals to modern techniques involving the construction of size-biased measures on the space of marked trees generated by the branching process. As a simple consequence we obtain existence and uniqueness of solutions (within a specified class) to a system of functional equations .

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1 Introduction

A multi-type branching random walk with p-types is defined as follows. An initial ancestor, $\mho$, of type $i \in \{1, ..., p\}$ resides at the origin of the real line. This individual gives birth to a random number of offspring scattered on $\mathbb{R}$ according to the point process $\mathbf{Z}_i = (Z_{i1}, ..., Z_{ip})$ where Z_{ij} is the point process counting the number of individuals of type $j \in \{1, ..., p\}$ born to the individual of type i. These offspring, the first generation, reproduce independently such that individuals of type j reproduce according to the point process $\mathbf{Z}_j$ $(j = 1, ..., p)$ and so on.

In this text we shall use a Ulam-Harris labelling notation. By counting siblings of the same type from left to right we can identify each individual, u $= (k_1 ..., k_n)$ with the understanding that u is the k_n-th child born to .....born to the k_1-th child born to the initial ancestor. With this formulation we write

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$|u|$ for the generation in which u lives, $\tau(u)$ for its type and $\zeta(u)$ for its position in $\mathbb{R}$ (thus $\tau(\mho) = i$ and $\zeta(\mho) = 0$). An individual is identified as uv if it is a descendent of u and on the tree growing from u, its line of decent looks like v.

Suppose now that μ_{ij} is the intensity measure of the point process Z_{ij} such that for any Borel measurable set A, $\mu_{ij}(A) = E(Z_{ij}(A))$. Define the matrix $M(\theta) = \{m_{ij}(\theta)\}$ satisfying

$$m_{ij}(\theta) = \int_{\mathbb{R}} e^{-\theta x} \mu_{ij}(dx)$$

where $\theta \in \mathbb{R}$. When the entries of $M(\theta)$ are finite and it is positive regular, the Perron-Frobenius Theorem tells us that there exists a positive maximum eigenvalue $\rho(\theta)$ and corresponding positive right and left eigenvectors $\mathbf{v}(\theta) = (v_1(\theta), ..., v_p(\theta))$ and $\mathbf{u}(\theta) = (u_1(\theta), ..., u_p(\theta))$ respectively whose entries are all finite and strictly positive. The following Assumptions will hold throughout this paper,

1. $\theta > 0$ and $M(\theta) < \infty$
2. $M(\theta)$ is positive regular for all $\theta \in \text{int}\{\phi : m_{ij}(\phi) \; < \infty\}$,
3. first derivatives of $M(\theta)$, $\mathbf{v}(\theta)$, $\mathbf{u}(\theta)$ and $\rho(\theta)$ all exist and are finite.
4. $M(0)$ is positive regular and has maximum eigenvalue strictly greater than one,
5. $P(Z_{ij}(\mathbb{R}) = \infty) = 0$ for all $i, j \in \{1, ..., p\}$.

Requiring that $\theta > 0$ is only a matter of convenience. The fourth condition implies that the process is supercritical and survives for an infinite number of generations with positive probability [see Athreya (1972) Chapter V for further details].

Without loss of generality, we can assume that the left and right eigenvectors of $M(\theta)$ are normalised such that

$$\sum_{i=1}^{p} v_i(\theta) = \sum_{i=1}^{p} u_i(\theta) = \sum_{i=1}^{p} u_i(\theta) v_i(\theta) = 1.$$

Defining for each $i \in 1, ..., p$

$$W_i^n(\theta) = \sum_{|u|=n} \frac{v_{\tau(u)}(\theta) e^{-\theta\zeta(u)}}{v_i(\theta) \rho(\theta)^n}$$

