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LOCALLY SUPPORTED, PIECEWISE POLYNOMIAL BIORTHOGONAL WAVELETS ON NON-UNIFORM MESHES

ROB STEVENSON

ABSTRACT. In this paper, biorthogonal wavelets are constructed on non-uniform meshes. Both primal and dual wavelets are explicitly given locally supported, continuous piecewise polynomials. The wavelets generate Riesz bases for the Sobolev spaces H^s for $|s| < \frac{3}{2}$. The wavelets at the primal side span standard Lagrange finite element spaces.

1. INTRODUCTION

This paper is concerned with the construction of locally supported biorthogonal wavelets on non-uniform meshes. We consider meshes that are generated by uniform refinements starting from an arbitrary initial triangulation of some domain Ω. In the wavelet literature this is also referred to as a semi-regular setting ([DGSS99]).

The wavelets at the primal side will span standard Lagrange (C^0) finite element spaces, with or without essential boundary conditions, of in principal any order. For any $|s| < \frac{3}{2}$, after a proper scaling the infinite union of the wavelets is a Riesz basis for the Sobolev space $H^s(\Omega)$ or $H_0^s(\Omega)$. The wavelet construction directly extends to Lipschitz' manifolds consisting of patches, where each patch can be described by a parametrization with a constant Jacobian determinant.

The wavelets satisfy all conditions to use them as ingredients in various wavelet-based algorithms for solving operator equations. For an overview of such algorithms, see [Dah97] and [Coh00]. Key aspects include optimal preconditioning, matrix compression, and adaptive schemes.

An alternative approach to construct wavelet bases on domains or manifolds that cannot be fitted with a uniform grid structure, is to write them as a disjoint union of parametric images of a unit cube, map wavelets living on the cube to the subdomains using the parametrizations, and finally stitch them together. Such constructions yielding wavelet bases suitable for solving operator equations can be found in [DS99a, CTU99, DS99b].

This work can be viewed as a continuation of [DS99c]. A novel aspect is that in the present paper also the dual wavelets are locally supported. As a consequence, the field of applications is extended to all 'classical' wavelet applications as signal analysis and image compression.

Another remarkable aspect is that the dual wavelets will be explicitly given, continuous piecewise polynomials. This allows the application of simple standard quadrature formulae for computing wavelet coefficients. Wavelet constructions, also of higher regularity, where

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the dual functions are piecewise polynomials were discussed earlier in [DGH99, DGH00, Goo00]. These constructions concern shift-invariant setting in one- or, in [DGH00], two dimensions. In [DGH99, DGH00] extensions are discussed to uniform meshes on bounded domains Ω. Yet, there the property of polynomial reproduction is lost, which means that the wavelets can only be shown to generate a Riesz basis for $L_2(\Omega)$, and wavelets near the boundary do not have cancellation properties.

Our construction distinguishes from other wavelet constructions on non-uniform meshes ('second generation wavelets') in the sense that, as in the shift-invariant case ('first generation wavelets'), the wavelets are proven to generate Riesz bases for a scale of Sobolev spaces. In this respect, note that any compression algorithm based on deleting small wavelet coefficients can only be meaningful when there is some notion of stability.

This paper is organized as follows: In §2, we recall theory concerning stability of biorthogonal space decompositions, which originates from [Dah96]. To construct bases of the subspaces that make up these space decompositions, that is, the wavelets, we follow the construction known as that of the 'stable completions' ([CDP96]), which is related to the 'lifting scheme' ([Swe97]). We give a new and short proof of stability of these bases, which is not based on matrix arguments, and therefore is fully separated from issues related to implementation.

In §3.1, we reduce the construction of biorthogonal bases on non-uniform meshes to a construction on a reference element. We give general criteria for locally biorthogonal bases so that they give rise continuous globally biorthogonal scaling functions and wavelets, all with supports that are restricted to a uniform bounded number of mesh-cells. Necessarily, these global functions depend on the (local) topology of the mesh. Yet, this dependence will be given explicitly.

In §3.2-3.5, we give four concrete realizations of biorthogonal bases on non-uniform meshes. With n denoting the space dimension and $d-1$, $\tilde{d}-1$ being the degrees of polynomial exactness at primal and dual side, these examples are characterized by $(n, d, \tilde{d}) = (1,2,4)$, $(1,5,4)$, $(2,2,4)$ and $(2,5,4)$. Although in two dimensions, the constructions are rather complex, we show how the wavelet and inverse wavelets transform can be implemented at relatively low costs.

2. General mechanism to construct stable wavelet bases

Let H be a separable Hilbert space with scalar product $\langle\ ,\ \rangle$ and norm $\|\ \|$. Let Φ be some countable collection of functions in H.

We start by recalling some convenient compact notations that for example can be found in [Dah97]. Let us formally view Φ as a column vector. Then for a column vector $\mathbf{c} = (c_\phi)_{\phi\in\Phi}$ of scalars, $\mathbf{c}^T\Phi := \sum_{\phi\in\Phi} c_\phi \phi$ is a natural notation. We always consider the spaces of scalar vectors as being equipped with the ℓ_2-norm, and consequently, the spaces of possibly infinite matrices as being equipped with the corresponding operator norm. For $x \in H$, with $\langle\Phi, x\rangle$ and $\langle x, \Phi\rangle$ we will mean the column- and row-vectors with coefficients $\langle\phi, x\rangle$ and $\langle x, \phi\rangle$, $\phi \in \Phi$. More generally, when $\tilde{\Phi}$ is another countable collection in H, with $\langle\Phi, \tilde{\Phi}\rangle$ is meant the matrix $(\langle\phi, \tilde{\phi}\rangle)_{\phi\in\Phi, \tilde{\phi}\in\tilde{\Phi}}$.

With these notations, a collection Φ is called a *Riesz system* when

$$\|\mathbf{c}^T\Phi\| \eqsim \|\mathbf{c}\|, \tag{2.1}$$

and Φ is called a *Riesz basis* when it is in addition a basis for H. Two collections Φ and $\tilde{\Phi}$ are called *biorthogonal*, or $\tilde{\Phi}$ is *dual* to Φ or vice versa, when

$$\langle \Phi, \tilde{\Phi} \rangle = \mathbf{I}. \tag{2.2}$$

Part (a) of the following lemma will be used in the forthcoming Theorem 2.3 concerning stability of biorthogonal space decompositions, whereas part (b) will be applied to construct Riesz bases for the subspaces that make up these space decompositions.

Lemma 2.1. *Let V and $\tilde{V}$ be closed subspaces of H.*
(a). *The following statements are equivalent:*
(i). *There exist Riesz bases Φ and $\tilde{\Phi}$ for V and $\tilde{V}$ such that $\langle \Phi, \tilde{\Phi} \rangle$ is bounded invertible.*
(ii).

$$\inf_{0\neq\tilde{v}\in\tilde{V}} \sup_{0\neq v\in V} \frac{|\langle \tilde{v}, v\rangle|}{\|\tilde{v}\|\|v\|} > 0, \tag{2.3}$$

and for any $v \in V$, there holds $\sup_{0\neq\tilde{v}\in\tilde{V}} \frac{|\langle \tilde{v}, v\rangle|}{\|\tilde{v}\|\|v\|} > 0$.
(iii). *There exists a (unique) bounded projector $Q : H \to H$ with $\operatorname{Im} Q = V$ and $\operatorname{Im}(I-Q) = \tilde{V}^\perp$.*
(iv). *To any Riesz basis for $\tilde{V}$ there corresponds a unique dual collection in V. Moreover, this collection is a Riesz basis for V.*

(b). *Let any of the equivalent conditions (i)-(iv) from (a) be satisfied. Let X, W be subspaces of H be such that $X = W + V$ and*

$$\cos\angle(W, V) := \sup_{0\neq w\in W, 0\neq v\in V} \frac{|\langle w, v\rangle|}{\|w\|\|v\|} < 1. \tag{2.4}$$

Then $(I-Q)|_W : W \to X \cap \tilde{V}^\perp$ is bounded invertible, see Figure 1.

Proof. (a). (i) $\to$ (ii): This follows easily by expressing v and $\tilde{v}$ in terms of the Riesz bases from (i).

(ii) $\to$ (iii): For this part we refer to [DS99c, Theorem 2.1(a)].

(iii) $\to$ (iv): Let $\tilde{\Phi}$ be a Riesz basis for $\tilde{V}$. Let $\tilde{V}'$ be the dual space of $\tilde{V}$ equipped with the operator norm. In [Dah91] it was proved that there exists a Riesz basis $\tilde{\Phi}'$ for $\tilde{V}'$ which is dual to $\tilde{\Phi}$, here in the sense that $\tilde{\Phi}'(\tilde{\Phi}) := (\tilde{\phi}'(\tilde{\phi}))_{\tilde{\phi}'\in\tilde{\Phi}', \tilde{\phi}\in\tilde{\Phi}} = \mathbf{I}$.

Let $\tilde{R} : \tilde{V}' \to \tilde{V}$ be the Riesz map, i.e., $\langle \tilde{v}, \tilde{R}\tilde{f}\rangle = \tilde{f}(\tilde{v})$ for all $\tilde{f} \in \tilde{V}'$, $\tilde{v} \in \tilde{V}$, and let Q be the projector onto V from (iii). From

$$\langle \tilde{\Phi}, Q\tilde{R}\tilde{\Phi}'\rangle = \langle \tilde{\Phi}, \tilde{R}\tilde{\Phi}'\rangle = \tilde{\Phi}'(\tilde{\Phi}),$$

we see that $\tilde{\Phi}$ and $Q\tilde{R}\tilde{\Phi}'$ are biorthogonal systems. Since $\tilde{R}$ is an isomorphism, we may conclude that $Q\tilde{R}\tilde{\Phi}'$ is a Riesz basis for V when $Q|_{\tilde{V}} : \tilde{V} \to V$ is a homeomorphism.

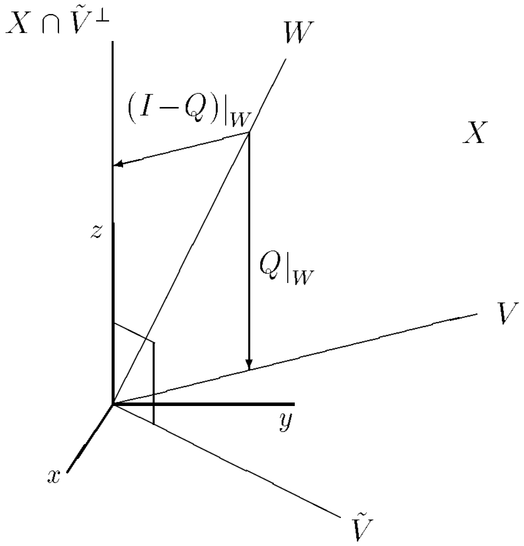

FIGURE 1. Illustration for Lemma 2.1(b). H and X are represented by $\mathbb{R}^3$ and the plane $x = 0$ respectively. $\tilde{V}$ is contained in the plane $z = 0$.

For $\tilde{v} \in \tilde{V}$, there holds $\|Q\tilde{v}\| \geq \frac{|\langle Q\tilde{v},\tilde{v}\rangle|}{\|\tilde{v}\|} = \|\tilde{v}\|$. Since $\tilde{V}$ is closed, this property of Q and its boundedness show that $\operatorname{Im}(Q|_{\tilde{V}})$ is closed. Now suppose that $\operatorname{Im}(Q|_{\tilde{V}}) \neq V$, then there would be a $0 \neq v \in V$, such that

$$0 = \langle Q\tilde{v}, v\rangle = \langle \tilde{v}, Q^* v\rangle \qquad (\tilde{v} \in \tilde{V}). \tag{2.5}$$

One easily verifies that $\operatorname{Im} Q^* = \tilde{V}$ and $\operatorname{Im}(I - Q^*) = V^{\perp}$. The first property together with (2.5) shows that $Q^*v = 0$, whereas the second property gives $\|Q^*v\| \geq \frac{|\langle Q^*v,v\rangle|}{\|v\|} = \|v\|$, which contradicts $v \neq 0$. We conclude that indeed $Q|_{\tilde{V}} : \tilde{V} \to V$ is a homeomorphism.

There remains to show that there is only one collection in V that is dual to $\tilde{\Phi}$. Suppose this is wrong. Then there would be a $0 \neq v \in V$ such that $\langle v, \tilde{\Phi}\rangle = 0$, and thus $\langle v, \tilde{v}\rangle = 0$ for all $\tilde{v} \in \tilde{V}$. Since $Q|_{\tilde{V}} : \tilde{V} \to V$ is a homeomorphism, there exists a $0 \neq \tilde{y} \in \tilde{V}$ with $Q\tilde{y} = v$. From $\operatorname{Im}(I - Q) = \tilde{V}^{\perp}$, we get $\langle \tilde{y}, \tilde{v}\rangle = 0$ for all $\tilde{v} \in \tilde{V}$, contradicting $\tilde{y} \neq 0$.

(iv) $\to$ (i): Any separable Hilbert space has an orthonormal basis. Starting with such a basis for $\tilde{V}$ and applying (iv) shows (i), where $\langle \Phi, \tilde{\Phi}\rangle$ is even the identity matrix.

(b). Write $x \in X$ as $x = w + v$ where $w \in W$, $v \in V$. Formula (2.4) shows that this decomposition is unique, and that $\|x\|^2 \eqsim \|w\|^2 + \|v\|^2$. Taking $x \in X \cap \tilde{V}^{\perp}$, we have $Qx = 0$, and so $v = Qv = -Qw$, i.e., $x = (I - Q)w$ and $\|x\|^2 \eqsim \|w\|^2 + \|Qw\|^2 \eqsim \|w\|^2$. $\square$

Remarks 2.2. (a). Since (i) is symmetric in V and $\tilde{V}$, so are (ii)-(iv), i.e., the roles of V and $\tilde{V}$ may everywhere be interchanged. As was already mentioned in the proof, the projector from (iii) obtained in that way is nothing else than Q^*. Pairs of spaces V, $\tilde{V}$ that satisfy any, and thus all of (i)-(iv) will be said to satisfy the *maximum angle condition.*

(b). Estimate (2.4) is known as the *strengthened Cauchy-Schwarz inequality*. Pairs of spaces W, V that satisfy (2.4) will be said to satisfy the *minimum angle condition.*

(c). If Φ, $\tilde{\Phi}$ are Riesz bases for V and $\tilde{V}$ such that $\langle \Phi, \tilde{\Phi} \rangle$ is bounded invertible, then the projector Q from (iii) can be computed by

$$Qx = \langle x, \tilde{\Phi} \rangle \langle \Phi, \tilde{\Phi} \rangle^{-1} \Phi,$$

and similarly $Q^* y = \langle y, \Phi \rangle \langle \tilde{\Phi}, \Phi \rangle^{-1} \tilde{\Phi}$

(d). Below we will apply Lemma 2.1 to an infinite *sequence* of pairs of closed subspaces V, $\tilde{V}$ of some Hilbert space H, together with corresponding sequences of spaces X and W. We will be interested in results that hold *uniformly* over these sequences. The proof of the lemma shows that if we replace in (i), (iii) and (b) 'bounded' by '*uniformly* bounded', and the conditions for being a Riesz system or satisfying (2.3) or (2.4) by corresponding conditions that hold *uniformly* over the sequences, then the resulting lemma remains valid. In this respect, we will speak about *uniform Riesz systems*, *uniform Riesz bases* and *uniform maximum* or *minimum angle conditions.*

In the following, let $\mathcal{H}^s$ for $s \in I\!R$ or $|s| \leq t$, denote a scale of Sobolev spaces, possibly incorporating essential boundary conditions, on an n-dimensional domain or sufficiently smooth manifold. We will denote $\mathcal{H}^0$ also as L_2, and when $s < 0$ the space $\mathcal{H}^s$ is understood to be the *dual* of $\mathcal{H}^{-s}$. From now on, the role of the general Hilbert space H will be played by L_2, and so $(\ )^*$ will mean an adjoint with respect to the L_2-scalar product, and $\perp$ denotes orthogonality with respect to this scalar product.

Theorem 2.3 ('Biorthogonal space decompositions'). *Let $V_0 \subset V_1 \subset V_2 \subset \cdots$ and $\tilde{V}_0 \subset \tilde{V}_1 \subset \tilde{V}_2 \subset \cdots$ be sequences of nested closed subspaces of L_2, and let $\rho > 1$ be some constant, that in applications will be the* refinement factor.

Assume that $(V_j, \tilde{V}_j)_j$ satisfies the uniform maximum L_2-angle condition. Let (Q_j) be the sequence of uniformly bounded projectors $Q_j : L_2 \to L_2$ with $\mathrm{Im}\, Q_j = V_j$ and $\mathrm{Im}\,(I - Q_j) = \tilde{V}_j^{\perp}$ from Lemma 2.1(a) (iii).

Assume that there exist $0 < \gamma < d$ such that

$$(\mathfrak{J}) \qquad \inf_{v_j \in V_j} \|v - v_j\|_{L_2} \lesssim \rho^{-sj} \|v\|_{\mathcal{H}^s} \quad (v \in \mathcal{H}^s,\ 0 \leq s \leq d)$$

*(*direct *or* Jackson estimate*), and*

$$(\mathfrak{B}) \qquad \|v_j\|_{\mathcal{H}^s} \lesssim \rho^{sj} \|v_j\|_{L_2} \quad (v_j \in V_j,\ 0 \leq s < \gamma) \quad \text{(inverse } \textit{or} \text{ Bernstein estimate)},$$

and that analogous assumptions $(\tilde{\mathfrak{J}})$ and $(\tilde{\mathfrak{B}})$ with constants $0 < \tilde{\gamma} < \tilde{d}$ hold for $(\tilde{V}_j)$.

