

# A NOTE ON ROTATIONS AND INTERVAL EXCHANGE TRANSFORMATIONS ON 3-INTERVALS

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ABSTRACT. We prove the conjecture that an interval exchange transformation on 3-intervals with corresponding permutation  $(1, 2, 3) \rightarrow (3, 2, 1)$ , and rationally independent discontinuity points, is never measure theoretically isomorphic to an irrational rotation.

## 1. INTRODUCTION

Interval exchange transformations were first introduced by Keane in [K1], and are defined as follows. Let  $I = [0, 1)$ ,  $n \geq 2$  and  $\alpha = (\alpha_1, \dots, \alpha_n)$  a probability vector with  $\alpha_i > 0$ . Define  $\beta_0 = 0$  and  $\beta_i = \sum_{k=1}^i \alpha_k$ , and set  $I_i = [\beta_{i-1}, \beta_i)$ . Let  $\tau$  be a permutation of  $\{1, 2, \dots, n\}$ , and consider the probability vector  $\alpha^\tau = (\alpha_{\tau^{-1}(1)}, \dots, \alpha_{\tau^{-1}(n)})$ . Note that  $\alpha_{\tau^{-1}(i)} > 0$  for all  $i$ . Let  $\beta_0^\tau = 0$  and  $\beta_i^\tau = \sum_{k=1}^i \alpha_{\tau^{-1}(k)}$ , and set  $I_i^\tau = [\beta_{i-1}^\tau, \beta_i^\tau)$ .

Define  $T : I \rightarrow I$  by

$$Tx = x - \beta_{i-1} + \beta_{\tau(i)-1}^\tau$$

if  $x \in I_i$ .  $T$  is called an  $(\alpha, \tau)$  interval exchange transformation on  $n$  intervals. It is clear that  $T$  is invertible,  $T\beta_{i-1} = \beta_{\tau(i)-1}^\tau$  and  $T$  maps  $I_i$  isometrically onto  $I_{\tau(i)}^\tau$ . Further,  $T$  is continuous except possibly at  $\{\beta_1, \dots, \beta_{n-1}\}$ . At these points  $T$  is right continuous. Note that  $T$  is continuous at  $\beta_i$  if and only if  $\tau(i+1) = \tau(i) + 1$ . In other words,  $T$  is discontinuous at  $\beta_i$  if and only if  $T\beta_{i-1}, T\beta_i$  do not appear in this order as consecutive terms in the ordered set  $\{\beta_0^\tau, \dots, \beta_{2n}^\tau\}$ . We say  $T$  is in *standard form* if  $T$  is discontinuous at  $\beta_i$  for all  $i = 1, 2, \dots, n-1$  or equivalently, if  $\tau(i+1) \neq \tau(i) + 1$  for all  $i = 1, 2, \dots, n-1$ . Notice that any interval exchange transformation on  $n$  intervals can be written in standard form as an interval exchange transformation on  $m$  intervals with  $m \leq n$ . Since if  $T$  is not in standard form, then  $T$  is continuous at  $\beta_i$  for some  $i$ , then  $\tau(i+1) = \tau(i) + 1$ , and so  $T$  maps the interval  $[\beta_{i-1}, \beta_{i+1})$  isometrically onto  $[\beta_{\tau(i)-1}^\tau, \beta_{\tau(i)+1}^\tau)$ . Thus, we can redefine  $T$  on intervals with end points

$$\{\beta_0, \dots, \beta_{i-1}, \beta_{i+1}, \dots, \beta_n\}.$$

We repeat this process until all the remaining  $\beta$ 's are discontinuity points of  $T$ .

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The permutation  $\tau$  corresponding to  $T$  is said to be irreducible if

$$\tau(\{1, 2, \dots, k\}) \neq \{1, 2, \dots, k\}, \text{ for all } k = 0, 1, \dots, n-1.$$

Note that if  $\tau$  is reducible, then  $T$  can be decomposed into two interval exchange transformations, one on  $[0, \beta_k)$  and the other on  $[\beta_k, 1)$ . We assume throughout this paper that  $T$  is irreducible.