it can be shown that $\{W_i^n(\theta)\}_{n\geq 0}$ is a mean one martingale with respect to $\{\mathcal{F}_i^n\}_{n\geq 0}$, the sigma algebras generated by the first $n\,(\geq 0)$ levels of the multi-type branching random walk initiated by an individual of type i. See for example Rahimzadeh Sani (1997) or Bramson *et al.* (1992). When there is just one type, then we have a branching random walk and this martingale is the same as the one which is the subject of the martingale convergence theorem first proved by Biggins (1977). Biggins' theorem gives necessary and sufficient conditions for the aforementioned martingale to converge in mean (note that almost sure convergence is automatic since it is positive). Further, it generalizes an older result of Kesten and Stigum which says that if $\{\alpha_n\}_{n\geq 0}$ is a finite mean Galton-Watson process then the positive martingale $E(\alpha_1)^{-n}\alpha_n$ converges in mean if and only if $E(\alpha_1 \log \alpha_1) < \infty$ and when this condition fails then it converges almost surely to zero [see Athreya and Ney (1972) has a full account]. Both the results of Kesten and Stigum and Biggins were recently re-proved by Lyons *et al.* (1995) and Lyons (1997) respectively using a method of changing measure on the space of marked trees in which realizations of the branching process exist. The change of measure corresponds to size-biasing the reproduction distribution on a randomly chosen line of decent. This method improved on the existing proofs by shortening their length and using probabilistic considerations alone. The robustness of this method has also proved itself given the number of other Kesten-Stigum type theorems for other types of branching processes that have since been re-proved using changes of measures on trees. See Kurtz *et al.* (1997), Olofsson (1998), Athreya (1999), Kyprianou (2000a,b,c) and Kyprianou and Biggins (2000). This presentation now adds to this list with the following Theorem.

Theorem 1 *Let $W_i(\theta)$ be the almost sure limit of the martingale $\{W_i^n(\theta)\}_{n\geq 0}$. Given the Assumptions 1-5, $W_i(\theta)$ is also the limit in mean if and only if the following two conditions hold,*

$$\log \rho(\theta) - \theta \frac{\rho'(\theta)}{\rho(\theta)} > 0 \text{ and } E\left[W_i^1(\theta) \log^+ W_i^1(\theta)\right] < \infty \text{ for } i = 1, ..., p.$$

Moreover, if either of these two conditions fail, then $W_i(\theta)$ is almost surely zero.

This Theorem has already been proved in Rahimzadeh Sani (1997) using methods that generalize the techniques that appeared in the original proof of Biggins' Martingale Convergence Theorem. The new proof we offer here is considerably quicker.

The connection between the limit of this martingale and a certain system of functional equations should not go unmentioned. We include this as a Corollary to the Theorem. Consider the class of vector functions of the form

$$\Phi = \left\{(\phi_1, ..., \phi_p) : \mathcal{L}_\mu \ni \phi_i : \mathbb{R}^+ \to [0,1] \text{ for all } i = 1, ..., p \text{ and some } \mu > 0\right\}$$

where $\mathcal{L}_\mu$ is the class of Laplace Transforms of positive variables with finite mean μ.

Corollary 2 *When all the conditions of Theorem 1 are satisfied, then*

$$\phi_i(x) = E\left(\exp\{-xW_i(\theta)\}\right) \text{ for } i = 1, ..., p$$

is the unique solution in Φ *(up to a multiplicative constant in the argument) to the system of functional equations*

$$\phi_i(x) = E\left[\prod_{|u|=1} \phi_{\tau(u)}\left(x\frac{v_{\tau(u)}(\theta)\, e^{-\theta\zeta(u)}}{v_i(\theta)\,\rho(\theta)}\right)\right] \text{ for } i = 1, ..., p. \qquad (1)$$

Remark 3 *If follows from this Corollary that for any* $(\phi_1, ..., \phi_p) \in \Phi$ *that*

$$\frac{1-\phi_i(x)}{x} \to k_i \text{ as } x \downarrow 0$$

where k_i *is a constant for each* $i \in \{1, ..., p\}$.

This system of functional equations can be thought of as a discrete time analogue of the ordinary differential equations giving travelling wave solutions to a coupled system of K-P-P equations, see Champneys *et al.* (1995). In principle the method of size-biasing we use here is equally applicable to constructing alternitive solutions the problems discussed there.

We conclude this section by giving a brief outline of the paper. In the next section we consider the multi-type branching random walk as a process having sample paths on a measurable space of trees with an associated probability measure. It is shown that there exists a new probability measure on this space of trees whose Radon-Nikodym derivative (restricted to $\mathcal{F}_i^n$) with respect to the original measure is precisely $W_i^n(\theta)$. Consequently the problem of L^1-convergence is transformed to studying the martingale under the new probability measure. As mentioned earlier, the change of measure corresponds to size-biasing the reproduction distribution along a randomly chosen line of descent. It turns out that under the new measure, the behaviour of the martingale is dominated by the asymptotic behaviour of the spine. In section 3 we show that the position and type of individuals on the spine follows a Markov additive process and further discuss some of its basic asymptotic properties. The final section is devoted to the proof of the Theorem and its Corollary.

