History and Future of Superconvergence in Three-Dimensional Finite Element Methods

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Abstract

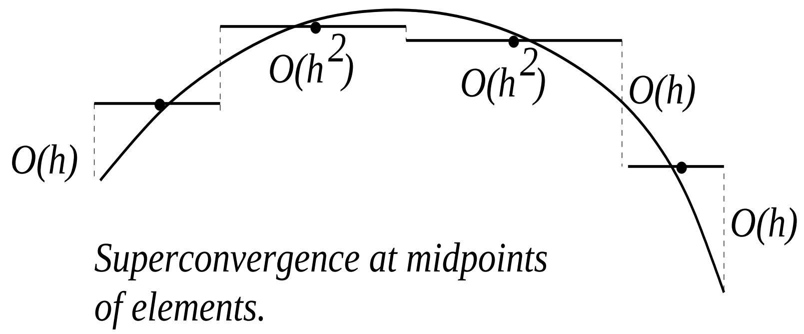
We will give an overview of superconvergence results for finite element methods applied to problems in three space dimensions. Apart from that, we sketch techniques that could be applied to three dimensional superconvergence questions, and indicate what exactly makes the three-dimensional case so much harder to tackle than the two-dimensional case, for which many more results are known.

Keywords: superconvergence, 3d-problem, local orthogonality, local symmetry, computer based proof

AMS subject classification: 65N30.

1 Introduction

This paper deals with superconvergence phenomena in finite element methods for three-dimensional problems. Two types of superconvergence will be considered here. The first is *pointwise superconvergence*, by which we mean that at special a priori known points, the rate of convergence of a finite element approximation is higher than what is globally possible, as was, for example, proved for specific situations in the papers [3, 10, 13, 21, 26, 28, 31, 32].



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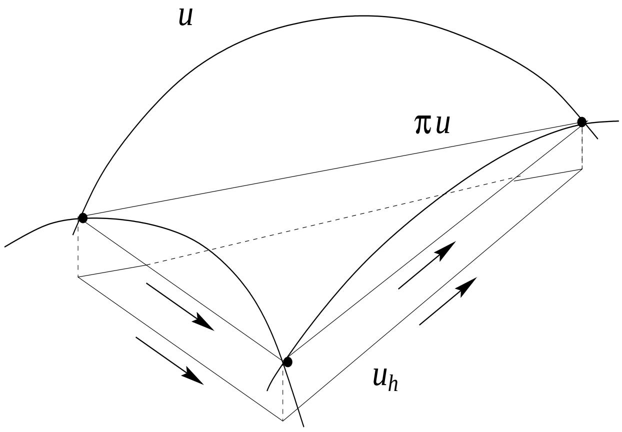
The second type of superconvergence we will refer to as *interpolantwise*. By this, we mean that the finite element method is a higher order perturbation of a local interpolation or projection scheme. Papers dealing with interpolantwise superconvergence are, for example, [1, 2, 4, 5, 8, 9, 12, 15, 16, 17, 19, 20, 23, 29].



Interpolantwise superconvergence of the gradient.

Both types of superconvergence can be exploited to post-process the approximate solution, for example with the aim to obtain an *a posteriori error estimate* of the error in the original approximation. In the case of pointwise superconvergence this can be done by interpolation at the superconvergence points with higher degree (piecewise-) polynomials, and in the case of interpolantwise superconvergence by using a post-processing mechanism of which it is known that it works successfully for the interpolant to which the finite element solution is close.

Example I

Consider the two-point boundary value problem $-u'' = f$ on the unit interval with homogeneous Dirichlet boundary conditions. Its finite element formulation using the space V_h of continuous piecewise linear functions relative to some partitioning of $[0,1]$ is to find $u_h \in V_h$ such that for all $v_h \in V_h$ we have $(u_h', v_h') = (f, v_h)$. It is a well-known fact that u_h equals u at the partitioning points of the domain.

This is an extreme example of pointwise superconvergence at the nodal points, but at the same time it serves as an example of interpolantwise superconvergence: u_h is locally equal to the linear Lagrange interpolant of u.

