Relative and Modified Relative Realizability

Lars Birkedal
The IT University of Copenhagen
birkedal@itu.dk*

Jaap van Oosten
Department of Mathematics, Utrecht University
jvoosten@math.uu.nl †

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Introduction

The notion of Relative Realizability was defined in [2] (see also [1, 4]). The idea is, that instead of doing realizability with one partial combinatory algebra A one uses an inclusion of partial combinatory algebras $A_\sharp \subseteq A$ (such that there are combinators $k, s \in A_\sharp$ which also serve as combinators for A); the principal point being that "($A_\sharp$-) computable" functions may also act on data (in A) that need not be computable. Of course this is reminiscent of Turing's computability with oracles and Kleene's definition ([12] and later papers) of a recursive functional of higher type, which, for example in the case of type 2, has to act on any (possibly non-recursive) function.

In itself, relative realizability was not new; Kleene's 1957–realizability ([11]), a precursor of his later function realizability, was probably of this type (we shall make this conjecture precise in section 3), and relative realizability also occurs in Thomas Streicher's "Topos for Computable Analysis" ([17]). However, in [2] there is an analysis of the relationships between relative realizability over $A_\sharp \subseteq A$ and the ordinary realizabilities over $A_\sharp$ and A. Let $\mathrm{RT}(A_\sharp, A)$ be the relative realizability topos, and $\mathrm{RT}(A_\sharp)$, $\mathrm{RT}(A)$ the ordinary (effective topos-like) realizability toposes; then

- *There is a local geometric morphism from* $\mathrm{RT}(A_\sharp, A)$ *to* $\mathrm{RT}(A_\sharp)$;
- *there is a logical functor from* $\mathrm{RT}(A_\sharp, A)$ *to* $\mathrm{RT}(A)$

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The motivation for the present paper was the observation that there is a general pattern underlying relative realizability. Basically, an inclusion $A_\sharp \subseteq A$ is seen as an *internal partial combinatory algebra in the topos* $\mathrm{Set}^{\to}$ (sheaves over Sierpinski space), and in fact we have three such internal algebras to consider (also $A_\sharp \to A_\sharp$ and $A \to A$). It turns out that $\mathrm{RT}(A_\sharp, A)$ is a sheaf subtopos of the ordinary realizability topos constructed over $\mathrm{Set}^{\to}$ with the internal pca $A_\sharp \to A$. In order to retrieve $\mathrm{RT}(A_\sharp, A)$ we have to take the $\neg\neg$-topology into account.

Therefore, in section 1, we embark on a general theory of triposes on a topos $\mathcal{E}$, connected to an internal partial combinatory algebra and an internal topology. One of the key notions appears to be that of an *elementary subobject* (definition 1.2) in $\mathcal{E}$. We recover, in a very general context, the theorems highlighted above: if, for internal pca's A and B in $\mathcal{E}$, we have an embedding such that A is an elementary subobject of B, then there is a local geometric morphism from the standard realizability topos (over $\mathcal{E}$) on B to the one on A. This restricts to a local geometric morphism between those toposes which are built using only the j-closed subsets of A (and B) as truth-values. Denoting these by $\mathcal{E}ff_{A,j}$, $\mathcal{E}ff_{B,j}$ we have moreover: if A is a j-dense subobject of B, then $\mathcal{E}ff_{B,j}$ is a filter quotient of $\mathcal{E}ff_{A,j}$. Recall, that the canonical functor from a topos to a filter quotient is always logical.

Section 2 explores the relationship with the topos of sheaves for j. We obtain some pullback results. Moreover, the general situation gives rise to a very general definition of "modified realizability": in the case that j is an *open topology*, the inclusion $\mathcal{E}ff_{A,j} \to \mathcal{E}ff_A$ is also open, and it makes sense to look at its *closed complement*, which we define as the *modified realizability topos* on $\mathcal{E}$ w.r.t. A and j.

Finally in section 3 we discuss a number of examples known in the literature. We find that the general description allows a comparison between several notions that was not available before; moreover it opens the search for more examples.

1 Triposes over Internal Pca's

1.1 Internal Partial Combinatory Algebras

In this section we intend to lay down some basic definitions and to fix notation.

We shall work, throughout this chapter, in an arbitrary topos $\mathcal{E}$. We shall employ the internal language and logic freely, and assume the reader is familiar with its use.

Let A be an object of $\mathcal{E}$, and $f : A \times A \rightharpoonup A$ a partial map. We shall write D_A for its domain, i.e. the object defined by the pullback diagram

$$\begin{tikzcd} D_A \arrow[r] \arrow[d] & A \times A \arrow[d, "f"] \\ A \arrow[r, "\eta_A"'] & \tilde{A} \end{tikzcd}$$

where $A \overset{\eta_A}{\to} \tilde{A}$ is the partial map classifier of A.

We see this as a structure for a language with just a partial binary function symbol, which we write as juxtaposition: $a, b \mapsto ab$. In composite expressions we assume association to the left, i.e. abc is short for $(ab)c$. In manipulating terms in this language we use the symbol "$\downarrow$" ("is defined"). For a term t, composed from variables $x_1, \ldots, x_n$ of type A and juxtaposition, we define its meaning $t[\vec{u}] = t[u_1, \ldots, u_n]$ and the formula $t[\vec{u}]\downarrow$ by a simultaneous induction (here $u_1, \ldots, u_n$ denote generalized elements of type A, i.e. morphisms $U \to A$ for some parameter object U):

$$\begin{array}{ll} x\downarrow = \top & x[u] = u \\ ts[\vec{u}]\downarrow = t[\vec{u}]\downarrow \wedge s[\vec{u}]\downarrow \wedge (t[\vec{u}], s[\vec{u}]) \in D_A & ts[\vec{u}] = f \circ \langle t[\vec{u}], s[\vec{u}] \rangle \end{array}$$

When we use the expression $t\downarrow$ we understand this to imply $t'\downarrow$ for any subterm t' of t; so, for example, $ac(bc)\downarrow$ is shorthand for $ac\downarrow \wedge bc\downarrow \wedge ac(bc)\downarrow$. Given two terms t and s, we use the expression $t \sim s$ as an abbreviation for

$$(t\downarrow \leftrightarrow s\downarrow) \wedge (t\downarrow \to t = s)$$

.