2 Measures on trees with spines

The set of possible realizations of the multi-type branching random walk with initial ancestor of type i generates a space of trees with nodes marked in $\mathbb{R} \times \{0, 1, ..., p\}$. Call this space $\mathcal{T}_i$ for $i = 1, ..., p$ and note that if $\mathcal{T}_i^n$ is the subspace of $\mathcal{T}_i$ consisting of all trees truncated at the n-th generation, then $\mathcal{T}_i^n$ is an $\mathcal{F}_i^n$-measurable space for all $n \geq 0$. The probability measure on this space of trees, η_i, corresponding to the reproduction laws outlined at the beginning of the previous section, satisfies the decomposition

$$d\eta_i^{n+1}(t) = d\eta_i^1(t) \prod_{|u|=1} d\eta_{\tau(u)}^n(t(u))$$

where $t \in \mathcal{T}_i$, η_i^n is the restriction of η_i to $\mathcal{F}_i^n$ and $\{t(u) : |u| = 1\}$ are the independent subtrees of t initiated by individuals in the first generation.

For each $t \in \mathcal{T}_i$, starting from the initial ancestor, we can distinguish (possibly finite) ancestral lines of decent $\xi = (\mho = \xi_0, \xi_1, ...)$ each of which we shall call a *spine*. Let $\tilde{\mathcal{T}}_i$ be the space of trees in $\mathcal{T}_i$ with a distinguished spine ξ and $\tilde{\mathcal{T}}_i^n$, the subspace of trees with spines truncated at generation n. Consider the measure on $\tilde{\mathcal{T}}_i^n$ such that for $(t, \xi) \in \tilde{\mathcal{T}}_i^n$

$$d\tilde{\eta}_i^n(t,\xi) = \prod_{k=0}^{n-1} d\eta^1_{\tau(\xi_k)}(t(\xi_k)) \prod_{\substack{|v|=1 \\ u=\xi_k v \neq \xi_{k+1}}} d\eta^{n-k-1}_{\tau(u)}(t(u))$$

which decomposes the probability measure η_i^n such that

$$d\eta_i^n(t) = \sum_{|u|=n} I(\xi_n = u)\, d\tilde{\eta}_i^n(t,\xi). \tag{2}$$

For $n \geq 0$, let $W^1_{\tau(\xi_n)}(\theta, \xi_n)$ be the version of $W^1_j(\theta)$ on the tree growing from ξ_n when $\tau(\xi_n) = j$. Construct a new bivariate probability measure on $\tilde{\pi}_i$ on $\tilde{\mathcal{T}}_i$ whose restriction to $\mathcal{F}_i^n$ satisfies

$$\begin{aligned} d\tilde{\pi}_i^n(t,\xi) &= \frac{v_{\tau(\xi_n)}(\theta)\, e^{-\theta\zeta(\xi_n)}}{v_i(\theta)\,\rho(\theta)^n} d\tilde{\eta}_i^n(t,\xi) \\ &= \prod_{k=0}^{n-1} \left\{ p(\xi_k) \times W^1_{\tau(\xi_k)}(\theta,\xi_k) \right\} d\tilde{\eta}_i^n(t,\xi) \end{aligned} \tag{3}$$

where $(t, \xi) \in \tilde{\mathcal{T}}_i^n$ and

$$p(\xi_k) = \frac{v_{\tau(\xi_{k+1})} e^{-\theta(\zeta(\xi_{k+1}) - \zeta(\xi_k))}}{\sum_{|v|=1} v_{\tau(\xi_k v)} e^{-\theta(\zeta(\xi_k v) - \zeta(\xi_k))}} \text{ for } 0 \leq k \leq n-1.$$

[The decomposition (2) can be used to show that $\tilde{\pi}_i$ is really a probability measure] Let π_i^n be its projection onto $\mathcal{T}_i^n$. In view of (2) this is a probability measure that satisfies

$$\frac{d\pi_i^n}{d\eta_i^n}(t) = W_i^n(\theta).$$

The construction (3) suggests that $\tilde{\pi}_i^n$ corresponds to a multi-type branching random walk that evolves generation by generation as in the introduction except for the following modification: along the spine, given the node ξ_n in generation $n \geq 0$, the law of its reproduction with respect to the law of $\mathbf{Z}_{\tau(\xi_n)}$ has Radon-Nikodym derivative $W^1_{\tau(\xi_n)}(\theta, \xi_n)$. Note that this implies the probability of no offspring is zero and therefore under the measure $\tilde{\pi}_i$, spines which are finite form a null set. This is what is understood in the literature as size-biasing.