Then, with $Q_{-1} := 0$, one has

$$(2.6) \qquad \| \sum_{j=0}^{\infty} w_j \|_{\mathcal{H}^s}^2 \eqsim \sum_{j=0}^{\infty} \rho^{2sj} \|w_j\|_{L_2}^2 \qquad (w_j \in \mathrm{Im}\,(Q_j - Q_{j-1}),\ s \in (-\tilde{d}, \gamma))$$

and

$$\sum_{j=0}^{\infty} \rho^{2sj} \|(Q_j - Q_{j-1})v\|_{L_2}^2 \lesssim \|v\|_{\mathcal{H}^s}^2 \qquad (v \in \mathcal{H}^s,\, s \in (-\tilde{\gamma}, d)). \tag{2.7}$$

For $s \in (-\tilde{\gamma}, \gamma)$, the mappings $(w_j) \mapsto \sum_{j=0}^{\infty} w_j$ and $v \mapsto ((Q_j - Q_{j-1})v)$, which are bounded in the sense of (2.6) and (2.7), are each others inverse.

Analogous results are valid with (Q_j) replaced by (Q_j^*) and with interchanged roles of (γ, d) and $(\tilde{\gamma}, \tilde{d})$.

Remark 2.4. An earlier theorem demonstrating stability of biorthogonal space decompositions in an even more general context can be found in [Dah96]. See also [Dah97, Coh00] and the references cited there, for example for generalizations to Besov norms. A proof of the theorem in its present form can be found in [DS99c, Theorem 2.1].

The essential point of the present formulation is that explicit knowledge of some biorthogonal bases for V_j and $\tilde{V}_j$ is *not* assumed. In [DS99c] the conditions of Theorem 2.3 were verified for both (V_j) and $(\tilde{V}_j)$ being sequences of standard finite element spaces.

In the remainder of this section, we will assume that we are in the situation as indicated in Theorem 2.3. The nesting $\tilde{V}_j \subset \tilde{V}_{j+1}$ gives $Q_j^* = Q_{j+1}^* Q_j^*$ or $Q_j = Q_j Q_{j+1}$, from which we deduce that

$$\operatorname{Im}(Q_{j+1} - Q_j) = V_{j+1} \cap \tilde{V}_j^{\perp}.$$

A direct consequence of Theorem 2.3 is that *if* we have uniform L_2-Riesz bases Ψ_j for the spaces $V_{j+1} \cap \tilde{V}_j^{\perp}$, and an L_2-Riesz basis Φ_0 for V_0, then for $s \in (-\tilde{\gamma}, \gamma)$,

$$\Phi_0 \cup \cup_{j=0}^{\infty} \rho^{-sj} \Psi_j$$

is a Riesz basis for $\mathcal{H}^s$. The elements of the Ψ_j are called *wavelets.*

Remark 2.5. Since in particular $\Psi := \Phi_0 \cup \cup_j \Psi_j$ is a Riesz basis for L_2, an application of Lemma 2.1(a) with 'V'='$\tilde{V}$'='H'$= L_2$ shows that there exists a unique dual collection $\tilde{\Psi} := \tilde{\Phi}_0 \cup \cup_j \tilde{\Psi}_j$ in L_2, which moreover is a Riesz basis for L_2. Exploiting biorthogonality shows that the $\tilde{\Psi}_j$ are uniform L_2-Riesz bases for the spaces $\tilde{V}_j \cap V_{j-1}^{\perp}$, and that $\tilde{\Phi}_0$ is an L_2-Riesz basis for $\tilde{V}_0$. From Theorem 2.3, we conclude that for $s \in (-\gamma, \tilde{\gamma})$, $\tilde{\Phi}_0 \cup \cup_j \rho^{-sj} \tilde{\Psi}_j$ is a Riesz basis for $\mathcal{H}^s$. The elements of the $\tilde{\Psi}_j$ are called *dual wavelets.*

For $s \in (-\tilde{\gamma}, \gamma)$ and $v \in \mathcal{H}^s$, the unique expansion of v in terms of Ψ is given by

$$v = \langle v, \tilde{\Psi} \rangle \Psi. \tag{2.8}$$

Remark 2.6. The fact that the dual sequence $(\tilde{V}_j)$ satisfies a Jackson estimate is closely related to the fact that integration of a resulting biorthogonal wavelet against a smooth function produces something small. Indeed, for simplicity restricting ourselves to the domain case (for the manifold case, see e.g. [DS99c, Prop. 4.7]), the Jackson estimate $(\tilde{\mathfrak{J}})$ is usually enforced by demanding that $\tilde{V}_j$ contains all piecewise polynomials up to degree $\tilde{d} - 1$ satisfying some global smoothness conditions with respect to a quasi-uniform mesh with mesh-size $\sim \rho^{-j}$. Now the fact that $\psi_j \in \Psi_j$ satisfies $\psi_j \perp_{L_2} \tilde{V}_j$ shows that for smooth v, there holds $\langle v, \psi_j \rangle_{L_2} = \langle v - p, \psi_j \rangle_{L_2}$, where p is a Taylor polynomial of v of order $\tilde{d} - 1$ around some point in suppψ_j. Assuming that the wavelets are *uniformly local*, meaning that diam(suppψ_j) $\eqsim \rho^{-j}$, by estimating the remainder term we find that

$$|\langle v, \psi_j \rangle_{L_2}| \lesssim \rho^{-(\tilde{d}+n/2)j} \|v\|_{W^{\infty,\tilde{d}}(\mathrm{supp}\psi_j)},$$

which property of the wavelets is referred to as the *cancellation property of order* $\tilde{d}$.

Obviously, assuming that the dual wavelets are also uniformly local, they will satisfy the cancellation property of order d.

The cancellation property of the wavelets (or dual wavelets) is essential for finding sparse approximate wavelet representations of operators (or functions).

Usually, it is not a problem to equip V_0 with some L_2-Riesz basis Φ_0. Below we discuss the *construction of the wavelets*. Suppose that we can identify some spaces $W_j, \hat{V}_j \subset V_{j+1}$, where uniform L_2-Riesz bases $\check{\Psi}_j$ are available for the spaces W_j, such that

(2.9) $\quad V_{j+1} = W_j + \hat{V}_j,$

(2.10) $\quad (\hat{V}_j, \tilde{V}_j)_j$ satisfies the uniform maximum L_2-angle condition,

(2.11) $\quad (W_j, \hat{V}_j)_j$ satisfies the uniform minimum L_2-angle condition.

Then Lemma 2.1 shows that there exist unique uniformly L_2-bounded projectors $\hat{Q}_j$ with $\mathrm{Im}\,\hat{Q}_j = \hat{V}_j$ and $\mathrm{Im}\,(I - \hat{Q}_j) = \tilde{V}_j^{\perp}$, where moreover $(I - \hat{Q}_j)|_{W_j} : W_j \to V_{j+1} \cap \tilde{V}_j^{\perp}$ is invertible, with a uniformly L_2-bounded inverse. We conclude that these $(I - \hat{Q}_j)|_{W_j}$ map uniform L_2-Riesz bases to uniform L_2-Riesz bases, and thus that

$$\Psi_j := (I - \hat{Q}_j)\check{\Psi}_j \tag{2.12}$$

are uniform L_2-Riesz bases for the spaces $V_{j+1} \cap \tilde{V}_j^{\perp}$.

For computing these collections of wavelets Ψ_j, Remarks 2.2(c) shows that if $\hat{\Phi}_j$, $\tilde{\Phi}_j$ are biorthogonal L_2-Riesz bases of $\hat{V}_j$, $\tilde{V}_j$, then

$$\Psi_j = \check{\Psi}_j - \langle \check{\Psi}_j, \tilde{\Phi}_j \rangle_{L_2} \hat{\Phi}_j. \tag{2.13}$$

Remarks 2.7. (a). Note that Ψ_j depends on $\check{\Psi}_j$ and $\hat{V}_j$ and $\tilde{V}_j$, but not on the choice of $\hat{\Phi}_j$ and $\tilde{\Phi}_j$.

(b). For most applications, one is interested in having wavelets that are uniformly local. In (2.13), each $\check{\psi} \in \check{\Psi}_j$ is corrected by a number of terms of the form $\langle \check{\psi}, \tilde{\phi} \rangle_{L_2} \hat{\phi}$, where $\tilde{\phi} \in \tilde{\Phi}_j$, $\hat{\phi} \in \hat{\Phi}_j$ with $\langle \tilde{\phi}, \hat{\phi} \rangle_{L_2} = 1$. Since $\langle \check{\psi}, \tilde{\phi} \rangle_{L_2} \neq 0$ only if $\mathrm{supp}\,\check{\psi} \cap \mathrm{supp}\,\tilde{\phi} \neq \emptyset$, and furthermore $\mathrm{supp}\,\tilde{\phi} \cap \mathrm{supp}\,\hat{\phi} \neq \emptyset$, we conclude that the Ψ_j are uniformly local when the $\check{\Psi}_j$, $\tilde{\Phi}_j$ and $\hat{\Phi}_j$ are uniformly local.

(c). An important special case of the wavelet construction (2.12)/(2.13) is given by $\hat{V}_j = V_j$, since, as we will discuss later on, it may lead to dual wavelets which are also uniformly local. Note that in this case, (2.10) was already assumed in Theorem 2.3.

For $\hat{V}_j = V_j$, the wavelet construction (2.12)/(2.13) is known as the construction via 'stable completions' ([CDP96]), which is related to the so-called 'lifting scheme' ([Swe97]). Our derivation of the fact that the Ψ_j are uniform L_2-Riesz systems is new in the sense that is not based on matrix arguments, which means that it is fully separated from issues related to the implementation.

With $\hat{V}_j = V_j$, $\hat{\Phi}_j$ is a basis for V_j, and so the conditions for getting uniformly local wavelets we derived in above Part (b), now read as assuming that we have uniformly local, biorthogonal L_2-Riesz bases for the spaces V_j and $\tilde{V}_j$ at our disposal. In practice, this condition is much more restrictive than (2.10), which lead us in [DS99c] to consider the generalization $\hat{V}_j \neq V_j$, which suffices for all applications for which uniformly local dual wavelets are not needed. Examples of such applications are wavelet-based algorithms for solving operator equations (see [Dah97]). On the other hand, for 'classical' wavelet applications like signal analysis and image compression, having uniformly local dual wavelets is essential.

In many applications, one needs to switch from a representation of a function $v \in V_J$ with respect to the 'multi-scale basis' $\Phi_0 \cup \cup_{j=0}^{J-1} \Psi_j$, to a representation with respect to some 'single-scale' basis Φ_J.

Since $V_{j+1} = V_j \oplus (V_{j+1} \cap \tilde{V}_j^{\perp})$, there exist matrices $\mathbf{M}_{j,0}$ and $\mathbf{M}_{j,1}$ such that $\Phi_j^T = \Phi_{j+1}^T \mathbf{M}_{j,0}$ and $\Psi_j^T = \Phi_{j+1}^T \mathbf{M}_{j,1}$, and

$$\mathbf{M}_j = \begin{bmatrix} \mathbf{M}_{j,0} & \mathbf{M}_{j,1} \end{bmatrix}$$

is invertible. Writing $v \in V_J$ in both forms $\mathbf{c}_0^T \Phi_0 + \sum_{j=0}^{J-1} \mathbf{d}_\ell^T \Psi_j^T$ and $\mathbf{c}_J^T \Phi_J$, the basis transformation $\mathbf{T}_J$ mapping the 'multi-scale coefficients' $(\mathbf{c}_0^T, \mathbf{d}_0^T, \ldots, \mathbf{d}_{J-1}^T)^T$ to the 'single-scale coefficients' $\mathbf{c}_J$, satisfies

$$\mathbf{T}_J = \begin{bmatrix} \mathbf{M}_{J-1,0} \mathbf{T}_{J-1} & \mathbf{M}_{J-1,1} \end{bmatrix} = \mathbf{M}_{J-1} \begin{bmatrix} \mathbf{T}_{J-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{2.14}$$

and $\mathbf{T}_0 = \mathbf{I}$. So, assuming a geometrical increase of $\dim V_J$ as function of J, we see that $\mathbf{T}_J$ can be performed in $\mathcal{O}(\dim V_J)$ operations when the $\mathbf{M}_j$ are *uniformly sparse*.

Writing $\hat{\Phi}_j^T = \Phi_{j+1}^T \hat{\mathbf{M}}_{j,0}$, $\check{\Psi}_j^T = \Phi_{j+1}^T \check{\mathbf{M}}_{j,1}$, $\tilde{\Phi}_j^T = \tilde{\Phi}_{j+1}^T \tilde{\mathbf{M}}_{j,0}$ for some matrices $\hat{\mathbf{M}}_{j,0}$, $\check{\mathbf{M}}_{j,1}$ and $\tilde{\mathbf{M}}_{j,0}$, we infer that (2.13) is equivalent to

$$\mathbf{M}_{j,1} = (\mathbf{I} - \hat{\mathbf{M}}_{j,0} \tilde{\mathbf{M}}_{j,0}^* \langle \Phi_{j+1}, \tilde{\Phi}_{j+1} \rangle_{L_2}^T) \check{\mathbf{M}}_{j,1}.$$

We conclude that the $\mathbf{M}_j$ are uniformly sparse, whenever this holds for $\mathbf{M}_{j,0}$, $\hat{\mathbf{M}}_{j,0}$, $\tilde{\mathbf{M}}_{j,0}$, $\langle \Phi_{j+1}, \tilde{\Phi}_{j+1} \rangle_{L_2}$ and $\check{\mathbf{M}}_{j,1}$.

Formula (2.14) shows if one also needs an implementation of optimal complexity of $\mathbf{T}_J^{-1}$, mapping the 'single-scale coefficients' to the 'multi-scale coefficients', then it is necessary that also the $\mathbf{M}_j^{-1}$ are uniformly sparse. Only under special circumstances, the inverse of a sparse matrix is again sparse, and with the construction (2.13), $\mathbf{M}_j^{-1}$ will generally be a densely populated matrix.

We now focus on the special case $\hat{V}_j = V_j$. In this case, $\hat{\Phi}_j$ is a basis for V_j, and we take $\Phi_j = \hat{\Phi}_j$. With $\hat{\mathbf{M}}_{j,0} = \mathbf{M}_{j,0}$ and $\langle \Phi_{j+1}, \tilde{\Phi}_{j+1} \rangle_{L_2} = \mathbf{I}$, we now get

$$\mathbf{M}_j = \begin{bmatrix} \mathbf{M}_{j,0} & \check{\mathbf{M}}_{j,1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\tilde{\mathbf{M}}_{j,0}^* \check{\mathbf{M}}_{j,1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{2.15}$$

and we conclude that the $\mathbf{M}_j^{-1}$ are uniformly sparse under the additional condition that the initial supplements $\check{\Psi}_j$ are selected such that the $\begin{bmatrix} \mathbf{M}_{j,0} & \check{\mathbf{M}}_{j,1} \end{bmatrix}^{-1}$ are uniformly sparse.

In the wavelet literature, $\mathbf{T}_J^{-1}$ and $\mathbf{T}_J$ are called *wavelet transform* and *inverse wavelet transform* respectively.

A closely related advantage of having $\mathbf{M}_j^{-1}$ that are uniformly sparse is that uniformly local dual wavelets become available: In Remark 2.5 the set of dual wavelets $\tilde{\Psi}_j$ was defined as the unique collection in $\tilde{V}_{j+1} \cap V_j^{\perp}$ that is dual to Ψ_j. From $\begin{bmatrix} \Phi_j^T & \Psi_j^T \end{bmatrix} = \Phi_{j+1}^T \mathbf{M}_j$ and $\langle \mathbf{M}_j^T \Phi_{j+1}, (\overline{\mathbf{M}}_j)^{-1} \tilde{\Phi}_{j+1} \rangle_{L_2} = \mathbf{I}$, we infer that

$$\begin{bmatrix} \tilde{\Phi}_j^T & \tilde{\Psi}_j^T \end{bmatrix} = \tilde{\Phi}_{j+1}^T (\mathbf{M}_j^*)^{-1}.$$

We conclude that the $\tilde{\Psi}_j$ are uniformly local when the $\tilde{\Phi}_{j+1}$ are uniformly local, and the $\mathbf{M}_j^{-1}$ are uniformly sparse.

3. Biorthogonal scaling functions on non-uniform meshes

In the remainder of this paper, we will construct biorthogonal, uniformly local, uniform L_2-Riesz bases Φ_j, $\tilde{\Phi}_j$ for spaces V_j, $\tilde{V}_j$, that are nested as function of j, and that satisfy Bernstein estimates with $\gamma = \tilde{\gamma} = \frac{3}{2}$ and Jackson estimates for certain values $d, \tilde{d} > \frac{3}{2}$. The fact that such biorthogonal bases are available implies that $(V_j, \tilde{V}_j)_j$ also satisfies the uniform maximum L_2-angle condition, and thus that all the conditions of Theorem 2.3 are satisfied. By applying the wavelet construction from the previous section with $\hat{V}_j = V_j$, we are able to construct wavelets and dual wavelets that both exhibit *all* possibly desired properties concerning locality and optimal transforms discussed in the previous section. That is, in contrast to our earlier joint paper with W. Dahmen ([DS99c]), here we obtain also uniformly local dual wavelets, at the cost of getting wavelets with larger supports. Properly scaled, the wavelets and dual wavelets generate Riesz bases for $\mathcal{H}^s$ for $|s| < \frac{3}{2}$.