Definition 1.1 a) The structure $(A, D_A \overset{f}{\to} A)$ is called a *partial combinatory algebra* in $\mathcal{E}$, if the statements:

$$\begin{array}{ll} \mathbf{k} & \exists k{:}A \forall xy{:}A. kxy\downarrow \wedge kxy = x \\ \mathbf{s} & \exists s{:}A \forall xyz{:}A. sxy\downarrow \wedge sxyz \sim xz(yz) \end{array}$$

are both true in the internal logic of $\mathcal{E}$.

b) Given two partial combinatory algebras $(A, D_A \overset{f}{\to} A)$ and $(B, D_B \overset{g}{\to} B)$ in $\mathcal{E}$, a monic map $\mu : A \to B$ is an *embedding* if the following conditions hold:

i) the map $D_A \overset{\mu\times\mu}{\to} B \times B$ factors through D_B

ii) the diagram

$$\begin{tikzcd} D_A \arrow[r, "\mu\times\mu"] \arrow[d, "f"'] & D_B \arrow[d, "g"] \\ A \arrow[r, "\mu"'] & B \end{tikzcd}$$

is a pullback in $\mathcal{E}$ (in particular, it commutes!)

iii) the formulas

$$\begin{array}{l} \exists k{:}A \forall xy{:}B. kxy\downarrow \wedge kxy = x \\ \exists s{:}A \forall xyz{:}B. sxy\downarrow \wedge sxyz \sim xz(yz) \end{array}$$

are true in $\mathcal{E}$ (under the identification of elements of A with their μ-images).

Note the following points in definition 1.1: we do *not* require that A has global elements (as we don't need it), we have formulated the "combinator axioms" as *properties* rather than *structure*; furthermore, we shall, in the case of an embedding $A \to B$, identify A with its image in B. Note that for two elements x, y of A: $xy\downarrow$ in A if and only if $xy\downarrow$ in B.

The standard facts about partial combinatory algebras (see, e.g.,[3]) that we need, are all constructively valid, and carry over to internal partial combinatory algebras in a topos $\mathcal{E}$. In particular, we shall use

- Schönfinkel's *Combinatory Completeness*: for any term t and any variable x, there is a term $\Lambda x.t$ such that for any term s, $(\Lambda x.t)s \sim t[s/x]$ holds;
- *Pairing*: the sentence

  $$\exists p, p_0, p_1{:}A.\forall xy{:}A.pxy\downarrow \wedge p_0x\downarrow \wedge p_1x\downarrow \wedge p_0(pxy) = x \wedge p_1(pxy) = y$$

  is true in $\mathcal{E}$. In fact, any choice of k and s in A give p, p_0, p_1 definable in k,s.

Given a partial combinatory algebra A we define the two maps

$$\wedge_A, \Rightarrow_A : \Omega^A \times \Omega^A \to \Omega^A$$

internally by:

$$\begin{array}{lcl} X \wedge_A Y & = & \{x \in A \,|\, p_0x \in X \text{ and } p_1x \in Y\} \\ X \Rightarrow_A Y & = & \{a \in A \,|\, \forall b \in X(ab\downarrow \wedge ab \in Y)\} \end{array}$$

The notations $\wedge, \Rightarrow$ will be extended to morphisms: $X \to \Omega^A$ by composition; and the subscript will be used only if confusion is possible.

Definition 1.2 A subobject A of an object B of $\mathcal{E}$ is said to be *elementary*[1] if for any subobject C of B: if $C \to 1$ is an epimorphism, so is $A \cap C \to 1$.

Note that if the subobject A of B is elementary, the internal logic of $\mathcal{E}$ obeys the following rule:

$$\mathcal{E} \models \exists x{:}B.R(x) \Rightarrow \mathcal{E} \models \exists x{:}A.R(X)$$

for any closed formula $\exists x{:}R(x)$ of the internal language, with x a variable of type B.

1.2 Realizability Triposes on $\mathcal{E}$

Let $(A, D_A \xrightarrow{f} A)$ be a partial combinatory algebra in $\mathcal{E}$. We shall not define the notion of a tripos (instead, refer the reader to [7]), but just for definiteness we recall the definition of the *standard realizability tripos* on $\mathcal{E}$ with respect to

[1] The term "elementary", reminiscent of a familiar criterion in Model Theory, was suggested to us by Tibor Beke.

A, which we shall denote by P_A. $P_A(X)$ is the set of arrows: $X \to \Omega^A$ in $\mathcal{E}$. $P_A(X)$ is preordered by: for $\varphi, \psi \in P_A(X)$, $\varphi \leq \psi$ if and only if the sentence

$$\exists a{:}A.\forall x{:}X.a \in \varphi(x) \Rightarrow \psi(x)$$

is true in $\mathcal{E}$.

$P_A(X)$ is a Heyting prealgebra, and the (extensions of) maps $\wedge, \Rightarrow$ serve as *meet* and *Heyting implication*, respectively.

For any arrow $f{:}X \to Y$ we have $P_A(f){:}P_A(Y) \to P_A(X)$ by composition. This map is a morphism of Heyting prealgebras and has both adjoints $\exists_f$ and $\forall_f$:

$$\begin{array}{rcl} \exists_f(\varphi)(y) & = & \{a \in A \mid \exists x{:}X.f(x) = y \wedge a \in \varphi(x)\} \\ \forall_f(\varphi)(y) & = & \{a \in A \mid \forall x{:}X.f(x) = y \to a \in (A \Rightarrow \varphi(x))\} \end{array}$$

Our first proposition concerns *geometric morphisms* between realizability triposes (again, the reader is referred to [7] for a definition). Recall from [10], that a geometric morphism between toposes is called *local* if it is bounded and its direct image part has a full and faithful right adjoint. Since any geometric morphism which arises from a geometric morphism of triposes is automatically bounded (indeed, localic; see [2] for a proof) we shall say that a geometric morphism between triposes is local if its direct image has a full and faithful right adjoint. The following proposition is essentially already in [2].

