Perpetual Options and Canadization Through Fluctuation Theory

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Abstract

In this article it is shown that one is able to evaluate the price of perpetual calls, puts, Russian and integral options directly as the Laplace transform of a stopping time of an appropriate diffusion using standard fluctuation theory. This approach is offered in contrast to the approach of optimal stopping through free boundary problems [see volume 39,1 of Theory of Probability and its Applications]. Following ideas in [5], we discuss the Canadization of these options as a method of approximation to their finite time counterparts. Fluctuation theory is again used in this case.

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1 Introduction

We begin by introducing the standard stochastic model of a complete arbitrage free market. The market consists of a bond and a risky asset. The value of the bond $B = \{B_t : t \geq 0\}$ evolves in time deterministically such that

$$B_t = B_0 e^{rt}, \quad B_0 > 0, \quad r \geq 0, \quad t \geq 0. \tag{1.1}$$

The value of the risky asset $S = \{S_t : t \geq 0\}$ is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with the following components. Ω is the space of continuous functions $\omega = \{\omega_t\}_{t\geq 0}$, from $[0, \infty)$ to $\mathbb{R}$ with $\omega_0 = 0$. $\mathcal{F}$ is the smallest σ-algebra on Ω such that for every $t \geq 0$, the map $\omega \mapsto \omega_t$ of Ω to $\mathbb{R}$ is $\mathcal{F}/\mathcal{B}$-measurable, where $\mathcal{B}$ is the Borel-σ-algebra on $\mathbb{R}$. The probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is such that $W = W(\omega) = \{\omega_t : t \geq 0\} = \{W_t : t \geq 0\}$ is a Wiener process starting from the origin. Let $\mathcal{F}_t^0$ be the σ-algebra generated by W up to time t, then the filtration $\mathbf{F}$ is a flow of σ-algebras $\{\mathcal{F}_t : t \geq 0\}$, which are equal to the closure of $\cap_{s>t}\mathcal{F}_s^0$ by the $\mathbb{P}$-null sets of $\mathcal{F}$. The dynamics of the risky asset under $\mathbb{P}$ are given by an exponential of a Brownian motion with drift

$$S_t = s \exp\{\sigma W_t + \mu t\}$$

where $s > 0$, $\sigma > 0$ and $\mu \in \mathbb{R}$.

An option is a contract between the seller and the buyer, in which the buyer receives payments of the seller if certain events happen. Options may be divided two classes: *American* type options, which can be exercised at any time before the expiration date and *European* type options, which have exercise only at expiration. A *perpetual* option is an American type option with no expiration date. The buyer of a perpetual has the right to exercise it at any time t and receive then a payment π_t, which depends in some way on the underlying stock price S. Note that the zero time point is always taken to be the instant at which the contract commences. Examples of perpetual options are the call, the put, the Russian option [20, 21], and the integral option [11], with payments $\pi^c, \pi^p, \pi^r, \pi^i$ respectively:

$$\pi_t^p = e^{-\lambda t}(K - S_t)^+ \qquad \pi_t^c = e^{-\lambda t}(S_t - K)^+ \tag{1.2}$$

$$\pi_t^r = e^{-\lambda t} \max\left\{\max_{u\leq t} S_u, s\psi\right\} \qquad \pi_t^i = e^{-\lambda t}\left[\int_0^t S_u \mathrm{d}u + s\varphi\right] \tag{1.3}$$

where $\lambda, K, \psi, \varphi > 0$ are constants.

Remark 1.1 The parameter K is called the *strike* price, s is usually taken as the value of the stock at time zero and we use y^+ to denote $\max\{y, 0\}$. The parameter λ can be considered as a continuous dividend rate. In order for the arbitrage free price of the Russian and integral perpetual option to be finite, λ has to be positive, whereas the price of put and call remain finite for $\lambda = 0$. See also [21, 6, 19]. Note $s\psi$ can be understood to be the supremum of the risky asset price process over some pre-contract period. Likewise, $s\varphi$ can be understood to be the integral of the stock price over some pre-contract period.

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The payoffs of the perpetual call and put differ fundamentally from that of the Russian and integral option. The payoff of call and put only depend on the value of the underlying stock S at the exercise time, whereas the Russian and integral options are path dependent options. That is to say, that the payoff π_t depends on the whole path of the stock price S from some instant at or before the contract begins and up to time t.

Two fundamental questions that can be asked of American-type and perpetual options are:

Q1. *What is the arbitrage free price of the option?* and

Q2. *What is an optimal time to exercise?*

Theorems 1.2 and 1.3 (see also for example [19] and [9]) give answers to these questions, but in a form that is not handy from an applied perspective. In order to state these theorems, we must first introduce a little more notation.

Throughout this article we shall use the letters s and x with the assumed relation

$$s = \exp\{\sigma x\}$$

to represent the relationship between the starting points of S and W. We introduce the measure $\mathbb{P}_x$ which is a translation of the measure $\mathbb{P}$ such that under $\mathbb{P}_x$, W is a Wiener process with inital position $W_0 = x$. Now introduce the measure $\mathbb{P}_x^\gamma$ under which $W_t - \gamma t$ is a Wiener process starting from x. The measures $\mathbb{P}_x^\gamma$ and $\mathbb{P}_x$ are related through the Girsanov change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}_x^\gamma}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \exp\left\{\gamma W_t - \frac{1}{2}\gamma^2 t\right\}.$$

Henceforth it is understood that $\mathbb{E}_x^\gamma$ refers to expectation with respect to $\mathbb{P}_x^\gamma$. Note the value of the risky asset under $\mathbb{P}^{\mu/\sigma}$ satisfies $S_t = \exp\{\sigma W_t\}$.

Finally let $\mathcal{T}_{t,\infty}$ be the set of $\mathbf{F}$-stopping times valued in $[t, \infty)$ and $\overline{\mathcal{T}}_{t,\infty}$ the set of $\mathbf{F}$-stopping times valued in $[t, \infty]$ where $t \geq 0$.

Suppose now that $\pi = \{\pi_t : t \geq 0\}$ is an $\mathbf{F}$-adapted sequence of payments. The following well established theorem addresses Q1 when the option holder has even the right never to exercise, corresponding to the case that their exercise time is infinite with possibly positive probability.

