STABILITY IN HAMILTONIAN SYSTEMS: Applications to the Restricted Three-Body Problem

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1 Introduction

As participants of the MASIE-project, we attended the summer school *Mechanics and Symmetry* in Peyresc, France, during the first two weeks of September 2000. This article was inspired by the notes we took there from Prof. Meyer's lectures on the *N-Body Problem*.

The N-body problem is a classical famous problem which has attracted a lot of attention. It consists of describing the complete behavior of all solutions of the equations of motions for a given initial condition. Still related to this kind of problem Euler in 1772 describe the three-body problem in his effort to study the motion of the moon. Later on Jacobi in 1836 brought forward the main mathematical interest in an even more specific part of the three body problem, namely the one which is reduced to a conservative two degrees of freedom problem. This has somehow brought up an extensive study on mechanics. Despite of all this effort of great mathematicians, in general the N-body problem for N>2 is still unsolved.

In 1858, Dirichlet told Kronecker that he has found a general method to treat the problem in mechanics. In particular, he claimed that he has proved the stability of the planetary system. This statement is still behind a big question mark due to the fact that he passed away without leaving any notes to prove his statement. Nevertheless, it initiate Weierstrass and his students S. Kovalevski and Mittag-Leffler to re-discover the method mentioned by Dirichlet. Mittag-Leffler even managed to convince the King of Sweden and Norway to establish a prize for the solution to the problem of finding a series expansion for coordinates of the N-body problem which is valid for all time, as indicated by Dirichlet statement. In 1889, the prize was awarded to a great mathematicians Poincaré, although he did not solve the problem. His essay, however, produced a lot of original ideas which have proven them selves to be very important in the development of the theory of mechanics. Moreover, some of them even stimulated other branches of mathematics (for instance topology) to be born and later on gain extensive study.[1]

In the first section we shall introduce the so-called *restricted three-body problem*. It describes the motion of a test particle in the combined gravitational field of two planets and it could serve for instance as a model for the motion of a satellite in the Earth-Moon system or a comet in the Sun-Jupiter system. The restricted three-body problem has a number of relative equilibria, which we compute. The remaining text will mainly be concerned with general Hamiltonian equilibria. Stability criteria for these equilibria will be derived, as well as detection methods for bifurcations of periodic solutions. Classical as well as more advanced mathematical techniques are used, such as spectral analysis, Liapunov functions, Birkhoff-Gustavson normal forms, Poincaré sections, and Kolmogorov twist stability. All help to study the motion of the test particle near the relative equilibria of the restricted problem.

[1] Summarized from [10], [11] and [8]

2 The restricted three-body problem

Before introducing the restricted three-body problem, let us study the two-body problem, the motion of two planets interacting via gravitation. Denote by $\boldsymbol{X_1}, \boldsymbol{X_2} \in \mathbb{R}^3$ the positions of the planets 1 and 2 respectively. Let us assume that planet 1 has mass $0 < \mu < 1$, planet 2 has mass $1 - \mu$ and the gravitational constant is equal to 1. These assumptions are not very restrictive, because they can always be arranged by a rescaling of time. The equations of motion for the two-body problem then read:

$$\frac{d^2\boldsymbol{X_1}}{dt^2} = -\frac{(1-\mu)}{||\boldsymbol{X_1} - \boldsymbol{X_2}||^3}(\boldsymbol{X_1} - \boldsymbol{X_2}) \; , \; \frac{d^2\boldsymbol{X_2}}{dt^2} = -\frac{\mu}{||\boldsymbol{X_1} - \boldsymbol{X_2}||^3}(\boldsymbol{X_2} - \boldsymbol{X_1}) \; . \tag{2.1}$$

Let us denote the center of mass

$$\boldsymbol{Z} := \mu\boldsymbol{X_1} + (1-\mu)\boldsymbol{X_2} \; . \tag{2.2}$$

Then we derive from (2.1) and (2.2) that $\frac{d^2\boldsymbol{Z}}{dt^2} = 0$, expressing that the center of mass moves with constant speed. Now we transform to comoving coordinates

$$\boldsymbol{Y_i} = \boldsymbol{X_i} - \boldsymbol{Z} \text{ for } i = 1, 2, \tag{2.3}$$

and we write down the equations of motions in these new variables:

$$\frac{d^2\mathbf{Y_1}}{dt^2} = -\frac{(1-\mu)^3}{||\mathbf{Y_1}||^3}\mathbf{Y_1} \; , \; \frac{d^2\mathbf{Y_2}}{dt^2} = -\frac{\mu^3}{||\mathbf{Y_2}||^3}\mathbf{Y_2} \; . \tag{2.4}$$

Let us analyse these equations a bit more. First of all, we see from the definitions (2.2) and (2.3) that $\mu\boldsymbol{Y_1} + (1-\mu)\boldsymbol{Y_2} = 0$, so $\boldsymbol{Y_1}$ and $\boldsymbol{Y_2}$ lie on a line through the origin of $\mathbb{R}^3$, both at another side of the origin, and their length ratio $\frac{||\mathbf{Y_1}||}{||\mathbf{Y_2}||}$ is fixed to the value $\frac{1-\mu}{\mu}$. The line connecting $\boldsymbol{Y_1}, \boldsymbol{Y_2}$ and the origin is called the *line of syzygy*. Because $\boldsymbol{Y}_2 = -\frac{\mu}{1-\mu}\boldsymbol{Y}_1$, we in fact only need to study the first equation of (2.4). The motion of the second planet then follows automatically.

Secondly, by differentiation one finds that the angular momentum $\boldsymbol{Y_1} \times \frac{d\mathbf{Y_1}}{dt}$ is independent of time. Indeed, $\frac{d}{dt}(\boldsymbol{Y_1} \times \frac{d\mathbf{Y_1}}{dt}) = \frac{d\mathbf{Y_1}}{dt} \times \frac{d\mathbf{Y_1}}{dt} + \boldsymbol{Y_1} \times \frac{d^2\mathbf{Y_1}}{dt^2} = 0$, because both terms are the cross-products of collinear vectors.

In the case that $\boldsymbol{Y_1} \times \frac{d\mathbf{Y_1}}{dt} = 0$, and assuming that $\boldsymbol{Y_1}(0) \neq 0$, we have that $\frac{d\mathbf{Y_1}}{dt}$ has the same direction as $\boldsymbol{Y_1}$, so the motion takes place in a one-dimensional subspace: $\boldsymbol{Y_1}, \frac{d\mathbf{Y_1}}{dt}, \boldsymbol{Y_2}, \frac{d\mathbf{Y_2}}{dt} \in \boldsymbol{Y_1}(0)\mathbb{R} = \boldsymbol{Y_2}(0)\mathbb{R}$. It is not difficult to derive the following scalar second order differential equation for the motion in this subspace: $\frac{d^2}{dt^2}||\boldsymbol{Y_1}|| = -(1-\mu)^3/||\boldsymbol{Y_1}||^2$. It turns out that in this case $\boldsymbol{Y_1}$ and $\boldsymbol{Y_2}$ fall into the origin in a finite time.

In the case that $\boldsymbol{Y_1} \times \frac{d\mathbf{Y_1}}{dt} \neq 0$, the motion takes place in the plane perpendicular to $\boldsymbol{Y_1} \times \frac{d\mathbf{Y_1}}{dt}$, because both $\boldsymbol{Y_1}$ and $\frac{d\mathbf{Y_1}}{dt}$ are perpendicular to the constant vector $\boldsymbol{Y_1} \times \frac{d\mathbf{Y_1}}{dt}$. By rotating our coordinate frame, we can arrange that $\boldsymbol{Y_1} \times \frac{d\mathbf{Y_1}}{dt}$ is some multiple of the third basis vector. Thus we can consider the equations (2.4) as two second order planar equations. It is well-known that the planar solutions of $\frac{d^2\mathbf{Y_1}}{dt^2} = -\frac{(1-\mu)^3}{||\mathbf{Y_1}||^3}\boldsymbol{Y_1}$ with $\boldsymbol{Y_1} \times \frac{d\mathbf{Y_1}}{dt} \neq 0$

describe one of the conic sections: a circle, an ellipse, a parabola or a hyperbola. $\boldsymbol{Y_2}$ clearly describes a similar conic section.