Proposition 1.3 *Let $i : A \to B$ be an embedding of partial combinatory algebras in $\mathcal{E}$. If A is an elementary subobject of B, there is a local geometric morphism of triposes: $P_B \to P_A$.*

Proof. Define $\Phi^* : P_B \to P_A$ by composition with the map $\Omega^i : \Omega^B \to \Omega^A$ (i.e., intersection with A). To show that this is order-preserving we use that $A \to B$ is elementary: if $\varphi \leq \psi$ in $P_B(X)$, then

$$\exists a{:}B\forall x{:}X.a \in \varphi(x) \Rightarrow_B \psi(x)$$

hence, by elementariness,

$$\exists a{:}A\forall x{:}X.a \in \varphi(x) \Rightarrow_B \psi(x)$$

and since i is an embedding we have

$$\exists a{:}A\forall x{:}X.a \in (\varphi(x) \cap A \Rightarrow_A \psi(x) \cap A)$$

We define $\Phi_! : P_A \to P_B$ by composition with the map $\exists_i : \Omega^A \to \Omega^B$. Clearly, if $\varphi : X \to \Omega^A$ and $\psi : X \to \Omega^B$ then $\Phi_!\Phi^*(\psi) \leq \psi$ and $\varphi \leq \Phi^*\Phi_!(\varphi)$, so $\Phi_! \dashv \Phi^*$. Moreover, $\Phi_!$ preserves finite meets: since i is an embedding, internally a choice for the pairing combinators exists in A which are also pairing combinators for B. And since A is inhabited, $\Phi_!$ preserves the top element.

We define, moreover, $\Phi_* : P_A \to P_B$ by putting, for $\varphi \in P_A(X)$,

$$\Phi_*(\varphi)(x) = \{a \in B \mid \exists \alpha{:}\Omega^B.a \in \alpha \wedge ((\alpha \cap A) \Rightarrow_B \varphi(x))\}$$

(here we assume that the pairing combinators in B are chosen from A). To see that Φ_* is order-preserving, reason internally. Let $a : A$ testify $\varphi \leq \psi$, that is,

$$\forall x{:}X.a \in \varphi(x) \Rightarrow_A \psi(x)$$

Suppose $a' \in \Phi_*(\varphi)(x)$, so for some $\alpha \in \Omega^B$,

$$p_0 a' \in \alpha \text{ and } p_1 a' \in ((\alpha \cap A) \Rightarrow_B \varphi(x))$$

Clearly then,

$$\Lambda y.a((p_1 a')y) \in (\alpha \cap A) \Rightarrow_B \psi(x)$$

so

$$\Lambda a'.\langle p_0 a', \Lambda y.a((p_1 a')y)\rangle \in \Phi_*(\varphi)(x) \Rightarrow_B \Phi_*(\psi)(x)$$

The proof that $\Phi^* \dashv \Phi_*$ is left to the reader. Note that by elementary category theory, full and faithfulness of Φ_* follows from full and faithfulness of $\Phi_!$, which follows again from elementariness. ∎

1.3 Realizability Triposes and Internal Topologies

Let A be a partial combinatory algebra in $\mathcal{E}$. Now suppose that $j{:}\Omega \to \Omega$ is an internal topology in $\mathcal{E}$, i.e. the following axioms are true in $\mathcal{E}$:

$$\begin{array}{l}\forall p{:}\Omega.p \to j(p)\\ \forall pq{:}\Omega.(p \to q) \to (j(p) \to j(q))\\ \forall p{:}\Omega.j(j(p)) \to j(p)\end{array}$$

We call the partial combinatory algebra A *j-regular* if the subobject $D_A \to A \times A$ is j-closed; this means:

$$\forall xy{:}A.j(xy\downarrow) \to xy\downarrow$$

holds in $\mathcal{E}$. Henceforth we shall always assume that our partial combinatory algebras are j-regular.

As usual, Ω_j denotes the image of j; Ω_j^A is the object of j-closed subsets of A and $j^A : \Omega^A \to \Omega_j^A$ is the internal closure map. In the logic, $j^A(\alpha) = \{x \mid j(x \in \alpha)\}$. Note, that if A is a j-regular partial combinatory algebra, we have

$$\forall \alpha\beta \in \Omega^A.(j^A(\alpha) \Rightarrow j^A(\beta)) = (\alpha \Rightarrow j^A(\beta))$$

for the inclusion from left to right is obvious, and if $a \in (\alpha \Rightarrow j^A(\beta))$, $b \in j^A(\alpha)$ then $j(ab\downarrow)$ hence $ab\downarrow$ by regularity, and $j(ab \in j^A(\beta))$ so $ab \in j^A(\beta)$ since j is idempotent. Note, that also

$$\forall \alpha\beta{:}\Omega^A.j^A(\alpha \wedge_A \beta) = j^A(\alpha) \wedge_A j^A(\beta)$$

holds in $\mathcal{E}$.

We define the realizability tripos $P_{A,j}$ by: $P_{A,j}(X)$ is the set of arrows $X \to \Omega_j^A$ in $\mathcal{E}$. We regard this as a subset of $P_A(X)$, and give $P_{A,j}(X)$ the sub-preorder. Using the above remarks, the verification that this is a tripos is straightforward. The following easy proposition occurs in [18]:

Proposition 1.4 *$P_{A,j}$ is a tripos and there is a geometric inclusion of triposes: $P_{A,j} \to P_A$.*

Proposition 1.5 *If $A \to B$ is an embedding of partial combinatory algebras, and $A \subseteq B$ an elementary subobject, the local geometric morphism $P_B \to P_A$ restricts to a local geometric morphism $P_{B,j} \to P_{A,j}$. That is, there is a commutative diagram*

$$\begin{tikzcd} P_{B,j} \arrow[r] \arrow[d] & P_{A,j} \arrow[d] \\ P_B \arrow[r] & P_A \end{tikzcd}$$

of geometric morphisms of triposes.