Theorem 1.2 *The artbitrage free price $\Pi(t, s)$ for an American type perpetual option at time t into the contract, with payments π and S starting at s satisfies*

$$\Pi(t, s) = \operatorname*{ess\,sup}_{\tau \in \overline{\mathcal{T}}_{t,\infty}} \mathbb{E}_x^{(r/\sigma - \sigma/2)} \left[\mathrm{e}^{-r(\tau - t)} \pi_\tau \middle| \mathcal{F}_t \right].$$

In particular, the arbitrage free price of the option is given by

$$\sup_{\tau \in \overline{\mathcal{T}}_{0,\infty}} \mathbb{E}_x^{(r/\sigma - \sigma/2)} \left[\mathrm{e}^{-r\tau} \pi_\tau \right] \tag{1.4}$$

If we formulate the problem insisting that the buyer must exercise within an almost surely finite time then exactly the same result holds except that $\overline{\mathcal{T}}_{t,\infty}$ should be replaced by $\mathcal{T}_{t,\infty}$.

The next Theorem, taken from [19], addresses Q2.

Theorem 1.3 *Suppose that the payments π are $\mathcal{F}_t$-measurable, càdlàg, without negative jumps and*

$$\{\mathrm{e}^{-r\tau}\pi_\tau : \tau \in \overline{\mathcal{T}}_{0,\infty}\}$$

is uniformly integrable with respect to $\mathbb{P}^{(r/\sigma - \sigma/2)}$. Then

$$\tau^* = \inf\{t \geq 0 : \Pi(t, s) \leq \pi_t\}$$

is an optimal exercise time for (1.4).

Again, when the problem of pricing is reformulated so that the buyer must exercise within an almost surely finite time, in the above Theorem we can replace $\overline{\mathcal{T}}_{0,\infty}$ by $\mathcal{T}_{0,\infty}$.

In reviewing the litterature concerning perpetual options one finds two dominant methods that are used for their evaluation given that Theorems 1.2 and 1.3 hold.

Free boundary problem approach. The first method has been nicely characterized in a series of papers [19, 21, 11] that appeared all together in volume 39,1 of Theory of Probability and its Applications. However its origin can be traced back as far as McKean's paper [14] in 1965. In these papers an approach based on free boundary problems, sometimes called Stephan problems, is applied to perpetual American call and put options, Russian options and integral options. Based on heuristic reasoning, the solution to an appropriate free boundary problem is taken as a candidate price for the option at hand. Then this solution is shown to be equal to the supremum (1.4) as a consequence of it being a solution to the free boundary problem.

Fluctuation theory appoach. The second approach [19, 12], used for evaluating American call and put options, consists of proving that the optimal stopping time has the form of a hitting time of the stock price at some level, say a. Given that $(K-S_t)^+$ (or indeed $(S_t-K)^+$) is constant at such a hitting time, the price of the option is essentially proportional to the Laplace transform of the hitting time optimized over the level a. The computations for this procedure are very elementary once the optimal stopping time is realized as a hitting time.

In the case of the Russian perpetual option, it is also worth mentioning the paper of [7]. In this proof the authors use two important properties to recover the price of the Russian perpetual. The first is that for a continuous positive Markov processes Z with associated operator L and starting point $z \leq v$, if τ_v is a hitting time then, the expectation $E_z(\mathrm{e}^{-\lambda\tau_v}Z_{\tau_v})$ is a solution to the system $Lu = \lambda u$ with $u(v) = v$. (Note we have used obvious notation). The second fact is the strong Markov property. These two essentially are enough to show that the optimal stopping time is that of a hitting time of an appropriate diffusion and also give the analytical form of the solution.

Below we give the conclusion of both the fluctuation theory and free boundary methods for perpetual calls and puts and the conclusion achieved by the first of these two methods for perpetual Russian and integral options. Recall that r and σ are parameters of the market (B,S) and λ is a parameter appearing in the claims outlined in (1.2) and (1.3).

Let $x_1 < 0 < x_2$ be the two roots of the quadratic equation

$$x^2 - \left(1 - \frac{2r}{\sigma^2}\right)x - \left(\frac{2\lambda + 2r}{\sigma^2}\right) = 0. \tag{1.5}$$

Theorem 1.4 *The arbitrage free price of a perpetual call and put at time t into the contract,$\Pi^{\mathrm{call}}(t,s)$ and $\Pi^{\mathrm{put}}(t,s)$, with payoff π^c and π^p respectively, are given by*

$$\Pi^{\mathrm{call}}(t,s) = \mathrm{e}^{-\lambda t}\Pi^C(S_t) \text{ and } \Pi^{\mathrm{put}}(t,s) = \mathrm{e}^{-\lambda t}\Pi^P(S_t) \tag{1.6}$$

where

$$\Pi^C(s) = \begin{cases} (s_2 - K)\,(s/s_2)^{x_2} & \text{if } s < s_2 \\ s - K & \text{if } s \geq s_2 \end{cases}$$

and

$$\Pi^P(s) = \begin{cases} (K - s_1)\,(s/s_1)^{x_1} & \text{if } s > s_1 \\ K - s & \text{if } s \leq s_1. \end{cases}$$

Here

$$s_1 = K\frac{x_1}{x_1 - 1} < K\frac{x_2}{x_2 - 1} = s_2$$

are the optimal exercise boundaries. That is to say that the holder should exercise if the value of the asset exceeds or falls below s_2 and s_1 in the case of the call and put respectively.

Consider now the equation

$$y^2 - \left(1 + \frac{2r}{\sigma^2}\right)y - \left(\frac{2\lambda}{\sigma^2}\right) = 0 \tag{1.7}$$

with roots $y_1 < 0 < y_2$.

Theorem 1.5 *The arbitrage free price $\Pi^{\mathrm{russ}}(t,s,\psi)$ of a perpetual Russian option at time t into the contract, with payoff π^r satisfies*

$$\Pi^{\mathrm{russ}}(t,s,\psi) = \mathrm{e}^{-\lambda t}S_t\Pi^{\mathrm{R}}(\Psi_t)$$

where $\Psi_t := (\sup_{0\leq u\leq t} S_u \vee s\psi)/S_t$ and

$$\Pi^{\mathrm{R}}(\psi) = \begin{cases} \tilde{\psi}\cdot\frac{y_2\psi^{y_1} - y_1\psi^{y_2}}{y_2\tilde{\psi}^{y_1} - y_1\tilde{\psi}^{y_2}}, & 1 \leq \psi < \tilde{\psi}, \\ \psi, & \psi \geq \tilde{\psi}. \end{cases} \tag{1.8}$$

Here

$$\tilde{\psi} = \left|\frac{y_2}{y_1}\cdot\frac{y_1 - 1}{y_2 - 1}\right|^{\frac{1}{y_2 - y_1}}$$

is the optimal exercise boundary. That is to say that the holder should exercise if the process Ψ_t exceeds or equals ψ.