Let us now assume that a certain solution of the two-body problem is given to us. We want to study the motion of a *test particle* in the gravitational field of the two main bodies, which we call *primaries*. The test particle is assumed to have zero mass. Therefore it does not affect the primaries, but it does feel the gravitational force of the primaries acting on it. The resulting problem is called the restricted three-body problem. It could serve as a model for a satellite in the Earth-Moon system or a comet in the Sun-Jupiter system. Let $\boldsymbol{X} \in \mathbb{R}^3$ denote the position of the test particle. Then the restricted three-body problem is given by

$$\frac{d^2\boldsymbol{X}}{dt^2} = -\frac{\mu}{||\boldsymbol{X}-\boldsymbol{X_1}||^3}(\boldsymbol{X}-\boldsymbol{X_1}) - \frac{(1-\mu)}{||\boldsymbol{X}-\boldsymbol{X_2}||^3}(\boldsymbol{X}-\boldsymbol{X_2}) \ , \tag{2.5}$$

in which $(\boldsymbol{X_1}, \boldsymbol{X_2})$ is the given solution of the two-body problem. One can again transform to comoving coordinates, setting $\boldsymbol{Y} = \boldsymbol{X} - \boldsymbol{Z}$, which results in the system

$$\frac{d^2\boldsymbol{Y}}{dt^2} = -\frac{\mu}{||\boldsymbol{Y}-\mathbf{Y_1}||^3}(\boldsymbol{Y}-\mathbf{Y_1}) - \frac{(1-\mu)}{||\boldsymbol{Y}-\mathbf{Y_2}||^3}(\boldsymbol{Y}-\mathbf{Y_2}) \ . \tag{2.6}$$

At this point we start making assumptions. Let us assume that the primaries move in a circular orbit around their center of mass with constant angular velocity . This is approximately true for the Earth-Moon system and the Sun-Jupiter system. We set the angular velocity equal to 1. Without loss of generality, we can assume that the motion takes place in the plane perpendicular to the third basis-vector. Thus, after translating time if necessary,

$$\mathbf{Y_1} = R(t)\begin{pmatrix} 1-\mu \\ 0 \\ 0 \end{pmatrix} \ , \quad \mathbf{Y_2} = R(t)\begin{pmatrix} -\mu \\ 0 \\ 0 \end{pmatrix} \ , \tag{2.7}$$

in which $R(t)$ is the rotation matrix:

$$R(t) := \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{2.8}$$

Note that we have introduced a rotating coordinate frame in which the motion of the primaries has become stationary. At this point we put in our test particle and again we make an assumption, namely that it moves in the same plane as the primaries do. So we set

$$\boldsymbol{Y} = R(t)\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \ . \tag{2.9}$$

Let $(\boldsymbol{x},0)^T = (x_1,x_2,0)^T$ be the coordinates of the test particle in the rotating coordinate frame. By inserting (2.7), (2.8) and (2.9) into (2.6), multiplying the resulting equation from the left by $R(t)^{-1}$ and using two following identities

$$\frac{d^2}{dt^2}\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix} = -\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix},$$

$$\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}^{-1}\frac{d}{dt}\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

we deduce the planar equations of motion for $\boldsymbol{x}$:

$$\frac{d^2\boldsymbol{x}}{dt^2} - \boldsymbol{x} + \begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix}\frac{d\boldsymbol{x}}{dt} = -\frac{\mu}{||\boldsymbol{x}-\binom{1-\mu}{0}||^3}(\boldsymbol{x}-\tbinom{1-\mu}{0}) - \frac{1-\mu}{||\boldsymbol{x}-\binom{-\mu}{0}||^3}(\boldsymbol{x}-\tbinom{-\mu}{0}),\ \boldsymbol{x}\in\mathbb{R}^2.$$

Finally, setting $\boldsymbol{y} = \frac{d\boldsymbol{x}}{dt} + \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)\boldsymbol{x}$, we find that these are Hamiltonian equations of motion on $\mathbb{R}^4\backslash\{\boldsymbol{x} = \binom{1-\mu}{0},\ \binom{-\mu}{0}\}$ with Hamiltonian

$$H = \frac{1}{2}(y_1^2+y_2^2) - (x_1y_2 - x_2y_1) - \frac{\mu}{||\boldsymbol{x}-\binom{1-\mu}{0}||} - \frac{1-\mu}{||\boldsymbol{x}-\binom{-\mu}{0}||}, \tag{2.10}$$

where we have equipped $\mathbb{R}^4$ with the canonical symplectic form $dx_1\wedge dy_1 + dx_2\wedge dy_2$, i.e. the equations of motion are given by $\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}$, $\frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}$.

3 Relative equilibria

Let us look for equilibrium solutions of the Hamiltonian vector field induced by (2.10). These correspond to stationary motion of the test particle relative to the rotating coordinate frame and are therefore called *relative equilibria*. In the original coordinates they correspond to the test particle rotating around the center of mass of the primaries with angular velocity 1.

First of all, to facilitate notation, we introduce the potential energy function

$$V(\boldsymbol{x}) := -\frac{\mu}{||\boldsymbol{x}-\binom{1-\mu}{0}||} - \frac{1-\mu}{||\boldsymbol{x}-\binom{-\mu}{0}||}$$

To find the equilibrium solutions of (2.10) we set all the partial derivatives of H equal to zero and find

$$y_1 + x_2 = 0\ ,\ y_2 - x_1 = 0\ ,\ -y_2 + \frac{\partial V}{\partial x_1}(\boldsymbol{x}) = 0\ ,\ y_1 + \frac{\partial V}{\partial x_2}(\boldsymbol{x}) = 0\ ,$$

or equivalently,

$$\frac{\partial V}{\partial x_1}(\boldsymbol{x}) = x_1\ ,\ \frac{\partial V}{\partial x_2}(\boldsymbol{x}) = x_2\ , \tag{3.1}$$

where $\boldsymbol{y}$ at the equilibrium point can easily be found once we solved (3.1) for $\boldsymbol{x}$ at the equilibrium point. Note that $\boldsymbol{x}$ solves (3.1) if and only if $\boldsymbol{x}$ is a stationary point of the function

$$U(\boldsymbol{x}) := \frac{1}{2}(x_1^2 + x_2^2) - V(\boldsymbol{x}) \ ,$$

called the *amended potential.*

Let us first look for equilibrium points of the amended potential that lie on the line of syzygy, i.e. for which $x_2 = 0$. Note that $\frac{\partial U}{\partial x_2}(\boldsymbol{x}) = 0$ is automatically satisfied in this case since $\frac{\partial V}{\partial x_2}\,|_{x_2=0} \equiv 0$. $\frac{\partial U}{\partial x_1}(\boldsymbol{x}) = 0$ reduces to

$$\frac{d}{dx_1}U(x_1,0) = \frac{d}{dx_1}\left(\frac{1}{2}x_1^2 + \frac{\mu}{|x_1+\mu-1|} + \frac{1-\mu}{|x_1+\mu|}\right) = 0 \ . \tag{3.2}$$

Clearly, $U(x_1,0)$ goes to infinity if x_1 approaches $-\infty, -\mu, 1-\mu$ or ∞, so $U(x_1,0)$ has at least one critical point on each of the intervals $(-\infty,-\mu)$, $(-\mu, 1-\mu)$ and $(1-\mu,\infty)$. But we also calculate that $\frac{d^2}{dx_1^2}U(x_1,0) = 1 + 2\frac{\mu}{|x_1+\mu-1|^3} + 2\frac{1-\mu}{|x_1+\mu|^3} > 0$. So $U(x_1,0)$ is convex on each of these intervals and we conclude that there is exactly one critical point in each of the intervals. The three relative equilibria on the line of syzygy are called the *Eulerian* equilibria. They are denoted by $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$, where $\mathcal{L}_1 \in (-\infty,-\mu)\times\{0\}$, $\mathcal{L}_2 \in (-\mu,1-\mu)\times\{0\}$ and $\mathcal{L}_3 \in (1-\mu,\infty)\times\{0\}$.