Interval exchange transformations have been studied by several authors. Here we mention few of the known results. In [K1], Keane studied the minimality of such transformations, and in [K2] questions concerning unique ergodicity were investigated. It is easy to see that if  $n = 2$ ,  $T$  corresponds to a rotation and if  $n = 3$ , then  $T$  can be seen as an induced transformation of a rotation. Thus, if the  $\beta$ 's are rationally independent, then in both cases  $T$  is uniquely ergodic. Keynes and Newton [KN], and also Keane [K2] gave examples of interval exchange transformations that are not uniquely ergodic. Masur [M], and independently Veech [V1, V2, V3, V4, V5] showed that almost every minimal interval exchange transformation is uniquely ergodic. Later Boshernitzan [B] gave another proof of this result by more elementary means. Some of the spectral properties were studied by Veech in a series of papers [V3, V4, V5]. Oseledets [O] and Goodson [G] constructed ergodic interval exchange transformations with simple spectrum. Recently, Berthé, Chekhova and Ferenczi [BCF] proved that every ergodic interval exchange transformation on three intervals has simple spectrum. The first interval exchange transformation with continuous spectrum was given by Katok and Stepin [KS], their example is also an exchange on three intervals. In [BCF], the authors gave other examples of exchanges on three intervals with continuous spectrum, and they conjectured that no non-trivial exchange on three intervals is measure theoretically isomorphic to an irrational rotation. In section 2 we prove this conjecture as a corollary of a recent result by Simin Li [S], where he gave necessary and sufficient conditions for an interval exchange transformation to be conjugate to an irrational rotation.

## 2. NON-TRIVIAL EXCHANGES ON 3-INTERVALS

Let  $0 < l < m < 1$  with  $1, l, m$  rationally independent. Consider the interval exchange transformation  $T$  given by

$$Tx = \begin{cases} x + 1 - l & x \in [0, l), \\ x + 1 - l - m & x \in [l, m), \\ x - m & x \in [m, 1). \end{cases}$$

$T$  corresponds to the permutation  $(1, 2, 3) \rightarrow (3, 2, 1)$ . Notice that  $T$  is the only interval exchange transformation on 3-intervals which is irreducible and in standard form. Moreover, by a result of Keane [K1],  $T$  is minimal. We call  $T$  a non-trivial exchange transformation on 3-intervals. It is well known that  $T$  is an induced transformation of the interval exchange transformation  $S$  defined on  $[0, 1 - l + m)$  by

$$Sx = \begin{cases} x + 1 - l & x \in [0, m), \\ x - m & x \in [m, 1 - l + m). \end{cases}$$

Since after normalization  $S$  is isomorphic to an irrational rotation,  $S$  is minimal and uniquely ergodic, and hence so is  $T$ .

Let  $\alpha = \frac{1-l}{1-l+m}$  and  $\beta = \frac{1}{1-l+m}$ . In [KS], the authors proved that if  $\alpha$  has unbounded partial quotients and if for some subsequence  $q_n$  of denominators of convergents of  $\alpha$ , we have

$$|\alpha - \frac{p_n}{q_n}| < o(\frac{1}{q_n^2}), \text{ and } |\beta - \frac{r}{q_n}| > \frac{c}{q_n}$$

for all  $r$  and some constant  $c > 0$ , then  $T$  is not measure theoretically isomorphic to an irrational rotation. In [BCF], it is proved that when  $\alpha$  has bounded partial quotients, and  $\beta \in K(\alpha)$  for some Cantor set  $K(\alpha)$ , then  $T$  is not measure theoretically isomorphic to an irrational rotation.

Simin Li [Li] gave recently necessary and sufficient conditions for an interval exchange transformation to be conjugate to an irrational rotation.

