Fixed Points and Contraction Factor Functions

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Abstract: In a complete metric space (X, d), we define w-distance functions $p : X \times X \to [0, \infty)$, of which the metric d is a special case, and contraction factor functions $r : X \times X \to [0, \infty)$ such that if

$$p(Tx, Ty) \leq r(x, y)p(x, y)$$

for all $x, y \in X$, then $T : X \to X$ has a (unique) fixed point.

Keywords: contraction, w-distance, fixed point.

1 Introduction

Banach's Contraction Principle states that if (X, d) is a complete metric space and $T : X \to X$ is a contraction, i.e., there exists a number $0 \leq r < 1$ such that for every two points $x, y \in X$:

$$d(Tx, Ty) \leq rd(x, y), \tag{1}$$

then T has a unique fixed point. There exist numerous extensions of this result; Rakotch [1], for instance, considers the problem of defining contraction factor functions such that the Banach Contraction Principle remains valid when the constant r in (1) is replaced by a function $r(x, y)$. This allows $\sup_{(x,y)} r(x, y) = 1$, in which case T is no longer a contraction. The purpose of this note is to define:

- functions $p : X \times X \to [0, \infty)$, of which the distance function d is a special case, and
- contraction factor functions $r : X \times X \to [0, \infty)$, including those of Rakotch [1],

such that if

$$p(Tx, Ty) \leq r(x, y)p(x, y)$$

for all $x, y \in X$, then $T : X \to X$ has a (unique) fixed point. The functions p are so-called w-distances, introduced and studied in a recent sequence of papers by Kada, Suzuki, and Takahashi [2], Suzuki and Takahashi [3], and Suzuki [4].

2 Preliminaries

Denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let X be a metric space with metric d. Following [2, p. 381], we call a function $p : X \times X \to [0, \infty)$ a *w-distance* on X if the following conditions hold:

- p satisfies the triangle inequality, i.e., $\forall x, y, z \in X : p(x, z) \leq p(x, y) + p(y, z)$;
- $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous for every $x \in X$, i.e., if a sequence (y_m) in X converges to $y \in X$, then $p(x, y) \leq \liminf_{m \to \infty} p(x, y_m)$;
- for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $x, y, z \in X$: if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$, then $d(x, y) \leq \varepsilon$.

The metric d is a w-distance. Examples of many other w-distances are found in [2] and [3, Lemma 1]. Kada *et al.* [2, Lemma 1] prove:

Lemma 2.1 *Let (X, d) be a metric space and let p be a w-distance on X. Consider points $x, y, z \in X$, a sequence (x_n) in X, and sequences (α_n) and (β_n) in $[0, \infty)$ converging to zero. The following claims hold:*

(a) *If $p(x_n, x_m) \leq \alpha_n$ for all $m, n \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence in (X, d).*

(b) *If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = p(x, z) = 0$, then $y = z$.*

Generalizing Rakotch [1], we define a family of functions that take over the role of the contraction factors in the original statement of Banach's Contraction Principle and its variants.

Definition 2.2 Let (X, d) be a metric space and let p be a w-distance on X. A function $r : X \times X \to [0, \infty)$ is a *contraction factor function* if there exists a function $f : (0, \infty) \to [0, 1)$ such that $p(x, y) \geq \varepsilon$ for some $x, y \in X$ and $\varepsilon > 0$ implies $r(x, y) \leq f(\varepsilon)$. The set of all contraction factor functions is denoted $F(p)$.

Definition 2.2 implies that $r(x, y) \in [0, 1)$ for all $x, y \in X$ with $p(x, y) > 0$. Rakotch [1, Def. 2] takes $p = d$, and considers only functions β on $X \times X$ for which there exists a function $\alpha : [0, \infty) \to \mathbb{R}$ such that

(a) $\forall x, y \in X : \beta(x, y) = \alpha(d(x, y))$, i.e., $\beta(x, y)$ only depends on the distance between x and y;

(b) $\alpha(\varepsilon) \in [0, 1)$ for all $\varepsilon > 0$;

(c) α is a decreasing function, i.e., if $\varepsilon_1 \geq \varepsilon_2 \geq 0$, then $\alpha(\varepsilon_1) \leq \alpha(\varepsilon_2)$.

Notice that $\beta(x, y) = \alpha(d(x, y)) \in [0, 1)$ if $d(x, y) > 0$. If $d(x, y) = 0$, then for arbitrary $\varepsilon > 0$, decreasingness of α implies that $\beta(x, y) = \alpha(d(x, y)) \geq \alpha(\varepsilon) \geq 0$. Hence β is a function into $[0, \infty)$. Defining $f(\varepsilon) = \alpha(\varepsilon)$ for all $\varepsilon > 0$, it follows that $d(x, y) \geq \varepsilon$ for some $x, y \in X$ and $\varepsilon > 0$ implies $\beta(x, y) = \alpha(d(x, y)) \leq \alpha(\varepsilon) = f(\varepsilon)$. Consequently, the functions β considered by Rakotch are contained in $F(d)$, the class of contraction factor functions for the w-distance $p = d$.