Now we shall look for equilibrium points that do not lie on the line of syzygy. Let us use $d_1 = ||\boldsymbol{x} - \binom{1-\mu}{0}|| = \sqrt{(x_1+\mu-1)^2 + x_2^2}$ and $d_2 = ||\boldsymbol{x} - \binom{-\mu}{0}|| = \sqrt{(x_1+\mu)^2 + x_2^2}$ as coordinates in each of the half-planes $\{x_2 > 0\}$ and $\{x_2 < 0\}$. Then U can be written as $U = \frac{\mu}{2}d_1^2 + \frac{1-\mu}{2}d_2^2 - \frac{\mu(1-\mu)}{2} + \frac{\mu}{d_1} + \frac{1-\mu}{d_2}$. So the critical points of U are given by $d_i = d_i^{-2}$ i.e. $d_1 = d_2 = 1$. This gives us the two *Lagrangean* equilibria which lie at the third vertex of the equilateral triangle with the primaries at its base-points: $\mathcal{L}_4 = (\frac{1}{2}-\mu, \frac{1}{2}\sqrt{3})^T$ and $\mathcal{L}_5 = (\frac{1}{2}-\mu, -\frac{1}{2}\sqrt{3})^T$.

This paper discusses some useful tools for the study of the flow of Hamiltonian vector fields near equilibrium points. We will for instance establish stability criteria for Hamiltonian equilibria and study bifurcations of periodic solutions near Hamiltonian equilibria. The Eulerian and Lagrangean equilibria of the restricted three-body problem will serve as an instructive and inspiring example.

4 Linear Hamiltonian Systems

One of the techniques to prove stability for an equilibrium of a system of differential equations, is to analyze the linearized system around that equilibrium. Stability or instability then follows from the eigenvalues of the matrix of the linearized system. In Hamiltonian systems, these eigenvalues have a special structure which implies that the linear theory can only be used to prove instability, not stability. We will start by giving a brief introduction to linear Hamiltonian systems. We then conclude this section with a lemma which shows why one can not conclude stability from the linear analysis.

Consider a symplectic vector space $\mathbb{R}^{2n}$ with coordinates $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y})^T$ and the symplectic form is $d\boldsymbol{x} \wedge d\boldsymbol{y} := \sum_{j=1}^{n} dx_j \wedge dy_j$. Then every continuously differentiable function $H : \mathbb{R}^{2n} \to \mathbb{R}$ induces the Hamiltonian vector field X_H on $\mathbb{R}^{2n}$ defined by $X_H(z) = J(\nabla H(z))^T$, in which the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

is called the *standard symplectic matrix.* Note that X_H gives rise to the Hamiltonian system of differential equations $\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}$, $\frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}$. The function H is called Hamiltonian function

Suppose that for $\boldsymbol{z}_\circ \in \mathbb{R}^{2n}$ we have $\nabla H(\boldsymbol{z}_\circ) = 0$, then $\boldsymbol{z}_\circ$ is called a *rest point, equilibrium point, fixed point,* or *critical point* of H. Note that $X_H(\boldsymbol{z}_\circ) = 0$ so $\boldsymbol{z}_\circ$ is fixed by the flow of X_H. By translating our coordinate frame, we can arrange that $\boldsymbol{z}_\circ = 0$. We will assume that H is a sufficiently smooth function in a neighborhood of its equilibrium 0, so that we can write $H(\boldsymbol{z}) = H_2(\boldsymbol{z}) + \mathcal{O}(||\boldsymbol{z}||^3)$ as $\boldsymbol{z} \to 0$, where H_2 is a quadratic form on $\mathbb{R}^{2n}$. The linearized vector field of X_H at 0 is the Hamiltonian vector field X_{H_2} induced by the quadratic Hamiltonian H_2. This encourages us to study quadratic Hamiltonians and their induced linear Hamiltonian vector fields.

Let $H_2 : \mathbb{R}^{2n} \to \mathbb{R}$ be a quadratic form, determined by the symmetric $2n \times 2n$ matrix Q, i.e. $H_2(\boldsymbol{z}) = \frac{1}{2}\boldsymbol{z}^T Q \boldsymbol{z}$ with $Q^T = Q$. H_2 generates a linear Hamiltonian vector field:

$$X_{H_2}(\boldsymbol{z}) = J(\nabla H_2(\boldsymbol{z}))^T = JQ\boldsymbol{z}, \tag{4.1}$$

Matrices S of the form $S = JQ$ for some symmetric matrix Q are called *infinitesimally symplectic* or *Hamiltonian.* The set of all infinitesimally symplectic matrices is denoted by

$$\mathrm{sp}(n) := \{S \in \mathbb{R}^{2n\times 2n} \mid S = JQ \text{ for some } Q = Q^T\} = \{S \in \mathbb{R}^{2n\times 2n} \mid S^T J + JS = 0\} .$$

Note that the standard symplectic matrix J satisfies $J^{-1} = J^T = -J$. Now take any infinitesimally symplectic matrix S of the form $S = JQ$, with Q symmetric. Then the simple calculation

$$J^{-1}(-S^T)J = J^{-1}(-JQ)^T J = -J^{-1}(QJ^T)J = -J^{-1}Q = JQ = S,$$

shows that S and $-S^T$ are similar. But similar matrices have equal eigenvalues. And because S has real coefficients, this observation leads to the following lemma:

Lemma 4.1 *If $S \in \mathrm{sp}(n)$ and λ is an eigenvalue of S, then also $-\lambda, \overline{\lambda}$ and $-\overline{\lambda}$ are eigenvalues of S.*

Now let us consider the exponential of a infinitesimally symplectic matrix, $\exp(S) = \exp(JQ)$, which is the fundamental matrix for the flow of the linear Hamiltonian vector field $\boldsymbol{z} \mapsto S\boldsymbol{z} = JQ\boldsymbol{z}$. It is a nice exercise to show that it satisfies $(\exp(S))^T J \exp(S) = J$.

In general, a matrix $P \in \mathbb{R}^{2n\times 2n}$ satisfying $P^T J P = J$ is called *symplectic*. The set of symplectic matrices is denoted

$$\mathrm{Sp}(n) := \{P \in \mathbb{R}^{2n\times 2n} \mid P^T J P = J\}.$$

For a symplectic matrix P one easily derives that $J^{-1}P^{-T}J = P$, so P^{-T} and P are similar. This leads to:

Lemma 4.2 *If $P \in \mathrm{Sp}(n)$ and λ is an eigenvalue of P, then also $\frac{1}{\lambda}, \overline{\lambda}$ and $\overline{\frac{1}{\lambda}}$ are eigenvalues of P.*

We remark here that $\mathrm{Sp}(n)$ is a Lie-group with matrix multiplication. Its Lie-algebra is exactly $\mathrm{sp}(n)$.

Remember that we studied linear Hamiltonian systems to determine stability or instability of an equilibrium from the spectrum of its linearized vector field. From lemma 4.1 we see if one eigenvalue has a nonzero real part, then there must be an eigenvalue with positive real part. In this case the equilibrium is unstable. The other possibility is that all eigenvalues are purely imaginary. In this case, adding nonlinear terms could destabilize the equilibrium. So lemma 4.1 states that for Hamiltonian systems, the linear theory can only be useful to prove instability of an equilibrium.

Lemma 4.2 states a similar thing for symplectic maps: if $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectic diffeomorphism with a fixed point, then the linearisation of Ψ at that fixed point can only be used to prove instability of the fixed point, not stability.

The reader should be convinced now that we need more sophisticated mathematical techniques if we want to have stability results. Some of them will be explained in the following section.

5 Liapunov's and Chetaev's Theorem

We will now describe a direct method to determine stability of an equilibrium. We will give references for the proofs and explain the interpretation of the theory instead. In section 6 we shall apply the obtained results to the relative equilibria of the restricted three-body problem.

