An Axiomatization of the Euclidean Compromise Solution

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Abstract: The Euclidean compromise solution in multicriteria optimization is a solution concept that assigns to a feasible set the alternative with minimal Euclidean distance to the utopia point. The purpose of this paper is to provide a characterization of the Euclidean compromise solution.

Keywords: Compromise solutions, axiomatization, multicriteria optimization.

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1 Introduction

Multicriteria optimization extends optimization theory by permitting several — possibly conflicting — objective functions, which are to be 'optimized' simultaneously. By now an important branch of Operations Research (see Steuer *et al.*, 1996), it ranges from highly verbal approaches like Larichev and Moshkovich (1997) to highly mathematical approaches like Sawaragi *et al.* (1985), and is known by various other names, including Pareto optimization, vector optimization, efficient optimization, and multiobjective optimization. Formally, a multicriteria optimization problem can be formulated as

$$\begin{aligned} \text{Optimize} \quad & f_1(x), \ldots, f_n(x) \\ \text{subject to} \quad & x \in F, \end{aligned} \tag{1}$$

where F denotes the feasible set of alternatives and $n \in \mathbb{N}$ the number of separate objective functions $f_k : F \rightarrow \mathbb{R}$ $(k = 1, \ldots, n)$.

The simultaneous optimization of multiple objective functions suggests the question: what does it mean to optimize, i.e., *what is a good outcome*? Different answers to this question lead to different ways of solving multicriteria optimization problems. For a detailed description and good introductions to the area, see White (1982), Yu (1985), and Zeleny (1982).

Yu (1973) introduced compromise solutions, based on the idea of finding a feasible point that is as close as possible to an ideal outcome. Zeleny (1976) even states this informally as an axiom of choice:

> "Alternatives that are closer to the ideal are preferred to those that are farther away. To be as close as possible to the perceived ideal is the rationale of human choice."

The ideal point, or utopia point, specifies for each objective function separately the optimal feasible value. Assume, for instance, that in the optimization problem (1) higher values of the objective functions correspond with better outcomes. In that case, the utopia point $u \in \mathbb{R}^n$ is defined by taking

$$\forall k \in \{1, \ldots, n\} : \quad u_k = \max_{x \in F} \; f_k(x).$$

Whereas Yu (1973) concentrates on distance functions defined by ℓ_p-norms, possible extensions include the use of different norms (cf. Gearhart, 1979) or penalty functions (cf. White, 1984).

In a manifesto, Bouyssou *et al.* (1993) observe that within multicriteria decision making '[a] systematic axiomatic analysis of decision procedures and algorithms is yet to be carried out'. Yu (1973, 1985) and Freimer and Yu (1976) already indicate several properties of compromise solutions. In this paper we concentrate on the Euclidean compromise solution, selecting the feasible point that minimizes the Euclidean distance to the utopia point, and provide a list of properties characterizing this solution: the Euclidean compromise solution is shown to be the unique solution concept satisfying these properties on a domain of multicriteria optimization problems.

Most of the axioms can be found in Yu (1973, 1985) and Freimer and Yu (1976). Two new axioms are introduced: a projection property and a scaling property. The projection axiom indicates that if all likely solution candidates, i.e., all Pareto optimal points, have the same value according to a certain criterion, then attention can be restricted to the remaining coordinates. The scaling axiom tells how the solution reacts to rescaling the coordinates of certain symmetric choice sets by a positive constant.

The set-up of the paper is as follows. Section 2 contains preliminary results and definitions. The Euclidean compromise solution and the domain of choice problems are defined in Section 3. In Section 4, the axioms are stated and it is shown that the Euclidean compromise solution indeed satisfies these properties. Our main result, Theorem 5.4, is given in Section 5, where the Euclidean compromise solution is shown to be the *unique* solution concept satisfying these properties. Section 6 contains remarks on possible modifications of our characterization and related literature.

2 Preliminaries

Let $n \in \mathbb{N}$. For vectors $a, b \in \mathbb{R}^n$, write

$$\begin{aligned} a = b &\quad\Leftrightarrow\quad \forall k \in \{1, \ldots, n\} : a_k = b_k \\ a \geqq b &\quad\Leftrightarrow\quad \forall k \in \{1, \ldots, n\} : a_k \geqq b_k \\ a \geq b &\quad\Leftrightarrow\quad a \geqq b, \text{ and } a \neq b \end{aligned}$$

$$a > b \quad \Leftrightarrow \quad \forall k \in \{1,\ldots,n\} : a_k > b_k$$

Relations $\leqq, \leq, <$ are defined analogously. Denote $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x \geqq 0\}$ and $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n \mid x > 0\}$. For two sets $A, B \subseteq \mathbb{R}^n$, define $A + B = \{a + b \mid a \in A, b \in B\}$. Let $a \in \mathbb{R}^n$. With a slight abuse of notation, we sometimes write $a + B$ instead of $\{a\} + B$.