Apart from the fact that Definition 2.2 allows contraction factor functions to be defined for an arbitrary w-distance, it extends the functions of Rakotch in two directions: the values do not just depend on the distance between two points, and the monotonicity assumption is omitted.

3 The contraction theorem

After proving auxiliary results in Proposition 3.1, a contraction result is provided in Theorem 3.2.

Proposition 3.1 *Let (X, d) be a metric space, let p be a w-distance on X, and $T : X \to X$ a function from X into itself. Assume there exists a function $r \in F(p)$ such that*

$$\forall x, y \in X: \quad p(Tx, Ty) \leq r(x, y)p(x, y). \tag{2}$$

Let $x_0 \in X$ and $x_n = T^n x_0$ for all $n \in \mathbb{N}$. The following claims hold:

(a) *The sequence $(p(x_0, x_n))$ in $[0, \infty)$ is bounded.*

(b) *There exists a sequence (α_n) in $[0, \infty)$ converging to zero, such that*

$$p(x_n, x_m) \leq \alpha_n \tag{3}$$

for all $m, n \in \mathbb{N}$ with $m > n$.

Proof. Fix a function f as in Definition 2.2. Then $r(x, y) \in [0, 1)$ for all $x, y \in X$ with $p(x, y) > 0$. Hence, repeated application of (2) yields that

$$\forall n \in \mathbb{N}: \quad p(x_{n+1}, x_n) \leq p(x_1, x_0), \tag{4}$$

and

$$\forall k, n, p \in \mathbb{N} \cup \{0\}: \quad \text{if } n > k, \text{ then } p(x_n, x_{n+p}) \leq p(x_k, x_{k+p}). \tag{5}$$

Proof of (a): Let $\varepsilon > 0$. Then $f(\varepsilon) \in [0, 1)$ by Definition 2.2, so $R := \max\{\varepsilon, \frac{p(x_0,x_1)+p(x_1,x_0)}{1-f(\varepsilon)}\}$ is well-defined. Let $n \in \mathbb{N}$. We prove that $p(x_0, x_n) \leq R$. This is clear if $p(x_0, x_n) < \varepsilon$, so assume that $p(x_0, x_n) \geq \varepsilon$. Then $0 \leq r(x_0, x_n) \leq f(\varepsilon) < 1$ by Definition 2.2. Using this inequality, the triangle inequality, (2) and (4), it follows that

$$\begin{aligned} p(x_0, x_n) &\leq p(x_0, x_1) + p(x_1, x_{n+1}) + p(x_{n+1}, x_n) \\ &\leq p(x_0, x_1) + r(x_0, x_n)p(x_0, x_n) + p(x_1, x_0) \\ &\leq p(x_0, x_1) + f(\varepsilon)p(x_0, x_n) + p(x_1, x_0), \end{aligned}$$

which implies

$$p(x_0, x_n) \leq \frac{p(x_0, x_1) + p(x_1, x_0)}{1 - f(\varepsilon)} \leq R.$$

Proof of (b): We prove:

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } n \geq N, p \in \mathbb{N} \text{ implies } p(x_n, x_{n+p}) < \varepsilon. \tag{6}$$

This implies (3): for each $k \in \mathbb{N}$, (6) implies the existence of $N(k) \in \mathbb{N}$ such that

$$\forall n \geq N(k), \ \forall p \in \mathbb{N}: \quad p(x_n, x_{n+p}) < \frac{1}{k}. \tag{7}$$

Without loss of generality, one can take $N(k) < N(k+1)$ for each $k \in \mathbb{N}$. Take R such that $p(x_0, x_n) \leq R$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define

$$\alpha_n = \begin{cases} R & \text{if } n < N(1), \\ \frac{1}{k} & \text{if } N(k) \leq n < N(k+1) \text{ for some } k \in \mathbb{N}. \end{cases}$$

Then (α_n) is a sequence in $[0, \infty)$ converging to zero. To see that $p(x_n, x_{n+p}) \leq \alpha_n$ for all $n, p \in \mathbb{N}$, notice: firstly, if $n < N(1)$, then $p(x_n, x_{n+p}) \leq p(x_0, x_p) \leq R = \alpha_n$ by (5), and secondly, if $N(k) \leq n < N(k+1)$ for some $k \in \mathbb{N}$, then $p(x_n, x_{n+p}) \leq \frac{1}{k} = \alpha_n$ by (7).