Consider a general system of differential equations,

$$\dot{\boldsymbol{v}} = f(\boldsymbol{v}) \ , \tag{5.1}$$

where f is a C^r vector field on $\mathbb{R}^m$ and $f(0) = 0$. Let $V : \mathbb{R}^m \to \mathbb{R}$ be a positive definite C^1 function on a neighborhood U of the origin, i.e. $V(0) = 0$ and $V(\boldsymbol{z}) > 0, \ \forall \boldsymbol{z} \in U\backslash\{0\}$. If $\boldsymbol{u}$ is a solution of (5.1), then the derivative of V along $\boldsymbol{u}$ is $\frac{d}{dt}V(\boldsymbol{u}(t)) = \nabla V(\boldsymbol{u}(t)) \cdot \dot{\boldsymbol{u}}(t) = \nabla V(\boldsymbol{u}(t)) \cdot f(\boldsymbol{u}(t))$. So let us define the *orbital derivative* $\dot{V} : U \to \mathbb{R}$ of V as

$$\dot{V}(\boldsymbol{v}) := \nabla V(\boldsymbol{v}) \cdot f(\boldsymbol{v}) \ .$$

Theorem 5.1 Liapunov's Theorem *Given such a function V for the system of equations (5.1), we have:*

1. *If $\dot{V}(\boldsymbol{v}) \leq 0, \ \ \forall \boldsymbol{v} \in U \backslash \{0\}$ then the origin is stable.*
2. *If $\dot{V}(\boldsymbol{v}) < 0, \ \ \forall \boldsymbol{v} \in U \backslash \{0\}$ then the origin is asymptotically stable.*
3. *If $\dot{V}(\boldsymbol{v}) > 0, \ \ \forall \boldsymbol{v} \in U \backslash \{0\}$ then the origin is unstable.*

The function V is called a *Liapunov function.*

Let us see what this means for $m = 2$. Since V is a positive definite function, 0 is a local minimum of V. This implies that there exists a small neighborhood U' of 0 such that the level sets of V lying in U' are closed curves. Recall that $\nabla V(\boldsymbol{u}_c)$ is a normal vector to the level set C of V at $\boldsymbol{u}_c$ pointing outward. If an orbit $\boldsymbol{u}(t)$ crosses this level curve C at $\boldsymbol{u}_c$, then the velocity vector of the orbit and the gradient $\nabla V(\boldsymbol{u}_c)$ will form an angle θ for which

$$\cos(\theta) = \frac{\dot{V}(\boldsymbol{u}_c)}{||\nabla V(\boldsymbol{u}_c)||||f(\boldsymbol{u}_c)||} \ .$$

$\dot{V}(\boldsymbol{u}) < 0$ implies that $\pi/2 < \theta < 3\pi/2$. It follows that the orbit is moving inwards the level curve C in this case. If $\dot{V}(\boldsymbol{u}) = 0$, the orbit follows C. If $\dot{V}(\boldsymbol{u}) > 0$ we see the orbit moving outwards of C, that is away from the origin. See [7] for proof of Liapunov's theorem.

An immediate implication of Liapunov's theorem is the following. Consider a Hamiltonian system

$$\dot{\boldsymbol{z}} = J(\nabla H(\boldsymbol{z}))^T \ . \tag{5.2}$$

A good candidate for the Liapunov function in this Hamiltonian system would be the Hamiltonian function itself, because the orbits of a Hamiltonian system lie in the level set of the Hamiltonian. So $\dot{V} = \dot{H} = 0$. Thus, if H is positive definite then Liapunov's theorem applies. And if H is negative definite, one can choose $-H$ as a Liapunov function. We have:

Theorem 5.2 Dirichlet's Theorem *The origin is a stable equilibrium of (5.2), if it is an isolated local maximum or local minimum of the Hamiltonian H.*

Example 5.3 Consider a two-degrees of freedom Hamiltonian H which can be expanded as $H = \frac{1}{2}\omega_1({x_1}^2 + {y_1}^2) + \frac{1}{2}\omega_2({x_2}^2 + {y_2}^2) + \mathcal{O}(||\boldsymbol{z}||^3)$, for certain nonzero reals ω_1 and ω_2. The eigenvalues of the linearized vector field are $\pm i\omega_1, \pm i\omega_2$, so we can not conclude stability or instability from the eigenvalues. Nevertheless, if ω_1 and ω_2 are of the same sign, then H is definite near 0. So then Dirichlet's theorem implies that 0 is a stable equilibrium.

Where do we meet this situation in the restricted three-body problem? One can calculate that for the parameter values $0 < \mu < \mu_1 := (1 - \sqrt{23/27})/2$ and $1 - \mu_1 < \mu < 1$, the

Taylor expansion of the Hamiltonian (2.10) around $\mathcal{L}_4$ and $\mathcal{L}_5$ can be written in the above form. Briefly, this goes as follows: expand H around its equilibrium point up to second order. This gives the quadratic part H_2 of H near the equilibrium. Then after an appropriate linear symplectic transformation H_2 can be written in the above form. However, Dirichlet's theorem turns out not to be applicable, because ω_1 and ω_2 have different signs, whatever the value of μ. More sophisticated tools are needed here.

The condition for instability in Liapunov's theorem is very strong since it requires the orbital derivative to be positive everywhere in U. The following theorem is a way to conclude instability under somewhat weaker conditions.

Theorem 5.4 Chetaev's theorem *Let U be a small neighborhood of the origin where the C^1 Chetaev function $V : U \to \mathbb{R}$ is defined. Let Ω be an open subset of U such that*

1. $0 \in \partial\Omega$,
2. $V(\boldsymbol{v}) = 0, \forall\, \boldsymbol{v} \in \partial\Omega \cap U$,
3. $V(\boldsymbol{v}) > 0$ *and* $\dot{V}(\boldsymbol{v}) > 0, \forall\, \boldsymbol{v} \in \Omega \cap U$.

Then the origin is an unstable equilibrium of (5.2).

The interpretation of this theorem is the following. An orbit $\boldsymbol{u}(t;\boldsymbol{u}_\circ)$ starting in $\Omega \cap U$, will never cross $\partial\Omega$ due to the properties (2) and (3) of the Chetaev function. From the second part of property (3) it now follows that $V(\boldsymbol{u}(t;\boldsymbol{u}_\circ))$ is increasing whenever $\boldsymbol{u}(t;\boldsymbol{u}_\circ)$ in $\Omega \cap U$. This orbit can not stay in $\partial\Omega \cap U$ due to the fact that U is open. Thus, $\boldsymbol{u}(t)$ moves away from the origin. Hence the origin is unstable.

6 Applications to the restricted problem

In this section we apply the theory of the previous sections to the relative equilibria of the restricted three-body problem.

Consider a Hamiltonian system with Hamiltonian

$$H = \frac{1}{2}\omega({x_1}^2 + {y_1}^2) + \lambda x_2 y_2 + \mathcal{O}(||\boldsymbol{z}||^3), \tag{6.1}$$

where $\omega, \lambda \neq 0$ are reals. One can calculate that the Hamiltonians of the restricted three-body problem at the Eulerian equilibria $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ can be written in this form for all parameter values μ. The eigenvalues of the linearized system are $\pm i\omega$ and $\pm\lambda$, so the origin is an unstable equilibrium for the system induced by (6.1). But we can say more about the flow near this equilibrium.

We will first make a little excursion to a theorem on the existence of periodic solutions, known as *Liapunov's Center Theorem.*

Theorem 6.1 Liapunov's Center Theorem *Consider a Hamiltonian system of differential equations on* $\mathbb{R}^{2m}$, $\dot{\boldsymbol{u}} = f(\boldsymbol{u})$ *with* $f(0) = 0$. *Suppose that the eigenvalues of the linearized system around* 0 *are nonzero and given as* $\pm i\omega, \lambda_3, \ldots, \lambda_{2m}$, *where* $\omega \in \mathbb{R}$ *and* $\lambda_j \in \mathbb{C}$. *If* $\lambda_j / i\omega \notin \mathbb{Z}$ *for all* j, *then there is a smooth* $2-$*dimensional surface through the origin, tangent to the eigenspace corresponding to* $\pm i\omega$, *filled with periodic solutions with period close to* $2\pi/\omega$ *(as* $\boldsymbol{u} \to 0$*).*

This surface of periodic solutions is called the *Liapunov center*. Consider now the Hamiltonian system (6.1), for which $m = 4$. The eigenvalues of the linearized system are $\pm i\omega$, and $\pm\lambda$ where λ is real. Therefore Liapunov's Center Theorem holds: there exists such a Liapunov center through the origin of the system (6.1). In fact, we have the following result.