**Theorem 1** (Li). *Let  $T$  be an interval exchange transformation, and let  $d(T^n)$  be the number of discontinuities of  $T^n$ . Then,  $T$  is conjugate to an irrational rotation if and only if (i)  $T^n$  is minimal for all  $n \geq 1$ , (ii)  $\{d(T^n)\}$  is bounded by some integer  $N > 0$  and (iii) there exists  $k > 0$  and  $M \geq 2^{N^3+3N^2}$  such that  $d(T^k) = d(T^{2k}) = \dots = d(T^{M^k})$ .*

Since a non-trivial interval exchange on 3-intervals is uniquely ergodic, to show that it is not measure theoretically isomorphic to an irrational rotation, we prove that  $\{d(T^n)\}$  is an unbounded sequence.

**Theorem 2.** *Let  $T$  be a non-trivial interval exchange transformation on 3-intervals with rationally independent discontinuity points. Let  $D(T^n)$  be the set of discontinuity points of  $T^n$ , and let  $d(T^n)$  denote the cardinality of  $D(T^n)$ . Then*

$$D(T^n) = \{T^{-i}l, T^{-j}m : 0 \leq i, j \leq n-1\},$$

and hence,  $d(T^n) = 2n$ .

**Proof:** The proof is done by induction on  $n$ . The result is true for  $n = 1$ . Suppose

$$D(T^k) = \{T^{-i}l, T^{-j}m : 0 \leq i, j \leq k-1\},$$

for  $k = 1, 2, \dots, n$ . We prove the result for  $k = n + 1$ . Let

$$0 < \beta_1 < \beta_2 < \dots < \beta_{2n} < 1$$

be the discontinuities of  $T^n$  written in increasing order. By the induction hypothesis,

$$D(T^n) = \{\beta_i : 1 \leq i \leq 2n\} = \{T^{-i}l, T^{-j}m : 0 \leq i, j \leq n-1\}.$$

Let  $\beta_0 = 0$  and  $\beta_{2n+1} = 1$ . The underlying partition of  $T^n$  is given by

$$\mathcal{P}(T^n) = \{[\beta_i, \beta_{i+1}) : i = 0, 1, \dots, 2n\}.$$

Let  $\tau_n$  be the permutation corresponding to  $T^n$  (notice that  $T^n$  is an interval exchange transformation). Then,

$$T^n\{\beta_0, \beta_1, \dots, \beta_{2n}\} = \{\beta_0^{\tau_n}, \beta_1^{\tau_n}, \dots, \beta_{2n}^{\tau_n}\}$$

with  $\beta_0 = \beta_0^{\tau_n} = 0$ , and  $T^n\beta_i = \beta_{\tau_n(i+1)-1}^{\tau_n}$  for  $i = 0, 1, \dots, 2n$ . Furthermore, since  $1, l$  and  $m$  are rationally independent, and each  $\beta_i^{\tau_n}$  is a linear combination of  $1, l$  and  $m$  with integer coefficients, it follows that  $l, m \notin \{\beta_0^{\tau_n}, \beta_1^{\tau_n}, \dots, \beta_{2n}^{\tau_n}\}$ . Now invertibility of  $T$  implies that  $T\beta_0^{\tau_n}, \dots, T\beta_{2n}^{\tau_n}, Tm, Tl$  are all distinct.

Suppose  $l \in (\beta_{r-1}^{\tau_n}, \beta_r^{\tau_n})$ , and  $m \in (\beta_{s-1}^{\tau_n}, \beta_s^{\tau_n})$ . We consider three cases.

*Case 1.* If  $r = s$ , then  $T^{-n}l, T^{-n}m \in (\beta_{p-1}, \beta_p)$  where  $p = \tau_n^{-1}(r)$ . Since  $T$  is an order preserving isometry on  $[\beta_{p-1}, \beta_p)$ , it follows that  $T^{-n}l < T^{-n}m$ . The underlying partition of  $T^{n+1}$  is then given by

$$\mathcal{P}_1(T^{n+1}) = \{[\beta_0, \beta_1), \dots, [\beta_{p-2}, \beta_{p-1}), [\beta_{p-1}, T^{-n}l), [T^{-n}l, T^{-n}m), [T^{-n}m, \beta_p), \\ [\beta_p, \beta_{p+1}), \dots, [\beta_{2n}, \beta_{2n+1})\}.$$

