Fixed Points in a Group of Isometries

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Abstract: The Bruhat-Tits fixed point theorem states that a group of isometries on a complete metric space with negative curvature possesses a fixed point if it has a bounded orbit. This theorem is extended by a relaxation of the negative curvature condition in terms of the w-distance functions introduced by Kada *et al.* [Non-convex minimization theorems and fixed point theorems in complete metric spaces, *Math. Japon.* 44 (1996), 381-391].

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1 Introduction

According to the Bruhat-Tits fixed point theorem, a group of isometries on a complete metric space with negative curvature — a complete metric space satisfying a relaxation of the parallelogram law — possesses a fixed point if it has a bounded orbit; see [2] for the original statement, [1] for an extensive overview of the theory on Bruhat-Tits buildings, and [5] for a very accessible treatment of this fixed point theorem.

The result and some of its special cases have wide applicability, witnessing their use in for instance the Bruhat-Tits theory of buildings, group theory (cf. Cartan's fixed point theorem, [3, Ch. I, Theorem 13.5]), and the theory of trees [6, Section I.4.3, Proposition 19]. Several applications are discussed in [1, Ch. VI]. Perhaps surprisingly, it has also been used in the study of communication to establish that individuals speaking different languages that allow sufficient freedom to express nuances have a common interpretation of at least some phrases in their vocabulary; see [9].

The purpose of this note is to extend the Bruhat-Tits fixed point theorem by stating the negative curvature condition not in terms of distance functions, but using the more general notion of w-distances, as introduced and studied in a recent sequence of papers by Kada, Suzuki, and Takahashi [4], Suzuki and Takahashi [8], and Suzuki [7].

Section 2 recalls the definition of w-distances. Generalized Bruhat-Tits spaces are considered in Section 3. Finally, the fixed point theorem is provided in Section 4.

2 Preliminaries

This section settles some standard matters of notation and defines the w-distance functions introduced in Kada *et al.* [4]. Denote by $\mathbb{N}$ the set of positive integers. Let X be a metric space with distance d. Following [4, p. 381], we call a function $p : X \times X \to [0, \infty)$ a *w-distance* on X if the following conditions hold:

- p satisfies the triangle inequality, i.e., $\forall x, y, z \in X : p(x, z) \leq p(x, y) + p(y, z)$;
- $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous for every $x \in X$, i.e., if a sequence $(y_m)_{m \in \mathbb{N}}$ in X converges to $y \in X$, then $p(x, y) \leq \liminf_{m \to \infty} p(x, y_m)$;
- for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $x, y, z \in X$: if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$, then $d(x, y) \leq \varepsilon$.

The metric d is a w-distance. Examples of many other w-distances are found in [4] and [8, Lemma 1]. Kada *et al.* [4, Lemma 1] prove:

Lemma 2.1 *Let (X, d) be a metric space and let p be a w-distance on X. Consider points $x, y, z \in X$, a sequence $(x_n)_{n \in \mathbb{N}}$ in X, and a sequence $(\alpha_n)_{n \in \mathbb{N}}$ in $[0, \infty)$ converging to zero. The following claims hold:*

(a) *If $p(x_n, x_m) \leq \alpha_n$ for all $m, n \in \mathbb{N}$ with $m > n$, then $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in (X, d).*

(b) *If $p(x, y) = p(x, z) = 0$, then $y = z$.*

Balls are defined with respect to the w-distance p. The ball around $x \in X$ with radius $r \geq 0$ is the set

$$B(x, r) := \{y \in X \mid p(y, x) \leq r\}.$$

A set $S \subseteq X$ is *p-bounded* if there exists a ball $B(x, r)$ such that $S \subseteq B(x, r)$.

The composition of two functions $g, h : X \to X$ is denoted by $gh : x \mapsto g(h(x))$. For $S \subseteq X$, write $g(S) = \{g(s) \mid s \in S\}$.

3 Generalized Bruhat-Tits spaces

A *generalized Bruhat-Tits space* (X, d, p) is a complete metric space (X, d) with a w-distance p satisfying the following property: for any two points $x_1, x_2 \in X$ there is a point $z \in X$ such that for all $y \in X$,

$$p(x_1, x_2)^2 + 4p(y, z)^2 \leq 2p(y, x_1)^2 + 2p(y, x_2)^2. \tag{1}$$

This is a generalization in terms of the w-distance p of the well-known negative curvature inequality, sometimes referred to as the semi-parallelogram law [cf. 5]. For a brief motivation, we resort to planar geometry. Consider Figure 1. The parallelogram law states that, using the Euclidean distance d_2, the sum of the squared lengths of the diagonals equals the sum of the squared lengths of the sides of the parallelogram:

$$d_2(x_1, x_2)^2 + d_2(y, x_3)^2 = 2d_2(y, x_1)^2 + 2d_2(y, x_2)^2.$$

Let z be the midpoint between x_1 and x_2. Since $2d(y, z) = d(y, x_3)$, substitution in the parallelogram law yields:

$$d_2(x_1, x_2)^2 + 4d_2(y, z)^2 = 2d_2(y, x_1)^2 + 2d_2(y, x_2)^2.$$

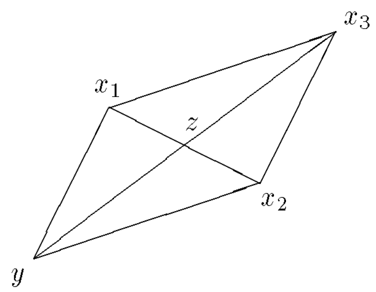

Figure 1: The (semi-)parallelogram law

Generalizing this to an arbitrary metric space (X,d) and allowing for a weak inequality, rather than equality, (X,d) is said to satisfy the semi-parallelogram law if for any two points $x_1, x_2 \in X$ there exists a point $z \in X$ such that for all $y \in X$,

$$d(x_1,x_2)^2 + 4d(y,z)^2 \leq 2d(y,x_1)^2 + 2d(y,x_2)^2.$$

Replacing the distance function d with the more general notion of a w-distance p, one finds condition (1).

Theorem 3.1 *Let (X,d,p) be a generalized Bruhat-Tits space and let $S \subseteq X$ be p-bounded. There exists a unique ball of minimum radius containing S.*

Proof.
Existence: Since S is p-bounded, the set $\mathcal{B} = \{B(x,r) \mid S \subseteq B(x,r)\}$ of balls containing S is nonempty. Set $r = \inf_{B(x,r') \in \mathcal{B}} r'$ and let $B(x_n, r_n)$ be a sequence in $\mathcal{B}$ such that $(r_n)_{n\in\mathbb{N}}$ is a non-increasing sequence converging to r. We proceed to show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d). Let $m,n \in \mathbb{N}$ with $m > n$. Then (1) implies the existence of a point $z_{mn} \in X$ such that

$$\forall y \in X: \quad p(x_n,x_m)^2 + 4p(y,z_{mn})^2 \leq 2p(y,x_n)^2 + 2p(y,x_m)^2. \tag{2}$$

By definition of r, there exists a point $y \in S$ with $p(y, z_{mn}) \geq r^2 - \frac{1}{n}$. Substituting this in (2) and using that $y \in B(x_k, r_k)$ for all $k \in \mathbb{N}$ and $(r_k)_{k\in\mathbb{N}}$ is a non-increasing sequence, yields:

$$\begin{aligned} p(x_n,x_m)^2 &\leq 2p(y,x_n)^2 + 2p(y,x_m)^2 - 4p(y,z_{mn})^2 \\ &\leq 2r_n^2 + 2r_m^2 - 4(r^2 - \frac{1}{n}) \\ &\leq 2r_n^2 + 2r_n^2 - 4(r^2 - \frac{1}{n}) \\ &= 4(r_n^2 - r^2) + \frac{4}{n}. \end{aligned}$$

Lemma 2.1(a) with $\alpha_n = \sqrt{4(r_n^2 - r^2) + \frac{4}{n}}$ implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d).

Since (X, d) is a complete metric space, the Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ has a limit $x \in X$. We proceed to show that the ball $B(x, r)$ is a ball of minimum radius containing S. The definition of r implies that the radius is indeed minimal. Let $y \in S$. The triangle inequality and the lower semicontinuity property of p imply that for each $n \in \mathbb{N}$:

$$\begin{aligned} p(y, x) &\leq p(y, x_n) + p(x_n, x) \\ &\leq r_n + \liminf_{m\to\infty} p(x_n, x_m) \\ &\leq r_n + \alpha_n, \end{aligned}$$

where the last inequality follows from the fact that $p(x_n, x_m) \leq \alpha_n$ for each $m > n$, as shown earlier in the proof. Taking limits as $n \to \infty$ yields that $p(y, x) \leq r$, i.e., $y \in B(x, r)$. Hence $S \subseteq B(x, r)$.