Let $n \in \mathbb{N}$ and $S \subseteq \mathbb{R}^n$. A point $x \in S$ is *Pareto optimal in* S if there is no feasible alternative $y \in S$ such that $y \geq x$. The set of Pareto optimal points of S is denoted by $PO(S)$:

$$\forall n \in \mathbb{N}, \forall S \subseteq \mathbb{R}^n : \quad PO(S) = \{x \in S \mid \nexists y \in S : y \geq x\}.$$

Lemma 2.1 *Let $n \in \mathbb{N}$ and $S \subseteq \mathbb{R}^n$ be nonempty, compact. For each $x \in S$ there exists a vector $y \in PO(S)$ such that $y \geqq x$.*

Proof. Consider $T = (\{x\} + \mathbb{R}^n_+) \cap S$. Let $y \in \arg\max_{z \in T} \sum_{i=1}^n z_i$, which exists by compactness of T and continuity of the function $z \mapsto \sum_{i=1}^n z_i$. Then $y \in PO(S)$ and $y \geqq x$ by definition of T. $\square$

The inner product is denoted by $\langle \cdot, \cdot, \rangle$:

$$\forall n \in \mathbb{N}, \forall x, y \in \mathbb{R}^n : \quad \langle x, y \rangle = \sum_{i=1}^n x_i y_i.$$

The Euclidean norm is denoted by $\| \cdot \|$:

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{R}^n : \quad \|x\| = \sqrt{\langle x, x \rangle}.$$

The ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$ is denoted $B(x, r)$:

$$B(x, r) = \{y \in \mathbb{R}^n : \|y - x\| \leqq r\}.$$

Remark 2.2 Let $y \in B(x, r)$ with $\|y - x\| = r$. We often use the fact that

$$\{z \in \mathbb{R}^n : \langle y - x, z \rangle = \langle y - x, y \rangle\}$$

is the unique hyperplane supporting the ball $B(x, r)$ at the point y. $\triangleleft$

Let $n \in \mathbb{N}, n \geqq 2$, and consider a coordinate $i \in \{1, \ldots, n\}$. The function that projects each $x \in \mathbb{R}^n$ to the point in $\mathbb{R}^{n-1}$ obtained by omitting the i-th coordinate is denoted by p_i. Formally,

$$\forall x = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \in \mathbb{R}^n : \quad p_i(x) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

We say that $p_i(x) \in \mathbb{R}^{n-1}$ is the vector obtained from $x \in \mathbb{R}^n$ by *projecting away* the i-th coordinate. If $S \subseteq \mathbb{R}^n$, then $p_i(S) = \{p_i(s) \mid s \in S\}$.

For $x, y \in \mathbb{R}^n$, define $x * y = (x_1 y_1, \ldots, x_n y_n)$, the vector obtained by coordinatewise multiplication. For a set $S \subseteq \mathbb{R}^n$, $x * S = \{x * s \mid s \in S\}$. For $x \in \mathbb{R}^n_{++}$, define $x^{-1} = (\frac{1}{x_1}, \ldots, \frac{1}{x_n})$, the vector obtained by taking coordinatewise reciprocals.

For a normal $h \in \mathbb{R}^n$ and a number $a \in \mathbb{R}^n$, the hyperplane $H(h, a)$ and corresponding halfspace $H^-(h, a)$ are defined as follows:

$$\begin{aligned} H(h, a) &= \{x \in \mathbb{R}^n \mid \langle h, x \rangle = a\}, \\ H^-(h, a) &= \{x \in \mathbb{R}^n \mid \langle h, x \rangle \leqq a\}. \end{aligned}$$

Lemma 2.3 *Let $n \in \mathbb{N}, h, b \in \mathbb{R}^n_{++}, a \in \mathbb{R}$. Then $b * H^-(h, a) = H^-(h * b^{-1}, a)$.*

Proof. Let $y \in b * H^-(h, a)$. Then $y = b * x$ for some $x \in H^-(h, a)$, so $\langle h * b^{-1}, y \rangle = \sum_{i=1}^n \frac{h_i}{b_i} b_i x_i = \sum_{i=1}^n h_i x_i = \langle h, x \rangle \leqq a$, so $y \in H^-(h * b^{-1}, a)$.