Theorem 1.6 *The arbitrage free price $\Pi^{\mathrm{int}}(t,s,\varphi)$ of a perpetual integral option at time t into the contract with payoff π^i satisfies*

$$\Pi^{\mathrm{int}}(t,s,\varphi) = \mathrm{e}^{-\lambda t} S_t \Pi^{\mathrm{I}}(\Phi_t)$$

where $\Phi_t := \left(\int_0^t S_u du + \varphi s\right) / S_t$ and

$$\Pi^I(\varphi) = \begin{cases} \varphi^* \frac{u(\varphi)}{u(\varphi^*)}, & 0 \le \varphi < \varphi^*, \\ \varphi, & \varphi \ge \varphi^* \end{cases} \tag{1.9}$$

where

$$u(\varphi) = \int_0^\infty \mathrm{e}^{-2z/\sigma^2} z^{-y_2} (1+\varphi z)^{y_1} \mathrm{d}z$$

and φ^* is the root of the equation $\varphi u'(\varphi) = u(\varphi)$. Here φ^* is the optimal exercise boundary, such that the holder should exercise once the process Φ_t exceeds or equals φ^*.

In this paper we shall show that the pricing of Russian and integral perpetual options can also be reduced to evaluating a Laplace transform of the hitting time of an appropriate diffusion, followed by a simple optimization over the hitting level. These new proofs will rely heavily on fluctuation theory of Brownian motion and Bessel processes thus remaining loyal to ideas used in pricing perpetual calls and puts as explained in the second method above.

Several different proofs for pricing perpetual Russian options and one proof for the pricing of integral options already exist, [20, 21, 6, 11, 7]. One might therefore question the motivation behind providing alternative proofs. The first reason is that the method of proof exposed in this paper can, in principle, be applied (in particular in the case of the Russian option) in markets where the underlying is assumed to be driven by a spectrally one sided Lévy process. The interested reader is refered to [1]. The free boundary problem approach should also be applicable in this latter case. However knowledge of solutions to integro-differential equations is needed as opposed to fluctuation theory of Lévy processes. The second of these two has enjoyed a considerable amount of attention in recent years. Secondly, the fluctuation techniques also give us an approach to deal with the issue of Canadization. As a subsiduary reason, these proofs give supplementary material that may be used for teaching purposes in a subject area which is becoming more firmly embedded within university mathematics curricula.

The rest of this paper is organized as follows. In the next section, for the sake of completeness and later reflection, we review the derivation of the arbitrage free price of perpetual calls and puts in the context of fluctuation theory. Continuing in this vane, in section 3 we show how the value of the Russian perpetual option can be established in a similar way. The strength of section 3 centres around Theorem 3.2 which evaluates the stopping time of a process representing the excursions of Brownian motion away from its supremum. Section 4 deals with the integral option. In this case the stopping time turns out to be that of a Bessel squared process with drift. This follows from the close relationship between exponential Brownian motion and Bessel squared processes. This connection appears in the study of Asian options in [8]. Integral options can be considered in some sense as perpetual Asian options and thus it is not suprising that the use of Bessel processes is a necessary tool as far as a fluctuation theory approach is concerned. Finally in section 5 we discuss the Canadization of Russian and integral options. Recently it has been proposed by Carr in [5] that finite expiry American type options can be approximated by a randomization of the expiry date using an independent exponential distribution. This is what Carr refers to as Canadian type options. The importance of Canadizing American call and put options follows from the lack of memory property of exponential distribution. The effect of randomization is to make the optimal exercise boundary a constant, just like in the perpetual case. A better approximation to a fixed time expiry than this can be made by randomizing using a sum of n independent exponential distributions (hence an Erlang distribution) whose total mean is the length of the contract. As n tends to infinity, it is possible to show convergence to the price of the finite expiry American option. These ideas work equally well for the Russian and integral option.

On a final note we should say that the use of fluctuation theory, as indicated in the title of this paper, in effect constitutes only half of the pricing procedure. There is still a strength of optimal stopping theory found in Theorems 1.2 and 1.3 which give the foundation on which we build. For standard references in the context of these the reader is referred to [18], [16] and [12].

2 Perpetual Call and Put Options

Combining Theorem 1.2 with the actual form of the system of payments for call and put (1.2), we find by a simple Markovian decompostion of the process S_t that the the price $\Pi^{\text{call}}, \Pi^{\text{put}}$ of a perpetual call and put satisfy (1.6) where

$$\Pi^{\text{call}}(t,s) = \mathrm{e}^{-\lambda t}\Pi^{\mathrm{C}}(S_t) = \sup_{\tau\in\overline{\mathcal{T}}_{0,\infty}} \mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\mathrm{e}^{-(r+\lambda)\tau}(S_\tau - K)^+\right] \tag{2.1}$$

$$\Pi^{\text{put}}(t,s) = \mathrm{e}^{-\lambda t}\Pi^{\mathrm{P}}(S_t) = \sup_{\tau\in\overline{\mathcal{T}}_{0,\infty}} \mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\mathrm{e}^{-(r+\lambda)\tau}(K - S_\tau)^+\right]. \tag{2.2}$$

Corollary 2.1 *The optimal stopping times in (2.1) and (2.2) are of the form*

$$\inf\{t \geq 0 : S_t \geq \mathrm{e}^{\sigma h}\} \text{ and } \inf\{t \geq 0 : S_t \leq \mathrm{e}^{\sigma l}\}$$

respectively where h and l are real constants.

PROOF. By choosing $\tau = 0$, we see that $\Pi^{\text{call}}(s) \geq (s-K)^+$, $\Pi^{\text{put}}(s) \geq (K-s)^+$, that is, perpetual calls and puts are always as least as valuable as the direct payoff. Noting that the function $x \mapsto (x-K)^+$ is increasing and convex, we see $\Pi^{\text{call}}(\cdot)$ is increasing and convex, since integration and taking the supremum preserve monotonicity and convexity. Furthermore, Π^{call} is bounded above by $\sup_\tau \mathbb{E}_x^{(r/\sigma-\sigma/2)}[\mathrm{e}^{-r\tau}S_\tau] \leq s$. Similarly, by the properties of $x \mapsto (K-x)^+$, $\Pi^{\text{put}}(\cdot)$ is bounded by K, decreasing and convex. Theorem 1.3 implies the optimal stopping times for the call and put are given by $\inf\{t \geq 0 : \Pi^{\text{call}}(S_t) = (S_t - K)^+\}$ and $\inf\{t \geq 0 : \Pi^{\text{put}}(S_t) = (K - S_t)^+\}$ respectively, which combined with above remarks completes the proof. □

Remark 2.2 If we define for any Borel set B

$$\tau_B^W = \inf\{t \geq 0 : W_t \in B\},$$

then both the stopping times in the above Corollary can be expressed respectively as $\tau^W_{[h,\infty)}$ and $\tau^W_{(-\infty,l]}$ under $\mathbb{P}_x^{(r/\sigma-\sigma/2)}$.