Proof. Adapt the proof of 1.3 by inserting j's at the appropriate points, to obtain j-closed predicates. For example define $\Phi_* : P_{A,j} \to P_{B,j}$ by

$$\Phi_*(\varphi)(x) = \{a{:}B \mid j(\exists\alpha{:}\Omega_j^B.a \in \alpha \wedge ((\alpha \cap A \Rightarrow_B j^B(\varphi(x)))\}$$

$\Phi_! : P_{A,j} \to P_{B,j}$ sends $\varphi : X \to \Omega_j^A$ to its closure in B. To see that $\Phi_!$ is full and faithful one employs the same reasoning as used in the proof of 1.3 to show that Φ^* was order-preserving: if $\Phi_!(\varphi) \le \Phi_!(\psi)$ in $P_{B,j}(X)$, so

$$\exists a{:}B\forall x{:}X.a \in j^B(\varphi(x)) \Rightarrow_B j^B(\psi(x))$$

one deduces by elementariness and the property of an embedding, that

$$\exists a{:}A\forall x{:}X.a \in (j^B(\varphi(x)) \cap A \Rightarrow_A j^B(\psi(x)) \cap A)$$

But always, $j^B(\varphi(x)) \cap A = j^A(\varphi(x)) = \varphi(x)$ for $\varphi \in P_{A,j}(X)$; hence, $\Phi_!$ reflects the order.

Finally, it is easy to see that the diagram in the statement of the proposition commutes. ∎

Recall that a topology j is *open* if there is a global element u of Ω such that $j(x) = u \to x$ for all $x \in \Omega$. By analogy we say that a geometric inclusion $\Phi^* \vdash \Phi_*$ of triposes: $P \to Q$ is open, if there is an element α of $Q(1)$ such that for every $\varphi \in Q(X)$, $\Phi_*\Phi^*(\varphi)$ is isomorphic to $Q(!)(\alpha) \Rightarrow \varphi$ where $!$ denotes $X \to 1$, and $\Rightarrow$ is the Heyting implication of $Q(X)$.

It is an easy exercise to show that open inclusions of triposes yield open inclusions between the corresponding toposes.

Proposition 1.6 *If j is an open topology, then the inclusion $P_{A,j} \to P_A$ is open.*

Proof. Let $j(p) = u \to p$ for some $u \in \Omega$; let U be the subobject of 1 classified by u. In $P_A(1)$ we have the image A' of the projection $A \times U \to A$, so $A' = \{a{:}A \mid u\}$.

We calculate, for $\varphi \in P_A(X)$, the element $A' \Rightarrow \varphi$:

$$\begin{array}{rcl} A' \Rightarrow \varphi(x) & = & \{a \mid \forall b{:}A.u \to (ab\downarrow \wedge ab \in \varphi(x))\} \\ & = & \{a \mid \forall b{:}A.ab\downarrow \wedge (u \to ab \in \varphi(x)))\} \\ & = & A \Rightarrow j^A(\varphi(x)) \end{array}$$

Now clearly, $\lambda x{:}X.A \Rightarrow \varphi(x)$ is isomorphic to φ in $P_A(X)$; so $\lambda x{:}X.A' \Rightarrow \varphi(x)$ is isomorphic to $\lambda x{:}X.j^A(\varphi(x))$. Hence, the inclusion $P_{A,j} \to P_A$ is open. ∎

Next, we turn to the situation of an embedding $A \to B$ of partial combinatory algebras in $\mathcal{E}$ where A is a j-dense subobject of B, but not necessarily elementary. Generally, we don't have geometric morphisms any more. However, there is an interesting functor: $P_{A,j} \to P_{B,j}$.

By a "functor" between triposes we mean an $\mathcal{E}$-indexed functor; equivalently, a cartesian functor between fibrations over $\mathcal{E}$.

In order to explain the situation, we recall from Pitts' thesis ([16]) that for any tripos P on $\mathcal{E}$ and any filter Φ on the Heyting pre-algebra $P(1)$, one can consider the *filter quotient* tripos P_Φ: $P_\Phi(X)$ is the same *set* as $P(X)$, but the order is defined by:

$$\varphi \leq_\Phi \psi \text{ iff } \forall_!(\varphi \Rightarrow \psi) \in \Phi$$

where $! : X \to 1$ and $\Rightarrow$ is the Heyting implication in $P(X)$.

Every filter Φ on $P(1)$ gives a filter $\hat{\Phi}$ of subobjects of 1 in the topos $\mathcal{E}[P]$, and the topos $\mathcal{E}[P_\Phi]$ is the filter quotient $\mathcal{E}[P]_{\hat{\Phi}}$ ([16]). The filter quotient construction (which,by the way, is called "filter power" in [8]) is well explained in [14]. For us is important, that for any filter quotient there is a *logical functor* from the topos to the quotient.

We make the following definition.

Definition 1.7 A functor $F : P \to Q$ between $\mathcal{E}$-triposes is called *logical* if the following conditions hold:

i) For any object X of $\mathcal{E}$ and $\varphi, \psi \in P(X)$,

$$F_X(\varphi \Rightarrow \psi) \cong F_X(\varphi) \Rightarrow F_X(\psi)$$

ii) For any map $f : X \to Y$ in $\mathcal{E}$ and any $\varphi \in P(X)$,

$$F_Y(\forall_f(\varphi)) \cong \forall_f(F_X(\varphi))$$

iii) If $\sigma \in P(\Sigma)$ is a generic element for P, then $F_\Sigma(\sigma) \in Q(\Sigma)$ is a generic element for Q.