Conversely, let $y \in H^-(h * b^{-1}, a)$. Take $x = b^{-1} * y \in \mathbb{R}^n$. Then $\langle h, x \rangle = \sum_{i=1}^n h_i \frac{y_i}{b_i} = \langle h * b^{-1}, y \rangle \leqq a$, so $x \in H^-(h, a)$ and $y = b * x \in b * H^-(h, a)$. □

3 The Euclidean compromise solution

The Euclidean compromise solution assigns to a feasible set the alternative with minimal Euclidean distance to the utopia point. Each feasible set is assumed to be a nonempty, compact, and convex subset of a finite dimensional Euclidean space (endowed with the standard topology). Let $n \in \mathbb{N}$ denote the number of criteria or coordinates and define

$$\Sigma^n = \{S \subset \mathbb{R}^n \mid S \text{ is nonempty, convex, compact}\},$$

the collection of choice sets in $\mathbb{R}^n$. As usual, for a choice set $S \in \Sigma^n$ and a feasible alternative $x \in S$, the coordinate x_k $(k = 1, \ldots, n)$ indicates how alternative x is evaluated according to the k-th criterion. It is assumed throughout that larger values are preferred to smaller values. The collection of all choice sets is denoted Σ:

$$\Sigma = \cup_{n=1}^{\infty} \Sigma^n.$$

Let $n \in \mathbb{N}, S \in \Sigma^n$. The *utopia point* $u(S)$ of S is the point in $\mathbb{R}^n$ that specifies for each criterion separately the highest achievable value:

$$u(S) = (\max_{s \in S} s_1, \ldots, \max_{s \in S} s_n).$$

By compactness of S, the utopia point is well-defined. In the proof of Theorem 5.3, we also use the *disagreement point* $d(S)$, defined as

$$d(S) = (\min_{s \in S} s_1, \ldots, \min_{s \in S} s_n).$$

A *solution concept* on Σ is a function φ on Σ that assigns to each choice set $S \in \Sigma$ a feasible point $\varphi(S) \in S$. The *Euclidean compromise solution* is the solution concept Y that assigns to each $S \in \Sigma$ the feasible point closest to the utopia point $u(S)$:

$$\forall S \in \Sigma : \quad Y(S) = \arg\min_{x \in S} \|u(S) - x\|.$$

Since S is nonempty, compact, and convex and the function $\|\cdot\|$ is strictly convex, the function Y is well-defined.

The choice sets with utopia point equal to the zero vector deserve special mention.

$$\forall n \in \mathbb{N} : \quad \Sigma_0^n = \{S \in \Sigma^n \mid u(S) = 0\},$$

$$\Sigma_0 = \cup_{n=1}^{\infty} \Sigma_0^n.$$

The following lemma indicates that Σ is closed under rescaling of its coordinates and projections and also that utopia vectors and Pareto optima are in a sense robust against projections. The proofs are trivial exercises; we suffice with proving one of them.

Lemma 3.1 *Let $n \in \mathbb{N}, n \geqq 2, S \in \Sigma^n, i \in \{1, \ldots, n\}$, and $x \in \mathbb{R}^n_{++}$. The following claims hold:*

(a) $p_i(S) \in \Sigma^{n-1}$;

(b) *If* $PO(S) \subseteq \{x \in \mathbb{R}^n \mid x_i = 0\}$, *then* $p_i(PO(S)) = PO(p_i(S))$;

(c) $p_i(u(S)) = u(p_i(S))$;

(d) $x * S \in \Sigma^n$.

Proof. We only prove (b). Assume that $PO(S) \subseteq \{x \in \mathbb{R}^n \mid x_i = 0\}$.

Let $v \in p_i(PO(S))$. Then there exists a $\tilde{v} \in PO(S)$ such that $p_i(\tilde{v}) = v$. Suppose $v \notin PO(p_i(S))$. Then $w \geq v$ for some $w \in p_i(S)$. Let $\tilde{w} \in S$ be such that $p_i(\tilde{w}) = w$. By Lemma 2.1, there exists a $\tilde{x} \in PO(S)$ such that $\tilde{x} \geqq \tilde{w}$. Then $\tilde{v}, \tilde{x} \in PO(S)$ implies $\tilde{v}_i = \tilde{x}_i = 0$ and $p_i(\tilde{x}) \geqq p_i(\tilde{w}) = w \geq v = p_i(\tilde{v})$, so $\tilde{x} \geq \tilde{v}$, contradicting $\tilde{v} \in PO(S)$. Hence $v \in PO(p_i(S))$. Conclude that $p_i(PO(S)) \subseteq PO(p_i(S))$.