The proof of the theorem in Section 4 will follow by considering the following dichotomy relating the behaviour of $W_i^n(\theta)$ under the measures π_i and η_i. Let $\bar{W}_i(\theta) = \limsup_{n\uparrow\infty} W_i^n(\theta)$ (which also equals $W_i(\theta)$ η_i-a.s.) then

$$\bar{W}_i(\theta) = \infty \quad \pi_i\text{-a.s.} \Rightarrow \bar{W}_i(\theta) = 0 \quad \eta_i\text{-a.s} \tag{4}$$

$$\bar{W}_i(\theta) < \infty \quad \pi_i\text{-a.s.} \Rightarrow \int \bar{W}_i(\theta)\, d\eta_i = 1. \tag{5}$$

3 Process on the spine

In this section we justify the claim that the process on the spine

$$S_n(\theta) := \{\theta\zeta(\xi_n) + n\log m(\theta), \tau(\xi_n)\} \text{ for } n \geq 0$$

is $\widetilde{\pi}$-Markov additive. This property of the spine will prove to be very important in proving the Theorem its Corollary. To begin with we shall briefly recall the definition of a Markov additive process and demonstrate some of its properties.

Suppose we have a family of independent random variables $\{X_1, ..., X_p\}$ and an ergodic Markov process $\Lambda = \{\Lambda_n\}_{n\geq 0}$ on the integers $\{1, ..., p\}$. Define the process S as follows,

$$\begin{aligned} S_0 &= 0 \\ S_n &= \sum_{i=1}^n Y_i \text{ where } Y_i \overset{\text{i.i.d.}}{\sim} X_j \text{ if } \Lambda_i = j. \end{aligned}$$

The pair $\{(S_n, \Lambda_n)\}_{n\geq 0}$ is called a Markov additive process. The following result follows from an easy application of the classical properties of random walks and renewal processes.

Lemma 4 *Suppose that $E|X_i| < \infty$ and let $\mu_i = E(X_i)$ for all $i = 1, ..., p$. Suppose that $(\Pi_1, ..., \Pi_p)$ is the stationary distribution of k and define $\chi = \sum_{i=1}^p \mu_i \Pi_i$. Then $\chi > 0$ implies that*

$$\limsup_{n\uparrow\infty} S_n = \infty$$

and $\chi \leq 0$ implies that

$$\liminf_{n\uparrow\infty} S_n = -\infty.$$

Note that this is not the strongest statement we can say about the limiting behaviour of the spatial part of a Markov additive process, but it will suffice for our purposes.

Proof. Consider first the ratio

$$\frac{S_n}{n} = \sum_{j=1}^p \left\{ \frac{\sum_{i=1}^n Y_i I(\Lambda_i = j)}{\sum_{i=1}^n I(\Lambda_i = j)} \right\} \left\{ \frac{\sum_{i=1}^n I(\Lambda_i = j)}{n} \right\}.$$

The law of large numbers and ergodicity of Λ implies that

$$\frac{S_n}{n} \to \chi := \sum_{i=1}^{p} \mu_i \Pi_i \text{ a.s.}$$

as n tends to infinity.

