# Fixed Points in a Group of Isometries

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**Abstract:** The Bruhat-Tits fixed point theorem states that a group of isometries on a complete metric space with negative curvature possesses a fixed point if it has a bounded orbit. This theorem is extended by a relaxation of the negative curvature condition in terms of the $w$-distance functions introduced by Kada *et al.* [Non-convex minimization theorems and fixed point theorems in complete metric spaces, *Math. Japon.* 44 (1996), 381-391].

**Keywords:** Bruhat-Tits fixed point theorem, $w$-distances.

# 1 Introduction

According to the Bruhat-Tits fixed point theorem, a group of isometries on a complete metric space with negative curvature — a complete metric space satisfying a relaxation of the parallelogram law — possesses a fixed point if it has a bounded orbit; see [2] for the original statement, [1] for an extensive overview of the theory on Bruhat-Tits buildings, and [5] for a very accessible treatment of this fixed point theorem.

The result and some of its special cases have wide applicability, witnessing their use in for instance the Bruhat-Tits theory of buildings, group theory (cf. Cartan's fixed point theorem, [3, Ch. I, Theorem 13.5]), and the theory of trees [6, Section I.4.3, Proposition 19]. Several applications are discussed in [1, Ch. VI]. Perhaps surprisingly, it has also been used in the study of communication to establish that individuals speaking different languages that allow sufficient freedom to express nuances have a common interpretation of at least some phrases in their vocabulary; see [9].

The purpose of this note is to extend the Bruhat-Tits fixed point theorem by stating the negative curvature condition not in terms of distance functions, but using the more general notion of $w$-distances, as introduced and studied in a recent sequence of papers by Kada, Suzuki, and Takahashi [4], Suzuki and Takahashi [8], and Suzuki [7].

Section 2 recalls the definition of $w$-distances. Generalized Bruhat-Tits spaces are considered in Section 3. Finally, the fixed point theorem is provided in Section 4.

# 2 Preliminaries

This section settles some standard matters of notation and defines the $w$-distance functions introduced in Kada *et al.* [4]. Denote by $\mathbb{N}$ the set of positive integers. Let $X$ be a metric space with distance $d$. Following [4, p. 381], we call a function $p : X \times X \to [0, \infty)$ a *$w$-distance* on $X$ if the following conditions hold:

- $p$ satisfies the triangle inequality, i.e., $\forall x, y, z \in X : p(x, z) \leq p(x, y) + p(y, z)$;
- $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous for every $x \in X$, i.e., if a sequence $(y_m)_{m \in \mathbb{N}}$ in $X$ converges to $y \in X$, then $p(x, y) \leq \liminf_{m \to \infty} p(x, y_m)$;
- for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $x, y, z \in X$: if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$, then $d(x, y) \leq \varepsilon$.

The metric $d$ is a $w$-distance. Examples of many other $w$-distances are found in [4] and [8, Lemma 1]. Kada *et al.* [4, Lemma 1] prove:

**Lemma 2.1** *Let $(X, d)$ be a metric space and let $p$ be a $w$-distance on $X$. Consider points $x, y, z \in X$, a sequence $(x_n)_{n\in\mathbb{N}}$ in $X$, and a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $[0, \infty)$ converging to zero. The following claims hold:*

**(a)** *If $p(x_n, x_m) \leq \alpha_n$ for all $m, n \in \mathbb{N}$ with $m > n$, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, d)$.*

**(b)** *If $p(x, y) = p(x, z) = 0$, then $y = z$.*

Balls are defined with respect to the $w$-distance $p$. The ball around $x \in X$ with radius $r \geq 0$ is the set

$$B(x, r) := \{y \in X \mid p(y, x) \leq r\}.$$

A set $S \subseteq X$ is *$p$-bounded* if there exists a ball $B(x, r)$ such that $S \subseteq B(x, r)$.

The composition of two functions $g, h : X \to X$ is denoted by $gh : x \mapsto g(h(x))$. For $S \subseteq X$, write $g(S) = \{g(s) \mid s \in S\}$.

# 3 Generalized Bruhat-Tits spaces

A *generalized Bruhat-Tits space* $(X, d, p)$ is a complete metric space $(X, d)$ with a $w$-distance $p$ satisfying the following property: for any two points $x_1, x_2 \in X$ there is a point $z \in X$ such that for all $y \in X$,

$$p(x_1, x_2)^2 + 4p(y, z)^2 \leq 2p(y, x_1)^2 + 2p(y, x_2)^2. \tag{1}$$

This is a generalization in terms of the $w$-distance $p$ of the well-known negative curvature inequality, sometimes referred to as the semi-parallelogram law [cf. 5]. For a brief motivation, we resort to planar geometry. Consider Figure 1. The parallelogram law states that, using the Euclidean distance $d_2$, the sum of the squared lengths of the diagonals equals the sum of the squared lengths of the sides of the parallelogram:

$$d_2(x_1, x_2)^2 + d_2(y, x_3)^2 = 2d_2(y, x_1)^2 + 2d_2(y, x_2)^2.$$

Let $z$ be the midpoint between $x_1$ and $x_2$. Since $2d(y, z) = d(y, x_3)$, substitution in the parallelogram law yields:

$$d_2(x_1, x_2)^2 + 4d_2(y, z)^2 = 2d_2(y, x_1)^2 + 2d_2(y, x_2)^2.$$

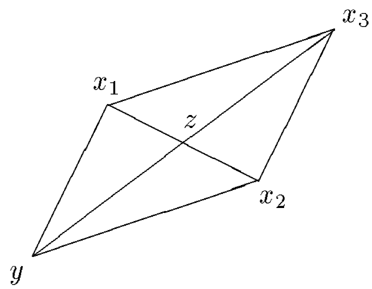


Figure 1: The (semi-)parallelogram law

Generalizing this to an arbitrary metric space $(X, d)$ and allowing for a weak inequality, rather than equality, $(X, d)$ is said to satisfy the semi-parallelogram law if for any two points $x_1, x_2 \in X$ there exists a point $z \in X$ such that for all $y \in X$,

$$d(x_1, x_2)^2 + 4d(y, z)^2 \leq 2d(y, x_1)^2 + 2d(y, x_2)^2.$$

Replacing the distance function $d$ with the more general notion of a $w$-distance $p$, one finds condition (1).

**Theorem 3.1** *Let $(X, d, p)$ be a generalized Bruhat-Tits space and let $S \subseteq X$ be $p$-bounded. There exists a unique ball of minimum radius containing $S$.*

**Proof.**
**Existence:** Since $S$ is $p$-bounded, the set $\mathcal{B} = \{B(x, r) \mid S \subseteq B(x, r)\}$ of balls containing $S$ is nonempty. Set $r = \inf_{B(x,r') \in \mathcal{B}} r'$ and let $B(x_n, r_n)$ be a sequence in $\mathcal{B}$ such that $(r_n)_{n \in \mathbb{N}}$ is a non-increasing sequence converging to $r$. We proceed to show that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(X, d)$. Let $m, n \in \mathbb{N}$ with $m > n$. Then (1) implies the existence of a point $z_{mn} \in X$ such that

$$\forall y \in X: \quad p(x_n, x_m)^2 + 4p(y, z_{mn})^2 \leq 2p(y, x_n)^2 + 2p(y, x_m)^2. \tag{2}$$

By definition of $r$, there exists a point $y \in S$ with $p(y, z_{mn}) \geq r^2 - \frac{1}{n}$. Substituting this in (2) and using that $y \in B(x_k, r_k)$ for all $k \in \mathbb{N}$ and $(r_k)_{k \in \mathbb{N}}$ is a non-increasing sequence, yields:

$$\begin{aligned} p(x_n, x_m)^2 &\leq 2p(y, x_n)^2 + 2p(y, x_m)^2 - 4p(y, z_{mn})^2 \\ &\leq 2r_n^2 + 2r_m^2 - 4(r^2 - \frac{1}{n}) \\ &\leq 2r_n^2 + 2r_n^2 - 4(r^2 - \frac{1}{n}) \\ &= 4(r_n^2 - r^2) + \frac{4}{n}. \end{aligned}$$

Lemma 2.1(a) with $\alpha_n = \sqrt{4(r_n^2 - r^2) + \frac{4}{n}}$ implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, d)$.

Since $(X, d)$ is a complete metric space, the Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ has a limit $x \in X$. We proceed to show that the ball $B(x, r)$ is a ball of minimum radius containing $S$. The definition of $r$ implies that the radius is indeed minimal. Let $y \in S$. The triangle inequality and the lower semicontinuity property of $p$ imply that for each $n \in \mathbb{N}$:

$$\begin{aligned} p(y,x) &\leq p(y,x_n) + p(x_n,x) \\ &\leq r_n + \liminf_{m\to\infty} p(x_n,x_m) \\ &\leq r_n + \alpha_n, \end{aligned}$$

where the last inequality follows from the fact that $p(x_n, x_m) \leq \alpha_n$ for each $m > n$, as shown earlier in the proof. Taking limits as $n \to \infty$ yields that $p(y, x) \leq r$, i.e., $y \in B(x, r)$. Hence $S \subseteq B(x, r)$.