By Corollary 2.1, the supremum over all stopping times in $\overline{\mathcal{T}}_{0,\infty}$ in equations (2.1) and (2.2) is equal to the supremum over all hitting times $\{\tau^W_{[h,\infty)} : h \in \mathbb{R}\}$ and $\{\tau^Z_{(-\infty,l]} : l \in \mathbb{R}\}$ respectively. Thanks to the continuity of Brownian motion, there is no overshoot at these stopping times. Thus the prices Π^{call}, Π^{put} are given by $\Pi^{\text{call}}(s) = \sup_{h\in\mathbb{R}} V_h^{(1)}(s)$ and $\Pi^{\text{put}}(s) = \sup_{l\in\mathbb{R}} V_l^{(2)}(s)$ where

$$V_h^{(1)}(s) = \begin{cases} \mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\mathrm{e}^{-(r+\lambda)\tau^W_{[h,\infty)}}\right](\mathrm{e}^{\sigma h} - K)^+ & \log s < \sigma h, \\ (s-K)^+ & \log s \geq \sigma h, \end{cases} \tag{2.3}$$

and

$$V_l^{(2)}(s) = \begin{cases} \mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\mathrm{e}^{-(r+\lambda)\tau^W_{(-\infty,l]}}\right](K - \mathrm{e}^{\sigma l})^+ & \log s > \sigma l, \\ (K-s)^+ & \log s \leq \sigma l. \end{cases} \tag{2.4}$$

Remark 2.3 The functions $V_h^{(1)}$ and $V_l^{(2)}$ in equations (2.3) and (2.4) have a clear financial interpretation. $V_h^{(1)}$ is the value of an option that "knocks in" on exceedance of the level $\exp\sigma h$ with call rebate, that is, the option expires as soon as the stocks exceeds the level $\exp\sigma h$ and pays out then the amount $(\exp\sigma h - K)^+$. By optimizing over all possible values of h we find the value of the perpetual call. Similarly, $V_l^{(2)}$ is the value function of an option which expires if the stock value falls below the level $\exp\sigma l$ and then pays out the amount $(K - \exp\sigma l)^+$.

Thus, the computation of the prices $\Pi^{\text{call}}, \Pi^{\text{put}}$ boils down to the computation of the Laplace transform of a hitting time of Brownian motion at a certain (constant) level, followed by an optimization over that level. This Laplace transform has a well known explicit formula to be found in any standard text on Brownian motion and can for example easily be derived using the Wald martingale. We thus quote without reference that

$$\mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\mathrm{e}^{-(r+\lambda)\tau^W_{[h,\infty)}}\right] = \mathrm{e}^{-\sigma x_2(h-x)} \text{ and } \mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\mathrm{e}^{-(r+\lambda)\tau^W_{(-\infty,l]}}\right] = \mathrm{e}^{\sigma x_1(x-l)}$$

when $h > x$ and $l < x$ respectively. Recall that x_1 and x_2 are the roots of the quadratic equation (1.5).

PROOF OF THEOREM 1.4. This follows as a simple optimization procedure in (2.3) and (2.4). □

Remark 2.4 Notice the optimal stopping times for the optimal stopping problem are not necessarily finite, depending on the sign of $r-\sigma^2/2$. If, for example, $r<\sigma^2/2$ and the risky asset starts below the optimal exercise value s_2, the optimal stopping time for a call is infinite with positive $\mathbb{P}^{(r/\sigma-\sigma/2)}$-probability. Had we insisted that the holder should exercise in an almost surely finite length of time, there would have been no optimal exercise strategy in this case.

3 Perpetual Russian Option

Following the lead of [19], the first step in solving this problem consists in recognizing that under $\mathbb{P}_x^{(r/\sigma-\sigma/2)}$, $s^{-1}\mathrm{e}^{-rt}S_t$ acts as a Girsanov change of measure, which adds an extra drift σ to the Wiener process W. If we insist now that the claimants of the Russian option must exercise within an almost surely finite time we can use the above change of measure together with Theorem 1.2 to get

$$\Pi^{\mathrm{russ}}(t,s,\psi)=S_t\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,\infty}}\mathbb{E}_x^{(r/\sigma+\sigma/2)}\left[\mathrm{e}^{-\lambda\tau}\frac{\overline{S}_\tau\vee\psi s}{S_\tau}\middle|\mathcal{F}_t\right]\tag{3.1}$$

where $\overline{S}_t:=\max_{0\le u\le t}S_u$. Introduce the new stochastic process $\Psi=\{\Psi_t,t\ge0\}$ with $\Psi_t=(\overline{S}_t\vee\psi s)/S_t$. Note that it can be easily verified that Ψ is a Markov process (see [21]). Suppose now that the underlying Brownian motion has been running not since time zero, but since some time $-M<0$ and further that, given $\mathcal{F}_0$, the exponential of the current distance of the Brownian motion from its previous maximum is ψ. In this instance Ψ can be understood to be the exponential of the excursions of a Brownian motion with drift away from its maximum given that at time zero its value is ψ. With this in mind, let us introduce a new measure for the process Ψ, namely $\overline{\mathbb{P}}_\psi^\gamma$, under which we assume that $W_t-\gamma t$ is a $\mathbb{P}_0^\gamma$-Brownian motion and $\Psi_0=\psi$. In light of the fact that Ψ is a Markov process we can thus re-write (3.1) as

$$\Pi^{\mathrm{russ}}(t,s,\psi)=\mathrm{e}^{-\lambda t}S_t\Pi^{\mathrm{R}}(\Psi_t)\tag{3.2}$$

with

$$\Pi^{\mathrm{R}}(\psi)=\sup_{\tau\in\mathcal{T}_{0,\infty}}\overline{\mathbb{E}}_\psi^{(r/\sigma+\sigma/2)}[\mathrm{e}^{-\lambda\tau}\Psi_\tau]$$

where $\overline{\mathbb{E}}_\psi^\gamma$ is expectation with respect to $\overline{\mathbb{P}}_\psi^\gamma$ and, in effect, we may now take $\Psi_t:=\overline{S}_t/S_t$ (which is not a function of s).