Example II

A second, more sophisticated example is given by the Poisson problem $-\Delta u = f$ on some domain $\Omega \subset I\!R^2$, discretized by a Raviart-Thomas [25] mixed finite element method, either of lowest order $k = 0$ or one but lowest order $k = 1$, using a so-called *three-line mesh*. It was proved in [4, 5] that the mixed approximation $\mathbf{p}_h$ to the gradient $\mathbf{p} = \mathbf{grad}\, u$ is interpolantwise superconvergent with respect to the local so-called *Fortin interpolant* of $\mathbf{p}$. Post-processing mechanisms were developed that are successful for the Fortin interpolant and, as a consequence of the superconvergence, also for improving the approximation quality of $\mathbf{p}_h$.

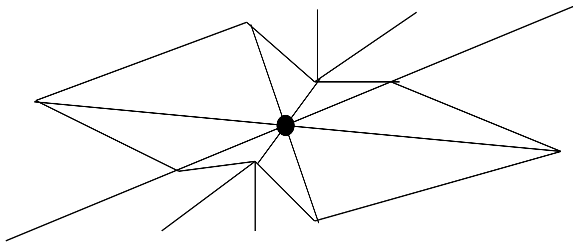
1.1 Superconvergence for three-dimensional elements

Superconvergence results for three-dimensional problems are relatively scarce. We give references to the most important ones below.

1.1.1 Standard elements for elliptic problems

The first results in three space dimensions go back to 1978 when Zlámal [30] considered pointwise superconvergence for a 20-degrees-of-freedom isoparametric quadratic elements of Serendipity type for an elliptic problem. Then, in 1980, Chen considered the linear tetrahedral element and proved interpolantwise superconvergence for the gradient in [6]. In the second half of the eighties, the same result was also obtained by Kantchev and Lazarov [18], who were at that time not aware of the developments in China. Pehlivanov [24] already in 1989 reflects on the quadratic tetrahedral case, but unfortunately without any (reference to a) proof. As far as we know, there is still no proof available in the literature. In 1994, Goodsell derived, for a recovered gradient, pointwise superconvergence results for linear tetrahedral elements in [14].

To conclude, we mention the recent (1996) superconvergence approach in [26]. It is very general in nature and yields pointwise superconvergence at symmetry points of the meshes.



Locally pointsymmetric mesh.

However, this approach yields superconvergence for the gradient only for odd degrees of polynomial approximation, and for function values only for even degrees of polynomial approximation.

1.1.2 Mixed finite elements

In the mixed finite element setting, superconvergence results were obtained by Monk [22] in 1993 for the discretization of the Maxwell-equations using Nédélec elements of lowest order on hexahedral meshes. For tetrahedral Raviart-Thomas elements of lowest order, Dupont and Keenan [11] witnessed superconvergence of the gradient approximation experimentally in 1998, but no proof is given. Superconvergence for the scalar variable in the mixed setting is basically independent of the dimension - and is therefore also present in three space dimensions. See [9] for a proof of superconvergence in the two-dimensional case.

2 Superconvergence techniques in 3D

Here we present an overview of the three most promising and up to date techniques to tackle superconvergence and in particular their potential in the three-dimensional setting. All three methods originate from the second half of the nineties.

2.1 Element Orthogonality Analysis

The *Element Orthogonality Analysis (EOA)* is a product of the Chinese school, i.e. Chen and Zhu et. al., see Chen [7] for an up-to-date treatment. The basic idea behind the EOA is as follows. Instead of the classical approach, in which an interpolant is taken and its difference to the finite element solution studied, it *locally constructs* an interpolant such that it is close to the finite element solution. For this, the local degrees of freedom are divided into two groups. The first group is used to ensure the required global regularity. The remaining degrees of freedom are used to *force local orthogonality* as much as possible. This local orthogonality is then used to prove bounds of the following type,

$$|u_h - \pi u|_1^2 = (\mathbf{grad}\,(u - \pi u), \mathbf{grad}\,(u_h - \pi u)) \leq Ch|u - \pi u|_1 |u_h - \pi u|_1, \quad (1)$$

which obviously results in interpolantwise superconvergence,

$$|u_h - \pi u|_1 \leq Ch|u - \pi u|_1. \quad (2)$$

Here, πu is the constructed interpolant, the equality in (1) is Galerkin orthogonality (for the Poisson equation) and the inequality is a *strengthened Cauchy-Schwartz inequality*, that hopefully follows from the local orthogonality properties of πu. And indeed, in for the three-line mesh in two space dimensions, this is the case.