Suppose now that $\{N_j^n\}_{n \geq 0}$ are the times that the Markov chain k is in state $j \in \{1, ..., p\}$. Note that $S^j = \left\{S_{N_j^n}\right\}_{n \geq 0}$ is a random walk. Further, by the previous observation and the ergodicity of Λ, we have the following law of large numbers,

$$\frac{S_{N_j^n}}{n} = \frac{S_{N_j^n}}{N_j^n} \frac{N_j^n}{n} \to \frac{\chi}{\Pi_j} \text{ a.s.}$$

for all $j = 1, ..., p$. Hence the mean increment of each S^j is χ / Π_j. Consequently the random walks $\{S^j : j = 1, ..., p\}$ are simultaneously transient or recurrent according to the value of χ. Since for any $j = 1, ..., p$, $\limsup_{n \uparrow \infty} S_n \geq \limsup_{n \uparrow \infty} S_{N_j^n}$ and $\liminf_{n \uparrow \infty} S_n \leq \liminf_{n \uparrow \infty} S_{N_j^n}$ the result follows. $\square$

Consider the increments $Y_n = \theta\zeta(\xi_n) - \theta\zeta(\xi_{n-1}) + \log\rho(\theta)$ under the measure $\tilde{\pi}_i$. By construction, these increments are independent. Further, given $\tau(\xi_{n-1}) = j$, the mean increment from type j

$$\begin{aligned}
\mu_j(\theta) &:= E_{\tilde{\pi}_j}(Y_1) \\
&= \theta E\left[\sum_{|u|=1} \zeta(u) \frac{v_{\tau(u)} e^{-\theta\zeta(u)}}{\sum_{|v|=1} v_{\tau(v)} e^{-\theta\zeta(v)}} W_j^1(\theta)\right] + \log\rho(\theta) \\
&= -\theta \sum_{k=1}^{p} \frac{m'_{jk}(\theta) v_k(\theta)}{\rho(\theta) v_j(\theta)} + \log\rho(\theta).
\end{aligned}$$

A similar calculation shows that that under Assupmtion 2, the absolute expectation of the increments are finite. Also if we consider the process of types along the spine, $\{\tau(\xi_n)\}_{n \geq 0}$ we see that it is a $\tilde{\pi}_i$-Markov chain with transition probability $p_{jk}(\theta)$ which is equal to

$$\begin{aligned}
P_{\tilde{\pi}_j}(\tau(\xi_1) = k) &= E\left[\sum_{|u|=1} I(\tau(u) = k) \frac{v_{\tau(u)} e^{-\theta\zeta(u)}}{\sum_{|v|=1} v_{\tau(v)} e^{-\theta\zeta(v)}} W_j^1(\theta)\right] \\
&= \frac{m_{jk}(\theta) v_k(\theta)}{\rho(\theta) v_j(\theta)}.
\end{aligned}$$

Hence it is easy to check that the stationary distribution $(\Pi_1(\theta), ..., \Pi_p(\theta))$ satisfies $\Pi_j(\theta) = v_j(\theta) u_j(\theta)$ for all $j = 1, ..., p$. Consequently we conclude that the process $\{\theta\zeta(\xi_n) + n\log\rho(\theta), \tau(\xi_n)\}_{n \geq 0}$ is a Markov additive. Its drift, $\chi(\theta)$, can be written more neatly as $\log\rho(\theta) - \theta\rho'(\theta)/\rho(\theta)$. To see this note that

$$\begin{aligned}\chi(\theta) &= \sum_{j=1}^{p} \mu_j(\theta)\, \Pi_j(\theta) \\ &= \log \rho(\theta) - \theta \frac{\mathbf{u}(\theta)^T M'(\theta)\, \mathbf{v}(\theta)}{\rho(\theta)}.\end{aligned}$$

Under Assuption 3 we can differentiate both sides of the equality $\mathbf{u}(\theta)^T M(\theta)\, \mathbf{v}(\theta) = \rho(\theta)$ andthe result follows.

4 Proofs

4.1 Proof of the Theorem

Suppose that $E\left(W_i^1(\theta) \log^+ W_i^1(\theta)\right) = \infty$ for some $i \in \{1, ..., p\}$ thus implying that for all $c > 0$

$$\sum_{k \geq 1} P_{\tilde{\pi}_i}\left(\log^+ W_i^1(\theta) > cn\right) = \infty. \tag{6}$$