To prove the result, we need to show that

$$\{\beta_1, \dots, \beta_{p-1}, T^{-n}l, T^{-n}m, \beta_p, \dots, \beta_{2n}\}$$

is the set of discontinuity points of  $T^{n+1}$ . Let

$$D_1 = \{\beta_0, \dots, \beta_{p-1}, T^{-n}l, T^{-n}m, \beta_p, \dots, \beta_{2n}\}$$

and

$$E_1 = \{\beta_0^{\tau_n}, \dots, \beta_{r-1}^{\tau_n}, l, m, \beta_r^{\tau_n}, \dots, \beta_{2n}^{\tau_n}\},$$

both considered as ordered sets. Then  $TD_1 = E_1$ , and by discontinuity of  $T^n$  at  $\beta_p$ , we have  $T^n\beta_p \neq \beta_r^{\tau_n}$ . Further,  $\beta_i^{\tau_n} \in (0, l)$  for  $1 \leq i \leq r-1$ , and  $\beta_i^{\tau_n} \in (m, l)$  for  $r \leq i \leq 2n$ . Hence,

$$\begin{aligned} T^{n+1}D_1 &= TE_1 \\ &= \{Tm = 0, T\beta_r^{\tau_n}, \dots, T\beta_{2n}^{\tau_n}, Tl = 1 - m, \\ &\quad T\beta_0^{\tau_n} = 1 - l, T\beta_1^{\tau_n}, \dots, T\beta_{r-1}^{\tau_n}\}. \end{aligned}$$

Here, the elements of  $TE_1$  are listed in increasing order.

We first show that  $T^{n+1}$  is discontinuous at  $\beta_i$  for  $i \neq p$ . To do this, we need to prove that  $T^{n+1}\beta_{i-1}$  and  $T^{n+1}\beta_i$  do not appear in this order as consecutive terms in  $TE_1$ . By assumption,  $T^n$  is discontinuous at  $\beta_i$ , hence  $T^n\beta_{i-1}$  and  $T^n\beta_i$  do not appear as consecutive terms of the form  $\beta_j^{\tau_n}, \beta_{j+1}^{\tau_n}$  in  $E_1$ . Let  $I_0 = [0, l)$ ,  $I_1 = [l, m)$  and  $I_2 = [m, 1)$ . If  $T^n\beta_{i-1}, T^n\beta_i \in I_j$  for some  $j = 0, 2$ , then since  $T$  maps  $I_j$  isometrically onto  $TI_j$ , it follows that  $T^{n+1}\beta_{i-1}$  and  $T^{n+1}\beta_i$  cannot appear as consecutive terms in  $TE_1$ . If  $T^n\beta_{i-1} \in I_j$  and  $T^n\beta_i \in I_k$  for  $j \neq k$ , then either  $T^n\beta_{i-1} \in I_0$  and  $T^n\beta_i \in I_2$ , or  $T^n\beta_{i-1} \in I_2$  and  $T^n\beta_i \in I_0$ . In the first case we get  $T^{n+1}\beta_i < 1 - m < T^{n+1}\beta_{i-1}$ , and in the second case, we get  $T^{n+1}\beta_{i-1} < 1 - m < T^{n+1}\beta_i$ . Hence,  $T^n\beta_{i-1}$  and  $T^n\beta_i$  do not appear as consecutive terms of the form  $\beta_j^{\tau_n}, \beta_{j+1}^{\tau_n}$  in  $E_1$ , and so  $T^{n+1}$  is discontinuous at  $\beta_i$ .

Now, the discontinuity of  $T^n$  at  $\beta_p$  implies that  $T^{n+1}\beta_p \neq T\beta_r^{\tau_n}$ , and

$$Tm = 0 < T\beta_r^{\tau_n} < T^{n+1}\beta_p.$$

Hence  $T^{n+1}(T^{-n}m) = Tm = 0$  and  $T^{n+1}\beta_p$  do not appear as consecutive terms in  $TE_1$ . So  $T^{n+1}$  is discontinuous at  $\beta_p$ .