Proposition 6.2 *The equilibrium at the origin for the Hamiltonian system with Hamiltonian (6.1) is unstable. There is a Liapunov Center through the origin. Furthermore, there is a neighborhood of the origin such that every solution which begins at an initial position away from the Liapunov center, leaves this neighborhood in either positive or negative time.*

It remains to prove the last statement. First of all, let us write $H = H_2 + H_r$, where H_r represents the higher order terms of H near 0. H_r starts with third order terms in $\boldsymbol{z}$. Secondly, to make life easier, let us assume that the Liapunov center is located at $x_2 = 0, y_2 = 0$. This implies that

$$\frac{\partial H_r}{\partial x_2}(x_1, 0, y_1, 0) = \frac{\partial H_r}{\partial y_2}(x_1, 0, y_1, 0) = 0\ . \tag{6.2}$$

Define $V(\boldsymbol{z}) = ({x_2}^2 - {y_2}^2)/2$. The orbital derivative of V is $\dot{V} = \lambda({x_2}^2 + {y_2}^2) + W(\boldsymbol{z})$ where

$$W(\boldsymbol{z}) := x_2 \frac{\partial H_r}{\partial y_2} - y_2 \frac{\partial H_r}{\partial x_2}.$$

From (6.2) we have that $W(\boldsymbol{z})$ is at least quadratic in (x_2, y_2). As a consequence we can choose a neighborhood U such that $|W(\boldsymbol{z})| \leq \lambda({x_2}^2 + {y_2}^2)/2$. Taking $\Omega = \{\boldsymbol{z} \mid {x_2}^2 > {y_2}^2\}$ and applying Chetaev's theorem, we conclude that every solution starting in $(U \backslash \{x_2 = y_2 = 0\}) \cap \Omega$ will leave U in positive time. Reversing the time, we conclude that taking $\Omega = \{\boldsymbol{z} \mid {x_2}^2 < {y_2}^2\}$, every solution starting in $(U \backslash \{x_2 = y_2 = 0\}) \cap \Omega$ will leave U in negative time.

Modulo small modifications if the Liapunov center is not flat, this concludes the proof of proposition 6.2. Recall that proposition 6.2 also completely describes the flow of the restricted three-body problem near the Eulerian equilibria $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$.

Last but not least, consider the Hamiltonian

$$H = \alpha(x_1 y_1 + x_2 y_2) + \beta(y_1 x_2 - x_1 y_2) + \mathcal{O}(||\boldsymbol{z}||^3), \tag{6.3}$$

where $\alpha, \beta \neq 0$ are real. The Hamiltonian of the restricted three-body problem at $\mathcal{L}_4$ and $\mathcal{L}_5$ is of this type for the parameter values $\mu_1 < \mu < 1 - \mu_1$. The eigenvalues of the linearized system are $\pm\alpha \pm i\beta$, so the origin is unstable. Moreover, choosing $V(\boldsymbol{z}) = ({x_1}^2 + {x_2}^2 - {y_1}^2 - {y_2}^2)/2$ as a Liapunov function we can verify the following result:

Proposition 6.3 *The Lagrangean equilibria $\mathcal{L}_4$ and $\mathcal{L}_5$ of the restricted three-body problem are unstable for $\mu_1 < \mu < 1 - \mu_1$. Furthermore there is a neighborhood of these points with the property that every nonzero solution starting in this neighborhood, will eventually leave it.*

By now we determined the stability of the equilibria of the restricted three-body problem except for the Lagrangean points at the parameter values $0 < \mu \leq \mu_1$ and $1 - \mu_1 \leq \mu < 1$. The theory that we developed until now, is not sufficient in these cases. The solution is to take into account also the nonlinear terms in the expansion of the system around its equilibrium. That is to look at the $\mathcal{O}(||\boldsymbol{z}||^3)$-terms of the Hamiltonian. A beautiful way to do that is using the theory of *normal forms*.

7 Normal forms

Let P_k be the space of homogeneous polynomials of degree k in the canonical variables $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, so $P_k := \mathrm{span}_{\mathbb{R}}\{x_1^{k_1} \cdots x_n^{k_n}\, y_1^{k_{n+1}} \cdots y_n^{k_{2n}} |\ \sum_{j=1}^{2n} k_j = k\}$. The space of all convergent power series without linear part, $P \subset \bigoplus_{k \geq 2} P_k$, is a Lie-algebra with the Poisson bracket

$$\{\cdot,\cdot\} : P \times P \to P,\ (f,g) \mapsto \{f,g\} := d\boldsymbol{x} \wedge d\boldsymbol{y}(X_f, X_g) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial y_j} - \frac{\partial f}{\partial y_j}\frac{\partial g}{\partial x_j} \right).$$

For each $h \in P$, its adjoint $\mathrm{ad}_h : P \to P$ is the linear operator defined by $\mathrm{ad}_h(H) = \{h, H\}$. Note that whenever $h \in P_k$, then $\mathrm{ad}_h : P_l \to P_{k+l-2}$.

Let us take an $h \in P$. It can be shown that for this h there is an open neighborhood U of the origin such that for every $|t| < 2$ each time-t flow $e^{tX_h} : U \to \mathbb{R}^{2n}$ of the Hamiltonian vector field X_h induced by h is a symplectic diffeomorphism on its image. These time-t flows define a family of mappings $(e^{tX_h})^* : P \to P$ by sending $H \in P$ to $(e^{tX_h})^* H := H \circ e^{tX_h}$. Differentiating the curve $t \mapsto (e^{tX_h})^* H$ with respect to t we find that it satisfies the linear differential equation $\frac{d}{dt}(e^{tX_h})^* H = dH \cdot X_h = -\mathrm{ad}_h(H)$ with initial condition $(e^{0X_h})^* H = H$. The solution reads $(e^{tX_h})^* H = e^{-t\mathrm{ad}_h} H$. In particular the symplectic transformation e^{-X_h} transforms H into

$$H' := (e^{-X_h})^* H = e^{\mathrm{ad}_h} H = H + \{h, H\} + \frac{1}{2!}\{h, \{h, H\}\} + \ldots \quad . \tag{7.1}$$

The diffeomorphism e^{-X_h} sends 0 to 0 (because $X_h(0) = 0$). If $h \in \bigoplus_{k \geq 3} P_k$, then $De^{-X_h}(0) = Id$. A diffeomorphism with these two properties is called a *near-identity* *transformation.*

An element $H \in P$ can be written as $H = \sum_{k=2}^{\infty} H_k$, where $H_k \in P_k$. Assume now, as will usually be the case for the problems we consider in this paper, that $\mathrm{ad}_{H_2} : P_k \to P_k$ is semisimple (i.e. complex-diagonalisable) for every $k \geq 3$. Then $P_k = \ker \mathrm{ad}_{H_2} \oplus \mathrm{im}\ \mathrm{ad}_{H_2}$, as is clear from the diagonalizability. In particular H_3 is uniquely decomposed as $H_3 = f_3 + g_3$, with $f_3 \in \ker \mathrm{ad}_{H_2}$, $g_3 \in \mathrm{im}\ \mathrm{ad}_{H_2}$. Now choose an $h_3 \in P_3$ such that $\mathrm{ad}_{H_2}(h_3) = g_3$. One could for example choose $h_3 = \tilde{g}_3 := (\mathrm{ad}_{H_2}|_{\mathrm{im}\ \mathrm{ad}_{H_2}})^{-1}(g_3)$. But clearly the choice $h_3 = \tilde{g}_3 + p_3$ suffices for any $p_3 \in \ker \mathrm{ad}_{H_2} \cap P_3$. For the transformed Hamiltonian $H' = (e^{-X_{h_3}})^* H$ we calculate from (7.1) that $H_2' = H_2$, $H_3' = f_3 \in \ker \mathrm{ad}_{H_2}$, $H_4' = H_4 + \{h_3, H_3 - \frac{1}{2} g_3\}$, etc. The reader should verify this! But now we can again write $H_4' = f_4 + g_4$ with $f_4 \in \ker \mathrm{ad}_{H_2}$, $g_4 \in \mathrm{im}\ \mathrm{ad}_{H_2}$ and it is clear that by a suitable choice of $h_4 \in P_4$ the Lie-transformation $e^{-X_{h_4}}$ transforms our H' into H'' for which $H_2'' = H_2$, $H_3'' = f_3 \in \ker \mathrm{ad}_{H_2}$ and $H_4'' = f_4 \in \ker \mathrm{ad}_{H_2}$. Continuing in this way, we can for any finite $r \geq 3$ find a sequence of symplectic near-identity transformations $e^{-X_{h_3}}, \ldots, e^{-X_{h_r}}$ with the property that $e^{-X_{h_k}}$ only changes the H_l with $l \geq k$, whereas the composition $e^{-X_{h_3}} \circ \ldots \circ e^{-X_{h_r}}$ transforms H into $\overline{H}$ with the property that $\overline{H}_k$ Poisson commutes with H_2 for every $2 \leq k \leq r$. The previous analysis culminates in the following