**Uniqueness:** Suppose there are two balls of minimum radius containing $S$: $B(x_1, r)$ and $B(x_2, r)$. Then (1) implies the existence of a point $z \in X$ such that for each $y \in X$:

$$p(x_1,x_2)^2 + 4p(y,z)^2 \leq 2p(y,x_1)^2 + 2p(y,x_2)^2. \tag{3}$$

Since $r$ is the minimum radius of a ball containing $S$, it follows that for each $\varepsilon > 0$ there exists a $y \in S$ such that $p(y, z) \geq r - \varepsilon$. Substituting this in (3) and using the fact that $y \in S \subseteq B(x_i, r)$ for $i = 1, 2$, yields that $\forall \varepsilon \in (0, r)$:

$$\begin{aligned} p(x_1,x_2)^2 &\leq 2p(y,x_1)^2 + 2p(y,x_2)^2 - 4p(y,z)^2 \\ &\leq 2r^2 + 2r^2 - 4(r-\varepsilon)^2 \\ &= 4r^2 - 4(r-\varepsilon)^2. \end{aligned}$$

Letting $\varepsilon \to 0$, this implies that $p(x_1, x_2)^2 \leq 0$. Consequently, $p(x_1, x_2) = 0$. Similarly, $p(x_2, x_1) = 0$. The triangle inequality implies that $0 \leq p(x_1, x_1) \leq p(x_1, x_2) + p(x_2, x_1) = 0$, so $p(x_1, x_1) = 0$. Lemma 2.1(b) and $p(x_1, x_1) = p(x_1, x_2) = 0$ imply that $x_1 = x_2$: the two balls are identical. □

# 4 Fixed point theorem

Let $(X, d, p)$ be a generalized Bruhat-Tits space. A *$p$-isometry* of $X$ is a bijection $g : X \to X$ such that $g$ preserves $p$-distances:

$$\forall x_1, x_2 \in X : \quad p(g(x_1), g(x_2)) = p(x_1, x_2).$$

Note that if $g$ and $h$ are $p$-isometries, so is their composition $gh$ and the inverse $g^{-1}$. Moreover, the identity function $\mathrm{id} : x \mapsto x$ is a $p$-isometry. The straightforward proofs of these claims are left to the reader. A *group* $G$ of $p$-isometries is a set of $p$-isometries such that

**(a)** the identity function $\mathrm{id} : x \mapsto x$ is an element of $G$;

**(b)** $G$ is closed under inversion: if $g \in G$, then $g^{-1} \in G$;

**(c)** $G$ is closed under composition: if $g_1, g_2 \in G$, then $g_1 g_2 \in G$.

Let $G$ be a group of isometries and $x \in X$. The *orbit* $O(x)$ of $x$ is defined to be the collection of images $g(x)$ with $g \in G$:

$$O(x) := \{g(x) \mid g \in G\}.$$

**Lemma 4.1** *Let $x \in X, g \in G$. Then $g(O(x)) = O(x)$.*

**Proof.**
($\subseteq$): Let $y \in g(O(x))$, i.e., there is a $h \in G$ such that $y = gh(x)$. Since $G$ is closed under composition: $gh \in G$, so $y \in O(x)$.
($\supseteq$): Let $y \in O(x)$, i.e., there is a $h \in G$ such that $y = h(x)$. Since $g \in G$, it follows that $g^{-1} \in G$ and $g^{-1}h \in G$. Consequently, $y = h(x) = (gg^{-1})h(x) = g(g^{-1}h(x)) \in g(O(x))$. $\square$

Let $G$ be a group of $p$-isometries in a generalized Bruhat-Tits space $(X, d, p)$. A *fixed point* of $G$ is a point $x \in X$ such that $g(x) = x$ for all $g \in G$, or equivalently, an $x \in X$ with $O(x) = \{x\}$. If there is a point with a $p$-bounded orbit, then $G$ has a fixed point.

**Theorem 4.2** *Let $(X, d, p)$ be a generalized Bruhat-Tits space, $G$ a group of $p$-isometries, and $y \in X$ such that $O(y)$ is $p$-bounded. Then $G$ has a fixed point.*

**Proof.** By Theorem 3.1 there is a unique ball $B(x, r)$ of minimum radius $r$ containing $O(y)$. The point $x \in X$ is shown to be a fixed point of $G$. Let $g \in G$. For each $z \in B(x, r)$ it holds that $p(g(z), g(x)) = p(z, x) \leq r$, so $g(z) \in B(g(x), r)$. Hence $g(B(x, r)) \subseteq B(g(x), r)$. Conversely, for every $z \in B(g(x), r)$, it holds that $p(g^{-1}(z), x) = p(z, g(x)) \leq r$, so $g^{-1}(z) \in B(x, r)$, so $z \in g(B(x, r))$. Consequently, $B(g(x), r) \subseteq g(B(x, r))$. Conclude that $g(B(x, r)) = B(g(x), r)$. But then Lemma 4.1 and the inclusion $O(y) \subseteq B(x, r)$ imply that $O(y) = g(O(y)) \subseteq g(B(x, r)) = B(g(x), r)$, i.e., $B(g(x), r)$ is also a ball of minimum radius containing $O(y)$. By uniqueness (Theorem 3.1), it follows that $g(x) = x$. $\square$

The original Bruhat-Tits fixed point theorem is recovered by restricting attention to the space $(X, d, d)$, i.e., by simply considering the $w$-distance $p = d$.

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