The discontinuity of  $T^{n+1}$  at  $T^{-n}l$  follows from the fact that  $T^{n+1}\beta_{p-1}$  is an interior point of  $TI_2$ , while  $T^{n+1}(T^{-n}l) = 1 - m$  is the left end-point of  $TI_1$ . Finally,  $T^{n+1}(T^{-n}m) = 0 < T\beta_r^{\tau_n} < 1 - m = T^{n+1}(T^{-n}l)$  implies that  $T^{n+1}$  is discontinuous at  $T^{-n}m$ . Therefore,  $D_1 = D(T^{n+1})$ .

*Case 2:* If  $r < s$  and  $p = \tau_n^{-1}r < \tau_n^{-1}s = q$ , then  $T^{-n}l \in (\beta_{p-1}, \beta_p)$  and  $T^{-n}m \in (\beta_{q-1}, \beta_q)$ . The discontinuity of  $T^n$  at  $\beta_p$  and  $\beta_q$  implies  $T^n\beta_p \neq \beta_r^{\tau_n}$  and  $T^n\beta_q \neq \beta_s^{\tau_n}$ . The underlying partition of  $T^{n+1}$  is easily seen to be

$$\mathcal{P}_2(T^{n+1}) = \{[\beta_0, \beta_1), \dots, [\beta_{p-2}, \beta_{p-1}), [\beta_{p-1}, T^{-n}l), [T^{-n}l, \beta_p),$$

$$[\beta_p, \beta_{p+1}), \dots, [\beta_{q-1}, T^{-n}m), [T^{-n}m, \beta_q), [\beta_q, \beta_{q+1}), \dots, [\beta_{2n}, 1)].$$

To show the discontinuity of  $T^{n+1}$  at  $\beta_1, \dots, \beta_{2n}, T^{-n}l, T^{-n}m$ , we consider the ordered sets

$$D_2 = \{\beta_0, \dots, \beta_{p-1}, T^{-n}l, \beta_p, \dots, \beta_{q-1}, T^{-n}m, \beta_q, \dots, \beta_{2n}\}$$

and

$$E_2 = \{\beta_0^{\tau_n}, \dots, \beta_{r-1}^{\tau_n}, l, \beta_r^{\tau_n}, \dots, \beta_{s-1}^{\tau_n}, m, \beta_s^{\tau_n}, \dots, \beta_{2n}^{\tau_n}\}.$$

Then,  $T^n D_2 = E_2$ . Notice that  $\beta_1^{\tau_n}, \dots, \beta_{r-1}^{\tau_n}$  are interior points of  $I_0$ ,  $\beta_r^{\tau_n}, \dots, \beta_{s-1}^{\tau_n}$  are interior points of  $I_1$  and  $\beta_s^{\tau_n}, \dots, \beta_{2n}^{\tau_n}$  are interior points of  $I_2$ . Thus,

$$\begin{aligned} T^{n+1}D_2 &= TE_2 \\ &= \{Tm = 0, T\beta_s^{\tau_n}, \dots, T\beta_{2n}^{\tau_n}, Tl = 1 - m, \\ &\quad T\beta_r^{\tau_n}, \dots, T\beta_{s-1}^{\tau_n}, T\beta_0^{\tau_n} = 1 - l, \dots, T\beta_{r-1}^{\tau_n}\}. \end{aligned}$$

Here, the elements of  $TE_2$  are listed in increasing order. We first prove that  $T^{n+1}$  is discontinuous at  $\beta_i$  for  $i \neq p, q$ . If  $T^n \beta_{i-1}, T^n \beta_i \in I_j$ , then since  $T^n \beta_{i-1}, T^n \beta_i$  do not appear as consecutive terms in  $E_1$  and since  $T$  is an isometry on  $I_j$ , we have that  $T^{n+1} \beta_{i-1}$  and  $T^{n+1} \beta_i$  are not consecutive terms of  $E_2$ , and thus  $T^{n+1}$  is discontinuous at  $\beta_i$ . If  $T^n \beta_i \in I_j$  and  $T^n \beta_{i-1} \in I_k$  for  $k \neq j$ , then we consider several cases.