Let $v \in PO(p_i(S))$. Then there exists a $\tilde{v} \in S$ such that $p_i(\tilde{v}) = v$. By Lemma 2.1, there exists a $\tilde{w} \in PO(S)$ such that $\tilde{w} \geqq \tilde{v}$. Then $p_i(\tilde{w}) \geqq p_i(\tilde{v}) = v \in PO(p_i(S))$, so the weak inequality must be an equality: $v \in p_i(PO(S))$. Conclude that $p_i(PO(S)) \supseteq PO(p_i(S))$. $\square$

Let $n \in \mathbb{N}, S \in \Sigma^n$. The choice set S is *closed with respect to cyclical rotation* (cf. Yu, 1973, p. 940) if for each $x \in S$ and each permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$: $(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in S$, i.e., if S is symmetric with respect to the line $\{(t, \ldots, t) \in \mathbb{R}^n \mid t \in \mathbb{R}\}$.

4 Properties of the Euclidean compromise solution

In this section, we list six properties of solution concepts, explain them, and indicate that the Euclidean compromise solution satisfies each of them. Let φ be a solution concept on Σ. Consider the following axioms:

Pareto Optimality (PO): $\forall n \in \mathbb{N}, \forall S \in \Sigma^n : \varphi(S) \in PO(S)$.

Independence of Irrelevant Alternatives (IIA): $\forall n \in \mathbb{N}, \forall S, T \in \Sigma^n :$ if $u(S) = u(T), S \subseteq T$, and $\varphi(T) \in S$, then $\varphi(S) = \varphi(T)$.

Symmetry (SYM): $\forall n \in \mathbb{N}, \forall S \in \Sigma^n :$ if S is closed w.r.t. cyclical rotation, then $\varphi_i(S) = \varphi_j(S)$ for all $i, j \in \{1, \ldots, n\}$.

Translation Invariance (TI): $\forall n \in \mathbb{N}, \forall S \in \Sigma^n, \forall x \in \mathbb{R}^n : \varphi(x+S) = x + \varphi(S)$.

Projection (PR): $\forall n \in \mathbb{N}, n \geqq 2, \forall S \in \Sigma_0^n :$ if there exists an $i \in \{1, \ldots, n\}$ such that $x_i = y_i$ for all $x, y \in PO(S)$, then $p_i(\varphi(S)) = \varphi(p_i(S))$.

Scaling (SC): Let $n \in \mathbb{N}, n \geqq 2, t \in \mathbb{R}_{++}$, and $a \in \mathbb{R}$ be such that the set

$$B = \{x \in \mathbb{R}^n \mid \forall i \in \{1, \ldots, n\} : x_i \in [-t, 0] \text{ and } \sum_{i=1}^{n} x_i \leqq a\}$$

has utopia point $u(B) = 0 \notin B$. Let $s \in \mathbb{R}^n_{++}$. Then

$$\forall i, j \in \{1, \ldots, n\} : \quad \frac{\varphi_i(s * B)}{\varphi_j(s * B)} = \frac{s_j \varphi_i(B)}{s_i \varphi_j(B)}.$$

Pareto optimality requires that φ selects a Pareto optimal alternative in each choice set. Independence of irrelevant alternatives states that if the utopia point remains unaffected and one only discards irrelevant alternatives (alternatives $x \in T$ with $x \neq \varphi(T)$), then the solution does not change. If φ satisfies symmetry, then it assigns equal value to each of the coordinates of a symmetric choice set. Translation invariance indicates that the only effect of translating a choice set is that the solution is translated to the same extent.

The projection axiom indicates that if all likely solution candidates, i.e., all Pareto optimal points, of a choice set $S \in \Sigma_0^n$ $(n \geqq 2)$ have the same value according to a certain criterion, then attention can be restricted to the remaining coordinates. Part (a) of Lemma 3.1 indicates that the projected problem is indeed a choice problem. Let $n \in \mathbb{N}, n \geqq 2, S \in \Sigma_0^n$, and $i \in \{1, \ldots, n\}$ such that $x_i = y_i$ for all $x, y \in PO(S)$. Let $v \in S$ be such that $v_i = u_i(S)$. Since $S \in \Sigma_0^n, u_i(S) = 0$. By Lemma 2.1, $v \leqq w$ for some $w \in PO(S)$. Then $0 = v_i \leqq w_i \leqq u_i(S) = 0$, so $w_i = 0$. By assumption, $x_i = w_i = 0$ for all $x \in PO(S)$. So the projection axiom can be equivalently stated as follows:

Projection (PR): $\forall n \in \mathbb{N}, n \geqq 2, \forall S \in \Sigma_0^n :$ if $PO(S) \subseteq \{x \in \mathbb{R}^n \mid x_i = 0\}$ for some $i \in \{1, \ldots, n\}$, then $p_i(\varphi(S)) = \varphi(p_i(S))$.

As opposed to independence of irrelevant alternatives, this axiom is a way to require independence of irrelevant criteria. Just like the previous axioms, this axiom is satisfied by many compromise solutions.