Remark 2.1 In Chen [7], pointwise superconvergence is derived from this interpolantwise superconvergence; in case the degree m of polynomial approximation is *even*, the *function values* are superconvergent, and in case m is *odd*, the superconvergence is for *averaged gradient values*.

The EOA has great potential to tackle three-dimensional superconvergence, simply by following the same procedure in 3D, although this will be technically more complicated. However, as we will in Section 3, local orthogonality properties of interpolants seem to disappear gradually as the dimension of the domain increases.

2.2 Locally pointsymmetric meshes

The approach of Schatz, Sloan and Wahlbin [26] based on locally pointsymmetric meshes is fully applicable to problems in three space dimensions. We will illustrate the main ingredients of this approach with their one-dimensional counterparts.

Observation 2.2 Let $p \in [1, \infty]$ and $B_d := [-d, d]$. Then, if $v \in W^1_\infty(B_d)$ is odd,

$$|v|_{L_p(B_d)} \leq d|v|_{W^1_\infty(B_d)}. \quad (3)$$

This simple fact follows immediately from Taylor expansion.

Observation 2.3 Let e be a function on B_d. Define

$$e_{\text{av}}(0) := \lim_{x \to 0} w(x), \quad \text{where} \quad w(x) := \frac{1}{2}\left(e(x) + e(-x)\right). \tag{4}$$

Then for all $\varepsilon > 0$,

$$e_{\text{av}}(0) \leq \|w\|_{\infty,[-\varepsilon,\varepsilon]}. \tag{5}$$

This last observation states that a function value (or its averaged value in case the function is discontinuous), can be bounded by considering only the even part of this function around the point of interest.

We will now loosely outline the analysis of superconvergence in four steps. The third step, the most sophisticated one, involves local supremum norm estimates coming from [27], which, in fact, have nothing to do with superconvergence at all. The other steps make clever use of the symmetry of the mesh in order to profit from the two observations above.

Step I. The local symmetry of the mesh at a point M yields, that the odd part e_{odd} around M of the error $e := u - u_h$ of a finite element approximation, can often be proved to *locally satisfy Galerkin orthogonality*, i.e., Galerkin orthogonality with respect to the local subset of basis functions living on the pointsymmetric part of the mesh only.

Step II. If we are interested in the (if necessary, averaged) *derivative* of e at M, we can, as seen in Observation 2.3, consider its even part around M. This even part of the derivative of e equals the derivative of e_{odd}.

Step III. The estimates in [27] for functions (such as e_{odd}) satisfying local Galerkin orthogonality provide a local supremum norm estimate for this derivative of e_{odd} in terms of a negative norm of e plus a local approximation problem for the odd part of u. The negative norm can often be controlled without too restrictive assumptions.

Step IV. If the proper degree of polynomial approximation is used, the local approximation problem yields an L_p norm of the k-th derivative of the odd part of u. Normally (i.e., for general u), this is the endpoint of the bounding process, but when k is even and having the odd part of u, we can apply Observation 2.2 and gain a superconvergent factor of order h. Depending on the bound of the global norm in Step III, this leads to superconvergence at M.

Remark 2.4 Similarly, one can start off with e instead of its derivative, in order to look for superconvergence for function values. Using the proper degree of polynomial approximation it is possible to end up with a norm of an odd function again - and gain a superconvergent factor of h, which might or might not survive the influences of the negative norm of e. See [26] for details.