The primal spaces V_j will be *standard Lagrange finite element spaces* with respect to meshes that are generated by *uniform dyadic refinements* starting with an *arbitrary initial mesh*. Both Φ_j and $\tilde{\Phi}_j$, and so Ψ_j and $\tilde{\Psi}_j$, will be defined *explicitly*.

Remark 3.1. Usually, at least the $\tilde{\Phi}_j$ are only given as solution of some refinement equation (cf. [CDF92]). Exceptions are given by [DGH99, DGH00, Goo00] dealing with uniform mesh cases. An advantage of knowing $\tilde{\Psi}_j$ explicitly is that there is much more freedom in making efficient and accurate numerical approximations of expansions like (2.8).

3.1. **Reduction to a reference element.** We will explain the general mechanism to reduce the construction of Φ_j, $\tilde{\Phi}_j$ to a construction on a reference (macro-)element.

Let τ_0 be a fixed collection of closed n-simplices, or elements, such that $\cup_{T\in\tau_0}T$ is a partition, also called triangulation, of the closure of some open domain $\Omega\subset I\!R^n$. We assume that the triangulation is conforming, i.e., the intersection of any two elements is either empty or a common face. Here with a face of T, we mean any k-simplex spanned by $k+1$ vertices of T, where $0\leq k<n$.

For $j>0$, let τ_j be the collection of n-simplices generated from τ_{j-1} by uniform, regular, dyadic refinement, i.e., each $T\in\tau_{j-1}$ is subdivided into 2^n congruent n-simplices. In this paper, we consider only examples with $n\leq 2$, which means that above refinement rule determines the τ_j uniquely.

For any n-simplex T, $\lambda_T(x)\in I\!R^{n+1}$ will denote the barycentric coordinates of $x\in I\!R^n$ with respect to the vertices of T ordered in some way. There holds $x\in T$ if and only if $\lambda_T(x)\in\boldsymbol{T}$, where

$$\boldsymbol{T}=\{\lambda\in I\!R^{n+1}:\sum_{i=1}^{n+1}\lambda_i=1,\lambda_i\geq 0\}.$$

Let $\boldsymbol{I}\subset\boldsymbol{T}$ be some finite set that is closed under permutations of the coordinates. We will consider collections of functions $\boldsymbol{\Phi}=\{\boldsymbol{\phi}_\lambda:\lambda\in\boldsymbol{I}\}$ that satisfy

($\mathcal{C}$) $\boldsymbol{\phi}_\lambda\in C(\boldsymbol{T})$,

($\mathcal{S}$) $\boldsymbol{\phi}_\lambda(\mu)=\boldsymbol{\phi}_{\pi(\lambda)}(\pi(\mu))$ for any permutation $\pi:I\!R^{n+1}\to I\!R^{n+1}$,

($\mathcal{V}$) $\boldsymbol{\phi}_\lambda$ vanishes on faces that do not include λ,

($\mathcal{I}$) For $\boldsymbol{e}=\boldsymbol{T}$, or for $\boldsymbol{e}$ being a face of $\boldsymbol{T}$, $\{\boldsymbol{\phi}_\lambda|_{\boldsymbol{e}}:\lambda\in\boldsymbol{I}\cap\boldsymbol{e}\}$ is independent.

These 'local' functions from such collections can be assembled to collections of 'global' functions in a way known from finite element methods: For $j\geq 0$ and with

$$I_{\tau_j}=\{x\in\Omega:\lambda_T(x)\in\boldsymbol{I}\text{ for some }T\in\tau_j\},$$

we define the collection $\Phi_j=\{\phi_{j,x}:x\in I_{\tau_j}\}$ of functions on Ω by

$$\phi_{j,x}(y)=\begin{cases}\mu(x;\tau_j)\boldsymbol{\phi}_{\lambda_T(x)}(\lambda_T(y)) & \text{if } x,y\in T\in\tau_j,\\ 0 & \text{elsewhere},\end{cases}\tag{3.1}$$

with scaling factor $\mu(x;\tau_j):=(\sum_{\{T\in\tau_j:T\ni x\}}\frac{\text{vol}(T)}{\text{vol}(\boldsymbol{T})})^{-\frac{1}{2}}$. The condition ($\mathcal{S}$) ensures that $\phi_{j,x}$ is well-defined also on faces that include x and are shared by elements, and by ($\mathcal{V}$) and ($\mathcal{C}$) it is continuous on Ω. Clearly, the Φ_j are sets of independent functions, and they are uniformly local. Below, we will collect some more properties of such (Φ_j) constructed in this way.

Suppose that we have two such sets $\boldsymbol{I}^{(1)}$ and $\boldsymbol{I}^{(2)}$, and collections $\boldsymbol{\Phi}^{(1)}$ and $\boldsymbol{\Phi}^{(2)}$. Then for the resulting $(\Phi_j^{(1)})$ and $(\Phi_j^{(2)})$, there holds $\text{span}\Phi_j^{(1)}\subset\text{span}\Phi_j^{(2)}$ $(j\in I\!N)$, if and only if

$$\text{span}\boldsymbol{\Phi}^{(1)}\subset\text{span}\boldsymbol{\Phi}^{(2)}.$$

To show the if-statement, let $\mathbf{Q} = (q_{\nu,\mu})_{\nu\in \boldsymbol{I}^{(2)},\mu\in\boldsymbol{I}^{(1)}}$ be such that $(\mathbf{\Phi}^{(1)})^T = (\mathbf{\Phi}^{(2)})^T\mathbf{Q}$, or

$$\phi^{(1)}_\mu = \sum_{\nu\in\boldsymbol{I}^{(2)}} q_{\nu,\mu}\phi^{(2)}_\nu. \tag{3.2}$$

Then, there holds that for $x \in I^{(1)}_{\tau_j}$,

$$\frac{\phi^{(1)}_{j,x}}{\mu(x;\tau_j)} = \sum_{\{y\in I^{(2)}_{\tau_j}:\exists T\in\tau_j,\,x,y\in T\}} q_{\lambda_T(y),\lambda_T(x)}\,\frac{\phi^{(2)}_{j,y}}{\mu(y;\tau_j)}\,. \tag{3.3}$$

Indeed, it is not difficult to verify that both sides (3.3) agree on supp$\phi^{(1)}_{j,x}$. Note that by $(\mathcal{S})$, the coefficient $q_{\lambda_T(y),\lambda_T(x)}$ in front of $\phi^{(2)}_{j,y}$ is uniquely defined, also when x and y are included on a face shared by elements in τ_j. Furthermore, the conditions $(\mathcal{V})$ on $\mathbf{\Phi}^{(1)}$ and $(\mathcal{I})$ on $\mathbf{\Phi}^{(2)}$ ensure that the right-hand side of (3.3) vanishes outside supp$\phi^{(1)}_{j,x}$.

Formula (3.3) shows that the representations of the inclusions $Incl$: span$\Phi^{(1)}_j \to$ span$\Phi^{(2)}_j$ with respect to $\Phi^{(1)}_j$ and $\Phi^{(2)}_j$ are uniformly sparse, and how they can be constructed from the representation $\mathbf{Q}$ of $Incl$: span$\mathbf{\Phi}^{(1)} \to$ span$\mathbf{\Phi}^{(2)}$ with respect to $\mathbf{\Phi}^{(1)}$ and $\mathbf{\Phi}^{(2)}$.

In particular, when span$\Phi^{(1)}_j$ = span$\Phi^{(2)}_j$ ($j \in \mathbb{N}$), or equivalently when span$\mathbf{\Phi}^{(1)}$ = span$\mathbf{\Phi}^{(2)}$, we conclude that the basis transformations in both directions are uniformly sparse.

The question whether for given $\mathbf{\Phi}$, there holds span$\Phi_j \subset$ spanΦ_{j+1} ($j \in \mathbb{N}$) can be reduced to a special case of the foregoing analysis. Indeed, let $\{\boldsymbol{T}_i : 1 \leq i \leq 2^n\}$ be the subdivision of $\boldsymbol{T}$ corresponding to dyadic refinement, and let $B_i : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be linear operators mapping $\boldsymbol{T}_i$ onto $\boldsymbol{T}$. With

$$\boldsymbol{I}^{(r)} := \cup_i B_i^{-1}(\boldsymbol{I}),$$

which is a set that is closed under permutations of the barycentric coordinates, we define $\mathbf{\Phi}^{(r)} = \{\phi^{(r)}_\lambda : \lambda \in \boldsymbol{I}^{(r)}\}$, satisfying $(\mathcal{C})$, $(\mathcal{S})$, $(\mathcal{V})$ and $(\mathcal{I})$, by

$$\phi^{(r)}_\lambda(\mu) = \begin{cases} \phi_{B_i(\lambda)}(B_i(\mu)) & \text{if } \lambda,\mu \in \boldsymbol{T}_i, \\ 0 & \text{elsewhere on } \boldsymbol{T}. \end{cases} \tag{3.4}$$

The resulting $(\Phi^{(r)}_j)$ satisfies $\Phi^{(r)}_j = 2^{-n/2}\Phi_{j+1}$, and so span$\Phi_j \subset$ spanΦ_{j+1} ($j \in \mathbb{N}$) if and only if

$$\text{span}\mathbf{\Phi} \subset \text{span}\mathbf{\Phi}^{(r)}. \tag{$\mathcal{R}$}$$

Such a collection $\mathbf{\Phi}$ will be called *refinable*, and $\mathbf{\Phi}^{(r)}$ the *refinement* of $\mathbf{\Phi}$. Formulas (3.2) and (3.3) show how the representation of $Incl$: span$\Phi_j \to$ spanΦ_{j+1} can be constructed from the representation of the local inclusion.

We note the trivial equality

$$\langle u,v\rangle_{L_2(\Omega)}=\sum_{T\in\tau_j}\tfrac{\mathrm{vol}(T)}{\mathrm{vol}(\boldsymbol{T})}\langle u\circ\lambda_T^{-1},v\circ\lambda_T^{-1}\rangle_{L_2(\boldsymbol{T})}. \tag{3.5}$$

From (3.5), and the fact that $\boldsymbol{\Phi}$ is an independent set and thus an $L_2(\boldsymbol{T})$-Riesz system, we obtain that

$$\begin{aligned}\|\mathbf{c}_j^T\Phi_j\|^2_{L_2(\Omega)} &= \sum_{T\in\tau_j}\tfrac{\mathrm{vol}(T)}{\mathrm{vol}(\boldsymbol{T})}\|\sum_{x\in I_j\cap T}c_{j,x}\mu(x;\tau_j)\boldsymbol{\phi}_{\lambda_T(x)}\|^2_{L_2(\boldsymbol{T})}\\ &\eqsim \sum_{T\in\tau_j}\tfrac{\mathrm{vol}(T)}{\mathrm{vol}(\boldsymbol{T})}\sum_{x\in I_j\cap T}|c_{j,x}|^2\mu(x;\tau_j)^2\\ &= \sum_{x\in I_j}|c_{j,x}|^2\mu(x;\tau_j)^2\sum_{\{T\in\tau_j:T\ni x\}}\tfrac{\mathrm{vol}(T)}{\mathrm{vol}(\boldsymbol{T})}\\ &= \|\mathbf{c}_j\|^2,\end{aligned}$$

i.e., the Φ_j are uniform $L_2(\Omega)$-Riesz systems.

Having two collections $\boldsymbol{\Phi}^{(1)}$, $\boldsymbol{\Phi}^{(2)}$, with index sets $\boldsymbol{I}^{(1)}$, $\boldsymbol{I}^{(2)}$, there holds for $x\in I^{(1)}_{\tau_j}$, $y\in I^{(2)}_{\tau_j}$, that

$$\langle\phi^{(1)}_{j,x},\phi^{(2)}_{j,y}\rangle_{L_2(\Omega)}=\mu(x;\tau_j)\mu(y;\tau_j)\sum_{\{T\in\tau_j:T\ni x,y\}}\tfrac{\mathrm{vol}(T)}{\mathrm{vol}(\boldsymbol{T})}\langle\boldsymbol{\phi}^{(1)}_{\lambda_T(x)},\boldsymbol{\phi}^{(2)}_{\lambda_T(y)}\rangle_{L_2(\boldsymbol{T})}, \tag{3.6}$$

where, when $\{T\in\tau_j:T\ni x,y\}\neq\emptyset$, the factors $\langle\boldsymbol{\phi}^{(1)}_{\lambda_T(x)},\boldsymbol{\phi}^{(2)}_{\lambda_T(y)}\rangle_{L_2(\boldsymbol{T})}$ in the sum on the right-hand side are independent of T. We see that the matrix $\langle\Phi_j^{(1)},\Phi_j^{(2)}\rangle_{L_2(\Omega)}$ can easily be constructed from $\langle\boldsymbol{\Phi}^{(1)},\boldsymbol{\Phi}^{(2)}\rangle_{L_2(\boldsymbol{T})}$ using some information about the geometry of τ_j.

In view of our aim to make biorthogonal scaling functions, we will construct examples of pairs of collections of functions on $\boldsymbol{T}$, which we will denote by $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$.

At the primal side, the collection $\boldsymbol{\Phi}$ will always be selected such that it satisfies ($\mathcal{C}$), ($\mathcal{S}$) and ($\mathcal{V}$), and such that for *some fixed d and m*,

$$\mathrm{span}\boldsymbol{\Phi}=P_{d-1,m}(\boldsymbol{T}),$$

being defined as *the space of continuous piecewise polynomials on $\boldsymbol{T}$ of degree $d-1$ with respect to an m-times repeated dyadic partition of $\boldsymbol{T}$*.

We define

$$\boldsymbol{I}_q=\{\lambda\in\boldsymbol{T}:\lambda_i/q\in I\!N\},$$

which is sometimes called the principal lattice of order q. It is well-known that

$$\mathrm{card}(\boldsymbol{I}_{(d-1)2^m})=\dim(P_{d-1,m}(\boldsymbol{T})).$$

We will always assume that the index set of $\boldsymbol{\Phi}$ is given by

$$\boldsymbol{I}=\boldsymbol{I}_{(d-1)2^m}, \tag{3.7}$$

which, as will turn out, guarantees that $\boldsymbol{\Phi}$ satisfies ($\mathcal{I}$) and ($\mathcal{R}$) as well.

Indeed, for $\boldsymbol{e} = \boldsymbol{T}$, or for $\boldsymbol{e}$ being a face of $\boldsymbol{T}$, by $(\mathcal{V})$ there holds

$$\operatorname{span}\{\boldsymbol{\phi}_\lambda|_e : \lambda \in \boldsymbol{I}_{(d-1)2^m} \cap e\} = \operatorname{span}\boldsymbol{\Phi}|_e = P_{d-1,m}(\boldsymbol{T})|_e = P_{d-1,m}(\boldsymbol{e}),$$

and so $\operatorname{card}(\boldsymbol{I}_{(d-1)2^m} \cap \boldsymbol{e}) = \dim(P_{d-1,m}(\boldsymbol{e}))$ shows $(\mathcal{I})$.

Furthermore, it is clear that $\operatorname{span}\boldsymbol{\Phi}^{(r)} \subset P_{d-1,m+1}(\boldsymbol{T})$. Now from $\boldsymbol{I}^{(r)}_{(d-1)2^m} = \boldsymbol{I}_{(d-1)2^{m+1}}$, we conclude that $\operatorname{span}\boldsymbol{\Phi}^{(r)} = P_{d-1,m+1}(\boldsymbol{T})$, and thus that $(\mathcal{R})$ is valid.

For the resulting sequence of collections (Φ_j) of functions on Ω defined by (3.1) corresponding to $\boldsymbol{\Phi}$, there holds $\operatorname{cl}_{L_2(\Omega)}\operatorname{span}\Phi_j = V_j$, being the space of *continuous piecewise polynomials of order $d-1$ with respect to τ_{j+m} having finite $L_2(\Omega)$-norm.* In view of this, the elements of τ_j will also be called *macro*-elements in case $m > 0$. The sequence (V_j) satisfies the Bernstein estimate $(\mathfrak{B})$ with $\gamma = \frac{3}{2}$ and the Jackson estimate $(\mathfrak{J})$ for this value of d, where ρ, being the refinement factor, is equal to 2.

A particular collection $\boldsymbol{\Phi}$ satisfying above conditions is the *nodal* one $\boldsymbol{\Phi} = \boldsymbol{\Delta}^{(d-1,m)} = \{\boldsymbol{\delta}^{(d-1,m)}_\lambda : \lambda \in \boldsymbol{I}_{(d-1)2^m}\} \subset P_{m-1,d}(\boldsymbol{T})$ defined by

$$\boldsymbol{\delta}^{(d-1,m)}_\lambda(\mu) = \begin{cases} 1 & \lambda = \mu, \\ 0 & \lambda \neq \mu \in \boldsymbol{I}_{(d-1)2^m}. \end{cases}$$

Note that $(\boldsymbol{\Delta}^{(d-1,m)})^{(r)} = \boldsymbol{\Delta}^{(d-1,m+1)}$.