The final property, the scaling axiom, is what makes the Euclidean compromise solution stand out from other compromise solutions. It tells how the solution reacts to rescaling the coordinates of a highly symmetric choice set. If each coordinate i of such a choice set B is rescaled by a positive factor s_i, then the ratio $\varphi_i(s * B)/\varphi_j(s * B)$ in the new choice set $s * B$ differs from the ratio $\varphi_i(B)/\varphi_j(B)$ in the original choice set B by a factor s_j/s_i for each pair of coordinates i, j. In the game theoretic literature on bargaining (cf. Nash, 1950, Roth, 1985), such proportionality properties, in combination with translation invariance, are common axioms to describe the effect of affine transformations on solutions to bargaining problems.

The following theorem indicates that the Euclidean compromise solution satisfies the six properties.

Theorem 4.1 *The Euclidean compromise solution Y satisfies PO, IIA, SYM, TI, PR, and SC.*

Proof. Yu (1973, pp. 939-940) indicates that the Euclidean compromise solution satisfies PO, IIA, and SYM. It is easy to see that it also satisfies TI.

To see that Y satisfies PR, let $n \in \mathbb{N}, n \geqq 2, S \in \Sigma_0^n$ and assume that for $i \in \{1, \ldots, n\}$: $PO(S) \subseteq \{x \in \mathbb{R}^n \mid x_i = 0\}$. According to Lemma 3.1, we have that $p_i(S) \in \Sigma_0^{n-1}$ and $p_i(PO(S)) = PO(p_i(S))$. That $p_i(Y(S)) = Y(p_i(S))$ follows from the following chain of equivalent statements:

$$\begin{aligned} Y(S) \text{ solves } \min_{x \in S} \|x\| &\Leftrightarrow Y(S) \text{ solves } \min_{x \in PO(S)} \|x\| \\ &\Leftrightarrow Y_i(S) = 0 \text{ and } p_i(Y(S)) \text{ solves } \min_{x \in PO(p_i(S))} \|x\| \\ &\Leftrightarrow Y_i(S) = 0 \text{ and } p_i(Y(S)) \text{ solves } \min_{x \in p_i(S)} \|x\| \\ &\Leftrightarrow Y_i(S) = 0 \text{ and } p_i(Y(S)) = Y(p_i(S)). \end{aligned}$$

The first and third equivalence follow from PO of Y, the second from the assumption that $x_i = 0$ for all $x \in PO(S)$, and the fourth by definition of $Y(p_i(S))$.

To see that Y satisfies SC, let $n \in \mathbb{N}, n \geqq 2, t \in \mathbb{R}_{++}$, and $a \in \mathbb{R}$ be such that the set

$$B = \{x \in \mathbb{R}^n \mid \forall i \in \{1, \ldots, n\} : x_i \in [-t, 0] \text{ and } \sum_{i=1}^{n} x_i \leqq a\}$$

has utopia point $u(B) = 0 \notin B$. Clearly $a < 0$. SYM and PO of Y on B imply that

$$Y(B) = \frac{a}{n}(1, \ldots, 1) < 0. \tag{2}$$

Let $s \in \mathbb{R}^n_{++}$ and $A := s * B$. Notice that $u(A) = s * u(B) = 0(\notin A)$. By Lemma 2.3,

$$A = \{x \in \mathbb{R}^n \mid \forall i \in \{1, \ldots, n\} : x_i \in [-ts_i, 0] \text{ and } \sum_{i=1}^{n} \frac{x_i}{s_i} \leqq a\}.$$

By definition of $Y(A)$, the ball $B(u(A), \|Y(A)\|)$ around the origin $u(A) = 0$ with radius $\|Y(A)\|$ and the choice set A have only the point $Y(A)$ in common. By the separating hyperplane theorem, there exists a hyperplane separating the ball and A, supporting the ball at $Y(A)$. By Remark 2.2, this hyperplane is unique. Since $PO(A) = \{x \in \mathbb{R}^n \mid \forall i \in \{1, \ldots, n\} : x_i \in [-ts_i, 0] \text{ and } \sum_{i=1}^{n} \frac{x_i}{s_i} = a\}$, its normal is (a multiple of) the vector $s^{-1} = (\frac{1}{s_1}, \ldots, \frac{1}{s_n}) \in \mathbb{R}^n_{++}$. This means that $Y(A) = \lambda s^{-1}$ for some $\lambda \in \mathbb{R}$ and that $Y(A)$ satisfies $\sum_{i=1}^{n} \frac{Y_i(A)}{s_i} = a$. Solving this yields $\lambda = \frac{a}{\|s^{-1}\|^2}$ and

$$\forall i \in \{1, \ldots, n\} : \;\; Y_i(A) = \frac{a}{\|s^{-1}\|^2 s_i}.$$

Combining this with (2) yields:

$$\forall i, j \in \{1, \ldots, n\} : \;\; \frac{Y_i(A)}{Y_j(A)} = \frac{s_j}{s_i} = \frac{s_j Y_i(B)}{s_i Y_j(B)}.$$

This proves that Y satisfies SC. □

5 Axiomatization of the Euclidean compromise solution

In this section, the Euclidean compromise solution is shown to be the *unique* solution concept on Σ satisfying PO, IIA, SYM, TI, PR, and SC. The proof is split up into several cases. Every solution concept that satisfies PO must select the utopia outcome, if this is feasible. This applies in particular to all one-dimensional choice problems $S \in \Sigma^1$.