Uniqueness: Suppose there are two balls of minimum radius containing S: $B(x_1, r)$ and $B(x_2, r)$. Then (1) implies the existence of a point $z \in X$ such that for each $y \in X$:

$$p(x_1, x_2)^2 + 4p(y, z)^2 \leq 2p(y, x_1)^2 + 2p(y, x_2)^2. \tag{3}$$

Since r is the minimum radius of a ball containing S, it follows that for each $\varepsilon > 0$ there exists a $y \in S$ such that $p(y, z) \geq r - \varepsilon$. Substituting this in (3) and using the fact that $y \in S \subseteq B(x_i, r)$ for $i = 1, 2$, yields that $\forall \varepsilon \in (0, r)$:

$$\begin{aligned} p(x_1, x_2)^2 &\leq 2p(y, x_1)^2 + 2p(y, x_2)^2 - 4p(y, z)^2 \\ &\leq 2r^2 + 2r^2 - 4(r - \varepsilon)^2 \\ &= 4r^2 - 4(r - \varepsilon)^2. \end{aligned}$$

Letting $\varepsilon \to 0$, this implies that $p(x_1, x_2)^2 \leq 0$. Consequently, $p(x_1, x_2) = 0$. Similarly, $p(x_2, x_1) = 0$. The triangle inequality implies that $0 \leq p(x_1, x_1) \leq p(x_1, x_2) + p(x_2, x_1) = 0$, so $p(x_1, x_1) = 0$. Lemma 2.1(b) and $p(x_1, x_1) = p(x_1, x_2) = 0$ imply that $x_1 = x_2$: the two balls are identical. □

4 Fixed point theorem

Let (X, d, p) be a generalized Bruhat-Tits space. A *p-isometry* of X is a bijection $g : X \to X$ such that g preserves p-distances:

$$\forall x_1, x_2 \in X : \quad p(g(x_1), g(x_2)) = p(x_1, x_2).$$

Note that if g and h are p-isometries, so is their composition gh and the inverse g^{-1}. Moreover, the identity function id : $x \mapsto x$ is a p-isometry. The straightforward proofs of these claims are left to the reader. A *group* G of p-isometries is a set of p-isometries such that

(a) the identity function id : $x \mapsto x$ is an element of G;

(b) G is closed under inversion: if $g \in G$, then $g^{-1} \in G$;

(c) G is closed under composition: if $g_1, g_2 \in G$, then $g_1 g_2 \in G$.

Let G be a group of isometries and $x \in X$. The *orbit* $O(x)$ of x is defined to be the collection of images $g(x)$ with $g \in G$:

$$O(x) := \{g(x) \mid g \in G\}.$$

Lemma 4.1 *Let $x \in X, g \in G$. Then $g(O(x)) = O(x)$.*

Proof.
($\subseteq$): Let $y \in g(O(x))$, i.e., there is a $h \in G$ such that $y = gh(x)$. Since G is closed under composition: $gh \in G$, so $y \in O(x)$.
($\supseteq$): Let $y \in O(x)$, i.e., there is a $h \in G$ such that $y = h(x)$. Since $g \in G$, it follows that $g^{-1} \in G$ and $g^{-1}h \in G$. Consequently, $y = h(x) = (gg^{-1})h(x) = g(g^{-1}h(x)) \in g(O(x))$. □

Let G be a group of p-isometries in a generalized Bruhat-Tits space (X, d, p). A *fixed point* of G is a point $x \in X$ such that $g(x) = x$ for all $g \in G$, or equivalently, an $x \in X$ with $O(x) = \{x\}$. If there is a point with a p-bounded orbit, then G has a fixed point.

Theorem 4.2 *Let (X, d, p) be a generalized Bruhat-Tits space, G a group of p-isometries, and $y \in X$ such that $O(y)$ is p-bounded. Then G has a fixed point.*

Proof. By Theorem 3.1 there is a unique ball $B(x, r)$ of minimum radius r containing $O(y)$. The point $x \in X$ is shown to be a fixed point of G. Let $g \in G$. For each $z \in B(x, r)$ it holds that $p(g(z), g(x)) = p(z, x) \leq r$, so $g(z) \in B(g(x), r)$. Hence $g(B(x, r)) \subseteq B(g(x), r)$. Conversely, for every $z \in B(g(x), r)$, it holds that $p(g^{-1}(z), x) = p(z, g(x)) \leq r$, so $g^{-1}(z) \in B(x, r)$, so $z \in g(B(x, r))$. Consequently, $B(g(x), r) \subseteq g(B(x, r))$. Conclude that $g(B(x, r)) = B(g(x), r)$. But then Lemma 4.1 and the inclusion $O(y) \subseteq B(x, r)$ imply that $O(y) = g(O(y)) \subseteq g(B(x, r)) = B(g(x), r)$, i.e., $B(g(x), r)$ is also a ball of minimum radius containing $O(y)$. By uniqueness (Theorem 3.1), it follows that $g(x) = x$. □

The original Bruhat-Tits fixed point theorem is recovered by restricting attention to the space (X, d, d), i.e., by simply considering the w-distance $p = d$.

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