Theorem 7.1 Birkhoff-Gustavson *Let $r > 2$ be a given natural number. If $H = \sum_{k=2}^{\infty} H_k \in P$ is such that $\mathrm{ad}_{H_2} : P_k \to P_k$ is semisimpel for each $k \geq 3$, then there is an open neighborhood $U \in \mathbb{R}^{2n}$ of the origin and an analytic symplectic diffeomorphism $\Psi : U \mapsto \Psi(U) \subset \mathbb{R}^{2n}$ such that $\Psi(0) = 0$, $D\Psi(0) = Id$ and $\overline{H} := H \circ \Psi = \sum_{k=2}^{\infty} \overline{H}_k \in P$ has the properties that $\overline{H}_2 = H_2$ and $\{\overline{H}_2, \overline{H}_k\} = 0$ for all $2 \leq k \leq r$.*

The near-identity transformation Ψ is the composition of $r-2$ time-1 flows of Hamiltonian vector fields, which can subsequently be determined. Note that it need not be unique.

The transformed Hamiltonian $\overline{H}$ is called a normal form of H of order r. It can explicitly be determined following the procedure of the paragraph that precedes theorem 7.1 and using formula (7.1). The study of $\overline{H}$ can give us useful information on solutions of the original Hamiltonian H near its equilibrium point 0. It helps for instance to detect bifurcations and to contruct approximations of solutions. More on normalisation by Lie-transformations can be found in [3].

Remark 7.2 Near-identity transformations Ψ with the properties of theorem 7.1 can be found in various ways. Lie-transformations, i.e. compositions of time-1 flows of Hamiltonian vector fields, are just one method. Other methods use power series expansions or averaging techniques. The method of Lie-transformations has the big advantage that the formula for the transformed Hamiltonian, (7.1), is fairly simple.

Normal form techniques also exist for non-Hamiltonian vector fields. Nothing changes dramatically, except that the near-identity transformations are of course no longer symplectic.

Remark 7.3 Let $S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a *linear symmetry* of the Hamiltonian $H \in P$, that is S is a linear symplectic transformation keeping H invariant: $S^*(d\boldsymbol{x} \wedge d\boldsymbol{y}) = d\boldsymbol{x} \wedge d\boldsymbol{y}$ and $S^*H = H \circ S = H$. It is not hard to show that this implies that the Hamiltonian vector field X_H induced by H is equivariant under S : $S \cdot X_H = X_H \circ S$. In other words: if $\gamma : \mathbb{R} \to \mathbb{R}^{2n}$ is an integral curve of X_H, then $S \circ \gamma : \mathbb{R} \to \mathbb{R}^{2n}$ is also an integral curve of X_H. This explains the name symmetry.

Similarly, let $R : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a *linear reversing symmetry* of the Hamiltonian H, i.e. R is a linear anti-symplectic transformation that keeps H invariant: $R^*(d\boldsymbol{x} \wedge d\boldsymbol{y}) = -d\boldsymbol{x} \wedge d\boldsymbol{y}$ and $R^*H = H \circ R = H$. One now shows that X_H is anti-equivariant under R : $R \cdot X_H = -X_H \circ R$. Thus, if $\gamma : \mathbb{R} \to \mathbb{R}^{2n}$ is an integral curve of X_H, then $R \circ \gamma \circ -Id : \mathbb{R} \to \mathbb{R}^{2n}$ is also an integral curve of X_H. This explains the name reversing symmetry.

The group generated by the linear symmetries and linear reversing symmetries of the Hamiltonian $H \in P$, is called the *reversing symmetry group* of H. It can be shown (cf. [3]) that the near-identity transformation Ψ in theorem 7.1 can always be chosen in such a way that $\overline{H} = H \circ \Psi$ is again invariant under the elements of the reversing symmetry group of H. Alternatively stated: one can construct normal forms $\overline{H}$ of H which have the same linear symmetries and linear reversing symmetries as H.

8 The Poincaré section

In this section we summarize the most important properties of the so-called Poincaré map. Although the Poincaré map is a very useful tool for the study of any ordinary differential equation, we will introduce it here for Hamiltonian systems only. More extensive information can be found in [1], ch. 7-8.

Theorem 8.1 *Let H be a Hamiltonian on a $2n$-dimensional symplectic manifold M with symplectic form ω. Suppose that $\gamma : \mathbb{R} \to M$ is a periodic solution of the Hamiltonian vector field X_H induced by H and that γ lies in a regular energy-surface of H, i.e. $H^{-1}(\{\gamma(0)\})$ is a manifold. Then there is a codimension 1 submanifold $S \subset H^{-1}(\{\gamma(0)\})$ and open submanifolds S_1 and S_2 of S with the following properties:*

- *$X_H(m) \notin T_mS$ for all $m \in S$.*
- *$\gamma(0) \in S_1 \cap S_2$*
- *S_1 and S_2 are codimension 2 symplectic submanifolds of M. If $\iota_i : S_i \to M$ are the inclusions, then the symplectic forms ω_i of S_i are given by $\omega_i = \iota_i^*\omega$.*
- *For every $m \in S_1$ there is a time $t(m) > 0$ such that m is mapped to S_2 by the time-$t(m)$ flow of X_H, i.e. $e^{t(m)X_H}(m) \in S_2$. There exists a unique smallest positive number $d(m)$ with this property. d is a smooth function on S_1.*
- *The flow of X_H defines a unique symplectic diffeomorphism $\mathcal{P} : S_1 \to S_2$. $\mathcal{P}$ is given by sending $m \in S_1$ to $e^{d(m)X_H}(m) \in S_2$.*

The proof is highly based on the implicit function theorem, cf. [1]. The property $X_H(m) \notin T_mS$ implies that $T_m(H^{-1}(\{m\})) = X_H(m) \oplus T_mS$. This explains why S is sometimes called a *local transversal section* to the flow of X_H at γ. But usually we call S a *Poincaré section* at γ. The mapping $\mathcal{P}$ is called a *Poincaré* map or *first return map.* We remark that any two Poincaré maps $\mathcal{P}^1 : S_1^1 \to S_2^1$ and $\mathcal{P}^2 : S_1^2 \to S_2^2$ at γ are locally conjugate, i.e. there is an open neighborhood U of $\gamma(0)$ in S_1^1 and a symplectic diffeomorphism $\Phi : U \to S_1^2$ such that $\Phi \circ \mathcal{P}^1 = \mathcal{P}^2 \circ \Phi$. For Φ one could take the mapping that takes $m \in U$ and let it follow X_H until it hits S_1^2 at $\Phi(m)$.