Proposition 5.1 *Let φ be a solution concept on Σ that satisfies PO. Let $S \in \Sigma$ be such that $u(S) \in S$. Then $\varphi(S) = u(S)$.*

Proof. Since $u(S) \geqq x$ for each $x \in S$, $u(S) \in S$ implies $PO(S) = \{u(S)\}$. By PO: $\varphi(S) = u(S)$. $\square$

In choice problems with utopia point zero and a Euclidean compromise solution which is smaller in each coordinate than the utopia point, every solution concept satisfying PO, IIA, SYM, and SC coincides with the Euclidean compromise solution.

Theorem 5.2 *Let φ be a solution concept on Σ that satisfies PO, IIA, SYM, and SC. Let $n \in \mathbb{N}, n \geqq 2$ and $S \in \Sigma_0^n$ such that $Y(S) < u(S)$. Then $\varphi(S) = Y(S)$.*

Proof. Since $S \in \Sigma_0^n : Y(S) < u(S) = 0$. By definition of $Y(S)$, the ball $B(0, \|Y(S)\|)$ around the utopia point $u(S) = 0$ with radius $\|Y(S)\|$ and the choice set S have only the point $Y(S)$ in common. By the separating hyperplane theorem, there exists a hyperplane that separates the ball $B(0, \|Y(S)\|)$ and S, supporting the ball at $Y(S)$. By Remark 2.2, this is the hyperplane $H(h, a)$ with

$$h = u(S) - Y(S) = -Y(S) > 0 \text{ and } a = \langle -Y(S), Y(S)\rangle = -\|Y(S)\|^2 < 0.$$

The choice set S lies in the halfspace $H^-(h,a) = \{x \in \mathbb{R}^n \mid \langle h, x\rangle \leqq a\}$. Choose $t \in \mathbb{R}_{++}$ sufficiently large, so that the set

$$A := \{x \in \mathbb{R}^n \mid \forall i \in \{1, \dots, n\} : x_i \in [-\frac{t}{h_i}, 0] \text{ and } \langle h, x\rangle \leqq a\}$$

satisfies

$$S \subseteq A \text{ and } u(S) = u(A) = 0.$$

Such a number $t \in \mathbb{R}_{++}$ exists, since $S \subseteq H^-(h,a), h > 0$, and S is bounded. By Lemma 2.3,

$$B := h * A = \{x \in \mathbb{R}^n \mid \forall i \in \{1, \dots, n\} : x_i \in [-t, 0] \text{ and } \sum_{i=1}^{n} x_i \leqq a\}.$$

Notice that $u(B) = h * u(S) = h * 0 = 0 \notin B$, since $a < 0$. Since Y and φ satisfy SYM and PO, it follows that

$$\varphi(B) = Y(B) = \frac{a}{n}(1, \dots, 1). \tag{3}$$

Since $A = h^{-1} * B$ and $h^{-1} > 0$, (3) and SC of Y and φ imply

$$\forall i, j \in \{1, \dots, n\} : \quad \frac{\varphi_i(A)}{\varphi_j(A)} = \frac{Y_i(A)}{Y_j(A)} = \frac{h_i(a/n)}{h_j(a/n)} = \frac{h_i}{h_j}.$$

So $\varphi(A) = \lambda h$ and $Y(A) = \mu h$ for some $\lambda, \mu \in \mathbb{R}$. PO of Y and φ implies that $\langle h, \varphi(A)\rangle = \langle h, Y(A)\rangle = a$, i.e., $\langle -Y(S), -\lambda Y(S)\rangle = \langle -Y(S), -\mu Y(S)\rangle = -\|Y(S)\|^2$. So $\lambda = \mu = -1$ and $\varphi(A) = Y(A) = Y(S)$.

Since $S \subseteq A, u(S) = u(A) = 0$, and $\varphi(A) = Y(S) \in S$, it follows from IIA of φ that $\varphi(S) = \varphi(A) = Y(S)$. □

The third result of this section considers choice sets in Σ_0 for which the Euclidean compromise solution has some, but not all, coordinates equal to the corresponding coordinates of the utopia point. On such choice sets, solution concepts satisfying PO, SYM, IIA, PR, and SC coincide with the Euclidean compromise solution.