Moreover, on account of Theorem 1.3, the optimal stopping time in (3.2) is given by

$$\inf\{s\ge0:\Pi^{\mathrm{R}}(\Psi_s)\le\Psi_s\}.\tag{3.3}$$

Corollary 3.1 *The optimal stopping time in (3.2) is given by*

$$\inf\{t\ge0:\Psi_s\ge\tilde{\psi}\}$$

for some constant $\tilde{\psi}\ge0$.

PROOF. By choosing the stopping time $\tau=0$ we see that $\Pi^{\mathrm{R}}(\psi)\ge\psi$. Now note that we can write

$$\Pi^{\mathrm{R}}(\psi)=\sup_{\tau\in\mathcal{T}_{0,\infty}}\overline{\mathbb{E}}_1^{(r/\sigma+\sigma/2)}[\mathrm{e}^{-\lambda\tau}(\overline{S}_\tau\vee\psi s)/S_\tau]$$

where the dependency on s is superficial as it disappears through cancelation in the ratio. Since for every $\omega\in\Omega$ the function $\psi\mapsto(\overline{S}_t\vee\psi s)/S_t$ is a convex increasing function, $\Pi^{\mathrm{R}}(\cdot)$ inherits these properties, as integration over ω and taking the supremum t preserve monotonicity and convexity. Combining these facts with Theorem 1.3 completes the proof. $\square$

It can now be seen that, just like the previous section, the valuation of the Russian option can be achieved by the evaluation of the Laplace transform of a crossing time. The following Theorem (proved later) tells us what we need to know.

Theorem 3.2 *For Borel sets B let*

$$\tau_B^\Psi=\inf\{s\ge0:\log\Psi_t\in B\}.$$

For $\log\psi\in[0,b]$ and $\gamma>0$ we have

$$\overline{\mathbb{E}}_\psi^\gamma[\mathrm{e}^{-\lambda\tau_{[b,\infty)}^\Psi}]=\mathrm{e}^{\gamma(\log\psi-b)/\sigma}\frac{\sqrt{2\lambda+\gamma^2}\cosh(\log\psi\sqrt{2\lambda+\gamma^2}/\sigma)-\gamma\sinh(\log\psi\sqrt{2\lambda+\gamma^2}/\sigma)}{\sqrt{2\lambda+\gamma^2}\cosh(b\sqrt{2\lambda+\gamma^2}/\sigma)-\gamma\sinh(b\sqrt{2\lambda+\gamma^2}/\sigma)}$$

Proof of Theorem 1.5. Corollary 3.1 and the continuity of Brownian motion imply that

$$\Pi^{\mathrm{R}}(\psi) = \sup_{m\in\mathbb{R}} V_m^{(3)}(\psi)$$

where

$$V_m^{(3)}(\psi) = \begin{cases} m\overline{\mathbb{E}}_\psi^{(r/\sigma+\sigma/2)}(\mathrm{e}^{-\lambda\tau^{\Psi}_{[\log m,\infty)}}) & 1\le\psi\le m, \\ \psi & \psi > m. \end{cases} \tag{3.4}$$

It follows as a matter of checking that

$$V_m^{(3)}(\psi) = \begin{cases} m\cdot\frac{y_2\psi^{y_1}-y_1\psi^{y_2}}{y_2 m^{y_1}-y_1 m^{y_2}} & 1\le\psi\le m, \\ \psi & \psi > m \end{cases}$$

where y_1 and y_2 are the two solutions to the quadratic equation (1.7). By elementary optimization we find, that Π^{R} is given by

$$\Pi^{\mathrm{R}}(\psi) = \begin{cases} \tilde\psi\cdot\frac{y_2\psi^{y_1}-y_1\psi^{y_2}}{y_2\tilde\psi^{y_1}-y_1\tilde\psi^{y_2}}, & 1\le\psi\le\tilde\psi, \\ \psi & \psi > \tilde\psi, \end{cases}$$

where

$$\tilde\psi = \left|\frac{y_1}{y_2}\cdot\frac{y_2-1}{y_1-1}\right|^{\frac{1}{y_2-y_1}}$$

is the optimal exercise boundary. □

We conclude this section by proving Theorem 3.2. We do this through the following two Lemmas.

Lemma 3.3 *Let $\sigma=1$. Define the events $\mathcal{A}(a,b) := \{\tau^W_{(-\infty,a]} < \tau^W_{[b,\infty)}\}$ for $a\le 0<b$, that is, $\mathcal{A}(a,b)$ is the part of Ω where the process W exits (a,b) at a. With $\mathbb{E}^\gamma$ as expectation with respect to $\mathbb{P}^\gamma=\mathbb{P}^\gamma_0$ we have*

$$\mathbb{E}^\gamma[\mathrm{e}^{-\lambda\tau^W_{[b,\infty)}}\mathbf{1}_{\mathcal{A}(a,b)^c}] = \mathrm{e}^{\gamma b}\cdot\frac{\sinh(-a\sqrt{2\lambda+\gamma^2})}{\sinh((b-a)\sqrt{2\lambda+\gamma^2})}$$

and

$$\mathbb{E}^\gamma[\mathrm{e}^{-\lambda\tau^W_{(-\infty,a]}}\mathbf{1}_{\mathcal{A}(a,b)}] = \mathrm{e}^{\gamma a}\cdot\frac{\sinh(b\sqrt{2\lambda+\gamma^2})}{\sinh((b-a)\sqrt{2\lambda+\gamma^2})}.$$

Proof. We only prove (i), the proof of (ii) is analogous. It can be found (or easily deduced) from any standard text on Brownian motion that

$$\mathbb{E}^\gamma[\mathrm{e}^{-\lambda\tau^W_{[b,\infty)}}] = \mathrm{e}^{-b\Phi(\lambda)} \text{ with } \Phi(\lambda) = -\gamma+\sqrt{2\lambda+\gamma^2}.$$