Using notation from previously, consider the $\mathcal{G}_{n+1}$-measurable events

$$\left\{\log^+ W_i^1\left(\theta, \xi_{N_i^n}\right) > cn\right\}$$

where $\mathcal{G}_{n+1} := \mathcal{F}_{N_i^{n+1}}$. In view of (6), an application of the Borel-Cantelli Lemma to this adapted sequence of events implies that

$$P_{\tilde{\pi}_i}\left(\log^+ W_i^1\left(\theta, \xi_{N_i^n}\right) > cn \text{ i.o.}\right) = 1$$

for all $c > 0$ and hence

$$\limsup_{n \uparrow \infty} \frac{\log^+ W^1_{\tau(\xi_n)}(\theta, \xi_n)}{n} \geq \limsup_{n \uparrow \infty} \frac{\log^+ W_i^1\left(\theta, \xi_{N_i^n}\right)}{n} = \infty$$

$\tilde{\pi}_i$-almost surely where, if $\tau(\xi_n) = j$, $W^1_{\tau(\xi_n)}(\theta, \xi_n)$ is the version of $W_j^1(\theta)$ on the tree growing from ξ_n.

Conversely, if we assume that $E\left(W_i^1(\theta) \log^+ W_i^1(\theta)\right) < \infty$ for all $i \in \{1, ..., p\}$ then a similar argument using again the Borel-Cantelli Lemma implies that

$$P_{\tilde{\pi}_i}\left(\log^+ W_i^1\left(\theta, \xi_{N_i^n}\right) > cn \text{ i.o.}\right) = 0$$

for all $c > 0$ and $i = 1, .., p$ so that

$$\limsup_{n \uparrow \infty} \frac{\log^+ W^1_{\tau(\xi_n)}(\theta, \xi_n)}{n} \leq \sum_{i=1}^{p} \limsup_{n \uparrow \infty} \frac{\log^+ W_i^1\left(\theta, \xi_{N_i^n}\right)}{n} = 0$$

$\tilde{\pi}_i$-almost surely.

Now that the $\tilde{\pi}_i$-almost sure asymptotic behaviour of $S_n(\theta)$ and $W^1_{\tau(\xi_n)}(\theta, \xi_n)$ have been established with respect to the conditions of the theorem the proof follows that of the case $p = 1$ given in Lyons (1997) (if not for some minor alterations). We shall include it here for completeness.

From the decomposition of $W_i^{n+1}(\theta)$ into contributions from the n-th generation we have

$$W_i^{n+1}(\theta) \geq \exp\left\{-\left(\theta\zeta(\xi_n) + n\log\rho(\theta)\right)\right\} \frac{\underline{v}(\theta)}{\overline{v}(\theta)} W^1_{\tau(\xi_n)}(\theta, \xi_n) \tag{7}$$

where

$$\underline{v}(\theta) = \min\left\{v_i(\theta) : i = 1, ..., p\right\} > 0 \text{ and } \overline{v}(\theta) = \max\left\{v_i(\theta) : i = 1, ..., p\right\} > 0.$$

From the previous discussion, if $\chi(\theta) \leq 0$ then the diverging exponential term in (7) is sufficient to guarentee that

$$\limsup_{n\uparrow\infty} W_i^n(\theta) = \infty \;\; \pi_i\text{-a.s.}$$

If however $\chi(\theta) > 0$ and $E\left(W_i^1(\theta)\log^+ W_i^1(\theta)\right) = \infty$ for some $i \in \{1, ..., p\}$ then the lim sup of the right hand side of (7) is dominated by the final term and we conclude again that $\bar{W}(\theta) = \infty$ π_i-almost surely.

Now suppose that both $\chi(\theta) > 0$ and $E\left(W_i^1(\theta)\log^+ W_i^1(\theta)\right) < \infty$ for all $i \in \{1, ..., p\}$. Let $\mathcal{G}$ be the sigma field generated by the sequence $\left\{\mathbf{Z}_{\tau(\xi_n)}\right\}_{n\geq 0}$ then

$$E_{\tilde{\pi}_i}\left(W_i^n(\theta)\,|\mathcal{G}\right) = \sum_{k=0}^{n-1} \frac{v_{\tau(\xi_k)}(\theta)e^{-\theta\zeta(\xi_k)}}{v_i(\theta)\rho(\theta)^k} W^1_{\tau(\xi_k)}(\theta, \xi_k) + \sum_{k=0}^{n-1} \frac{v_{\tau(\xi_k)}(\theta)e^{-\theta\zeta(\xi_k)}}{v_i(\theta)\rho(\theta)^k}.$$

Referring to the previous discussion, the summands in both terms above decay at most exponentially and thus Fatou's Lemma implies that

$$E_{\tilde{\pi}_i}\left(\liminf_{n\uparrow\infty} W_i^n(\theta)\,|\mathcal{G}\right) < \infty \;\; \tilde{\pi}_i\text{-a.s.}$$

As $\left[W_i^n(\theta)\right]^{-1}$ is a positive π_i-martingale we thus have that $\lim_{n\uparrow\infty} W_i^n(\theta)$ and hence $\bar{W}(\theta)$ are finite π_i-almost surely. $\square$

4.2 Proof of Corollary

The proof we will give uses ideas from Doney (1972) that were also employed in Biggins (1977) for the one-type branching random walk.