Remains to prove (6). Let $\varepsilon > 0$. Then $f(\varepsilon) \in [0, 1)$ by Definition 2.2. Choose $N \in \mathbb{N}$ such that

$$Rf(\varepsilon)^N < \varepsilon. \tag{8}$$

Let $n \geq N, p \in \mathbb{N}$. We prove that $p(x_n, x_{n+p}) < \varepsilon$. Repeated application of (2) yields

$$p(x_n, x_{n+p}) \leq p(x_0, x_p) \prod_{k=0}^{n-1} r(x_k, x_{k+p}).$$

Combining this with the assumption that R is a bound for the sequence $(p(x_0, x_n))$, we find

$$p(x_n, x_{n+p}) \leq R \prod_{k=0}^{n-1} r(x_k, x_{k+p}). \tag{9}$$

Discern two cases.

Case 1: If $p(x_k, x_{k+p}) < \varepsilon$ for some $k \in \{0, \ldots, n-1\}$, then (5) yields that $p(x_n, x_{n+p}) \leq p(x_k, x_{k+p}) < \varepsilon$.

Case 2: If $p(x_k, x_{k+p}) \geq \varepsilon$ for every $k \in \{0, \ldots, n-1\}$, then Definition 2.2 implies that $r(x_k, x_{k+p}) \leq f(\varepsilon)$ for every $k \in \{0, \ldots, n-1\}$. Using (9) and (8) yields

$$p(x_n, x_{n+p}) \leq R \prod_{k=0}^{n-1} r(x_k, x_{k+p}) \leq R(f(\varepsilon)^n \leq R(f(\varepsilon)^N < \varepsilon.$$

This finishes the proof. □

The preliminary work in Proposition 3.1 paves the road for our contraction result.

Theorem 3.2 *Let (X, d) be a complete metric space, let p be a w-distance on X, and $T : X \to X$ a function from X into itself. If there exists a function $r \in F(p)$ such that*

$$\forall x, y \in X : \quad p(Tx, Ty) \leq r(x, y)p(x, y), \tag{10}$$

then T has a unique fixed point $x \in X$. This fixed point satisfies $p(x, x) = 0$.

Proof. Let $x_0 \in X$ and $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Proposition 3.1(b) and Lemma 2.1(a) imply that (x_n) is a Cauchy sequence. Since (X, d) is complete, (x_n) has a limit $x \in X$. We show that $Tx = x$. Consider a sequence (α_n) as in Proposition 3.1(b). Since $p(x_n, \cdot)$ is lower semicontinuous and $x_m \to x$, it follows from (3) that

$$\forall n \in \mathbb{N} : \quad p(x_n, x) \leq \liminf_{m \to \infty} p(x_n, x_m) \leq \alpha_n, \tag{11}$$

and, using (10) and (11), that

$$\forall n \in \mathbb{N} : \quad p(x_n, Tx) = p(Tx_{n-1}, Tx) \leq p(x_{n-1}, x) \leq \alpha_{n-1}. \tag{12}$$

From (11), (12), and part (b) of Lemma 2.1, it follows that $Tx = x$, i.e., that x is a fixed point of T. To see that $p(x, x) = 0$, suppose — to the contrary — that $p(x, x) > 0$. Then $r(x, x) \in [0, 1)$ by Definition 2.2; by (10) and the fact that x is a fixed point, it follows that:

$$p(x, x) = p(Tx, Tx) \leq r(x, x)p(x, x) < p(x, x),$$

a contradiction. Finally, to prove that x is the *unique* fixed point of T, suppose that $y \in X$ satisfies $Ty = y$. Analogous to the proof that $p(x, x) = 0$, it follows that $p(x, y) = 0$, so part (b) of Lemma 2.1 implies that $x = y$. □

4 Concluding remarks

Theorem 3.2, replacing the metric d of a metric space (X, d) with w-distances recently introduced in Kada *et al.* [2], and the contraction factors by contraction factor functions as in Definition 2.2, provides generalizations of the classical Banach Contraction Principle, the contraction theorem of Rakotch [1, p. 463], and more recent results by Suzuki and Takahashi [3, Thm. 2].

If T itself does not satisfy (10), but some power T^n $(n \in \mathbb{N})$ of T does, the conclusion of Theorem 3.2 still holds: according to Theorem 3.2, T^n has a unique fixed point x, but

$$Tx = T(T^n x) = T^n(Tx)$$

indicates that Tx is also a fixed point of T^n. Hence $Tx = x$, i.e., x is a fixed point of T. The fact that x is the unique fixed point of T and $p(x, x) = 0$ follows in the same way as before.

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