Theorem 5.3 *Let φ be a solution concept on Σ that satisfies PO, IIA, SYM, SC, and PR. Let $n \in \mathbb{N}, n \geqq 2$ and $S \in \Sigma_0^n$ such that $Y(S) \leq u(S)$, but not $Y(S) < u(S)$. Then $\varphi(S) = Y(S)$.*

Proof. As before, the unique tangent hyperplane $H(h, a)$ separating the sets S and $B(0, \|Y(S)\|)$ has normal $h = -Y(S)$ and $a = -\|Y(S)\|^2$. Recall that $d(S)$ is the disagreement point of S. Take

$$T = \{x \in \mathbb{R}^n \mid \langle h, x\rangle \leqq a \text{ and } d(S) \leqq x \leqq 0\} \in \Sigma^n.$$

Then

$$S \subseteq T, u(S) = u(T) = 0, \text{ and } Y(S) = Y(T). \tag{4}$$

The equality $Y(S) = Y(T)$ follows from the fact that by construction the ball $B(0, \|Y(S)\|)$ and T have exactly the point $Y(S)$ in common. It suffices to prove that

$$\varphi(T) = Y(T), \tag{5}$$

since (4), (5), and IIA of φ then imply $\varphi(S) = \varphi(T) = Y(S)$, which was to be shown. By assumption, the set

$$\begin{aligned} I &= \{i \in \{1, \ldots, n\} \mid Y_i(S) = u_i(S)\} \\ &= \{i \in \{1, \ldots, n\} \mid Y_i(T) = u_i(T)\} \\ &= \{i \in \{1, \ldots, n\} \mid h_i = 0\} \end{aligned}$$

is nonempty. We claim that

$$\forall i \in I: \quad PO(T) \subseteq \{x \in \mathbb{R}^n \mid x_i = 0\}. \tag{6}$$

To see this, let $i \in I$ and $x \in PO(T)$. By definition, $x_i \leqq u_i(T) = 0$. Suppose that $x_i < 0$. Take $y = x - x_i e_i \geq x$, where $e_i \in \mathbb{R}^n$ denotes the i-th standard basis vector. Then $\langle h, y\rangle = \langle h, x\rangle - \langle h, x_i e_i\rangle = \langle h, x\rangle - h_i x_i = \langle h, x\rangle \leqq a$. Moreover, $d(S) \leqq x \leq y \leqq 0$. Hence $y \in T$ and $y \geq x$, contradicting $x \in PO(T)$. Conclude that (6) holds. By (6) and PO of φ and Y:

$$\forall i \in I: \varphi_i(T) = Y_i(T) = 0. \tag{7}$$

Lemma 3.1 and PR of Y imply that for each $i \in I$:

$$\begin{aligned} p_i(PO(T)) &= PO(p_i(T)), \\ p_i(u(T)) &= u(p_i(T)), \\ p_i(Y(T)) &= Y(p_i(T)). \end{aligned}$$

So even though the set T has $|I|$ coordinates i for which $Y_i(T) = u_i(T)$, the choice set $p_i(T)$ has only $|I| - 1$ such coordinates. Repeated application of projection reduces this number to zero: Write $I = \{i(1), \ldots, i(m)\}$ and take (with a slight abuse of notation)

$$V = p_{i(m)} \circ \cdots \circ p_{i(1)}(T),$$

the choice set in $\Sigma_0^{n-|I|}$ obtained from T by projecting away all coordinates in I. Then the set of coordinates j for which $Y_j(V) = u_j(V)$ is empty: $Y(V) < u(V)$. Theorem 5.2 and PR of φ and Y imply:

$$p_{i(m)} \circ \cdots \circ p_{i(1)}(Y(T)) = Y(V) = \varphi(V) = p_{i(m)} \circ \cdots \circ p_{i(1)}(\varphi(T)). \tag{8}$$

Equality (7) indicates that $Y_i(T) = \varphi_i(T)$ if $i \in I$ and equality (8) indicates that $Y_i(T) = \varphi_i(T)$ if $i \notin I$, which proves (5). $\square$

The results above combine into our main theorem, the axiomatization of the Euclidean compromise solution.

Theorem 5.4 *The Euclidean compromise solution Y is the unique solution concept on Σ satisfying PO, TI, SYM, SC, IIA, and PR.*

Proof. Y satisfies the axioms by Theorem 4.1. Let φ be a solution concept on Σ that also satisfies them. Let $S \in \Sigma$ and let $T = -u(S) + S \in \Sigma_0$. By TI of Y and φ, it suffices to show that $\varphi(T) = Y(T)$. If $u(T) \in T$, this follows from Proposition 5.1. If $Y(T) < u(T)$, it follows from Theorem 5.2; otherwise, it follows from Theorem 5.3. $\square$

6 Concluding remarks

Bouyssou *et al.* (1993) promote an axiomatic approach to the study of decision procedures in multicriteria optimization. Theorem 5.4 characterizes the Euclidean compromise solution by means of six properties. Five of these properties, PO, SYM, IIA, TI, and PR, are shared by many compromise solutions. The scaling axiom SC is a proportionality property as encountered in the literature on bargaining and is specific to the Euclidean compromise solution.