- If  $T^n \beta_i \in I_2$  and  $T^n \beta_{i-1} \in I_0$  or  $I_1$ , then since  $T^n \beta_{i-1} \neq l$  we have  $T^{n+1} \beta_i < 1 - m < T^{n+1} \beta_{i-1}$ .
- If  $T^n \beta_i \in I_1$  and  $T^n \beta_{i-1} \in I_2$ , then since  $T^n \beta_i \neq l$  it follows that  $T^{n+1} \beta_{i-1} < 1 - m < T^{n+1} \beta_i$ .
- If  $T^n \beta_i \in I_1$  and  $T^n \beta_{i-1} \in I_0$ , then  $T^{n+1} \beta_i < T^{n+1} \beta_{i-1}$ .
- If  $T^n \beta_i \in I_0$  and  $T^n \beta_{i-1} \in I_1$ , then since  $i \neq q$  we have  $T^{n+1} \beta_{i-1} < T\beta_{s-1}^{\tau_n} < T\beta_0^{\tau_n} \leq T^{n+1} \beta_i$ .
- If  $T^n \beta_i \in I_0$  and  $T^n \beta_{i-1} \in I_2$ , then  $T^{n+1} \beta_{i-1} < 1 - m < T^{n+1} \beta_i$ .

In all the above cases we see that  $T^{n+1}$  is not continuous at  $\beta_i$ .

The discontinuity of  $T^{n+1}$  at  $\beta_p$  and  $\beta_q$  follows from the fact that  $T^{n+1} \beta_p \neq T\beta_r^{\tau_n}$  and  $T^{n+1} \beta_q \neq T\beta_s^{\tau_n}$ , so that neither  $T^{n+1} \beta_p$  and  $T^{n+1}(T^{-n}l)$  nor  $T^{n+1} \beta_q$  and  $T^{n+1}(T^{-n}m)$  appear as consecutive terms in  $TE_2$ . Finally, from  $T^{n+1}(T^{-n}l) = 1 - m < 1 - l < T\beta_{r-1}^{\tau_n}$  and  $T^{n+1}(T^{-n}m) = 0 < 1 - m < T\beta_{s-1}^{\tau_n}$  we have that  $T^{n+1}$  is discontinuous at  $T^{n+1}(T^{-n}l)$  and  $T^{n+1}(T^{-n}m)$ . Thus,  $D_2 = D(T^{n+1})$ .

*Case 3:* If  $r < s$  and  $p = \tau_n^{-1}r > \tau_n^{-1}s = q$ , then the underlying partition of  $T^{n+1}$  is given by

$$\begin{aligned} \mathcal{P}_3(T^{n+1}) &= \{[\beta_0, \beta_1), \dots, [\beta_{q-2}, \beta_{q-1}), [\beta_{q-1}, T^{-n}m), [T^{-n}m, \beta_q), [\beta_q, \beta_{q+1}), \dots, \\ &\quad [\beta_{p-2}, \beta_{p-1}), [\beta_{p-1}, T^{-n}l), [T^{-n}l, \beta_p), [\beta_p, \beta_{p+1}), \dots, [\beta_{2n}, 1)\}. \end{aligned}$$

Let

$$D_3 = \{\beta_0, \dots, \beta_{q-1}, T^{-n}m, \beta_q, \dots, \beta_{p-1}, T^{-n}l, \beta_p, \dots, \beta_{2n}\}$$

and

$$E_3 = \{\beta_0^{\tau_n}, \dots, \beta_{r-1}^{\tau_n}, l, \beta_r^{\tau_n}, \dots, \beta_{s-1}^{\tau_n}, m, \beta_s^{\tau_n}, \dots, \beta_{2n}^{\tau_n}\}.$$