Girsanov's Theorem implies that

$$\begin{aligned}
\mathbb{E}^\gamma[\mathrm{e}^{-\lambda\tau^W_{[b,\infty)}}\mathbf{1}_{\mathcal{A}^c(a,b)}] &= \mathrm{e}^{-\Phi(\lambda)b}\mathbb{E}^\gamma[\mathrm{e}^{-\lambda\tau^W_{[b,\infty)}+\Phi(\lambda)b}\mathbf{1}_{\mathcal{A}^c(a,b)}] \\
&= \mathrm{e}^{-\Phi(\lambda)b}\mathbb{P}^{\Phi(\lambda)+\gamma}\left(\tau^W_{[b,\infty)} < \tau^W_{(-\infty,a)}\right) \\
&= \mathrm{e}^{-\Phi(\lambda)b}\cdot\frac{s(0)-s(a)}{s(b)-s(a)} \\
&= \mathrm{e}^{-\Phi(\lambda)b}\frac{\mathrm{e}^{-2a\sqrt{2\lambda+\gamma^2}}-1}{\mathrm{e}^{-2a\sqrt{2\lambda+\gamma^2}}-\mathrm{e}^{-2b\sqrt{2\lambda+\gamma^2}}} \\
&= \mathrm{e}^{\gamma b}\frac{\sinh(-a\sqrt{2\lambda+\gamma^2})}{\sinh((b-a)\sqrt{2\lambda+\gamma^2})}
\end{aligned}$$

where in the second equality $s(x)$ is the scale function of $W^{\gamma+\Phi(\lambda)}$ which is equal to $(1-\mathrm{e}^{-2x(\gamma+\Phi(\lambda))})/2(\gamma+\Phi(\lambda))$. For a proof of the double exit probability that appears in the second equality on can consult for example [17]. □

Now let $L(t)$ be the local time of the excursion process $\{\overline{W}_t - W_t : t\ge 0\}$ at zero where $\overline{W}_t := \sup_{0\le u\le t} W_u$.

$$L(t) = \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{[0,\varepsilon)}\left(\frac{\log\Psi_s}{\sigma}\right)ds$$

and denote $L^{-1}(t) = \inf\{s\ge 0 : L(s)\ge t\}$ its inverse.

Lemma 3.4 *Let $\sigma = 1$. Writing $\overline{\mathbb{P}}^\gamma$ as shorthand for $\overline{\mathbb{P}}^\gamma_1$ we have*

$$\overline{\mathbb{P}}^\gamma \left(\sup_{s \in [0, L^{-1}(t))} \log \Psi_s < b \right) = \exp \left\{ - \frac{2\gamma \mathrm{e}^{-2\gamma b} t}{1 - \mathrm{e}^{-2\gamma b}} \right\}.$$

Proof of Lemma 3.4. Under $\overline{\mathbb{P}}^\gamma$ with $\sigma = 1$, the process $\log \Psi_t$ can be written as the excursion process of W away from its supremum. From standard theory of Brownian motion, the suprema of excursions of $\log \Psi$ away from zero $\{h_t : t \geq 0\}$ form a Poisson point process indexed by the local time L with characteristic measure ν which can be found as follows. Under $\mathbb{P}^\gamma$ and for $x > 0$, the set $\mathcal{A}(-x, y) = \{\tau^W_{[y,\infty)} < \tau^W_{(-\infty,-x]}\}$ coincides with the set $\{h_t \leq t + x; 0 \leq t \leq y\}$ of excursions of W away from its supremum, upto local time y, which have height smaller than $x + t$ at local time t. Let $N_t(b)$ denote the number of excursions of maximal height greater or equal to b upto local time t. Then,

$$\begin{aligned} \mathbb{P}^\gamma (h_t \leq t + x; 0 \leq t \leq y) &= \mathbb{P}^\gamma (N_t(x + t) = 0, 0 \leq t \leq y) \\ &= \exp \left\{ - \int_0^y \nu([x + t, \infty)) \mathrm{d}t \right\}. \end{aligned} \tag{3.5}$$

On the other hand, we know

$$\mathbb{P}^\gamma \left(\tau^W_{[y,\infty)} < \tau^W_{(-\infty,-x]} \right) = \frac{s(0) - s(-x)}{s(y) - s(-x)}, \qquad x > 0, \tag{3.6}$$

where s denotes the scale function of a Brownian motion with drift γ $[s(x) = (1 - \mathrm{e}^{-2\gamma x})/2\gamma]$. Comparing (3.5) and (3.6) we find that $\nu([x, \infty)) = s'(x)/s(x)$ for $x > 0$. Performing a computation similar to (3.5) we now find

$$\begin{aligned} \overline{\mathbb{P}}^\gamma \left(\sup_{s \in [0, L^{-1}(t))} \log \Psi_s < b \right) &= \overline{\mathbb{P}}^\gamma (N_t(b) = 0) \\ &= \exp \{ -t\nu([b, \infty)) \} \\ &= \exp \left\{ - \frac{2\gamma \mathrm{e}^{-2\gamma b} t}{1 - \mathrm{e}^{-2\gamma b}} \right\}. \end{aligned}$$

□

Proof of Theorem 3.2. First we prove the identity for $\psi = 1$ and $\sigma = 1$. Begin by changing measure via the Girsanov Theorem to one under which W has no drift.

$$\begin{aligned} \overline{\mathbb{E}}^\gamma \left[\mathrm{e}^{-\lambda \tau^\Psi_{[b,\infty)}} \right] &= \overline{\mathbb{E}} \left[\mathrm{e}^{-(\lambda + \gamma^2/2)\tau^\Psi_{[b,\infty)} + \gamma W_{\tau^\Psi_{[b,\infty)}}} \right] \\ &= \overline{\mathbb{E}} \left[\mathrm{e}^{-(\lambda + \gamma^2/2)\tau^\Psi_{[b,\infty)} + \gamma (\overline{W}_{\tau^\Psi_{[b,\infty)}} - b)} \right] \end{aligned} \tag{3.7}$$

where $\overline{\mathbb{E}}$ is expectation with respect to $\overline{\mathbb{P}} := \overline{\mathbb{P}}^0$. Note that in order for this change of measure to make sense, it must be the case that $\tau^\Psi_{[b,\infty)}$ is almost surely finite. This follows as a result of the fact that excursions of Brownian motion from its supremum are recurrent when there is a non-negative drift. Next we note that $\overline{W}$ is identicle to L. It thus follows that we may write the left hand side of (3.7) as

$$\mathrm{e}^{-\gamma b} \overline{\mathbb{E}} \left[\mathrm{e}^{-(\lambda + \gamma^2/2)\tau^\Psi_{[b,\infty)} + \gamma L(\tau^\Psi_{[b,\infty)})} \right] \tag{3.8}$$