Remark 3.2. We included the possibility of $m > 0$ to introduce some freedom in the choice of $\boldsymbol{\Phi}$. Indeed, note that for $d = 2$ and $m = 0$, the only possibility is $\boldsymbol{\Phi} = \boldsymbol{\Delta}^{(1,0)}$ (or a scalar multiple of $\boldsymbol{\Delta}^{(1,0)}$).

At the dual side, we will select $\tilde{\boldsymbol{\Phi}}$ satisfying $(\mathcal{C})$, $(\mathcal{S})$, $(\mathcal{V})$, $(\mathcal{I})$ and $(\mathcal{R})$. Aiming at biorthogonality, for the resulting $(\tilde{\Phi}_j)$ defined by (3.1) corresponding to $\tilde{\boldsymbol{\Phi}}$, there should hold $\operatorname{card}(\tilde{\Phi}_j) = \operatorname{card}(\Phi_j)$, independent of τ_0. This means that the index set $\tilde{\boldsymbol{I}}$ of $\tilde{\boldsymbol{\Phi}}$ should satisfy $\operatorname{card}(\tilde{\boldsymbol{I}}) = \operatorname{card}(\boldsymbol{I}_{(d-1)2^m})$ and $\operatorname{card}(\tilde{\boldsymbol{I}} \cap \boldsymbol{e}) = \operatorname{card}(\boldsymbol{I}_{(d-1)2^m} \cap \boldsymbol{e})$ for any face $\boldsymbol{e}$ of $\boldsymbol{T}$, which means that it is no restriction to take $\tilde{\boldsymbol{I}} = \boldsymbol{I}_{(d-1)2^m}$.

Because of $(\mathcal{R})$, the sequence $(\tilde{V}_j)$, defined by $\tilde{V}_j := \operatorname{cl}_{L_2(\Omega)}\operatorname{span}\tilde{\Phi}_j$, is nested. Since the $\tilde{\phi}_{j,x}$ are continuous, standard arguments (see [Osw94, §2.4]) show that $(\tilde{V}_j)$ satisfies the Bernstein estimate $(\tilde{\mathfrak{B}})$ with $\tilde{\gamma} = \frac{3}{2}$. The set $\tilde{\boldsymbol{\Phi}}$ will selected such that for some $\tilde{d}$, its span includes $P_{\tilde{d}-1,0}(\boldsymbol{T})$, so that $(\tilde{V}_j)$ satisfies the Jackson estimate $(\tilde{\mathfrak{J}})$ for this value of $\tilde{d}$. In view of the cancellation property, we are aiming at making $\tilde{d}$ as large as possible. A dimension argument shows that $\tilde{d} - 1 \leq (d-1)2^m$, where in practice the upper-bound can not be attained because of the other requirements.

In some cases (§3.2, 3.3), we will be able to construct biorthogonal $\boldsymbol{\Phi}$, $\tilde{\boldsymbol{\Phi}}$. From (3.6), we conclude that then Φ_j, $\tilde{\Phi}_j$ are biorthogonal, uniformly local, uniform $L_2(\Omega)$-Riesz systems.

In the other cases (§3.4, 3.5), with respect to some partitioning of the index set $\boldsymbol{I}$ into $\boldsymbol{I}^{(1)}, \ldots, \boldsymbol{I}^{(q)}$, where each $\boldsymbol{I}^{(i)}$ is closed under permutations of the coordinates, $\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}} \rangle_{L_2(\boldsymbol{T})}$ will be a block lower triangular matrix, with diagonal blocks equal to identity matrices. Then, with respect to a corresponding partitioning of I_{τ_j} into $I^{(1)}_{\tau_j}, \ldots, I^{(q)}_{\tau_j}$, $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}$

is also a block lower triangular matrix, with diagonal blocks equal to identity matrices. We infer that both the $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}$ and their inverses are uniformly sparse and uniformly bounded matrices. So, we conclude that

$$\Phi_j, \ \langle \tilde{\Phi}_j, \Phi_j \rangle_{L_2(\Omega)}^{-1} \tilde{\Phi}_j$$

are biorthogonal, uniformly local, uniform $L_2(\Omega)$-Riesz systems. We will refer to this step as the *à posteriori biorthogonalization.*

Remark 3.3. The reason why we apply the à posteriori biorthogonalization, instead of biorthogonalizing $\boldsymbol{\Phi}$, $\tilde{\boldsymbol{\Phi}}$ before constructing the global scaling functions, is that in the cases in question such a 'local' biorthogonalization would violate $(\mathcal{V})$.

We have translated all conditions of Theorem 2.3 on (V_j), $(\tilde{V}_j)$, as well as those for equipping these sequences with biorthogonal bases, in terms of conditions on $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$. What is left is to specify uniform $L_2(\Omega)$-Riesz systems $\check{\Psi}_j$ such that with $W_j := \mathrm{cl}_{L_2(\Omega)} \mathrm{span} \check{\Psi}_j$, there holds $V_{j+1} = W_j + V_j$ ((2.9)), $(W_j, V_j)_j$ satisfies the uniform minimum L_2-angle condition ((2.11)), and such that both the basis transformations from $\Phi_j \cup \check{\Psi}_j$ to Φ_{j+1}, denoted by $\begin{bmatrix} \mathbf{M}_{j,0} & \check{\mathbf{M}}_{j,1} \end{bmatrix}$ in §2, as their inverses are uniformly sparse.

With

$$\check{\boldsymbol{\Psi}}^{(d-1,m)} := \{ \boldsymbol{\delta}_\lambda^{(d-1,m+1)} : \lambda \in \boldsymbol{I}_{(d-1)2^{m+1}} \backslash \boldsymbol{I}_{(d-1)2^m} \},$$

it is well-known that

$$P_{d-1,m+1}(\boldsymbol{T}) = \mathrm{span} \, \check{\boldsymbol{\Psi}}^{(d-1,m)} \oplus P_{d-1,m}(\boldsymbol{T}).$$

As a consequence, taking $\check{\Psi}_j$ as being the 'global' collection defined by (3.1) corresponding to $\check{\boldsymbol{\Psi}}^{(d-1,m)}$, using (3.5) we may conclude that $W_j = \mathrm{cl}_{L_2(\Omega)} \mathrm{span} \check{\Psi}_j$ satisfies aforementioned conditions. Note that $\check{\Psi}_j$ is nothing else than the *'hierarchical surplus'*, that is, the collection of all 'global' nodal basis functions corresponding to the 'new nodes'.

With the canonical application of $\boldsymbol{I}_{(d-1)2^{m+1}}$ as an index set for $\boldsymbol{\Phi} \cup \check{\boldsymbol{\Psi}}^{(d-1,m)}$, this collection satisfies $(\mathcal{C})$, $(\mathcal{S})$ and $(\mathcal{V})$ and, since it spans $P_{d-1,m+1}(\boldsymbol{T})$, also $(\mathcal{I})$ (and $(\mathcal{R})$). The collection $2^{n/2} \boldsymbol{\Phi}^{(r)}$ has the same properties, which means that the basis transformations in both directions between the corresponding global bases, which are $\Phi_j \cup \check{\Psi}_j$ and Φ_{j+1}, are uniformly sparse, and that they can be easily constructed from the local basis transformations.

Remark 3.4. To compute the wavelet and inverse wavelet transforms, formula (2.15) shows that, apart from $\begin{bmatrix} \mathbf{M}_{j,0} & \check{\mathbf{M}}_{j,1} \end{bmatrix}^{-1}$ and $\begin{bmatrix} \mathbf{M}_{j,0} & \check{\mathbf{M}}_{j,1} \end{bmatrix}$, one needs the application of the matrices $\mathbf{M}_{j,0}^* \check{\mathbf{M}}_{j,1}$. Taking into account the possibility that an à posteriori biorthogonalization is needed, meaning that the collections of dual scaling function are given by $\langle \tilde{\Phi}_j, \Phi_j \rangle_{L_2(\Omega)}^{-1} \tilde{\Phi}_j$, we have

$$\mathbf{M}_{j,0}^* \check{\mathbf{M}}_{j,1} = \langle \check{\Psi}_j, \langle \tilde{\Phi}_j, \Phi_j \rangle_{L_2(\Omega)}^{-1} \tilde{\Phi}_j \rangle_{L_2(\Omega)}^T = \langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}^{-T} \langle \check{\Psi}_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}^T.$$

In case $\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})} \neq \mathbf{I}$, the last equality in above display indicates an efficient way to apply $\mathbf{M}_{j,0}^* \check{\mathbf{M}}_{j,1}$ in a factorized way. Formula (3.6) shows how $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}$ and $\langle \check{\Psi}_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}$ can be computed from $\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})}$ and $\langle \check{\mathbf{\Psi}}^{(d-1,m)}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})}$. Since $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}$ is assumed to have a block lower triangular structure with diagonal blocks equal to identity matrices, $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}^{-1}$ can easily be constructed from $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}$, where its application takes as many operations as applying $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}$.

Remark 3.5. For the case that $\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})} \neq \mathbf{I}$, we applied a correction at the dual side, that is we considered the biorthogonal system Φ, $\langle \tilde{\Phi}_j, \Phi_j \rangle_{L_2(\Omega)}^{-1} \tilde{\Phi}_j$. The motivation not to consider the biorthogonal system $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}^{-1} \Phi_j$, $\tilde{\Phi}_j$ is that in that case $\begin{bmatrix} \mathbf{M}_{j,0} & \check{\mathbf{M}}_{j,1} \end{bmatrix}$ should be replaced by

$$\langle \Phi_{j+1}, \tilde{\Phi}_{j+1} \rangle_{L_2(\Omega)}^T \begin{bmatrix} \mathbf{M}_{j,0} & \check{\mathbf{M}}_{j,1} \end{bmatrix} \begin{bmatrix} \langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}^{-T} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{3.8}$$

being the basis transformation from $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}^{-1} \Phi_j \cup \check{\Psi}_j$ to $\langle \Phi_{j+1}, \tilde{\Phi}_{j+1} \rangle_{L_2(\Omega)}^{-1} \Phi_{j+1}$. Comparison with Remark 3.4 learns that for computing the inverse wavelet transform this correction at the primal side demands an additional application of $\langle \Phi_{j+1}, \tilde{\Phi}_{j+1} \rangle_{L_2(\Omega)}^T$. A similar observation holds for the wavelet transform. Note that since the supports of functions from $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}^{-1} \Phi_j$ extend to several macro-elements, one cannot expect to obtain a cheaper implementation by a 'direct' computation of above basis transformation, that is, not using the factorization (3.8).

Remark 3.6. Reversing the last argument from Remark 3.5 leads to the insight that, regardless whether $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$ are biorthogonal or not, for $m > 0$ particular efficient implementations of wavelet and inverse wavelet transforms can be expected, when as scaling functions at the primal side the collections of nodal basis functions $\Delta_j^{(d-1,m)}$ are applied, which are defined by (3.1) corresponding to $\mathbf{\Delta}^{(d-1,m)}$. Indeed, since the supports of functions from $\Delta_j^{(d-1,m)}$ are restricted to *elements* (i.e. $T \in \tau_{j+m}$) instead of *macro-elements*, and $\check{\Psi}_j$ is just a subset of $\Delta_{j+1}^{(d-1,m)}$, the basis transformations between $\Delta_j^{(d-1,m)} \cup \check{\Psi}_j$ and $\Delta_{j+1}^{(d-1,m)}$ can be implemented very efficiently. Let $\mathbf{G}_j$ now be the matrices such that $(\Delta_j^{(d-1,m)})^T = \Phi_j^T \mathbf{G}_j$. Both $\mathbf{G}_j$ and $\mathbf{G}_j^{-1}$ are uniformly bounded and uniformly sparse, and they can easily be constructed from the corresponding local transformations. The pairs $\Delta_j^{(d-1,m)}$, $\overline{\mathbf{G}}^{-1} \langle \tilde{\Phi}_j, \Phi_j \rangle_{L_2(\Omega)}^{-1} \tilde{\Phi}_j$ are biorthogonal, uniformly local, uniformly $L_2(\Omega)$-Riesz systems. With these systems applied, the matrix $\mathbf{M}_{j,0}^* \check{\mathbf{M}}_{j,1}$ reads as

$$\mathbf{G}_j^{-1} \langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}^{-T} \langle \check{\Psi}_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}^T.$$

The same arguments that were used in Remark 3.5 show that if the basis transformations between $\Phi_j \cup \check{\Psi}_j$ and Φ_{j+1} are most efficiently implemented as a composition of transformations from Φ_j to $\Delta_j^{(d-1,m)}$, $\Delta_j^{(d-1,m)} \cup \check{\Psi}_j$ to $\Delta_{j+1}^{(d-1,m)}$ and $\Delta_{j+1}^{(d-1,m)}$ to Φ_{j+1} or vice versa, then the approach of applying the nodal basis functions as scaling functions is more efficient.

So far we considered the construction of bases for the 'full' spaces. Homogeneous *Dirichlet conditions* on the boundary, or on a part of the boundary consisting of the union of $(n-1)$-dimensional faces of $T \in \tau_0$, can be incorporated in the construction by excluding those $\phi_{j,x}$, $\tilde{\phi}_{j,x}$ and $\check{\psi}_{j,x}$ from Φ_j, $\tilde{\Phi}_j$ and $\check{\Psi}_j$ for which x is on (that part of) the boundary. The conditions $(\mathcal{V})$ and $(\mathcal{I})$ ensure that the resulting sequences (V_j), $(\tilde{V}_j)$, defined by $V_j = \mathrm{cl}_{L_2(\Omega)}\mathrm{span}\Phi_j$ and $\tilde{V}_j = \mathrm{cl}_{L_2(\Omega)}\mathrm{span}\tilde{\Phi}_j$ are still nested. The space V_j is the standard Lagrange finite element space in which the boundary conditions are incorporated. Basis transformations between the 'reduced' sets $\Phi_j \cup \Psi_j$ and Φ_{j+1} and vice versa are obtained by simply deleting those rows and columns with indices corresponding basis functions that have been removed. By replacing the scale of Sobolev spaces by the scale of subspaces that incorporate the essential boundary conditions, the Jackson and Bernstein estimates remain valid, and so the wavelets generate Riesz bases for the same range in the scale. On the other hand, wavelets from the resulting Ψ_j or $\tilde{\Psi}_j$ with supports that intersect interiors of $T \in \tau_j$ will not have cancellation properties.

Finally, as demonstrated in [DS99c], a construction like this carries directly over to finite element type spaces on certain Lipschitz manifolds. More precisely, those manifolds are covered that consist of patches, each of them the parametric image of a domain with triangulations generated by uniform refinements, such that the images of the triangulations match at the interfaces, and on each domain the Jacobian determinant is piecewise constant with respect to the initial triangulation.

In the next subsections, for a number of examples of $(n, d, m, \tilde{d})$, we construct sets $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$. Using these two ingredients, the general theory presented in this subsection shows how the global scaling and dual scaling functions, and wavelets and dual wavelets can be constructed, and furthermore how the wavelet and inverse wavelet transforms can be computed.

3.2. **The case** $(n, d, m, \tilde{d}) = (1, 2, 2, 4)$**.** In order to easily formulate conditions $(\mathcal{S})$ and $(\mathcal{V})$, in §3.1 we used as an index set for $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ the subset $\boldsymbol{I}_{(d-1)2^m}$ of the barycentric coordinates. Yet, to view $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ as vectors, the index set $\{1, 2, \ldots, \#\boldsymbol{I}_{(d-1)2^m}\}$ would be more appropriate. Therefore, in Figure 2 we fix a numbering of $\boldsymbol{I}_{(d-1)2^m} = \boldsymbol{I}_4$, so that we can switch between both index sets at our convenience.

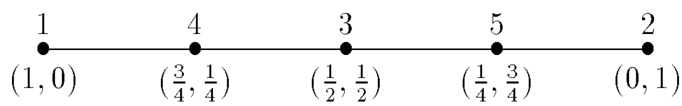

FIGURE 2. Numbering of $\boldsymbol{I}_4$.

We start with $\mathbf{\Phi}^{(0)} = \mathbf{\Delta}^{(1,2)}$, see Figure 3. It satisfies $(\mathcal{C})$, $(\mathcal{S})$, $(\mathcal{V})$, $(\mathcal{I})$ and $(\mathcal{R})$, and it spans $P_{1,2}(\boldsymbol{T})$.

Using a numbering of the elements of $\mathbf{\Delta}^{(3,0)}$ as indicated in Figure 4, at the dual side we start with $\tilde{\mathbf{\Phi}}^{(0)}$, where $\tilde{\boldsymbol{\phi}}_i^{(0)} = \boldsymbol{\delta}_i^{(3,0)}$ for $i \in \{1, 2, 4, 5\}$. Later, the missing $\tilde{\boldsymbol{\phi}}_3^{(0)}$ will be

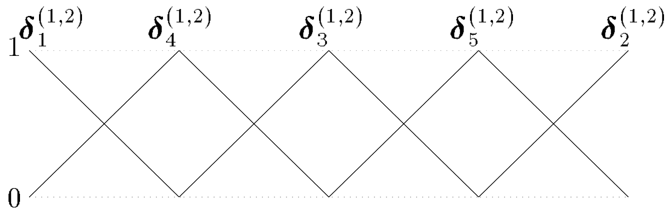

FIGURE 3. $\boldsymbol{\Delta}^{(1,2)}$.