Then,

$$\begin{aligned} T^{n+1}D_3 &= TE_3 \\ &= \{Tm = 0, T\beta_s^{\tau_n}, \dots, T\beta_{2n}^{\tau_n}, Tl = 1 - m, \\ &\quad T\beta_r^{\tau_n}, \dots, T\beta_{s-1}^{\tau_n}, T\beta_0^{\tau_n} = 1 - l, \dots, T\beta_{r-1}^{\tau_n}\}. \end{aligned}$$

The elements of  $D_3$ ,  $E_3$  and  $TE_3$  are listed in increasing order. A similar argument as in the above two cases shows that  $D_3 = D(T^{n+1})$ . Thus, the theorem is proved.

**Theorem 3.** *Any non-trivial interval exchange transformation on 3-intervals with rationally independent discontinuity points is not measure theoretically isomorphic to an irrational rotation.*

**Proof:** By theorem 2 and unique ergodicity, the result follows from Li's theorem.

In [BCF], the authors proved that every ergodic interval exchange transformation on three intervals has simple spectrum. Using this result and theorem 3, we have the following corollary.

**Corollary 1.** *Every non-trivial interval exchange transformation on three intervals with rationally independent discontinuity points has either rational or continuous spectrum.*

#### REFERENCES

- [BCF] Berthé, V., N. Chekhova and S. Ferenczi - *Covering numbers: arithmetics and dynamics for rotations and interval exchanges*, J. D'Analyse Math. **79** (1999), 1-31.
- [B] Boshernitzan, M. - *A condition for minimal interval exchange maps to be uniquely ergodic*, Duke Math. J. **52** (1985), 723-752.
- [G] Goodson, G.R. - *Functional equations associated with the spectral properties of compact group extensions*, Proceedings of Conference on Ergodic Theory and its connection with Harmonic Analysis, Alexandria 1993, Cambridge University Press, 1994, 309-327.
- [K1] Keane, M.S. - *Interval exchange transformations*, Math. Z. **141** (1975), 25-31.
- [K2] Keane, M.S. - *Non-ergodic interval exchange transformations*, Israel J. Math. **26** (1977), 188-196.
- [KN] Keynes, H. and D. Newton - *A minimal non-uniquely ergodic interval exchange transformation*, Math. Z. **148** (1976), 101-105.
- [KS] Katok, A.B. and A.M. Stepin - *Approximations in ergodic theory*, Uspekhi Math. Nauk **22**, 5 (1967), 81-106 (Russian), translated in Russian Mth. Surveys **22**, 5 (1967), 76-102.
- [Li] Li, Simin - *A Criterion for an Interval Exchange Map to be Conjugate to an Irrational Rotation*, J. Math. Sci. Univ. Tokyo **6** (1999), 679-690.
- [M] Masur, H. - *Interval exchange transformations and measured foliations*, Ann. of Math. **115** (1982), 169-200.
- [O] Oseledets, V.I. - *On the spectrum of ergodic automorphisms*, Doklady Akad. Nauk SSSR **168**, 5 (1966), 1009-1011 (in Russian), translated in Soviet Math. Doklady **7** (1966), 776-779.
- [R] Rauzy, G. - *Echanges d'intervalles et transformations induites*, Acta Arith. **34** (1979), 315-328.
- [V1] Veech, W.A. - *Interval exchange transformations*, J. D'Analyse Math. **33** (1978), 222-272.
- [V2] Veech, W.A. - *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. **115** (1982), 201-242.
- [V3] Veech, W.A. - *The metric theory of interval exchange transformations. I Generic spectral properties*, Amer. J. Math. **106** (1984), 1331-1359.
- [V4] Veech, W.A. - *The metric theory of interval exchange transformations. II Approximation by primitive exchange*, Amer. J. Math. **106** (1984), 1361-1387.
- [V5] Veech, W.A. - *The metric theory of interval exchange transformations. III The Sah-Arnoux-Fathi invariant*, Amer. J. Math. **106** (1984), 1389-1422.

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