The stopping time $\tau^\Psi_{[b,\infty)}$ under $\overline{\mathbb{P}}$ can be interpreted as the the first time that the excursion process $\log \Psi = \overline{W} - W$, starting from 0, hits the level b. That is to say, starting from a current supremum, the first time the Brownian motion W wanders a distance b from the current supremum. If we denote

$$g = \sup\{t \leq \tau^\Psi_{[b,\infty)} : \log \Psi_t = 0\}$$

as the time of the left end point of the excursion in which the the level b is first reached then we can write $\tau^\Psi_{[b,\infty)} = g + \tau(g)$ where $\tau(g)$ is the extra time to hit b from the begining of the successfull excursion. From standard theory it is known that excursions of $\log \Psi$ away from zero form a Poisson point process on $\mathcal{E}$, the space of excursions, with characteristic measure n indexed by local time L. Note that the similar comments made in the proof of Lemma 3.4 are a special case of this structure. Denote $\varepsilon_g := \{\varepsilon_g(t) : t \geq 0\} \in \mathcal{E}$ the excursion with left point at g and thus write $\tau(g) = \inf\{t > 0 : \varepsilon_g(t) \geq b\}$ the hitting time of b for the succesful excursion. It is known that g and $\tau(g)$ are independent variables such that

$$\overline{\mathbb{E}}\left[\mathrm{e}^{-\lambda\tau(g)}\right] = \frac{b\sqrt{2\lambda}}{\sinh(b\sqrt{2\lambda})}, \tag{3.9}$$

see for example [24]. One may also note that (3.9) is effectively the Laplace transform of the time it takes a Bessel-3 process starting from the origin to hit the level b. It thus follows that

$$\begin{aligned}\overline{\mathbb{E}}\left[\mathrm{e}^{-(\lambda+\gamma^2/2)\tau^{\Psi}_{[b,\infty)}+\gamma L(\tau^{\Psi}_{[b,\infty)})}\right] &= \overline{\mathbb{E}}\left[\mathrm{e}^{-(\lambda+\gamma^2/2)\tau(g)}\right]\overline{\mathbb{E}}\left[\mathrm{e}^{-(\lambda+\gamma^2/2)g+\gamma L(g)}\right] \\ &= \frac{b\sqrt{2\lambda}}{\sinh(b\sqrt{2\lambda})}\overline{\mathbb{E}}\left[\mathrm{e}^{-(\lambda+\gamma^2/2)g+\gamma L(g)}\right] \end{aligned}\tag{3.10}$$

where the second expectation on the right hand side can be evaluated by making use of the compensation formula for excursions (see for example [2] for a good exposition). Write $\overline{\varepsilon}$ for the supremum of an excursion $\varepsilon \in \mathcal{E}$. Recall from the proof of Lemma 3.4, that $\nu = s'\cdot s^{-1}$ is the characteristic measure of the point process of suprema of excursions (where s is the scale function). Then we have $n(\overline{\varepsilon} \geq b) = \nu([b,\infty)) = s'(b)/s(b) = 1/b$, since $s(b) = b$ is the scale function for Brownian motion. We have

$$\begin{aligned}&\overline{\mathbb{E}}\left[\mathrm{e}^{-(\lambda+\gamma^2/2)g+\gamma L(g)}\right] \\ &= \overline{\mathbb{E}}\left[\sum_g \mathrm{e}^{-(\lambda+\gamma^2/2)g+\gamma L(g)}\mathbf{1}_{\{\sup_{h<g}\overline{\varepsilon}_h<b\}}\mathbf{1}_{\{\overline{\varepsilon}_g\geq b\}}\right] \\ &= \overline{\mathbb{E}}\left[\int_0^\infty L(\mathrm{d}s)\mathbf{1}_{\{\sup_{h<s}\overline{\varepsilon}_h<b\}}\mathrm{e}^{-(\lambda+\gamma^2/2)s+\gamma L(s)}\int_{\mathcal{E}}\mathbf{1}_{\{\overline{\varepsilon}\geq b\}}n(\mathrm{d}\varepsilon)\right] \\ &= \overline{\mathbb{E}}\left[\int_0^\infty L(\mathrm{d}s)\mathrm{e}^{-(\lambda+\gamma^2/2)s+\gamma L(s)}\mathbf{1}_{\{\sup_{h<s}\overline{\varepsilon}_h<b\}}\right]n(\overline{\varepsilon}\geq b) \\ &= \frac{1}{b}\int_0^\infty \mathrm{d}t\overline{\mathbb{E}}\left[\mathrm{e}^{-(\lambda+\gamma^2/2)L^{-1}(t)+\gamma t}\mathbf{1}_{\{\sup_{0\leq s<L^{-1}(t)}\log\Psi_s<b\}}\right]\end{aligned}\tag{3.11}$$

In the first equality the sum is taken over all starting points g of excursions. The second equality follows directly by the compensation formula. The second factor in the final equality results from a simple variable change from local time to inverse local time. Since reflected Brownian motion is recurrent, we have $L(\infty) = \infty$ so that the change of variables in the final equality is justified.

In order to deal with the expectation in the final integral, it should be noted that $L^{-1}(t)$ is an $\overline{\mathbb{P}}$-almost surely finite stopping time. Hence, by an argument similar to one in the proof of Lemma 3.3,

$$\exp\{t\sqrt{2\lambda+\gamma^2} - (\lambda+\gamma^2/2)L^{-1}(t)\}$$

is a Girsanov density introducing a drift $\sqrt{2\lambda+\gamma^2}$ to W. Thus it follows with the use of Lemma 3.4 that the integral on the right hand side of (3.11) can be written as

$$\begin{aligned}&\int_0^\infty \mathrm{d}t\overline{\mathbb{E}}\left[\mathrm{e}^{-(\lambda+\gamma^2/2)L^{-1}(t)+\gamma t}\mathbf{1}_{\{\sup_{0\leq s<L^{-1}(t)}\log\Psi_s<b\}}\right] \\ &\quad= \int_0^\infty dt\exp t\left\{\gamma - \sqrt{2\lambda+\gamma^2}\frac{\cosh(b\sqrt{2\lambda+\gamma^2})}{\sinh(b\sqrt{2\lambda+\gamma^2})}\right\} \\ &\quad= \frac{\sinh(b\sqrt{2\lambda+\gamma^2})}{\sqrt{2\lambda+\gamma^2}\cosh(b\sqrt{2\lambda+\gamma^2}) - \gamma\sinh(b\sqrt{2\lambda+\gamma^2})}.\end{aligned}\tag{3.12}$$