Since, in a tripos, the whole structure is definable from implication, universal quantification and the generic element, any logical functor between triposes gives rise to a logical functor between the corresponding toposes. Moreover, the filter quotient functor: $P \to P_\Phi$ is a logical functor of triposes.

Proposition 1.8 *Suppose $A \to B$ is an embedding of partial combinatory algebras in $\mathcal{E}$, such that the inclusion $A \to B$ of objects is j-dense. Then there is a filter Φ on $P_{A,j}$ such that the triposes $P_{B,j}$ and $(P_{A,j})_\Phi$ are isomorphic; hence, there is a logical functor of triposes: $P_{A,j} \to P_{B,j}$.*

Proof. Let $\Phi \subseteq P_{A,j}(1)$ be the set of those j-closed subobjects α of A such that

$$\mathcal{E} \models \exists b{:}B.j(b \in \alpha)$$

It is easy to check that this is a filter; we define functors $F : (P_{A,j})_\Phi \to P_{B,j}$ and $G : P_{B,j} \to (P_{A,j})_\Phi$ which are each other's inverse.

$F_X : (P_{A,j})_\Phi(X) \to P_{B,j}(X)$ sends $\varphi : X \to \Omega_j^A$ to

$$\lambda x{:}X.j^B(\varphi(x)) : X \to \Omega_j^B$$

F is order preserving: in $(P_{A,j})_\Phi$, $\varphi \le \psi$ if and only if

$$\mathcal{E} \models \exists b{:}B.j(\forall x{:}X\forall a \in \varphi(x).ba{\downarrow} \wedge ba \in \psi(x))$$

Clearly, this implies

$$\mathcal{E} \models \exists b{:}B\forall x{:}X\forall a \in j^B(\varphi(x)).ba{\downarrow} \wedge ba \in j^B(\psi(x))$$

which is the definition of $F_X(\varphi) \le F_X(\psi)$.

$G : P_{B,j} \to (P_{A,j})_\Phi$ is defined by $G_X(\varphi) = \lambda x{:}X.\varphi(x) \cap A$. To show that G is order-preserving, reason internally. $\varphi \le \psi$ in $P_{B,j}(X)$ means

$$\mathcal{E} \models \exists b{:}B\forall x{:}X\forall a \in \varphi(x).ba{\downarrow} \wedge ba \in \psi(x)$$

so let $b{:}B$ satisfy this formula. Clearly, $b \in A$ implies

$$\forall x{:}X\forall a \in \varphi(x) \cap A.ba{\downarrow} \wedge ba \in \psi(x) \cap A$$

Since A is dense in B, we have therefore

$$\mathcal{E} \models \exists b{:}B.j(\forall x{:}X\forall a \in \varphi(x) \cap A.ba{\downarrow} \wedge ba \in \psi(x) \cap A)$$

so $G_X(\varphi) \le G_X(\psi)$ in $(P_{A,j})_\Phi(X)$.

Finally, since for $\alpha \in \Omega_j^A$ and $\beta \in \Omega_j^B$ we have the identities $j^B(\alpha) \cap A = j^A(\alpha) = \alpha$, and $j^B(\beta \cap A) = \beta$ (the last one because $A \to B$ is dense), we see that F and G are each other's inverse. ∎

2 Relations between the toposes

In this section we review connections between the toposes $\mathcal{E}$, $\mathcal{E}[P_A]$, $\mathcal{E}[P_{A,j}]$, and $\mathrm{Sh}_j(\mathcal{E})$ (the topos of j-sheaves in $\mathcal{E}$).

We write $i^* \dashv i_*$ for the geometric inclusion: $\mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$. From the theory of triposes we have, for each tripos P on $\mathcal{E}$, a "constant objects functor" $\Delta_P : \mathcal{E} \to \mathcal{E}[P]$.

Let us note, that $\mathrm{Sh}_j(\mathcal{E})$ is of form $\mathcal{E}[Q]$ where Q is the tripos corresponding to the internal locale Ω_j in $\mathcal{E}$, and that $i^* : \mathcal{E} \to \mathrm{Sh}_j(\mathcal{E})$ is the constant objects functor Δ_Q. This functor is a left adjoint, hence preserves epimorphisms, so Pitts' *iteration theorem* ([16], 6.2) applies: for any tripos R on $\mathrm{Sh}_j(\mathcal{E})$, we have that $P = R \circ (i^*)^{\mathrm{op}}$ is a tripos on $\mathcal{E}$, and there is a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\Delta_P"] \arrow[d, "i^*"'] & \mathcal{E}[P] \arrow[d, "K"] \\ \mathrm{Sh}_j(\mathcal{E}) \arrow[r, "\Delta_R"'] & \mathrm{Sh}_j(\mathcal{E})[R] \end{tikzcd}$$

where K is an equivalence of categories.

Now it is easy to see that if we compose $P_{A,j}$ with the embedding i_*, we get a tripos on $\mathrm{Sh}_j(\mathcal{E})$, because $P_{A,j}$ has a generic element living in the fibre over Ω_j^A, which is a j-sheaf. We see that if R is the $\mathrm{Sh}_j(\mathcal{E})$-tripos $P_{A,j} \circ (i_*)^{\mathrm{op}}$, the topos $\mathrm{Sh}_j(\mathcal{E})[R]$ is equivalent to

$$\mathcal{E}[P_{A,j} \circ (i_*)^{\mathrm{op}} \circ (i^*)^{\mathrm{op}}] \cong \mathcal{E}[P_{A,j}]$$

Hence, $\mathcal{E}[P_{A,j}]$ is also represented by the tripos R on $\mathrm{Sh}_j(\mathcal{E})$. In particular we have the constant objects functor $\Delta_R : \mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}[P_{A,j}]$.