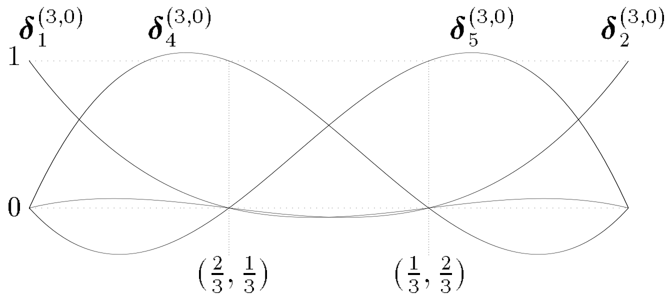

FIGURE 4. $\boldsymbol{\Delta}^{(3,0)}$.

selected from $P_{3,1}(\boldsymbol{T})\backslash P_{3,0}(\boldsymbol{T})$, such that it vanishes on $\partial\boldsymbol{T}$, and $\tilde{\boldsymbol{\phi}}_3^{(0)}(\lambda_1,\lambda_2) = \tilde{\boldsymbol{\phi}}_3^{(0)}(\lambda_2,\lambda_1)$. We infer that $\tilde{\boldsymbol{\Phi}}^{(0)}$ satisfies ($\mathcal{C}$), ($\mathcal{S}$), ($\mathcal{V}$) and ($\mathcal{I}$), and that

$$P_{3,0}(\boldsymbol{T}) \subset \operatorname{span}\tilde{\boldsymbol{\Phi}}^{(0)} \subset P_{3,1}(\boldsymbol{T})$$

showing ($\mathcal{R}$).

Remark 3.7. Note that refinements of the still unknown $\tilde{\boldsymbol{\phi}}_3^{(0)}$ are not used to ensure ($\mathcal{R}$). As a consequence, we will be able to construct the dual scaling functions explicitly.

On the other hand, allowing for implicitly defined dual scaling functions would introduce additional freedom in the construction, which might mean that smaller macro-elements can be used, resulting in wavelets with smaller support. However, in that case also $\tilde{d}$ will be smaller, giving weaker cancellation properties. We will discuss this approach in a forthcoming paper.

Together, above conditions mean that

$$\tilde{\boldsymbol{\phi}}_3^{(0)} \in \operatorname{span}\{\tilde{\boldsymbol{\phi}}_4^{(0)} + \tilde{\boldsymbol{\phi}}_5^{(0)}, \boldsymbol{\delta}^{(3,1)}_{(\frac{5}{6},\frac{1}{6})} + \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{6},\frac{5}{6})}, \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{2},\frac{1}{2})}\}, \tag{3.9}$$

see Figure 5.

Apart from fixing $\tilde{\boldsymbol{\phi}}_3^{(0)}$, in the following we apply some (invertible) basis transformations to both collections $\boldsymbol{\Phi}^{(0)}$ and $\tilde{\boldsymbol{\Phi}}^{(0)}$, which preserve ($\mathcal{S}$) and ($\mathcal{V}$). Obviously, a basis transformation always preserves ($\mathcal{C}$). Moreover, a basis transformation is represented by an invertible matrix. The fact that ($\mathcal{V}$) is preserved means that any principal sub-matrix of this matrix corresponding to all indices associated to some face is necessarily invertible, which means that ($\mathcal{I}$) is preserved as well. Since the basis transformations do not change

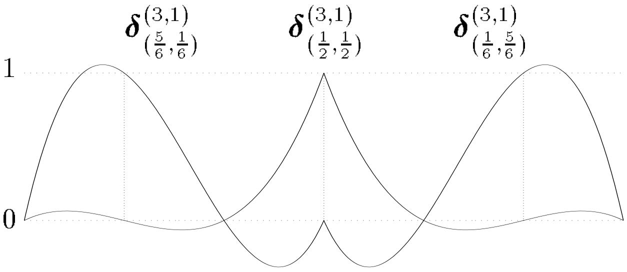

FIGURE 5

the spans and preserve $(\mathcal{S})$, $(\mathcal{V})$ and $(\mathcal{I})$, we conclude that also $(\mathcal{R})$ is preserved. We will end up with biorthogonal sets $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$.

Now we come to the description of the basis transformations and the selection of $\tilde{\boldsymbol{\phi}}_3^{(0)}$:

(I). We search

$$\boldsymbol{\phi}_1 \in \boldsymbol{\phi}_1^{(0)} + \operatorname{span}\{\boldsymbol{\phi}_3^{(0)}, \boldsymbol{\phi}_4^{(0)}, \boldsymbol{\phi}_5^{(0)}\},$$

such that $\boldsymbol{\phi}_1 \perp \tilde{\boldsymbol{\phi}}_2^{(0)}, \tilde{\boldsymbol{\phi}}_4^{(0)}, \tilde{\boldsymbol{\phi}}_5^{(0)}$. Obviously, $\boldsymbol{\phi}_2$ defined by $\boldsymbol{\phi}_2(\lambda_1,\lambda_2) = \boldsymbol{\phi}_1(\lambda_2,\lambda_1)$ then satisfies $\boldsymbol{\phi}_2 \perp \tilde{\boldsymbol{\phi}}_1^{(0)}, \tilde{\boldsymbol{\phi}}_4^{(0)}, \tilde{\boldsymbol{\phi}}_5^{(0)}$. For $i \in \{3,4,5\}$, we take $\boldsymbol{\phi}_i = \boldsymbol{\phi}_i^{(0)}$.

(II). We select $\tilde{\boldsymbol{\phi}}_3^{(0)}$ by imposing $\tilde{\boldsymbol{\phi}}_3^{(0)} \perp \boldsymbol{\phi}_1$ (and thus $\tilde{\boldsymbol{\phi}}_3^{(0)} \perp \boldsymbol{\phi}_2$). Since $\tilde{\boldsymbol{\phi}}_4^{(0)} + \tilde{\boldsymbol{\phi}}_5^{(0)} \perp \boldsymbol{\phi}_1$, the span of the resulting $\tilde{\boldsymbol{\Phi}}$ does not change if, instead of (3.9), we search $\tilde{\boldsymbol{\phi}}_3^{(0)}$ in the smaller space $\operatorname{span}\{\boldsymbol{\delta}^{(3,1)}_{(\frac{5}{6},\frac{1}{6})} + \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{6},\frac{5}{6})}, \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{2},\frac{1}{2})}\}$.

(III). With $\tilde{\boldsymbol{\Phi}} := \langle \tilde{\boldsymbol{\Phi}}^{(0)}, \boldsymbol{\Phi}\rangle_{L_2(\boldsymbol{T})}^{-1} \tilde{\boldsymbol{\Phi}}^{(0)}$, we get $\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}\rangle_{L_2(\boldsymbol{T})} = \mathbf{I}$. Since by the previous steps, in the first two columns of $\langle \tilde{\boldsymbol{\Phi}}^{(0)}, \boldsymbol{\Phi}\rangle_{L_2(\boldsymbol{T})}$ only the diagonal element is non-zero, this transformation preserves $(\mathcal{V})$.

By substituting

$$\langle \{\boldsymbol{\delta}_1^{(1,2)}, \boldsymbol{\delta}_3^{(1,2)}, \boldsymbol{\delta}_4^{(1,2)}\}, \{\boldsymbol{\delta}_1^{(3,0)}, \boldsymbol{\delta}_2^{(3,0)}, \boldsymbol{\delta}_4^{(3,0)}, \boldsymbol{\delta}_5^{(3,0)}, \boldsymbol{\delta}^{(3,1)}_{(\frac{5}{6},\frac{1}{6})} + \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{6},\frac{5}{6})}, \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{2},\frac{1}{2})}\}\rangle_{L_2(\boldsymbol{T})}$$

$$= \operatorname{vol}(\boldsymbol{T}) \begin{bmatrix} \frac{2413}{30720} & \frac{167}{30720} & \frac{687}{10240} & \frac{-267}{10240} & \frac{117}{1280} & \frac{17}{3840} \\ \frac{-5}{512} & \frac{-5}{512} & \frac{69}{512} & \frac{69}{512} & \frac{-27}{640} & \frac{193}{1920} \\ \frac{45}{1024} & \frac{7}{1024} & \frac{237}{1024} & \frac{-33}{1024} & \frac{15}{128} & \frac{1}{128} \end{bmatrix},$$

above procedure results in $\boldsymbol{\phi}_3 = \boldsymbol{\delta}_3^{(1,2)}$, $\boldsymbol{\phi}_4 = \boldsymbol{\delta}_4^{(1,2)}$,

$$\begin{aligned}
\boldsymbol{\phi}_1 &= \boldsymbol{\delta}_1^{(1,2)} + \tfrac{23}{150}\boldsymbol{\delta}_3^{(1,2)} - \tfrac{23}{60}\boldsymbol{\delta}_4^{(1,2)} - \tfrac{3}{100}\boldsymbol{\delta}_5^{(1,2)},\\
\tilde{\boldsymbol{\phi}}_3^{(0)} &= \boldsymbol{\delta}^{(3,1)}_{(\frac{5}{6},\frac{1}{6})} + \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{6},\frac{5}{6})} - \tfrac{657}{299}\boldsymbol{\delta}^{(3,1)}_{(\frac{1}{2},\frac{1}{2})},
\end{aligned}$$

$$\begin{bmatrix} \tilde{\boldsymbol{\phi}}_1 \\ \tilde{\boldsymbol{\phi}}_3 \\ \tilde{\boldsymbol{\phi}}_5 \end{bmatrix} = \frac{1}{\mathrm{vol}(\boldsymbol{T})} \begin{bmatrix} \frac{50}{3} & \frac{-299}{162} & \frac{-64}{27} & \frac{-2}{81} \\ 0 & \frac{-5083}{2025} & \frac{2552}{2025} & \frac{2552}{2025} \\ 0 & \frac{6877}{4050} & \frac{7196}{2025} & \frac{-484}{2025} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_1^{(3,0)} \\ \tilde{\boldsymbol{\phi}}_3^{(0)} \\ \boldsymbol{\delta}_4^{(3,0)} \\ \boldsymbol{\delta}_5^{(3,0)} \end{bmatrix},$$

see Figure 6.

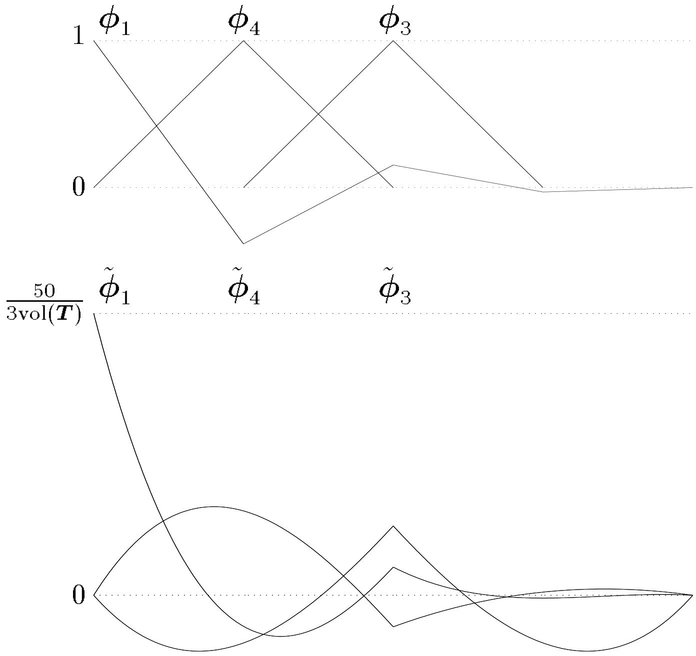

FIGURE 6. Biorthogonal $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$ ($\boldsymbol{\phi}_2$, $\boldsymbol{\phi}_5$, $\tilde{\boldsymbol{\phi}}_2$, $\tilde{\boldsymbol{\phi}}_5$ by permuting barycentric coordinates) .

The analysis from §3.1 shows that the resulting global sets Φ_j, $\tilde{\Phi}_j$ are biorthogonal, uniformly local, uniform $L_2(\Omega)$-Riesz systems. The collection Φ_j is a basis for the space of continuous piecewise linears with respect to τ_{j+2}. Furthermore, the spaces $\tilde{V}_j := \mathrm{cl}_{L_2(\Omega)}\mathrm{span}\tilde{\Phi}_j$ are nested, and satisfy $(\tilde{\mathfrak{B}})$ and $(\tilde{\mathfrak{J}})$ with $\tilde{\gamma} = \frac{3}{2}$ and $\tilde{d} = 4$.

3.3. **The case** $(n, d, m, \tilde{d}) = (1, 5, 0, 4)$. As in §3.2, $(d-1)2^m = 4$, and we use the same numbering from Figure 2 of the index set $\boldsymbol{I}_4$ for $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$. We now take $\boldsymbol{\Phi}^{(0)} = \boldsymbol{\Delta}^{(4,0)}$.

As in §3.2, at the dual side we take $\tilde{\boldsymbol{\phi}}_i^{(0)} = \boldsymbol{\delta}_i^{(3,0)}$ for $i \in \{1,2,4,5\}$, and search $\tilde{\boldsymbol{\phi}}_3^{(0)} \in \operatorname{span}\{\boldsymbol{\delta}^{(3,1)}_{(\frac{5}{6},\frac{1}{6})} + \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{6},\frac{5}{6})}, \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{2},\frac{1}{2})}\}$. To fix $\tilde{\boldsymbol{\phi}}_3^{(0)}$, and to biorthogonalize $\boldsymbol{\Phi}^{(0)}$, $\tilde{\boldsymbol{\Phi}}^{(0)}$, we follow the same procedure as described in §3.2.

By substituting

$$\langle\{\boldsymbol{\delta}_1^{(4,0)}, \boldsymbol{\delta}_3^{(4,0)}, \boldsymbol{\delta}_4^{(4,0)}\}, \{\boldsymbol{\delta}_1^{(3,0)}, \boldsymbol{\delta}_2^{(3,0)}, \boldsymbol{\delta}_4^{(3,0)}, \boldsymbol{\delta}_5^{(3,0)}, \boldsymbol{\delta}^{(3,1)}_{(\frac{5}{6},\frac{1}{6})} + \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{6},\frac{5}{6})}, \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{2},\frac{1}{2})}\}\rangle_{L_2(\boldsymbol{T})}$$
$$= \operatorname{vol}(\boldsymbol{T}) \begin{bmatrix} \frac{151}{2520} & 0 & \frac{1}{28} & \frac{-1}{56} & \frac{29}{560} & \frac{1}{336} \\ \frac{-13}{210} & \frac{-13}{210} & \frac{9}{70} & \frac{9}{70} & \frac{-3}{14} & \frac{23}{210} \\ \frac{2}{21} & \frac{2}{63} & \frac{2}{7} & \frac{-2}{35} & \frac{17}{70} & \frac{1}{210} \end{bmatrix},$$

this procedure now results in $\boldsymbol{\phi}_3 = \boldsymbol{\delta}_3^{(4,0)}$, $\boldsymbol{\phi}_4 = \boldsymbol{\delta}_4^{(4,0)}$,

$$\boldsymbol{\phi}_1 = \boldsymbol{\delta}_1^{(4,0)} - \tfrac{15}{128}\boldsymbol{\delta}_4^{(4,0)} + \tfrac{5}{128}\boldsymbol{\delta}_5^{(4,0)},$$
$$\tilde{\boldsymbol{\phi}}_3^{(0)} = \boldsymbol{\delta}^{(3,1)}_{(\frac{5}{6},\frac{1}{6})} + \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{6},\frac{5}{6})} - \tfrac{63}{5}\boldsymbol{\delta}^{(3,1)}_{(\frac{1}{2},\frac{1}{2})},$$
$$\begin{bmatrix} \tilde{\boldsymbol{\phi}}_1 \\ \tilde{\boldsymbol{\phi}}_3 \\ \tilde{\boldsymbol{\phi}}_5 \end{bmatrix} = \frac{1}{\operatorname{vol}(\boldsymbol{T})} \begin{bmatrix} 20 & \frac{-40}{27} & \frac{-56}{9} & \frac{-68}{27} \\ 0 & \frac{-5}{9} & \frac{4}{9} & \frac{4}{9} \\ 0 & \frac{5}{16} & \frac{163}{48} & \frac{23}{48} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_1^{(3,0)} \\ \tilde{\boldsymbol{\phi}}_3^{(0)} \\ \boldsymbol{\delta}_4^{(3,0)} \\ \boldsymbol{\delta}_5^{(3,0)} \end{bmatrix},$$

and $\boldsymbol{\phi}_2$, $\boldsymbol{\phi}_5$ and $\tilde{\boldsymbol{\phi}}_2$, $\tilde{\boldsymbol{\phi}}_5$ by permuting barycentric coordinates.

The resulting global sets Φ_j, $\tilde{\Phi}_j$ are biorthogonal, uniformly local, uniform $L_2(\Omega)$-Riesz systems. The collection Φ_j is a basis for the space of continuous piecewise quartics with respect to τ_j. Note that, in contrast to §3.2, for each $x \in I_{\tau_j}$, the basis function $\phi_{j,x}$ has the same support as the nodal basis function corresponding to that point.

3.4. **The case** $(n, d, m, \tilde{d}) = (2, 2, 2, 4)$. We number the index set $\boldsymbol{I}_{(d-1)2^m} = \boldsymbol{I}_4$ of $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$ as in Figure 7, and switch between these numbers and the corresponding barycentric coordinates at our convenience. We take $\boldsymbol{\Phi}^{(0)} = \boldsymbol{\Delta}^{(1,2)}$. It satisfies $(\mathcal{C})$, $(\mathcal{S})$, $(\mathcal{V})$, $(\mathcal{I})$ and $(\mathcal{R})$, and it spans $P_{1,2}(\boldsymbol{T})$.