2.3 Computer-based methods

Another recent method to tackle superconvergence was introduced by Babuška et. al. in [3]. In their approach, the use of the computer to find a priori superconvergence points in the mesh is central. Nevertheless, the basis of their approach is mathematical in nature and the computer is only needed to compute the location of the superconvergence points, since this cannot be done analytically.

The approach, as in [3], is worked out in two space dimensions and covers (locally) periodic triangular meshes such as the three-line mesh, the chevron, Union-Jack, and criss-cross meshes. The Poisson and Laplace problem are considered. The results do not restrict to lower order polynomial approximation, since it actually includes results up to degree seven piecewise polynomial approximation. Another special feature is that it is claimed that *all* superconvergence points for the derivatives can be found in this way.

Remark 2.5 Since the superconvergence points are proved to be intersections of certain zero-curves of polynomials in two variables, the computer is used to compute them. This is the only computational aspect in the approach.

The approach seems to allow a generalization to three space dimensions, even though the technical details will be much harder. Also, it is to be expected that when superconvergence points again coincide with intersections of polynomial zero-surfaces, their number will drastically decrease. Other reasons why superconvergence in three dimensions will probably be found less often than in two space dimensions we will give in the following section.

3 Typical difficulties in three dimensions

We will now indicate some convincing reasons why superconvergence in three space dimensions will be much more a rarity than in two dimensions. We start with one that has direct consequences in the EOA approach.

3.1 Loss of orthogonality

Consider as an example the quadratic finite elements on intervals, triangles, and tetrahedra. We will define, independent of the dimension of the domain, a local interpolation scheme Q_h that maps onto the space V_h^2 of continuous piecewise quadratic functions. By $\mathcal{P}_k(\cdot)$ we denote the space of polynomials of degree k on the domain between the brackets.

Definition 3.1 (Interpolation Q_h) For v smooth enough, its interpolant $Q_h v \in V_h^2$ is such that $Q_h v(N) = v(N)$ for all vertices N, and such that for all edges e in the partitioning,

$$\int_e Q_h v(s) ds = \int_e v(s) ds. \tag{6}$$

If we consider the orthogonality properties of the first derivative(s) of this interpolant, we find the following. In one space-dimension, it is easy to see, using integration by parts, that on each sub-interval e we have

$$\forall q \in \mathcal{P}_1(e), \quad \int_e \frac{\partial}{\partial x} \left(v(s) - Q_h v(s)\right) q(s) ds = 0. \tag{7}$$

As a matter of fact, this property implies that if $v \in H_0^1$,

$$\forall w_h \in V_h^2, \quad (\frac{\partial}{\partial x} v - \frac{\partial}{\partial x} Q_h(v), \frac{\partial}{\partial x} w_h) = 0, \tag{8}$$

so $Q_h v$ equals the quadratic finite element approximation of $-v'' = f$ with homogeneous Dirichlet boundary conditions. Or, in other words, Q_h equals the elliptic projection on V_h^2.

In two space-dimensions, this is not the case. Let T be a triangle, and define the six degrees of freedom as in Definition 3.1. Then there is no orthogonality with respect to gradients of quadratic polynomials, only with respect to (all) constant vector fields:

$$\forall \mathbf{q} \in [\mathcal{P}_0(T)]^2, \quad \int_T \mathbf{grad}\ (v(x) - Q_h v(x))^T\, \mathbf{q}(x) dx = 0. \tag{9}$$

This property, too, can be proved by using integration by parts. Going up one dimension more, we even loose *all* the orthogonality to polynomial spaces, i.e., defining a quadratic interpolant on a tetrahedron K by the values at the vertices and the averages along the edges as in Definition 3.1, we get

$$\forall \mathbf{q} \in [\mathcal{P}_0(K)]^3, \quad \int_K \mathbf{grad}\ (v(x) - Q_h v(x))^T\, \mathbf{q}(x) dx \neq 0. \tag{10}$$

In manipulating the Galerkin equations and hoping for cancellations, clearly this is a bad situation, and the cancellations need to be found in a different way, for example by combining several neighboring elements. But this too gets harder in higher dimensions.