It is clear that a study of the Poincaré map could provide us with very useful information on the flow of X_H in a neighborhood of the periodic solution γ, like stability and unstability. First of all, let us study the derivative of the Poincaré map, $T_{\gamma(0)}\mathcal{P} : T_{\gamma(0)}S \to T_{\gamma(0)}S$. Note that, since any two Poincaré maps are locally conjugate, their derivatives are similar linear mappings. Hence they have the same eigenvalues. This allows us to make the following definition:

Definition 8.2 *Let $\gamma : \mathbb{R} \to M$ be a periodic solution of a Hamiltonian vector field on a symplectic manifold. The* characteristic multipliers *of γ are the eigenvalues of $T_{\gamma(0)}\mathcal{P}$, where $\mathcal{P}$ is any Poincaré map at γ.*

In local Darboux coordinates, $T_{\gamma(0)}\mathcal{P}$ can be represented by a symplectic matrix. Thus, the characteristic multipliers of γ come in quadruples: if λ is a multiplier, then so are $\frac{1}{\lambda}$, $\bar{\lambda}$ and $\frac{\bar{1}}{\lambda}$. It is no surprise that whenever one of the multipliers does not lie on the unit circle in $\mathbb{C}$, there must be a multiplier outside the unit circle. This then can be used to prove that γ is an unstable periodic orbit. So γ can only be stable if all its multipliers have complex modulus 1. As usual, this is not sufficient to prove the stability of γ. Stability can sometimes be proved using variants of Liapunov's theorem. We will not go into this idea here.

Instead, we will focus on two-degrees of freedom Hamiltonian systems. In local Darboux coordinates, a Poincaré map near a periodic orbit γ in this case is an area-preserving planar map leaving the origin fixed. There are only two multipliers. γ is unstable if one of them does not lie on the unit circle. So suppose the multipliers are of the form $e^{\pm i\omega}$, with $\omega \in \mathbb{R}$. This expresses that the Poincaré map is, up to linear approximation, simply a rotation around 0 over an angle ω. This doesn't say anything yet about the stability of γ. But we shall see that under certain assumptions on the higher order approximations of $\mathcal{P}$ around 0, one can indeed prove that γ is stable. It turns out that $\mathcal{P}$ has to be a so-called *twist map.* The resulting type of stability goes under the name *Moser twist stability.*

9 The Twist Map and Arnold's Stability Theorem

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism in $\mathbb{R}^n$. Consider a sequence $\{\boldsymbol{z}_n\}_{n\in\mathbb{N}}$ constructed by a recursive formula, $\boldsymbol{z}_n := F(\boldsymbol{z}_{n-1})$. Given $\boldsymbol{z}_1$ as a starting point, we can construct the *discrete flow* of this dynamical system by calculating each term of the sequence. As an example, we have seen the Poincaré map in local coordinates.

For $\alpha, \omega \in \mathbb{R}$ with $\alpha \neq 0$, consider the special map in $\mathbb{R}^2$ given by

$$\begin{aligned} I_n &= I_{n-1} \\ \theta_n &= \theta_{n-1} + \omega + \alpha I_{n-1}, \end{aligned} \tag{9.1}$$

where we have used polar coordinates $x = \sqrt{2I}\cos(\theta), y = \sqrt{2I}\sin(\theta)$, so $I \in \mathbb{R}_{\geq 0}$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. It is easy to see that (9.1) maps I to itself. On the other hand (9.1) maps θ by (modulo 2π-)addition to a number depending on I. Thus, in $\mathbb{R}^2$, (9.1) rotates every point on a circle with rotation number depending on the distance to the origin. Such a map is called a *twist map*.

Consider now $\delta := (\omega + \alpha I_\circ)/2\pi$ where $I_\circ$ is the initial condition for I. If δ is rational, say $\delta = p/q$ with p and q relatively prime integers, then each point on the circle $x^2 + y^2 = 2I_\circ$ is a q-periodic point. If $\delta \notin \mathbb{Q}$, then the orbit of any point on the circle $x^2 + y^2 = 2I_\circ$ is dense in the circle. The latter type of dynamics is called *quasi-periodic*.

Finally, we look at perturbations of twist maps. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\begin{aligned} I_n &= I_{n-1} + \varepsilon^{r+s} f_1(I_{n-1}, \theta_{n-1}, \varepsilon) \\ \theta_n &= \theta_{n-1} + \omega + \varepsilon^s g(I) + \varepsilon^{s+r} f_2(I_{n-1}, \theta_{n-1}, \varepsilon), \end{aligned} \tag{9.2}$$

where $I \in \mathbb{R}_{\geq 0}$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}, \omega \in \mathbb{R}$. We require the following properties:

1. f_1 and f_2 are smooth functions for $0 \leq a \leq I < b < \infty$, $0 \leq \varepsilon \leq \varepsilon_\circ$, and all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

2. $r \geq 1$ and $s \geq 0$ are two integers.

3. g is a smooth function on $0 \leq a \leq I < b < \infty$.

4. $dg(I)/dI \neq 0$ for $0 \leq a \leq I < b < \infty$.

Theorem 9.1 Moser Twist Stability *Given such a map F with the following additional property. If Ξ is any closed curve of the form $\Xi = \{(I, \theta) \mid I = \Theta(\theta), \Theta : \mathbb{R}/2\pi\mathbb{Z} \to [a, b]$ continuous$\}$ then $\Xi \cap F(\Xi) \neq \emptyset$. Then, for sufficiently small ε, there is a continuous F-invariant curve Γ of the form $\Gamma = \{(I, \theta) \mid I = \Phi(\theta), \Phi : \mathbb{R}/2\pi\mathbb{Z} \to [a, b]$ continuous$\}$.*

This theorem was proposed by Kolmogorov and proved by Moser [9]. We note that assumption 4 implies that we are looking at a perturbation of a twist map restricted to an annulus. Another important remark is about the additional condition in the theorem. This condition excludes the situation where a closed curve of the prescribed form is mapped completely inside or outside itself. This is an important restriction and it prevents the perturbation from being arbitrary. The condition is satisfied for area-preserving maps.

Note that the unperturbed map is just a rotation, so its eigenvalues are $e^{\pm i\omega}$. We may now ask the question of stability of the fixed point 0 of the perturbed map F. Theorem 9.1 states now that if the restriction of F to any small annulus of the form $a \leq I < b$

satisfies the conditions of theorem 9.1, then there is an invariant curve in that annulus. In particular, we can choose this annulus as small as we like, so 0 is a stable fixed point of F.

We want to apply this to Poincaré maps at periodic solutions of two-degrees of freedom Hamiltonian systems. As was explained in the previous section, one can construct such a Poincaré map around a periodic solution and they are represented by an area-preserving map in $\mathbb{R}^2$ keeping the origin fixed. If the two multipliers of γ lie on the unit circle, then γ could be stable. Once we can show that the Poincaré map is in fact a perturbed twist map (in the sense of the previous theorem), this stability is proved.

We will use normal form theory to view the Poincaré map as a perturbed twist map. Let us assume that around 0 our Hamiltonian can be expanded as

$$H = \frac{1}{2}\omega_1(x_1^2 + y_1^2) - \frac{1}{2}\omega_2(x_2^2 + y_2^2) + H_3 + H_4 + \dots \quad ,$$

with $\omega_j \neq 0$ real numbers. The following result is pure 'algebra of normal forms':

Theorem 9.2 Birkhoff *Let H be of the above form and $\frac{|\omega_1|}{|\omega_2|} = \frac{p}{q}$ where p and q are relatively prime. Then any normal form of order smaller then or equal to $p + q - 1$ is of the following simple form:*

$$\overline{H}(\boldsymbol{x}, \boldsymbol{y}) = H_2(\boldsymbol{x}, \boldsymbol{y}) + \overline{H}_4(x_1^2 + y_1^2, x_2^2 + y_2^2) + \dots + \overline{H}_{2m}(x_1^2 + y_1^2, x_2^2 + y_2^2) + \mathcal{O}(||\boldsymbol{z}||^{2m+1}) \ , \tag{9.3}$$

with $m < (p + q - 1)/2$.