We define $\tilde{\boldsymbol{\phi}}^{(0)}_{1..3,7..12} = \boldsymbol{\delta}^{(3,0)}_{1..3,7..12}$ using a numbering of $\boldsymbol{I}_3$, and with that of the elements of $\boldsymbol{\Delta}^{(3,0)}$ as given in Figure 8. Later, we will define the missing $\tilde{\boldsymbol{\phi}}^{(0)}_{4..6,13..15}$ such that $\tilde{\boldsymbol{\Phi}}^{(0)} := \{\tilde{\boldsymbol{\phi}}^{(0)}_{1..15}\}$ satisfies $(\mathcal{C})$, $(\mathcal{S})$, $(\mathcal{V})$ and $(\mathcal{I})$, as well as

$$\boldsymbol{\delta}_{13}^{(3,0)} \in \operatorname{span}\{\tilde{\boldsymbol{\phi}}^{(0)}_{13..15}\}, \tag{3.10}$$

$$\tilde{\boldsymbol{\phi}}^{(0)}_{4..6} \in P_{3,1}(\boldsymbol{T}), \tag{3.11}$$

and

$$\tilde{\boldsymbol{\phi}}^{(0)}_{13..15} \in P_{3,1}(\boldsymbol{T}) \cup \operatorname{span}\{\tilde{\boldsymbol{\phi}}^{(0)}_{4..6}\}^{(r)}, \tag{3.12}$$

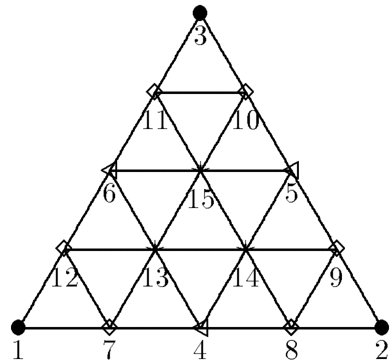

FIGURE 7. Numbering of $\boldsymbol{I}_4$, and its partitioning into $\{\bullet\} \cup \{\triangleleft\} \cup \{\diamond\} \cup \{\star\}$.

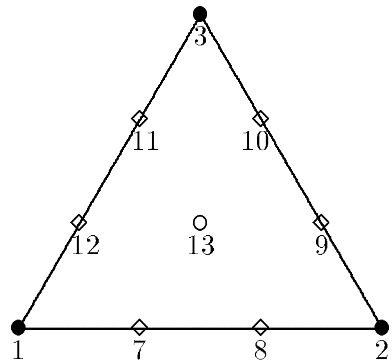

FIGURE 8. Numbering of $\boldsymbol{I}_3$.

where $\{\tilde{\boldsymbol{\phi}}^{(0)}_{4..6}\}^{(r)}$ is defined in (3.4) as the refinement of $\{\tilde{\boldsymbol{\phi}}^{(0)}_{4..6}\}$. From (3.10), we have $P_{3,0}(\boldsymbol{T}) \subset \operatorname{span}\{\tilde{\boldsymbol{\phi}}^{(0)}_{1..3,7..15}\}$, and so (3.11) and (3.12) show that $(\mathcal{R})$ is valid, and moreover that

$$P_{3,0}(\boldsymbol{T}) \subset \operatorname{span} \tilde{\boldsymbol{\Phi}}^{(0)} \subset P_{3,2}(\boldsymbol{T}). \tag{3.13}$$

Apart from specifying the missing $\tilde{\boldsymbol{\phi}}^{(0)}_{4..6,13..15}$, in the following we describe invertible basis transformations on both $\boldsymbol{\Phi}^{(0)}$ and $\tilde{\boldsymbol{\Phi}}^{(0)}$ that preserve $(\mathcal{S})$ and $(\mathcal{V})$. The same reasoning as in §3.2 shows that then $(\mathcal{C})$, $(\mathcal{I})$ and $(\mathcal{R})$ are preserved as well. As a consequence of $(\mathcal{S})$, we only have to specify $\boldsymbol{\phi}^{(0)}_i$ and $\tilde{\boldsymbol{\phi}}^{(0)}_i$ for i running over any element of the sets 1..3, 4..6, 7..12, 13..15 (corresponding to $\{\bullet\}$, $\{\triangleleft\}$, $\{\diamond\}$, $\{\star\}$ from Figure 7), since the other functions then follow by permuting the barycentric coordinates.

We will not be able to end up with biorthogonal $\boldsymbol{\Phi}$, $\tilde{\boldsymbol{\Phi}}$. Instead, we derive $\boldsymbol{\Phi}$, $\tilde{\boldsymbol{\Phi}}$, such that with respect to a partitioning of 1..15 into $\{\bullet\}$, $\{\triangleleft\}$, $\{\diamond\}$, $\{\star\}$, the matrix $\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}} \rangle_{L_2(\boldsymbol{T})}$

is of the form

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ * & * & \mathbf{I} & \mathbf{0} \\ * & * & * & \mathbf{I} \end{bmatrix}. \tag{3.14}$$

With respect to a corresponding partitioning of I_j, the matrix $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}$ of the global basis functions Φ_j, $\tilde{\Phi}_j$ defined by (3.1) then inherits the same block form. The pairs Φ_j, $\langle \tilde{\Phi}_j, \Phi_j \rangle_{L_2(\Omega)}^{-1} \tilde{\Phi}_j$ will be biorthogonal, uniformly local, uniform $L_2(\Omega)$-Riesz systems.

The sets $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$ are obtained by performing the steps **(I)**-**(VI)**:

(I). In view of $(\mathcal{S})$ and $(\mathcal{V})$, $\boldsymbol{\phi}_1$ is searched in

$$\boldsymbol{\phi}_1^{(0)} + \operatorname{span}\{\boldsymbol{\phi}_7^{(0)} + \boldsymbol{\phi}_{12}^{(0)}, \boldsymbol{\phi}_4^{(0)} + \boldsymbol{\phi}_6^{(0)}, \boldsymbol{\phi}_8^{(0)} + \boldsymbol{\phi}_{11}^{(0)}, \boldsymbol{\phi}_{13}^{(0)}, \boldsymbol{\phi}_{14}^{(0)} + \boldsymbol{\phi}_{15}^{(0)}\}$$

such that

$$\boldsymbol{\phi}_1 \perp \tilde{\boldsymbol{\phi}}_{2,7,8,9}^{(0)}, \boldsymbol{\delta}_{13}^{(3,0)}, \tag{3.15}$$

which determines $\boldsymbol{\phi}_1$ uniquely. Clearly, (3.15) is equivalent to $\boldsymbol{\phi}_1 \perp \tilde{\boldsymbol{\phi}}_{2,3,7..12}^{(0)}, \boldsymbol{\delta}_{13}^{(3,0)}$. Since $\boldsymbol{\delta}_{13}^{(3,0)} \in \operatorname{span}\{\tilde{\boldsymbol{\phi}}_{13..15}^{(0)}\}$ by (3.10), and forthcoming transformations at the dual side have to preserve $(\mathcal{V})$, condition (3.15) is necessary for obtaining the first row in (3.14). We define $\tilde{\boldsymbol{\phi}}_1^{(1)} = \tilde{\boldsymbol{\phi}}_1^{(0)} / \langle \tilde{\boldsymbol{\phi}}_1^{(0)}, \boldsymbol{\phi}_1 \rangle_{L_2(\boldsymbol{T})}$.

(II). In view of $(\mathcal{V})$, $\boldsymbol{\phi}_4$, $\boldsymbol{\phi}_7$ (and $\boldsymbol{\phi}_8$) are searched in $\operatorname{span}\{\boldsymbol{\phi}_{4,7,8,13..15}^{(0)}\}$, and, in view of $(\mathcal{S})$, in particular $\boldsymbol{\phi}_4 \in \boldsymbol{\phi}_4^{(0)} + \operatorname{span}\{\boldsymbol{\phi}_7^{(0)} + \boldsymbol{\phi}_8^{(0)}, \boldsymbol{\phi}_{13}^{(0)} + \boldsymbol{\phi}_{14}^{(0)}, \boldsymbol{\phi}_{15}^{(0)}\}$.

To get the zeros in the second row in (3.14), $\boldsymbol{\phi}_4$ must satisfy

$$\boldsymbol{\phi}_4 \perp \tilde{\boldsymbol{\phi}}_{9,10}^{(0)}, \boldsymbol{\delta}_{13}^{(3,0)}, \tag{3.16}$$

which determines $\boldsymbol{\phi}_4$ uniquely, and which is equivalent to $\boldsymbol{\phi}_4 \perp \tilde{\boldsymbol{\phi}}_{9..12}^{(0)}, \boldsymbol{\delta}_{13}^{(3,0)}$.

To get the zero in the third row in (3.14), it is necessary that $\boldsymbol{\phi}_7 \perp \boldsymbol{\delta}_{13}^{(3,0)}$. Furthermore, for obtaining the identity matrix in this row, $\boldsymbol{\phi}_7$ should be orthogonal to $\tilde{\boldsymbol{\phi}}_{9..12}$. If $\operatorname{span}\{\tilde{\boldsymbol{\phi}}_{9..12}\}$ would be equal to $\operatorname{span}\{\tilde{\boldsymbol{\phi}}_{9..12}^{(0)}\}$, then these conditions on $\boldsymbol{\phi}_7$ could only mean that $\boldsymbol{\phi}_7$ is a multiple of $\boldsymbol{\phi}_4$. Yet, since $\tilde{\boldsymbol{\phi}}_{9,10}^{(0)}$ ($\tilde{\boldsymbol{\phi}}_{7,8}^{(0)}$, $\tilde{\boldsymbol{\phi}}_{11,12}^{(0)}$) can be updated by a same multiple of $\tilde{\boldsymbol{\phi}}_5^{(0)}$ ($\tilde{\boldsymbol{\phi}}_4^{(0)}$, $\tilde{\boldsymbol{\phi}}_6^{(0)}$) that still has to be defined, it might be sufficient when only

$$\boldsymbol{\phi}_{7,8} \perp \tilde{\boldsymbol{\phi}}_9^{(0)} - \tilde{\boldsymbol{\phi}}_{10}^{(0)}, \tilde{\boldsymbol{\phi}}_{11}^{(0)} - \tilde{\boldsymbol{\phi}}_{12}^{(0)}, \boldsymbol{\delta}_{13}^{(3,0)}. \tag{3.17}$$

Indeed, in case $\tilde{\boldsymbol{\phi}}_5^{(0)}$ is selected such that

$$\frac{\langle \boldsymbol{\phi}_7, \tilde{\boldsymbol{\phi}}_9^{(0)} \rangle_{L_2(\boldsymbol{T})}}{\langle \boldsymbol{\phi}_7, \tilde{\boldsymbol{\phi}}_5^{(0)} \rangle_{L_2(\boldsymbol{T})}} = \frac{\langle \boldsymbol{\phi}_8, \tilde{\boldsymbol{\phi}}_9^{(0)} \rangle_{L_2(\boldsymbol{T})}}{\langle \boldsymbol{\phi}_8, \tilde{\boldsymbol{\phi}}_5^{(0)} \rangle_{L_2(\boldsymbol{T})}} =: \alpha, \tag{3.18}$$

then with

$$\tilde{\boldsymbol{\phi}}_9^{(1)} := \tilde{\boldsymbol{\phi}}_9^{(0)} - \alpha\tilde{\boldsymbol{\phi}}_5^{(0)}$$

(similarly $\tilde{\boldsymbol{\phi}}_{10}^{(1)}$, $\tilde{\boldsymbol{\phi}}_{7,8}^{(1)}$, $\tilde{\boldsymbol{\phi}}_{11,12}^{(1)}$), (3.17) gives

$$\boldsymbol{\phi}_{7,8} \perp \tilde{\boldsymbol{\phi}}_{9..12}^{(1)}, \boldsymbol{\delta}_{13}^{(3,0)}.$$

Together, (3.16) and (3.17), and the fact that $\{\boldsymbol{\phi}_{4,7,8}\}$ should be an independent set determine $\operatorname{span}\{\boldsymbol{\phi}_{4,7,8}\}$ uniquely. We fix $\boldsymbol{\phi}_7$ by selecting it from $\boldsymbol{\phi}_7^{(0)} + \operatorname{span}\{\boldsymbol{\phi}_{4,13..15}^{(0)}\}$.

Defining $\tilde{\boldsymbol{\phi}}_{7,8}^{(2)} \in \operatorname{span}\{\tilde{\boldsymbol{\phi}}_{7,8}^{(1)}\}$ (and with that $\tilde{\boldsymbol{\phi}}_{7..12}^{(2)}$) by imposing $\langle \boldsymbol{\phi}_{7,8}, \tilde{\boldsymbol{\phi}}_{7,8}^{(2)}\rangle_{L_2(\boldsymbol{T})} = \mathbf{I}$ now yields $\langle \boldsymbol{\phi}_{7..12}, \tilde{\boldsymbol{\phi}}_{7..12}^{(2)}\rangle_{L_2(\boldsymbol{T})} = \mathbf{I}$.

Remark 3.8. A consequence of above procedure is that $\tilde{\boldsymbol{\phi}}_5^{(0)} \not\perp \boldsymbol{\phi}_{7,8}$. Since orthogonality can not be restored by any transformation at the dual side that preserves $(\mathcal{V})$, we conclude that we cannot end up with biorthogonal $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$.

To ensure that (3.15) and (3.16) in which $\tilde{\boldsymbol{\phi}}_{7..12}^{(0)}$ are replaced by $\tilde{\boldsymbol{\phi}}_{7..12}^{(2)}$ remain valid, furthermore it is necessary that

$$\tilde{\boldsymbol{\phi}}_5^{(0)} \perp \boldsymbol{\phi}_{1,4}, \tag{3.19}$$

which is desirable on its own. Finally, since we also want $\boldsymbol{\phi}_4 \perp \tilde{\boldsymbol{\phi}}_7^{(2)}(, \tilde{\boldsymbol{\phi}}_8^{(2)})$, or equivalently $\boldsymbol{\phi}_5 \perp \tilde{\boldsymbol{\phi}}_9^{(2)}$, the function $\tilde{\boldsymbol{\phi}}_5^{(0)}$ should satisfy

$$\frac{\langle \boldsymbol{\phi}_5, \tilde{\boldsymbol{\phi}}_9^{(0)}\rangle_{L_2(\boldsymbol{T})}}{\langle \boldsymbol{\phi}_5, \tilde{\boldsymbol{\phi}}_5^{(0)}\rangle_{L_2(\boldsymbol{T})}} = \alpha. \tag{3.20}$$

(III). We take $\boldsymbol{\phi}_{13} = \boldsymbol{\phi}_{13}^{(0)}$.

At this point, we have fixed $\mathbf{\Phi}$. Further definitions and transformations take place at the dual side. First we specify $\tilde{\boldsymbol{\phi}}_{4..6}^{(0)}$ and $\tilde{\boldsymbol{\phi}}_{13..15}^{(0)}$.

(IV). We search $\tilde{\boldsymbol{\phi}}_{4..6}^{(0)} \in P_{3,1}(\boldsymbol{T})$. A basis for this space is given by

$$\{\tilde{\boldsymbol{\phi}}_{1..3,7..12}^{(0)}\} \cup \{\boldsymbol{\delta}_{13}^{(3,0)}\} \cup \{\boldsymbol{\delta}_{\lambda}^{(3,1)} : \lambda \in \boldsymbol{I}_6 \backslash \boldsymbol{I}_3\}.$$

To save some space in the expressions, we introduce a numbering of $\boldsymbol{I}_6\backslash\boldsymbol{I}_3$ given in Figure 9.

Because of $(\mathcal{S})$ and $(\mathcal{V})$, we may search

$$\tilde{\boldsymbol{\phi}}_5^{(0)} \in \operatorname{span}\{\boldsymbol{\delta}_2^{(3,1)}, \boldsymbol{\delta}_6^{(3,1)}+\boldsymbol{\delta}_7^{(3,1)}, \boldsymbol{\delta}_{11}^{(3,1)}+\boldsymbol{\delta}_{12}^{(3,1)}, \boldsymbol{\delta}_{15}^{(3,1)}+\boldsymbol{\delta}_{16}^{(3,1)}, \boldsymbol{\delta}_{14}^{(3,1)}+\boldsymbol{\delta}_{17}^{(3,1)}, \boldsymbol{\delta}_{13}^{(3,1)}+\boldsymbol{\delta}_{18}^{(3,1)}, \boldsymbol{\delta}_{10}^{(3,1)}\}.$$

In fact, we may also add $\boldsymbol{\delta}_{13}^{(3,0)}$ and $\tilde{\boldsymbol{\phi}}_9^{(0)} + \tilde{\boldsymbol{\phi}}_{10}^{(0)}$ to this set of generators. However, one may verify that both these functions satisfy all homogeneous linear conditions on $\tilde{\boldsymbol{\phi}}_5^{(0)}$ given below, and thus that adding these functions will not change the span of the resulting $\tilde{\mathbf{\Phi}}$. In

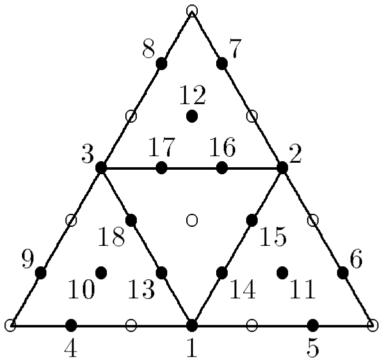

FIGURE 9. $\boldsymbol{\delta}^{(3,1)}_{1..18}$.