Theorem 2.1 *$\Delta_R : \mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}[P_{A,j}]$ is the direct image of a geometric inclusion. There is a commutative diagram*

$$\begin{tikzcd} \mathrm{Sh}_j(\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{E}[P_{A,j}] \arrow[d] \\ \mathcal{E} \arrow[r] & \mathcal{E}[P_A] \end{tikzcd}$$

which is a pullback in the category of toposes and geometric morphisms.

Proof. We shall denote a general object of $\mathcal{E}[P_A]$ or $\mathcal{E}[P_{A,j}]$ by (X, Eq) with Eq a morphism from $X \times X$ to Ω^A or Ω_j^A, respectively.

Δ_R sends a sheaf X to the object (X, Eq) where Eq is defined internally by

$$\mathrm{Eq}(x, y) = \{a{:}A \mid x = y\}$$

(this is a well-defined object of $\mathcal{E}[P_{A,j}]$, since X is a sheaf)

In the other direction, consider an object (X, Eq) of $\mathcal{E}[P_{A,j}]$. Let $X' = \{x{:}X \mid j(\exists a{:}A.a \in \mathrm{Eq}(x, x))\}$, and let $\sim$ be the equivalence relation on X' defined by

$$x \sim y \ \equiv \ j(\exists a{:}A.a \in \mathrm{Eq}(x, y))$$

Then $X'/\sim$ is a j-separated object of $\mathcal{E}$ which we denote by $G(X, \mathrm{Eq})$. Suppose $F{:}X \times Y \to \Omega_j^A$ represents a morphism $(X, \mathrm{Eq}) \to (Y, \mathrm{Eq})$. Then F determines a subobject $G(F)$ of $G(X, \mathrm{Eq}) \times G(Y, \mathrm{Eq})$ defined by

$$G(F) = \{([x], [y]) \mid j(\exists a{:}A.a \in F(x, y))\}$$

Clearly, the composite $G(F) \to G(X, \mathrm{Eq}) \times G(Y, \mathrm{Eq}) \to G(X, \mathrm{Eq})$ is a j-dense monic in $\mathcal{E}$, so if we apply i^* to it, we obtain a morphism $i^*G(X, \mathrm{Eq}) \to i^*G(Y, \mathrm{Eq})$.

We leave it to the reader to check that this defines a functor $\Gamma_R{:}\mathcal{E}[P_{A,j}] \to \mathrm{Sh}_j(\mathcal{E})$ whose object part sends (X, Eq) to $i^*G(X, \mathrm{Eq})$. Let us show that Γ_R is left adjoint to Δ_R.

Given a morphism $(Y, \mathrm{Eq}) \to \Delta_R(X)$, represented by $F{:}Y \times X \to \Omega_j^A$, the totality requirement of F with respect to the tripos $P_{A,j}$ means that

$$\begin{aligned}\mathcal{E} \models \exists a{:}A.\forall y{:}Y.\forall b{:}A.b \in \mathrm{Eq}(y, y) \to ab{\downarrow}\wedge \\ j(\exists x{:}X.ab \in F(y, x))\end{aligned}$$

Writing $G(Y, \mathrm{Eq})$ as $Y'/\sim$, we see that if we let

$$Y'' = \{y{:}Y \mid \exists x{:}X.\exists a{:}A.a \in F(y, x)\}$$

the inclusion $Y'' \to Y'$ is j-dense; so $(Y''/\sim) \to (Y'/\sim)$ is a j-dense monic. Since the quantifier $\exists x{:}X$ in the definition of Y'' is (by single-valuedness of F) in fact of form $\exists!x{:}X$, we have a morphism $(Y''/\sim) \to X$ in $\mathcal{E}$ which extends, since X is a sheaf, uniquely to a morphism $\Gamma_R(Y, \mathrm{Eq}) \to X$.

Conversely, given a morphism $G(Y, \mathrm{Eq}) \xrightarrow{f} X$ we define $F{:}Y \times X \to \Omega_j^A$ in $\mathcal{E}$ by

$$F(y, x) = \{a{:}A \mid f([y]) = x\}$$

Then F represents a morphism $(Y, \mathrm{Eq}) \to \Delta_R(X)$. The reader can verify that the two operations on morphisms are inverse to each other, that the correspondence obtained is natural, and that the composite $\Gamma_R \circ \Delta_R$ is naturally isomorphic to the identity on $\mathrm{Sh}_j(\mathcal{E})$.

It is straightforward to check from the explicit description of the geometric morphisms, that the diagram in the statement of the theorem commutes.

Finally, the mentioned pullback property amounts to the following. Let j_0, j_1, j_2 be the topologies in $\mathcal{E}[P_A]$ whose categories of sheaves are $\mathcal{E}$, $\mathcal{E}[P_{A,j}]$ and $\mathrm{Sh}_j(\mathcal{E})$, respectively. Then we must show that j_2 is the join of j_0 and j_1 in the lattice of internal topologies in $\mathcal{E}[P_A]$. This will be immediate from the observation that these maps are determined by morphisms $k_0, k_1, k_2 : \Omega^A \to \Omega^A$ in $\mathcal{E}$:

$$\begin{aligned} k_0(\alpha) &= \{a{:}A \mid \exists a'{:}A.a' \in \alpha\} \\ k_1(\alpha) &= j^A(\alpha) \\ k_2(\alpha) &= \{a{:}A \mid j(\exists a'{:}A.j(a' \in \alpha))\} \end{aligned}$$

∎

From theorem 2.1 we draw two inferences: firstly, the implication in Proposition 1.6 is actually an equivalence, because it is well known (e.g.,[9]) that open inclusions are stable under pullback along inclusions.