3.2 Loss of symmetry in the meshes

In three space dimensions and using simplectic elements, there will be less symmetry in the meshes than in two space dimensions. To start with, it is impossible to fill space with regular tetrahedra, while clearly the plane can be covered by triangles of the same shape (even of any shape). Therefore, in two space dimensions, it is easy to have meshes in which each two triangles sharing an edge form a point-symmetric domain. Two tetrahedra sharing a face however, do not form a point-symmetric domain. There are, however, simple ways to subdivide parallelepipeds into a number of tetrahedra (e.g., six, eight) giving rise to relatively small *pointsymmetric patches* of tetrahedra.

These considerations have their impact on the EOA if the EOA needs to consider patches of elements instead of single elements. Also, in the approach with locally pointsymmetric meshes, the consequences of having less symmetry are evident. The computer-based approach in 3D might suffer less from the loss of symmetry, but still, locally periodic meshes need to be formed.

4 Outlook

Apart from the three general approaches in Section 2, it will always be possible to try and tackle problems case by case. At the moment, the authors of this paper aim to prove superconvergence for the gradient in the quadratic tetrahedral setting, as Pehlivanov conjectured in [24]. Also, the three-dimensional hexahedral Raviart-Thomas elements (gradient approximations) as well as the lowest order tetrahedral elements (as conjectured by Dupont and Keenan in [11]) are candidates for superconvergence and work on that is in progress. For this, we use a three-dimensional version of the *Discrete Helmholtz Decomposition* for discrete vector fields, the pointsymmetry of (patches of) elements, the L_2-stability of the nodal basis, and the Bramble Hilbert lemma. Note that the approach in Section 2.2 does not apply in the mixed setting due to the lack of local supremum norm estimates.

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References

[1] A.B. Andreev. Error estimate of type superconvergence of the gradient for quadratic triangular elements. *C. R. Acad. Bulgare Sci.*, 36:1179–1182, 1984.

[2] A.B. Andreev and R.D. Lazarov. Superconvergence of the gradient for quadratic triangular finite elements. *Numer. Methods Partial Differential Equations*, 4:15–32, 1988.

[3] I. Babuška, T. Strouboulis, C.S. Upadhyay and S.K. Gangaray. Computer-based proof of existence of superconvergence points in the finite element method; superconvergence of derivatives in finite element solutions of Laplace's, Poisson's and the elasticity equations. *Numer. Methods Partial Differential Equations*, 12:347–392, 1996.

[4] J.H. Brandts. Superconvergence and a posteriori error estimation for triangular mixed finite elements. *Numer. Math.*, 68(3):311–324, 1994.

[5] J.H. Brandts. Superconvergence for triangular order $k = 1$ Raviart-Thomas mixed finite elements and for triangular standard quadratic finite element methods. *Appl. Numer. Anal.*, 34:39–58, 2000.

[6] C.M. Chen. Optimal points of stresses for tetrahedron linear element. *Natur. Sci. J. Xiangtan Univ.*, 3:16–24, 1980.

[7] C.M. Chen. Superconvergence for triangular finite elements. *Science in China*, 42(9):917–924, 1999.

[8] J. Douglas, T. Dupont, and M.F. Wheeler. An l^∞ estimate and a superconvergence result for a Galerkin method for elliptical equations based on tensor products of piecewise polynomials. *RAIRO Anal. Numér.*, 8:61–66, 1974.

[9] J. Douglas and J.E. Roberts. Global estimates for mixed methods for second order elliptic problems. *Math. Comp.*, 44(169):39–52, 1985.

[10] J. Douglas and J. Wang. Superconvergence of mixed finite element methods on rectangular domains. *Calcolo*, 26:121–133, 1989.

[11] T.F. Dupont and P.T. Keenan. Superconvergence and Postprocessing of fluxes from lowest-order mixed methods on triangles and tetrahedra. *SIAM J. Sci. Comp.*, 19(4):1322–1332, 1998.