(II), we already imposed on $\tilde{\boldsymbol{\phi}}^{(0)}_5$ the conditions (3.18), (3.19) (two conditions) and (3.20). Here we add the conditions

$$\tilde{\boldsymbol{\phi}}^{(0)}_5 \perp \boldsymbol{\phi}_2, \tag{3.21}$$

and (3.24) to be discussed below. Together, these six conditions determine span$\{\tilde{\boldsymbol{\phi}}^{(0)}_5\}$ uniquely.

We define $\tilde{\boldsymbol{\phi}}^{(1)}_5 = \tilde{\boldsymbol{\phi}}^{(0)}_5 / \langle \tilde{\boldsymbol{\phi}}^{(0)}_5, \boldsymbol{\phi}_5 \rangle_{L_2(\boldsymbol{T})}$. Note that (3.19) and (3.21) are equivalent to $\tilde{\boldsymbol{\phi}}^{(1)}_5 \perp \boldsymbol{\phi}_{1..4,6}$ resulting in the zero and the identity matrix in the second column of (3.14).

(V). We search $\tilde{\boldsymbol{\phi}}^{(0)}_{13..15}$ satisfying

$$\boldsymbol{\delta}^{(3,0)}_{13} \in \operatorname{span}\{\tilde{\boldsymbol{\phi}}^{(0)}_{13} + \tilde{\boldsymbol{\phi}}^{(0)}_{14} + \tilde{\boldsymbol{\phi}}^{(0)}_{15}\}, \tag{3.22}$$

which is equivalent to (3.10), and

$$\boldsymbol{\phi}_{2,4,7,8} \perp \tilde{\boldsymbol{\phi}}^{(0)}_{13}. \tag{3.23}$$

By ($\mathcal{S}$), $\boldsymbol{\phi}_2 \perp \tilde{\boldsymbol{\phi}}^{(0)}_{13}$ implies $\boldsymbol{\phi}_3 \perp \tilde{\boldsymbol{\phi}}^{(0)}_{13}$, and so $\boldsymbol{\phi}_1 \perp \tilde{\boldsymbol{\phi}}^{(0)}_{14,15}$. Since furthermore $\boldsymbol{\phi}_1 \perp \boldsymbol{\delta}^{(3,0)}_{13}$, we get

$$\langle \boldsymbol{\phi}_1, \tilde{\boldsymbol{\phi}}^{(0)}_{13} \rangle_{L_2(\boldsymbol{T})} = \langle \boldsymbol{\phi}_1, \tilde{\boldsymbol{\phi}}^{(0)}_{13} + \tilde{\boldsymbol{\phi}}^{(0)}_{14} + \tilde{\boldsymbol{\phi}}^{(0)}_{15} \rangle_{L_2(\boldsymbol{T})} - \langle \boldsymbol{\phi}_1, \tilde{\boldsymbol{\phi}}^{(0)}_{14} + \tilde{\boldsymbol{\phi}}^{(0)}_{15} \rangle_{L_2(\boldsymbol{T})} = 0.$$

By applying the same argument onto $\boldsymbol{\phi}_4 \perp \tilde{\boldsymbol{\phi}}^{(0)}_{13}, \boldsymbol{\delta}^{(3,0)}_{13}$ and $\boldsymbol{\phi}_{7,8} \perp \tilde{\boldsymbol{\phi}}^{(0)}_{13}, \boldsymbol{\delta}^{(3,0)}_{13}$, we see that (3.22) and (3.23) imply that

$$\boldsymbol{\phi}_{1..12} \perp \tilde{\boldsymbol{\phi}}^{(0)}_{13,14,15},$$

giving the zeros in the last column of (3.14).

It turns out not to be possible to find $\tilde{\boldsymbol{\phi}}^{(0)}_{13} \in P_{3,1}(\boldsymbol{T})$ satisfying (3.22) and (3.23). Therefore, we enlarge this space with the span of the refinement of $\{\tilde{\boldsymbol{\phi}}^{(0)}_{4..6}\}$, which is a collection of functions defined in (3.4), with index set $\boldsymbol{I}_4 \backslash \boldsymbol{I}_2$. Since $\tilde{\boldsymbol{\phi}}^{(0)}_{13}$ should vanish on $\partial \boldsymbol{T}$, it is sufficient to consider only those functions from this collection corresponding to 'interior points' of $\boldsymbol{I}_4\backslash\boldsymbol{I}_2$. We will denote these functions by $\boldsymbol{\eta}_{1..3}$, according to the numbering given in Figure 10.

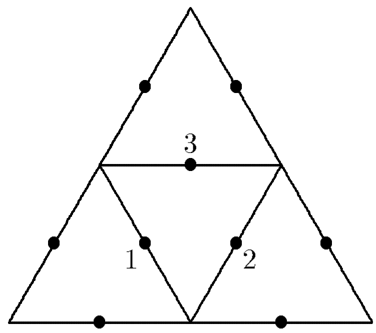

FIGURE 10. $\boldsymbol{I}_4\backslash\boldsymbol{I}_2$ and $\boldsymbol{\eta}_{1..3}$.

In view of ($\mathcal{S}$) and ($\mathcal{V}$), we may search

$$\tilde{\boldsymbol{\phi}}_{13}^{(0)} \in \operatorname{span}\{\boldsymbol{\delta}_{10}^{(3,1)}, \boldsymbol{\delta}_{11}^{(3,1)}+\boldsymbol{\delta}_{12}^{(3,1)}, \boldsymbol{\delta}_{13}^{(3,1)}+\boldsymbol{\delta}_{18}^{(3,1)}, \boldsymbol{\delta}_{14}^{(3,1)}+\boldsymbol{\delta}_{17}^{(3,1)}, \boldsymbol{\delta}_{15}^{(3,1)}+\boldsymbol{\delta}_{16}^{(3,1)},$$
$$\boldsymbol{\eta}_1, \boldsymbol{\eta}_2+\boldsymbol{\eta}_3, \boldsymbol{\delta}_{13}^{(3,0)}\}.$$

Any choice of $\tilde{\boldsymbol{\phi}}_{13}^{(0)}$ fixes $\tilde{\boldsymbol{\phi}}_{14,15}^{(0)}$ by permuting the barycentric coordinates. Since $\boldsymbol{\delta}_{13}^{(3,0)} \notin \operatorname{span}(\{\boldsymbol{\delta}_{1..18}^{(3,1)}\} \cup \{\boldsymbol{\eta}_{1..3}\})$, condition (3.22) can be rewritten as

$$\tilde{\boldsymbol{\phi}}_{13}^{(0)} \in \lambda\boldsymbol{\delta}_{13}^{(3,0)} + \operatorname{span}\boldsymbol{\Theta},$$

with a scalar $\lambda \neq 0$, and with $\boldsymbol{\Theta} = \{\boldsymbol{\theta}_{1..4}\}$ being defined by

$$\begin{aligned}
\boldsymbol{\theta}_1 &= \boldsymbol{\delta}_{11}^{(3,1)} + \boldsymbol{\delta}_{12}^{(3,1)} - 2\boldsymbol{\delta}_{10}^{(3,1)}\\
\boldsymbol{\theta}_2 &= \boldsymbol{\delta}_{13}^{(3,1)} + \boldsymbol{\delta}_{18}^{(3,1)} - \boldsymbol{\delta}_{14}^{(3,1)} - \boldsymbol{\delta}_{17}^{(3,1)}\\
\boldsymbol{\theta}_3 &= \boldsymbol{\delta}_{15}^{(3,1)} + \boldsymbol{\delta}_{16}^{(3,1)} - \boldsymbol{\delta}_{14}^{(3,1)} - \boldsymbol{\delta}_{17}^{(3,1)})\\
\boldsymbol{\theta}_4 &= \boldsymbol{\eta}_2 + \boldsymbol{\eta}_3 - 2\boldsymbol{\eta}_1.
\end{aligned}$$

Moreover, since $\boldsymbol{\delta}_{13}^{(3,0)}$ may not be a multiple of $\tilde{\boldsymbol{\phi}}_{13}^{(0)}$, since that would mean $\tilde{\boldsymbol{\phi}}_{13}^{(0)} = \tilde{\boldsymbol{\phi}}_{14}^{(0)} = \tilde{\boldsymbol{\phi}}_{15}^{(0)}$, and furthermore $\boldsymbol{\phi}_{2,4,7,8} \perp \boldsymbol{\delta}_{13}^{(3,0)}$, condition (3.23) now means that

$$\tilde{\boldsymbol{\phi}}_{13}^{(0)} = \lambda\boldsymbol{\delta}_{13}^{(3,0)} + \mathbf{c}^T\boldsymbol{\Theta},$$

where $0 \neq \mathbf{c} \in \operatorname{Ker}\langle\boldsymbol{\phi}_{2,4,7,8}, \boldsymbol{\Theta}\rangle_{L_2(\boldsymbol{T})}$. A computation shows that the first three columns of $\langle\boldsymbol{\phi}_{2,4,7,8}, \boldsymbol{\theta}_{1..3}\rangle_{L_2(\boldsymbol{T})}$ are independent, and so $\boldsymbol{\theta}_4$, and thus $\tilde{\boldsymbol{\phi}}_4^{(0)}$, should be selected such that

$$\operatorname{Ker}\langle\boldsymbol{\phi}_{2,4,7,8}, \boldsymbol{\Theta}\rangle_{L_2(\boldsymbol{T})} \neq \{0\}, \tag{3.24}$$

which condition on $\tilde{\boldsymbol{\phi}}_4^{(0)}$ was already announced in step **(II)**.

One may verify that $\operatorname{span}\{\tilde{\boldsymbol{\phi}}^{(0)}_{13..15}\}$ does not depend on the choice of $\lambda \neq 0$ and $\mathbf{c} \neq 0$ in the one-dimensional space $\operatorname{Ker}\langle \boldsymbol{\phi}_{2,4,7,8}, \boldsymbol{\Theta}\rangle_{L_2(\boldsymbol{T})}$. We define $\tilde{\boldsymbol{\phi}}^{(1)}_{13..15} \in \operatorname{span}\{\tilde{\boldsymbol{\phi}}^{(0)}_{13..15}\}$ by imposing $\langle \boldsymbol{\phi}_{13..15}, \tilde{\boldsymbol{\phi}}^{(1)}_{13..15}\rangle_{L_2(\boldsymbol{T})} = \mathbf{I}$.

By steps **(I)**-**(V)**, with $\tilde{\boldsymbol{\phi}}^{(2)}_{1..6,13..15} := \tilde{\boldsymbol{\phi}}^{(1)}_{1..6,13..15}$, the matrix $\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}^{(2)}\rangle_{L_2(\boldsymbol{T})}$ has the desired block-lower triangular form (3.14), which we more specifically denote by

$$\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}^{(2)}\rangle_{L_2(\boldsymbol{T})} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} & \mathbf{I} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} & \mathbf{F} & \mathbf{I} \end{bmatrix}.$$

As already was pointed out in Remark 3.8, it is not possible to obtain a biorthogonal system. Indeed $\langle \tilde{\boldsymbol{\Phi}}^{(2)}, \boldsymbol{\Phi}\rangle^{-1}_{L_2(\boldsymbol{T})} \tilde{\boldsymbol{\Phi}}^{(2)}$ will violate $(\mathcal{V})$, since by this transformation some $\tilde{\boldsymbol{\phi}}^{(2)}_i$ will be updated by $\tilde{\boldsymbol{\phi}}^{(2)}_j$ with j corresponding to points on edges that do not include point i. Yet, as will be shown in step **(VI)**, by performing some 'partial' transformations at the dual side, which do preserve $(\mathcal{C})$, $(\mathcal{S})$, $(\mathcal{V})$, $(\mathcal{I})$ and $(\mathcal{R})$, it is possible to introduce a number of zeros in the lower block triangular part.

(VI). With

$$\tilde{\boldsymbol{\Phi}}^{(3)} := \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & -\mathbf{D}^* \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & -\mathbf{E}^* \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{F}^* \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \tilde{\boldsymbol{\Phi}}^{(2)},$$

we have

$$\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}^{(3)}\rangle_{L_2(\boldsymbol{T})} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

In view of $(\mathcal{V})$, note that each $\tilde{\boldsymbol{\phi}}^{(3)}_i$ is obtained by adding to $\tilde{\boldsymbol{\phi}}^{(2)}_i$ a linear combination of $\tilde{\boldsymbol{\phi}}^{(2)}_{13..15}$, which functions vanish on $\partial \boldsymbol{T}$.

Let $\hat{\mathbf{A}}$ be the matrix obtained from $\mathbf{A} = (\langle \boldsymbol{\phi}_i, \tilde{\boldsymbol{\phi}}^{(3)}_j\rangle_{L_2(\boldsymbol{T})})_{i\in\{4..6\}, j\in\{1..3\}}$ by replacing those entries by zeros which correspond to pairs of points on different edges. With

$$\tilde{\boldsymbol{\Phi}}^{(4)} := \begin{bmatrix} \mathbf{I} & -\hat{\mathbf{A}}^* & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \tilde{\boldsymbol{\Phi}}^{(3)},$$

we get

$$\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}}^{(4)} \rangle_{L_2(\boldsymbol{T})} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A} - \hat{\mathbf{A}} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{G} & \mathbf{C} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix},$$

where $\mathbf{G} := \mathbf{B} - \mathbf{C}\hat{\mathbf{A}}$.

Finally, with $\hat{\mathbf{G}}$, $\hat{\mathbf{C}}$ being the matrices obtained from $\mathbf{G} = (\langle \boldsymbol{\phi}_i, \tilde{\boldsymbol{\phi}}_j^{(4)} \rangle_{L_2(\boldsymbol{T})})_{i \in \{7..12\}, j \in \{1..3\}}$, $\mathbf{C} = (\langle \boldsymbol{\phi}_i, \tilde{\boldsymbol{\phi}}_j^{(4)} \rangle_{L_2(\boldsymbol{T})})_{i \in \{7..12\}, j \in \{4..6\}}$ respectively by replacing those entries by zeros which correspond to pairs of points on different edges, and

$$\tilde{\mathbf{\Phi}} := \begin{bmatrix} \mathbf{I} & \mathbf{0} & -\hat{\mathbf{G}}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & -\hat{\mathbf{C}}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \tilde{\mathbf{\Phi}}^{(4)},$$

we get

$$\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A} - \hat{\mathbf{A}} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{G} - \hat{\mathbf{G}} & \mathbf{C} - \hat{\mathbf{C}} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

From the definitions of $\tilde{\mathbf{\Phi}}^{(3)}$, $\tilde{\mathbf{\Phi}}^{(4)}$ and $\tilde{\mathbf{\Phi}}$, it follows that the matrix $\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})}$ only contains possibly non-zero off-diagonal entries $\langle \boldsymbol{\phi}_i, \tilde{\boldsymbol{\phi}}_j \rangle_{L_2(\boldsymbol{T})}$ on the positions $(i, j) = (5, 1)$, $(9, 1)$, $(9, 4)$ and $(10, 4)$, as well as those that correspond to permuting barycentric coordinates. All these entries correspond to pairs of points that are included on different edges.

Remark 3.9. The fact that $\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})} \neq \mathbf{I}$ and thus $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)} \neq \mathbf{I}$ has clearly an adverse affect on the sizes of the supports of the dual scaling functions from $\langle \tilde{\Phi}_j, \Phi_j \rangle_{L_2(\Omega)}^{-1} \tilde{\Phi}$ and thus on that of the wavelets and dual wavelets. Yet, by computing the wavelet and inverse wavelet transforms in the way as exposed in Remark 3.4, the fact that$\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})} \neq \mathbf{I}$ only affects the computation of these transforms in the sense that on each level $j + 1$, in addition an application of the matrix $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}^{-T}$ has to be performed. Assuming a uniform square grid, a simple calculation using the fact that $\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})}$ has only a few non-zero off-diagonal entries shows that the total number of operations needed for these computations is less than half the number of degrees of freedom on the highest level.