In a recent article, Rubinstein and Zhou (1999) characterize the solution concept that assigns to each choice set the point closest to an exogenously given and fixed reference point e, rather than the utopia point, which varies as a function of the choice set. Their axiomatization involves a symmetry condition and independence of irrelevant alternatives. Whereas the symmetry condition in Section 4, taken from Yu (1973), requires symmetry only in the line through the origin with equal coordinates, the symmetry condition of Rubinstein and Zhou applies to choice sets that are symmetric with respect to *any* line through the reference point e.

The domain of our solution concepts was taken to be the collection of all nonempty, compact, convex subsets of finite-dimensional Euclidean spaces. The condition that choice sets are compact was used to guarantee the existence of utopia points. The boundedness condition inherent in compactness can be weakened: our axiomatization — with minor modifications in the proofs — also holds on the domain of nonempty, convex, closed, and *upper bounded* subsets of finite-dimensional Euclidean spaces.

There is an interesting duality between the multicriteria literature that suggests a compromise approach by finding a desirable alternative from a feasible set and the game-theoretic approach to bargaining. The compromise approach entails formulating a desirable, ideal point (the utopia point) and then 'working your way down' to a feasible solution as close as possible to the ideal. The bargaining approach entails formulating a typically undesirable disagreement point and then 'working your way up' to a feasible point dominating the disagreement outcome. Mixtures of the two approaches, like the Kalai-Smorodinsky (1975) solution, exists as well. Conley, McLean, and Wilkie (1999) give an interesting discussion of this duality between the bargaining and the multicriteria optimization approach and also provide an axiom that is related to (but more involved than) our scaling axiom SC. Unfortunately, their treatment of the multicriteria approach contains several imprecisions.

References

BOUYSSOU D., PERNY P., PIRLOT M., TSOUKIAS A., AND VINCKE P. (1993): "A manifesto for the new MCDA era", *Journal of Multi-Criteria Decision Analysis*, 2, 125-127.

CONLEY J.P., MCLEAN R., AND WILKIE S. (1999): "Axiomatic foundations for compromise theory: the duality of bargaining theory and multi-objective programming", working paper, University of Illinois.

FREIMER M. AND YU P.L. (1976): "Some new results on compromise solutions for group decision problems", *Management Science*, 22, 688-693.

GEARHART W.B. (1979): "Compromise solutions and estimation of the noninferior set", *Journal of Optimization Theory and Applications*, 28, 29-47.

KALAI E. AND SMORODINSKY M. (1975): 'Other solutions to Nash's bargaining problem", *Econometrica*, 43, 513-518.

LARICHEV O.I. AND MOSHKOVICH H.M. (1997): *Verbal decision analysis for unstructured problems*, Boston/Dordrecht/London: Kluwer Academic Publishers.

NASH J.F. (1950): "The bargaining problem", *Econometrica*, 18, 155-162.

ROTH A.E. (1985): *Game-theoretic models of bargaining*, Cambridge: Cambridge University Press.

RUBINSTEIN A. AND ZHOU L. (1999): "Choice problems with a 'reference' point", *Mathematical Social Sciences*, 17, 205-209.

SAWARAGI Y., NAKAYAMA H., AND TANINO T. (1985): *Theory of multiobjective optimization*, New York: Academic Press.

STEUER R.E., GARDINER L.R., AND GRAY J. (1996): "A bibliographic survey of the activities and international nature of multiple criteria decision making", *Journal of Multi-Criteria Decision Analysis*, 5, 195-217.

WHITE D.J. (1982): *Optimality and efficiency*, New York: John Wiley and Sons.

WHITE D.J. (1984): "Multiple objective programming and penalty functions", *Journal of Optimization Theory and Applications*, 43, 583-599.

YU P.L. (1973): "A class of solutions for group decision problems", *Management Science*, 19, 936-946.

YU P.L. (1985): *Multiple-criteria decision making: Concepts, techniques, and extensions*, New York: Plenum Pub. Corp.

ZELENY M. (1976): "The theory of the displaced ideal", in *Multi-Criteria Decision Making*, Zeleny M. (ed.), Berlin: Springer Verlag.

ZELENY M. (1982): *Multiple criteria decision making*, New York: McGraw-Hill.