Proof:
Assume that both ω_1 and ω_2 are positive. We transform to complex variables by $z_j = x_j + iy_j$, $j = 1, 2$ and re-scale the Hamiltonian by $2i$. The symplectic form in these coordinates are $dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2$. Using this symplectic form we can define the Poisson bracket

$$\{f, g\} = \sum_{j=1}^{2} \left(\frac{\partial f}{\partial z_j} \frac{\partial g}{\partial \overline{z}_j} - \frac{\partial f}{\partial \overline{z}_j} \frac{\partial g}{\partial z_j} \right),$$

for any two functions f and g in P, the Lie-algebra of convergent power series in $(z_1, z_2, \overline{z}_1, \overline{z}_2)$ with no linear terms. It is easy to check that $H_2 = i(pz_1\overline{z}_1 + qz_2\overline{z}_2)$ and

$$\mathrm{ad}_{H_2}({z_1}^{\alpha_1}, {z_2}^{\alpha_2}, {\overline{z}_1}^{\beta_1}, {\overline{z}_2}^{\beta_2}) = ((\alpha_1 - \beta_1)\, p + (\alpha_2 - \beta_2)\, q)\, {z_1}^{\alpha_1}, {z_2}^{\alpha_2}, {\overline{z}_1}^{\beta_1}, {\overline{z}_2}^{\beta_2}.$$

For $m < (|p| + |q| - 1)/2$, $\overline{H}_{2m} \in \ker \mathrm{ad}_{H_2}$. If $\overline{H}_{2m}$ has a term of the form $C{z_1}^{\alpha_1}, {z_2}^{\alpha_2}, {\overline{z}_1}^{\beta_1}, {\overline{z}_2}^{\beta_2}$ then we have

$$(\alpha_1 - \beta_1)\, p + (\alpha_2 - \beta_2)\, q = 0. \tag{9.4}$$

Since p and q are relatively prime there exists k_1 and k_2 in $\mathbb{Z}$ such that

$$k_1 p + k_2 q = 1. \tag{9.5}$$

Multiplying (9.4) by k_1, (9.5) by $\alpha_1 - \beta_1$, and subtract the resulting equations to have $((\alpha_1-\beta_1)k_2-(\alpha_2-\beta_2)k_1)q = \alpha_1-\beta_1$, which is simply means there exists $m_1 \in \mathbb{Z}$ such that $m_1 q = \alpha_1 - \beta_1$. Analog with this, we conclude there exists $m_2 \in \mathbb{Z}$ such that $m_2 p = \alpha_2 - \beta_2$. Hence $m_1 = m_2$. The conclusion is $\alpha_1 - \beta_1 + \alpha_2 - \beta_2 = m(p+q) \geq (p+q)$. Thus, if $m < (p+q-1)/2$ then (9.4) holds if and only if $\alpha_j = \beta_j, j = 1, 2$ which conclude the proof. For one of ω_1 or ω_2 is negative runs the same way. □

A Hamiltonian of the form in Birkhoff's theorem is said to be in *Birkhoff normal form.* Note that by $x_j = \sqrt{2I_j}\cos(\varphi_j)$ and $y_j = \sqrt{2I_j}\sin(\varphi_j)$ we can write the Birkhoff normal form of the Hamiltonian in the so-called *symplectic polar coordinate.* By rescaling the variables, we can introduce small parameter ε into the system. As a consequence, we can view the Hamiltonian H as a perturbation of two harmonic oscillator. Obviously, I_1 and I_2 are the integrals of the unperturbed Hamiltonian system. This is why people also call $(\boldsymbol{\varphi}, \boldsymbol{I}) = (\varphi_1, \varphi_2, I_1, I_2)$ the *action-angle* variables.

Let us now consider a Hamiltonian in Birkhoff normal form written in action-angle variables. We have also re scale the variables to introduce the small parameter ε. We remark that the corresponding Hamiltonian system generate by this normal form has two functionally independent integrals in involution namely $H_2 = \omega_1 I_1 + \omega_2 I_2$ and

$$\mathcal{H} = (\overline{H} - H_2)/\varepsilon^2 = A{I_1}^2 + BI_1I_2 + C{I_2}^2 + \varepsilon^2 P(\boldsymbol{I};\varepsilon) + \varepsilon^{p+q-4}R(\boldsymbol{\varphi}, \boldsymbol{I};\varepsilon).$$

We fix a value for one of the integral, for instance $H_2 = E_\circ \in \mathbb{R}$. We then eliminate one of the actions for instance by setting $I_2 = (E_\circ - \omega_1 I_1)/\omega_2$ and substitute it to $\mathcal{H}$ to have

$$\begin{aligned}\mathcal{H} = &\left(A + B\tfrac{\omega_1}{\omega_2} + C\left(\tfrac{\omega_1}{\omega_2}\right)^2\right){I_1}^2 + \left(B\tfrac{E_\circ}{\omega_2} - 2CE_\circ\tfrac{\omega_1}{\omega_2}\right)I_1 + C\left(\tfrac{E_\circ}{\omega_2}\right)^2 + \\ &\varepsilon^2\overline{P}(I_1;E_\circ,\varepsilon) + \varepsilon^{p+q-4}\overline{R}(\boldsymbol{\varphi}, I_1;E_\circ,\varepsilon).\end{aligned}$$

We then choose the section by setting the corresponding angle $\varphi_2 = 0$. Substitute this to $\mathcal{H}$ and then write equations of motion

$$\begin{aligned}\dot{I}_1 =&\varepsilon^{p+q-4}\frac{\partial\overline{R}}{\partial\varphi_1}\\ \dot{\varphi}_1 =&2\left(A + B\frac{\omega_1}{\omega_2} + C\left(\frac{\omega_1}{\omega_2}\right)^2\right)I_1 + \left(B\frac{E_\circ}{\omega_2} - 2CE_\circ\frac{\omega_1}{\omega_2}\right)\\ &+\varepsilon^2\frac{\partial P}{\partial I_1} + \varepsilon^{p+q-4}\frac{\partial\overline{R}}{\partial I_1}.\end{aligned}$$

This equations of motion is then describe the map of the form (9.2).

So a combination of normal form theory and Moser's stability theorem can be used to prove stability of periodic solutions in a neighborhood of an equilibrium point. The surprise is now, that the theory can be extended in order to actually prove the stability of the equilibrium itself:

Theorem 9.3 Arnold's Stability Theorem *Consider the Hamiltonian system with Hamiltonian (9.3). If there exists a $2 \leq k \leq m$ such that $D_{2k} := H_{2k}(\omega_2, \omega_1) \neq 0$ then the origin is stable. Moreover, arbitrarily close to the origin in $\mathbb{R}^4$ there are invariant tori filled with quasi-periodic solutions.*

The proof is deep and we refer to [2] or [7] for it. For the restricted three-body problem with parameter values $0 < \mu < \mu_1$ (with $\mu \neq \mu_2 := 0.0242938\ldots$, $\mu_3 := 0.0135116\ldots$ [2]) Deprit and Deprit-Bartholomé in 1967 calculated the normal form of the Hamiltonian at $\mathcal{L}_4$ and $\mathcal{L}_5$. They found that $D_4 \neq 0$ except for $\mu \approx 0.010$. Nowadays we know that $D_6 \neq 0$ at this parameter value. Thus, by Arnold's theorem we have the following result.

Proposition 9.4 *In the restricted three-body problem, the Lagrangean equilibria $\mathcal{L}_4$ and $\mathcal{L}_5$ are stable for $0 < \mu < \mu_1$ with $\mu \neq \mu_2, \mu_3$.* (**or** $1 - \mu$)

Thus, the stability of the equilibria of the restricted three-body problem has been established except for $\mathcal{L}_4$ and $\mathcal{L}_5$ if $\mu \in \{\mu_1, \mu_2, \mu_3\}$. Arnold's stability theorem can not be applied in those cases since we can only bring the H_2 into Birkhoff normal form. Neither does Dirichlet's theorem work. We refer to [6] for the analysis of those cases.

In the Sun-Jupiter system, the result of proposition 9.4 can really be observed: if we draw the equilateral triangles with the sun and Jupiter at its base points, then we find two groups of asteroids at the third vertex. They are called the Trojans and the Greeks.

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[2] μ_2 and μ_3 are real numbers which produce the relations $\omega_1 : \omega_2 = 1 : 2$ and $\omega_1 : \omega_2 = 1 : 3$, respectively.

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