The second inference is more important for our purposes. Suppose now that j is an open topology, $j(x) = u \to x$. Then j has a *complement* in the lattice of topologies in $\mathcal{E}$, the *closed complement* $k(x) = u \vee x$ (see, e.g.,[8]). By extension one also says $\mathrm{Sh}_k(\mathcal{E})$ is the closed complement of $\mathrm{Sh}_j(\mathcal{E})$ in $\mathcal{E}$. By 1.6, $\mathcal{E}[P_{A,j}]$ is an open subtopos of $\mathcal{E}[P_A]$. Now it is an easy exercise in internal locale theory to prove the following: if

$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d] & \mathcal{H} \arrow[d] \\ \mathcal{F} \arrow[r] & \mathcal{E} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{K} \arrow[r] \arrow[d] & \mathcal{L} \arrow[d] \\ \mathcal{F} \arrow[r] & \mathcal{E} \end{tikzcd}$$

are pullback squares of inclusions of toposes, $\mathcal{H} \to \mathcal{E}$ is open and $\mathcal{L} \to \mathcal{E}$ its closed complement, then $\mathcal{K} \to \mathcal{F}$ is the closed complement of $\mathcal{G} \to \mathcal{F}$.

Definition 2.2 Let $\mathcal{E}$ a topos, j an open topology in $\mathcal{E}$, A a j-regular internal partial combinatory algebra in $\mathcal{E}$. The *Modified Realizability Topos* $\mathcal{M}_{A,j}$ with respect to A and j, is defined as the closed complement of $\mathcal{E}[P_{A,j}]$ in $\mathcal{E}[P_A]$.

We shall see in the next section that this definition agrees with traditional usage of the term "modified realizability". Note that we do *not* claim that if k is the closed complement of j, $\mathcal{M}_{A,j}$ is $\mathcal{E}[P_{A,k}]$! In fact this is generally false (and it would make little sense anyway, since we cannot assume A is k-regular). The following proposition is now obvious.

Proposition 2.3 *Let j be an open topology in $\mathcal{E}$, A j-regular. Let k be j's closed complement. Then*

$$\begin{tikzcd} \mathrm{Sh}_k(\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{M}_{A,j} \arrow[d] \\ \mathcal{E} \arrow[r] & \mathcal{E}[P_A] \end{tikzcd}$$

is a pullback diagram of toposes.

Let us describe a tripos representing $\mathcal{M}_{A,j}$ explicitly. Suppose j is the open topology $x \mapsto u \to x$, then we saw in 1.6 that the inverse image of the inclusion $P_{A,j} \to P_A$ is given by

$$\varphi \mapsto \lambda x{:}X.A' \Rightarrow \varphi(x)$$

where $A' = \{a{:}A \mid u\}$. Therefore the tripos $Q_{A,j}$ representing $\mathcal{M}_{A,j}$ can be defined by

$$Q_{A,j}(X) = \{\varphi : X \to \Omega^A \mid (\lambda x{:}X.A') \leq \varphi\}$$

where $\leq$ refers to the order in $P_A(X)$. The reflection $P_A(X) \to Q_{A,j}(X)$ is given by $\varphi \mapsto (\lambda x{:}X.A') \vee \varphi$, where $\vee$ is the join in the Heyting algebra $P_A(X)$.

Proposition 2.4 *Let $A \to B$ be an embedding of internal pca's in $\mathcal{E}$ such that A is an elementary subobject of B. Then there is a geometric morphism of triposes*

$Q_{B,j} \to Q_{A,j}$ which gives rise to a surjective geometric morphism $\mathcal{M}_{B,j} \to \mathcal{M}_{A,j}$ such that the diagram

$$\begin{tikzcd}
\mathcal{M}_{B,j} \arrow[r] \arrow[d] & \mathcal{M}_{A,j} \arrow[d] \\
\mathcal{E}[P_B] \arrow[r] & \mathcal{E}[P_A]
\end{tikzcd}$$

commutes.

Proof. The direct image $\Phi^* : P_B(X) \to P_A(X)$ was given by pointwise intersection with A (1.3); now this restricts to a map $Q_{B,j}(X) \to Q_{A,j}(X)$ which is seen as follows: if, internally in $\mathcal{E}$, $\varphi : X \to \Omega^B$ is an element of $Q_{B,j}(X)$ then

$$\exists e{:}B\,\forall x{:}X\,\forall b{:}B\,(u \to eb{\downarrow} \wedge eb \in \varphi(x))$$

By elementariness,

$$\exists e{:}A\,\forall x{:}X\,\forall b{:}B\,(u \to eb{\downarrow} \wedge eb \in \varphi(x))$$

By the embedding property,

$$\exists e{:}A\,\forall x{:}X\,\forall a{:}A\,(u \to ea{\downarrow} \wedge ea \in (\varphi(x) \cap A))$$

In the other direction, the left adjoint $\Phi_!$ of Φ^* also restricts: if $\varphi \in Q_{A,j}(X)$ then

$$\exists e{:}A\,\forall x{:}X\,\forall a{:}A\,(u \to ea{\downarrow} \wedge ea \in \varphi(x))$$

Given such e, let $e' = \Lambda x.ek$. Then since $A \to B$ is an embedding,

$$\forall x{:}X\,\forall b{:}B\,(u \to e'b{\downarrow} \wedge e'b \in \varphi(x))$$

so $\varphi \in Q_{B,j}(X)$.

Note, that the fact that $\Phi_! : P_A(X) \to P_B(X)$ preserves finite meets implies that its restriction to $Q_{A,j}(X)$ also preserves them. The commutation is also clear. Finally, just as in 1.3 it follows from elementariness that $\Phi_!$ reflects the order, hence is full and faithful; from this it follows easily that $\mathcal{M}_{B,j} \to \mathcal{M}_{A,j}$ is surjective. ∎

3 Examples

3.1 An almost-example

N. Goodman ([5]) has the following situation: let T be a set of partial functions $\mathbb{N} \rightharpoonup \mathbb{N}$, ordered by inclusion. A is the internal pca in Set^T where at each partial function r, A_r is the ordinary pca of indices for partial functions recursive in r.

The realizability is defined as follows (we adapt notation to ours): for $\varphi, \psi : X \to \Omega^A$,

$\varphi \leq \psi$ is forced at r iff for some $a \in A_r$: for all $s \geq r$ and all $x \in X_s, b \in \varphi(x)_s$, there is $t \geq s$ such that ab is defined in A_t and an element of $\psi(x)_t$.