Together, steps **(I)**-**(VI)** fully describe the procedure to find $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$. A sufficient ingredient for the actual calculations is the matrix $\langle \mathbf{\Delta}^{(1,2)}, \mathbf{\Delta}^{(3,2)} \rangle_{L_2(\boldsymbol{T})}$. These calculations result in a collection $\mathbf{\Phi}$ defined by

$$
\begin{aligned}
\boldsymbol{\phi}_1 = {} & \boldsymbol{\delta}_1^{(1,2)} + \tfrac{101}{2490}(\boldsymbol{\delta}_4^{(1,2)} + \boldsymbol{\delta}_6^{(1,2)} + \boldsymbol{\delta}_{13}^{(1,2)}) - \tfrac{173}{996}(\boldsymbol{\delta}_7^{(1,2)} + \boldsymbol{\delta}_{12}^{(1,2)}) \\
& - \tfrac{9}{1660}(\boldsymbol{\delta}_8^{(1,2)} + \boldsymbol{\delta}_{11}^{(1,2)} + \boldsymbol{\delta}_{14}^{(1,2)} + \boldsymbol{\delta}_{15}^{(1,2)}) \\
\boldsymbol{\phi}_4 = {} & \boldsymbol{\delta}_4^{(1,2)} + \tfrac{361}{658}(\boldsymbol{\delta}_7^{(1,2)} + \boldsymbol{\delta}_8^{(1,2)}) - \tfrac{1219}{3290}(\boldsymbol{\delta}_{13}^{(1,2)} + \boldsymbol{\delta}_{14}^{(1,2)}) + \tfrac{8}{35}\boldsymbol{\delta}_{15}^{(1,2)} \\
\boldsymbol{\phi}_7 = {} & \boldsymbol{\delta}_7^{(1,2)} - \tfrac{353029}{564499}\boldsymbol{\delta}_4^{(1,2)} - \tfrac{1033547}{2822495}\boldsymbol{\delta}_{13}^{(1,2)} + \tfrac{131990}{564499}\boldsymbol{\delta}_{14}^{(1,2)} + \tfrac{342166}{2822495}\boldsymbol{\delta}_{15}^{(1,2)} \\
\boldsymbol{\phi}_{13} = {} & \boldsymbol{\delta}_{13}^{(1,2)}
\end{aligned}
$$

At the dual side, $\tilde{\boldsymbol{\Phi}}^{(2)}$ is defined by

$$
\begin{aligned}
\tilde{\boldsymbol{\phi}}_1^{(2)} = {} & \tfrac{1}{\mathrm{vol}(\boldsymbol{T})}\tfrac{415}{3}\boldsymbol{\delta}_1^{(3,0)} \\
\tilde{\boldsymbol{\phi}}_4^{(2)} = {} & \tfrac{1}{\mathrm{vol}(\boldsymbol{T})}\Big[\tfrac{-9301424162156}{1912996185027}\boldsymbol{\delta}_1^{(3,1)} + \tfrac{111448863524740}{17216965665243}(\boldsymbol{\delta}_4^{(3,1)} + \boldsymbol{\delta}_5^{(3,1)}) \\
& + \tfrac{120098054733160}{5738988555081}(\boldsymbol{\delta}_{10}^{(3,1)} + \boldsymbol{\delta}_{11}^{(3,1)}) + \tfrac{791219875405708}{17216965665243}(\boldsymbol{\delta}_{13}^{(3,1)} + \boldsymbol{\delta}_{14}^{(3,1)}) \\
& + \tfrac{349505115151472}{17216965665243}(\boldsymbol{\delta}_{18}^{(3,1)} + \boldsymbol{\delta}_{15}^{(3,1)}) + \tfrac{545882055813164}{17216965665243}(\boldsymbol{\delta}_{17}^{(3,1)} + \boldsymbol{\delta}_{16}^{(3,1)}) + \tfrac{29746337340748}{17216965665243}\boldsymbol{\delta}_{12}^{(3,1)}\Big] \\
\tilde{\boldsymbol{\phi}}_7^{(2)} = {} & \tfrac{1}{\mathrm{vol}(\boldsymbol{T})}\Big[\tfrac{162144418334740 60}{183117091220847}\boldsymbol{\delta}_7^{(3,0)} + \tfrac{9269556596061196}{183117091220847}\boldsymbol{\delta}_8^{(3,0)}\Big] - \tfrac{359961477817185491}{89252626683938760}\tilde{\boldsymbol{\phi}}_4^{(2)} \\
\tilde{\boldsymbol{\phi}}_{13}^{(2)} = {} & \tfrac{1}{\mathrm{vol}(\boldsymbol{T})}\Big[\tfrac{512}{135}\boldsymbol{\delta}_{13}^{(3,0)} - \tfrac{429691798688}{26453357865}(\boldsymbol{\delta}_{11}^{(3,1)} + \boldsymbol{\delta}_{12}^{(3,1)} - 2\boldsymbol{\delta}_{10}^{(3,1)}) \\
& + \tfrac{146540371984}{5290671573}(\boldsymbol{\delta}_{13}^{(3,1)} + \boldsymbol{\delta}_{18}^{(3,1)} - \boldsymbol{\delta}_{14}^{(3,1)} - \boldsymbol{\delta}_{17}^{(3,1)}) \\
& - \tfrac{403973483368}{26453357865}(\boldsymbol{\delta}_{15}^{(3,1)} + \boldsymbol{\delta}_{16}^{(3,1)} - \boldsymbol{\delta}_{14}^{(3,1)} - \boldsymbol{\delta}_{17}^{(3,1)})\Big] + \tfrac{637665395009}{1289356257420}(\boldsymbol{\eta}_2 + \boldsymbol{\eta}_3 - 2\boldsymbol{\eta}_1),
\end{aligned}
$$

where $\boldsymbol{\eta}_{1..3}$ are the functions that correspond to 'interior points' (cf. Figure 10) from the refinement of above $\{\tilde{\boldsymbol{\phi}}_{4..6}^{(2)}\}$ defined by (3.4). The transformations described in step **(VI)** yield the collection $\tilde{\boldsymbol{\Phi}}$ given by

$$
\begin{aligned}
\tilde{\boldsymbol{\phi}}_1 = {} & \tilde{\boldsymbol{\phi}}_1^{(2)} - \tfrac{10209}{21056}(\tilde{\boldsymbol{\phi}}_4^{(2)} + \tilde{\boldsymbol{\phi}}_6^{(2)}) - \tfrac{1107721691222002944137}{7372011065695958855 68}(\tilde{\boldsymbol{\phi}}_7^{(2)} + \tilde{\boldsymbol{\phi}}_{12}^{(2)}) \\
& - \tfrac{1934386505651739484 39}{7372011065695958855 68}(\tilde{\boldsymbol{\phi}}_8^{(2)} + \tilde{\boldsymbol{\phi}}_{11}^{(2)}) - \tfrac{269103595837}{10869175296}\tilde{\boldsymbol{\phi}}_{13}^{(2)} - \tfrac{140609892845}{5434587648}(\tilde{\boldsymbol{\phi}}_{14}^{(2)} + \tilde{\boldsymbol{\phi}}_{15}^{(2)}) \\
\tilde{\boldsymbol{\phi}}_4 = {} & \tilde{\boldsymbol{\phi}}_4^{(2)} + \tfrac{2496527831240624965}{17278150935224903568}(\tilde{\boldsymbol{\phi}}_7^{(2)} + \tilde{\boldsymbol{\phi}}_8^{(2)}) - \tfrac{2034877615278695}{36035450441065728}(\tilde{\boldsymbol{\phi}}_{13}^{(2)} + \tilde{\boldsymbol{\phi}}_{14}^{(2)}) \\
& + \tfrac{16741222248937735}{36035450441065728}\tilde{\boldsymbol{\phi}}_{15}^{(2)} \\
\tilde{\boldsymbol{\phi}}_7 = {} & \tilde{\boldsymbol{\phi}}_7^{(2)} + \tfrac{4978122426946063}{651082991007456}\tilde{\boldsymbol{\phi}}_{13}^{(2)} + \tfrac{6063260745291823}{651082991007456}\tilde{\boldsymbol{\phi}}_{14}^{(2)} + \tfrac{4163663044298017}{651082991007456}\tilde{\boldsymbol{\phi}}_{15}^{(2)} \\
\tilde{\boldsymbol{\phi}}_{13} = {} & \tilde{\boldsymbol{\phi}}_{13}^{(2)}.
\end{aligned}
$$

The non-zero off-diagonal entries of $\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})}$ are given by

$$\begin{array}{lrcl@{\qquad}rcl} (3.25) & \langle \phi_5, \tilde{\phi}_1 \rangle_{L_2(\boldsymbol{T})} &=& \frac{747}{2632} & \langle \phi_9, \tilde{\phi}_1 \rangle_{L_2(\boldsymbol{T})} &=& \frac{-769556495}{8164913536} \\ & \langle \phi_9, \tilde{\phi}_4 \rangle_{L_2(\boldsymbol{T})} &=& \frac{115709629}{9185527728} & \langle \phi_{10}, \tilde{\phi}_4 \rangle_{L_2(\boldsymbol{T})} &=& \frac{1601470997}{9185527728} \end{array}$$

with, as always, equal values for those entries that correspond to permuting barycentric coordinates.

The resulting collections Φ_j, $\langle \tilde{\Phi}_j, \Phi_j \rangle_{L_2(\Omega)}^{-1} \tilde{\Phi}$ are biorthogonal, uniformly local, uniform $L_2(\Omega)$-Riesz systems. The primal collection is a basis for the space of continuous piecewise linears with respect to τ_{j+2}. The spans of the dual collections are nested as function of j, and satisfy $(\tilde{\mathfrak{B}})$ and $(\tilde{\mathfrak{J}})$ with $\tilde{\gamma} = \frac{3}{2}$ and $\tilde{d} = 4$.

3.5. **The case** $(n, d, m, \tilde{d}) = (2, 5, 0, 4)$**.** As in §3.4, $(d-1)2^m = 4$, and to construct $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$, we follow exactly the same procedure from that section described in steps **(I)**-**(VI)**, except that we now start with $\mathbf{\Phi}^{(0)} = \mathbf{\Delta}^{(4,0)}$ instead of $\mathbf{\Delta}^{(1,2)}$. The actual computations using $\langle \mathbf{\Delta}^{(4,0)}, \mathbf{\Delta}^{(3,2)} \rangle_{L_2(\boldsymbol{T})}$ now result in a collected $\mathbf{\Phi}$ defined by

$$\begin{aligned} \phi_1 &= \boldsymbol{\delta}_1^{(4,0)} - \tfrac{1}{40}(\boldsymbol{\delta}_4^{(4,0)} + \boldsymbol{\delta}_6^{(4,0)} + \boldsymbol{\delta}_{13}^{(4,0)}) - \tfrac{3}{640}(\boldsymbol{\delta}_7^{(4,0)} + \boldsymbol{\delta}_{12}^{(4,0)}) \\ &\qquad + \tfrac{13}{640}(\boldsymbol{\delta}_8^{(4,0)} + \boldsymbol{\delta}_{11}^{(4,0)} + \boldsymbol{\delta}_{14}^{(4,0)} + \boldsymbol{\delta}_{15}^{(4,0)}) \\ \phi_4 &= \boldsymbol{\delta}_4^{(4,0)} + \tfrac{3}{4}(\boldsymbol{\delta}_7^{(4,0)} + \boldsymbol{\delta}_8^{(4,0)}) - \tfrac{1}{8}(\boldsymbol{\delta}_{13}^{(4,0)} + \boldsymbol{\delta}_{14}^{(4,0)}) \\ \phi_7 &= \boldsymbol{\delta}_7^{(4,0)} - \tfrac{224}{285}\boldsymbol{\delta}_4^{(4,0)} - \tfrac{259}{1140}\boldsymbol{\delta}_{13}^{(4,0)} - \tfrac{23}{380}\boldsymbol{\delta}_{14}^{(4,0)} + \tfrac{23}{190}\boldsymbol{\delta}_{15}^{(4,0)} \\ \phi_{13} &= \boldsymbol{\delta}_{13}^{(4,0)} \end{aligned}$$

At the dual side, $\tilde{\mathbf{\Phi}}^{(2)}$ is defined by

$$\begin{aligned} \tilde{\phi}_1^{(2)} &= \tfrac{150}{\mathrm{vol}(\boldsymbol{T})}\boldsymbol{\delta}_1^{(3,0)} \\ \tilde{\phi}_4^{(2)} &= \tfrac{1}{\mathrm{vol}(\boldsymbol{T})}\Big[\tfrac{10534545}{112976}\boldsymbol{\delta}_1^{(3,1)} - \tfrac{837515}{112976}(\boldsymbol{\delta}_4^{(3,1)} + \boldsymbol{\delta}_5^{(3,1)}) \\ &\quad - \tfrac{319865}{56488}(\boldsymbol{\delta}_{10}^{(3,1)} + \boldsymbol{\delta}_{11}^{(3,1)}) - \tfrac{1398915}{112976}(\boldsymbol{\delta}_{13}^{(3,1)} + \boldsymbol{\delta}_{14}^{(3,1)}) \\ &\quad - \tfrac{1385055}{56488}(\boldsymbol{\delta}_{18}^{(3,1)} + \boldsymbol{\delta}_{15}^{(3,1)}) - \tfrac{2232895}{112976}(\boldsymbol{\delta}_{17}^{(3,1)} + \boldsymbol{\delta}_{16}^{(3,1)}) - \tfrac{93205}{112976}\boldsymbol{\delta}_{12}^{(3,1)}\Big] \\ \tilde{\phi}_7^{(2)} &= \tfrac{1}{\mathrm{vol}(\boldsymbol{T})}\Big[\tfrac{90905}{3528}\boldsymbol{\delta}_7^{(3,0)} - \tfrac{32575}{3528}\boldsymbol{\delta}_8^{(3,0)}\Big] - \tfrac{5833}{12348}\tilde{\phi}_4^{(2)} \\ \tilde{\phi}_{13}^{(2)} &= \tfrac{1}{\mathrm{vol}(\boldsymbol{T})}\Big[\tfrac{35}{12}\boldsymbol{\delta}_{13}^{(3,0)} - \tfrac{67744}{12339}(\boldsymbol{\delta}_{11}^{(3,1)} + \boldsymbol{\delta}_{12}^{(3,1)} - 2\boldsymbol{\delta}_{10}^{(3,1)}) \\ &\quad - \tfrac{51068}{12339}(\boldsymbol{\delta}_{13}^{(3,1)} + \boldsymbol{\delta}_{18}^{(3,1)} - \boldsymbol{\delta}_{14}^{(3,1)} - \boldsymbol{\delta}_{17}^{(3,1)}) \\ &\quad + \tfrac{11380}{12339}(\boldsymbol{\delta}_{15}^{(3,1)} + \boldsymbol{\delta}_{16}^{(3,1)} - \boldsymbol{\delta}_{14}^{(3,1)} - \boldsymbol{\delta}_{17}^{(3,1)})\Big] - \tfrac{112976}{431865}(\boldsymbol{\eta}_2 + \boldsymbol{\eta}_3 - 2\boldsymbol{\eta}_1), \end{aligned}$$

where $\boldsymbol{\eta}_{1..3}$ are the functions that correspond to 'interior points' (cf. Figure 10) from the refinement of above $\{\tilde{\phi}^{(2)}_{4..6}\}$ defined by (3.4). Finally, the collection $\tilde{\boldsymbol{\Phi}}$ is given by

$$
\begin{aligned}
\tilde{\phi}_1 &= \tilde{\phi}^{(2)}_1 - \tfrac{10}{21}(\tilde{\phi}^{(2)}_4 + \tilde{\phi}^{(2)}_6) - \tfrac{162721}{40831}(\tilde{\phi}^{(2)}_7 + \tilde{\phi}^{(2)}_{12}) - \tfrac{14913}{5833}(\tilde{\phi}^{(2)}_8 + \tilde{\phi}^{(2)}_{11}) \\
&\qquad + \tfrac{128480}{7203}\tilde{\phi}^{(2)}_{13} + \tfrac{21800}{7203}(\tilde{\phi}^{(2)}_{14} + \tilde{\phi}^{(2)}_{15}) \\
\tilde{\phi}_4 &= \tilde{\phi}^{(2)}_4 + \tfrac{119012}{87495}(\tilde{\phi}^{(2)}_7 + \tilde{\phi}^{(2)}_8) - \tfrac{4807}{3087}(\tilde{\phi}^{(2)}_{13} + \tilde{\phi}^{(2)}_{14}) + \tfrac{57628}{71001}\tilde{\phi}^{(2)}_{15} \\
\tilde{\phi}_7 &= \tilde{\phi}^{(2)}_7 - \tfrac{3008}{1029}\tilde{\phi}^{(2)}_{13} + \tfrac{1108}{1029}\tilde{\phi}^{(2)}_{14} - \tfrac{29545}{94668}\tilde{\phi}^{(2)}_{15} \\
\tilde{\phi}_{13} &= \tilde{\phi}^{(2)}_{13}.
\end{aligned}
$$

As in §3.4, $\boldsymbol{\Phi}$, $\tilde{\boldsymbol{\Phi}}$ are not biorthogonal. The non-zero off-diagonal entries of $\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}} \rangle_{L_2(\boldsymbol{T})}$ are given by

$$
\begin{aligned}
\langle \phi_5, \tilde{\phi}_1 \rangle_{L_2(\boldsymbol{T})} &= \tfrac{10}{21} & \qquad \langle \phi_9, \tilde{\phi}_1 \rangle_{L_2(\boldsymbol{T})} &= -\tfrac{64}{171} \\
\langle \phi_9, \tilde{\phi}_4 \rangle_{L_2(\boldsymbol{T})} &= \tfrac{181}{570} & \qquad \langle \phi_{10}, \tilde{\phi}_4 \rangle_{L_2(\boldsymbol{T})} &= \tfrac{371}{570},
\end{aligned}
$$

with, as always, equal values for those entries that correspond to permuting barycentric coordinates.

The resulting collections Φ_j, $\langle \tilde{\Phi}_j, \Phi_j \rangle^{-1}_{L_2(\Omega)} \tilde{\Phi}$ are biorthogonal, uniformly local, uniform $L_2(\Omega)$-Riesz systems. The primal collection is a basis for the space of continuous piecewise quartics with respect to τ_j. The spans of the dual collections are nested as function of j, and satisfy $(\tilde{\mathfrak{B}})$ and $(\tilde{\mathfrak{J}})$ with $\tilde{\gamma} = \frac{3}{2}$ and $\tilde{d} = 4$.

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Department of Mathematics, Utrecht University, P.O. Box 80.010, NL-3508 TA Utrecht, The Netherlands.

E-mail address: `stevenso@math.uu.nl`