Decomposing $W_i^n(\theta)$ into contributions from individuals in the n-th generation and taking limits as n tends to infinity gives the distributional identity

$$W_i(\theta) = \sum_{|u|=1} \frac{v_{\tau(u)}(\theta) e^{-\theta\zeta(u)}}{v_i(\theta)\rho(\theta)} W_{\tau(u)}(\theta, u)$$

where, if $\tau(u) = j$, $W_{\tau(u)}(\theta, u)$ is the version of $W_j(\theta)$ on the tree rooted at u. Taking Laplace transforms of this identity under the Assumptions of the Theorem gives a solution to the functional equation (1) in Φ.

Suppose now we take two solutions $(\phi_1, ..., \phi_p)$ and $(\varphi_1, ..., \varphi_p)$ in Φ that satisfy the functional equation (1). Since we are to prove uniqueness up to a multiplicative constant in the argument, it suffices to assume that for $i = 1, ..., p$, ϕ_i and φ_i are in $\mathcal{L}_1$. For all $x > 0$, let $g_i(x) = x^{-1}|\phi_i(x) - \varphi_i(x)|$ and

$$g(x) = \max\left\{\frac{|\phi_i(x) - \varphi_i(x)|}{x} : i = 1, ..., p\right\}$$

so that g is bounded and positive with $g(0^+) = 0$. We have

$$\begin{aligned}
g_i(x) &\leq \frac{1}{x} E\left(\sum_{|u|=1}\left|\phi_{\tau(u)}\left(x\frac{v_{\tau(u)}(\theta)\, e^{-\theta\zeta(u)}}{v_i(\theta)\,\rho(\theta)}\right) - \varphi_{\tau(u)}\left(x\frac{v_{\tau(u)}(\theta)\, e^{-\theta\zeta(u)}}{v_i(\theta)\,\rho(\theta)}\right)\right|\right) \\
&\leq E\left(\sum_{|u|=1} g_{\tau(u)}\left(x\frac{v_{\tau(u)}(\theta)\, e^{-\theta\zeta(u)}}{v_i(\theta)\,\rho(\theta)^n}\right)\frac{v_{\tau(u)}(\theta)\, e^{-\theta\zeta(u)}}{v_i(\theta)\,\rho(\theta)^n}\right) \\
&= E_{\tilde{\pi}_i}\left(g_{\tau(\xi_1)}\left(x\frac{v_{\tau(\xi_1)}(\theta)\, e^{-\theta\zeta(\xi_1)}}{v_i(\theta)\,\rho(\theta)}\right)\right).
\end{aligned}$$

Iterating yields

$$\begin{aligned}
g_i(x) &\leq E_{\tilde{\pi}_i}\left(g_{\tau(\xi_n)}\left(x\frac{v_{\tau(\xi_n)}(\theta)\, e^{-\theta\zeta(\xi_n)}}{v_i(\theta)\,\rho(\theta)^n}\right)\right) \\
&\leq E_{\tilde{\pi}_i}\left(g\left(x\frac{v_{\tau(\xi_n)}(\theta)\, e^{-\theta\zeta(\xi_n)}}{v_i(\theta)\,\rho(\theta)^n}\right)\right)
\end{aligned}$$

for all $n \geq 1$. As has already been demonstrated, under the conditions of the theorem

$$\lim_{n\uparrow\infty} \frac{v_{\tau(\xi_n)}(\theta)}{v_i(\theta)} \exp\{-(\theta\zeta(\xi_n) + n\log\rho(\theta))\} = 0 \quad \tilde{\pi}_i\text{-a.s.}$$

and hence since g is bounded $0 \leq g_i(x) \leq g(0^+) = 0$ for all $x > 0$. □

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