In our tripos-theoretic context this means the following. Let j be the double-negation topology, A the given internal pca. $P(X)$ is the set of arrows: $X \to \Omega^A$ in set^T, and $\varphi \leq \psi$ holds iff

$$\exists a{:}A\, \forall x{:}X\, \forall b \in \varphi(x)\, j(ab{\downarrow} \wedge ab \in \psi(x))$$

is true in Set^T.

It is straightforward to prove that this gives a tripos on Set^T, and also that φ is isomorphic in $P(X)$ to $\lambda x{:}X\, j^A(\varphi(x))$. So P looks very much like our $P_{A,j}$. However, Goodman's pca is *not* $\neg\neg$-regular, and there is no inclusion in the tripos P_A. This is obviously a variation, and the exact connection with our set-up remains to be clarified. It is true that $\mathrm{Sh}_{\neg\neg}(\mathrm{Set}^T)$ is a subtopos of $\mathrm{Set}^T[P]$ ([18]), but we do not know whether it is equivalent to any of the toposes we consider.

A very similar example, where the topology is different from $\neg\neg$ and the pca *is* j-regular, is used in [19].

3.2 Relative Realizability

Given an embedding $A_\sharp \subseteq A$ in Set, [2] defines a tripos P on Set: $P(X) = \mathcal{P}(A)^X$ but $\varphi \leq \psi$ iff there is $a \in A_\sharp$ such that for all $x \in X, b \in \varphi(x)$, ab is defined and an element of $\psi(x)$.

Regard $A_\sharp \to A$ as an internal pca $\mathcal{A}$ in the topos $\mathrm{Set}^\to$. This topos has a point $0 : \mathrm{Set} \to \mathrm{Set}^\to$, corresponding to the open point of Sierpinski space: $0_*(X) = (X \xrightarrow{\mathrm{id}} X)$, $0^*(X \to Y) = Y$. Moreover, 0_* embeds Set as $\neg\neg$-sheaves into $\mathrm{Set}^\to$.

In $\mathrm{Set}^\to$, the power object $\Omega^{\mathcal{A}}$ is $(R \xrightarrow{\pi_2} \mathcal{P}(A))$ where

$$R = \{(U, V) \mid U \in \mathcal{P}(A_\sharp), V \in \mathcal{P}(A), U \subseteq V\}$$

and π_2 is the second projection.

$(\Omega_{\neg\neg})^{\mathcal{A}}$ is $(R' \xrightarrow{\pi_2} \mathcal{P}(A))$ where

$$R' = \{(U, V) \mid V \in \mathcal{P}(A), U = V \cap A_\sharp\}$$

We see that there is a natural 1-1 correspondence between maps $X \xrightarrow{\varphi} \mathcal{P}(A)$ in Set, and morphisms $0_*(X) \xrightarrow{\tilde{\varphi}} (\Omega_{\neg\neg})^{\mathcal{A}}$ in $\mathrm{Set}^\to$, and we have $\varphi \leq \psi$ in $P(X)$ iff

$$\mathrm{Set}^\to \models \exists a{:}\mathcal{A}\, \forall x{:}0_*(X)\, \forall b \in \tilde{\varphi}(x)\, (ab{\downarrow} \wedge ab \in \tilde{\psi}(x))$$

So in fact, P is $P_{\mathcal{A},\neg\neg} \circ (0_*)^{\mathrm{op}}$ and we are in the situation described just above Theorem 2.1.

Quite similarly, the standard realizability tripos over a pca A in Set is equivalent to $P_{\mathcal{A},\neg\neg} \circ (0_*)^{\mathrm{op}}$ where now $\mathcal{A} = (A \xrightarrow{\mathrm{id}} A)$.

Note, that the requirement of $A_\sharp \to A$ to be an embedding in Set, makes the inclusion of $(A_\sharp \xrightarrow{\mathrm{id}} A_\sharp)$ into $(A_\sharp \to A)$ an embedding in $\mathrm{Set}^\to$; and it is trivial to see that this is also the inclusion of an elementary subobject.

Moreover, there is a $\neg\neg$-dense inclusion of $(A_\sharp \to A)$ into $(A \to A)$. So our propositions 1.5 and 1.8 generalize the theorems in [2] on the existence of a local map of toposes, and a logical functor between toposes.

3.3 Kleene's 1957-realizability

Our conjecture here is the following: this notion of realizability (formulated in terms of partial recursive application with oracle functions) is a relative realizability situation $A_\sharp \to A$, where A is the pca for (Kleene's) function realizability, and $A_\sharp$ its sub-pca of total recursive functions.

Kleene later abandoned his 1957 concept in favour of function realizability, which he said was "equivalent". Now these realizabilities were, at the time, looked at from a classical point of view, so for every sentence ϕ, either ϕ or its negation is realizable. In this sense, the equivalence should be a consequence of the logical functor

$$\mathrm{RT}(A_\sharp, A) \to \mathrm{RT}(A)$$

3.4 Modified and Relative Modified Realizability

In the special case of the pca $A = (\mathbb{N} \to \mathbb{N})$ in $\mathrm{Set}^\to$ and the open $\neg\neg$-topology there, the fact that $\mathcal{M}_{A,\neg\neg}$ is the closed complement of $\mathrm{Set}^\to[P_{A,\neg\neg}]$ in $\mathrm{Set}^\to[P_A]$ (that is, the *modified realizability topos $\mathcal{M}od$ is the closed complement of the* effective topos $\mathcal{E}ff$ in this topos), was demonstrated in [20].

An example of Relative Modified Realizability occurs in [15]. Here one has $\mathcal{M}_{\mathcal{A},\neg\neg}$ where $\mathcal{A} = (A_\sharp \to A)$ is again the inclusion of total recursive functions into the pca for function realizability.

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