[12] R. Durán. Superconvergence for rectangular mixed finite elements. *Numer. Math.*, 58:287–298, 1990.

[13] R.E. Ewing, R.D. Lazarov, and J. Wang. Superconvergence of the velocity along the Gauss lines in mixed finite element methods. *SIAM J. Numer. Anal.*, 28:1015–1029, 1991.

[14] G. Goodsell. Pointwise superconvergence of the gradient for the linear tetrahedral element *Numer. Methods Partial Differential Equations*, 10:651–666, 1994.

[15] G. Goodsell and J.R. Whiteman. A unified treatment of superconvergent recovered gradient functions for piecewise linear finite element approximations. *Internat. J. Numer. Methods Engrg.*, 27:469–481, 1989.

[16] G. Goodsell and J.R. Whiteman. Superconvergence of recovered gradients of piecewise quadratic finite element approximations, Part I, L^2-error estimates. *Numer. Methods Partial Differential Equations*, 7:61–83, 1991.

[17] G. Goodsell and J.R. Whiteman. Superconvergence of recovered gradients of piecewise quadratic finite element approximations, Part 2, L^∞-error estimates. *Numer. Methods Partial Differential Equations*, 7:85–99, 1991.

[18] V. Kantchev and R.D. Lazarov. *Superconvergence of the gradient of linear finite elements for 3D Poisson equation. Proc. Conf. Optimal Algorithms (B. Sendov ed.)*, Publ. Bulg. Acad. Sci., Sofia, 172:182, 1986.

[19] M. Křížek and P. Neittaanmäki. On a global superconvergence of the gradient of linear triangular elements. *J. Comput. Appl. Math.*, 18:221-233, 1987.

[20] N. Levine. Superconvergent recovery of the gradient from piecewise linear finite element approximations. *IMA J. Numer. Anal.*, 5:407-427, 1985.

[21] Q. Lin and J.C. Xu. Linear finite elements with high accuracy. *J. Comp. Math.*, 3:115–133, 1985.

[22] P. Monk, An analysis of Nédélec's method for the spatial discretization of Maxwell's equation. *J. of Comp. Appl. Math.* 47:292–315, 1993.

[23] L.A. Oganesjan and L.A. Ruhovets. Study of the rate of convergence of variational difference schemes for second-order elliptic equations in a two-dimensional field with a smooth boundary. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 9:1102-1120, 1969.

[24] A. Pehlivanov. *Superconvergence of the gradient for quadratic 3D simplex finite elements. Proc. Conf. Numer. Methods and Appl.*, Publ. Bulgar. Acad. Sci., Sofia, 362–366, 1989.

[25] P.A. Raviart and J.M. Thomas. A mixed finite element method for second order elliptic problems. *LN in Math.*, 606:292–315, 1977.

[26] A.H. Schatz, I.H. Sloan and L.B. Wahlbin. Superconvergence in finite element methods and meshes that are locally symmetric with respect to a point. *SIAM J. Numer. Anal.*, 33:505-521, 1996.

[27] A.H. Schatz and L.B. Wahlbin. Interior maximum-norm estimates for finite element methods, Part II. *Math. Comp.*, 64:907–928, 1995.

[28] J. Wang. Superconvergence and extrapolation for mixed finite element methods on rectangular domains. *Math. Comp.,* 56:477-503, 1991.

[29] M.F. Wheeler and J.R. Whiteman. Superconvergent recovery of gradients on subdomains from piecewise linear finite-element approximations. *Numer. Methods Partial Differential Equations*, 3:357–374, 1987.

[30] M. Zlámal. Superconvergence and reduced integration in the finite element method. *Math. Comp.*, 32:663–685, 1978.

[31] Q.D. Zhu. The derivative good points for the finite element method with 2-degree triangular element. *Natur. Sci. J. Xiangtan Univ.*, 4:36–45, 1981.

[32] Q.D. Zhu. *Natural inner superconvergence for the finite element method.* In: Proc. China-France Sympos. on the FEM, Bejing, pp. 935–960, 1982.