Piecing everything together from (3.7), (3.8), (3.10) and (3.11) we recover

$$\overline{\mathbb{E}}^{\gamma}[\mathrm{e}^{-\lambda\tau_{(b,\infty)}}] = \frac{\mathrm{e}^{-\gamma b}\sqrt{2\lambda+\gamma^2}}{\sqrt{2\lambda+\gamma^2}\cosh(b\sqrt{2\lambda+\gamma^2}) - \gamma\sinh(b\sqrt{2\lambda+\gamma^2})}. \tag{3.13}$$

To compute the Laplace transform for $\log\psi \in (0,b)$ and $\sigma = 1$, we split the probability space in two parts, $\mathcal{A} := \mathcal{A}(-(b-\log\psi),\log\psi) = \{\tau^W_{(-\infty,-(b-\log\psi)]} < \tau^W_{[\log\psi,\infty)}\}$ and its complement respectively. Note, that, on $\mathcal{A}$, $\tau^{\Psi}_{[b,\infty)}$ has the same $\overline{\mathbb{P}}^{\gamma}_{\psi}$-law as $\tau^W_{(-\infty,-(b-\log\psi)]}$ does under $\mathbb{P}^{\gamma}$. Further, on $\mathcal{A}^c$, $\tau^{\Psi}_{[b,\infty)}$ has the same

$\overline{\mathbb{P}}^\gamma_\psi$-law as $\tau^W_{[\log\psi,\infty)} + \tilde{\tau}^\Psi_{[b,\infty)}$ where $\tau^W_{[\log\psi,\infty)}$ and $\tilde{\tau}^\Psi_{[b,\infty)}$ are indepedent $\mathbb{P}^\gamma$ and $\overline{\mathbb{P}}^\gamma$-stopping times respectively. Hence,

$$\begin{aligned}\overline{\mathbb{E}}^\gamma_\psi\left[\mathrm{e}^{-\lambda\tau^\Psi_{[b,\infty)}}\right] &= \mathbb{E}^\gamma\left[\mathrm{e}^{-\lambda\tau^W_{(-\infty,-(b-\log\psi)]}}\mathbf{1}_{\mathcal{A}}\right] \\ &\quad+\mathbb{E}^\gamma\left[\mathrm{e}^{-\lambda\tau^W_{[\log\psi,\infty)}}\mathbf{1}_{\mathcal{A}^c}\right]\cdot\overline{\mathbb{E}}^\gamma\left[\mathrm{e}^{-\lambda\tau^\Psi_{[b,\infty)}}\right].\end{aligned}$$

The first expectation in the right hand side above are given by Lemma 3.3. The second and third expectations are given by Lemma 3.3 and (3.13) respectively. A simple algebraic exercise combining these expressions gives

$$\begin{aligned}&\overline{\mathbb{E}}^\gamma_\psi[\mathrm{e}^{-\lambda\tau^\Psi_{[b,\infty)}}] \\ &= \mathrm{e}^{\gamma(\log\psi-b)}\frac{\sqrt{2\lambda+\gamma^2}\cosh(\log\psi\sqrt{2\lambda+\gamma^2})-\gamma\sinh(\log\psi\sqrt{2\lambda+\gamma^2})}{\sqrt{2\lambda+\gamma^2}\cosh(b\sqrt{2\lambda+\gamma^2})-\gamma\sinh(b\sqrt{2\lambda+\gamma^2})}.\end{aligned}$$

In order to remove the condition $\sigma = 1$, it suffices to consider the Laplace transform of the first time the process $\Psi_t^{1/\sigma}$ enters the interval $[b/\sigma,\infty)$. □

Remark 3.5 The proof of Theorem 3.2 we have offered here uses reasonably elementary properties of fluctuation theory essentially motivated by the decomposition discussed in the previous paragraph. However, as the reader in this field may already suspect, simpler proofs could be at hand depending on the depth and complexity of the Theorems used. Here is another possibility that is in this respect a shorter proof. It is assumed that $\sigma = 1$.

In [10] it is possible to find the following interesting change of measure for η and γ positive,

$$\left.\frac{d\overline{\mathbb{P}}^\eta_\psi}{d\overline{\mathbb{P}}^\gamma_\psi}\right|_{\mathcal{F}_t} = \exp\left\{-(\eta-\gamma)(\log(\Psi_t/\psi)-L(t))-\frac{1}{2}(\eta^2-\gamma^2)t\right\}.$$

This density can be used in the same way that the Wald martingale density is used to evaluate Laplace transforms of hitting times of Brownian motion. Namely, for $1\le\log\psi<b$, since $\tau^\Psi_{[b,\infty)}$ is $\overline{\mathbb{P}}^\gamma_\psi$-almost surely finite,

$$\overline{\mathbb{E}}^\gamma_\psi\left[\mathrm{e}^{-\lambda\tau^\Psi_{[b,\infty)}}\right] = \mathrm{e}^{(\eta-\gamma)(b-\log\psi)}\overline{\mathbb{E}}^\eta_\psi\left[\mathrm{e}^{-(\eta-\gamma)L(\tau^\Psi_{[b,\infty)})}\right] \tag{3.14}$$

where $\eta=\sqrt{2\lambda+\gamma^2}$. The last expectation on the right hand side above can be evaluated using the strong Markov property as follows,

$$\begin{aligned}\overline{\mathbb{E}}^\eta_\psi\left[\mathrm{e}^{(\eta-\gamma)L(\tau^\Psi_{[b,\infty)})}.\right] &= \overline{\mathbb{P}}^\eta_\psi\left(\tau^\Psi_{[b,\infty)}<\tau^\Psi_{(-\infty,0)}\right) \\ &\quad+\overline{\mathbb{P}}^\eta_\psi\left(\tau^\Psi_{[b,\infty)}\ge\tau^\Psi_{(-\infty,0)}\right)\overline{\mathbb{E}}^\eta\left[\mathrm{e}^{(\eta-\gamma)L(\tau^\Psi_{[b,\infty)})}.\right] \\ &= \frac{s(\log\psi)-s(0)}{s(b)-s(0)} \\ &\quad+\frac{s(b)-s(\log\psi)}{s(b)-s(0)}\frac{s'(b)}{(\eta-\gamma)s(b)+s'(b)}\end{aligned} \tag{3.15}$$

where s is the scale function of $\log\Psi$ and the last factor on the right hand side represents the expectation in the previous equality. To see where this factor comes from note that under $\overline{\mathbb{P}}^\eta$, $L(\tau^\Psi_{[b,\infty)})$ is the start point of the first excursion in which a height greater or equal to b is reached. Since suprema of excursions in the interval $[b,\infty)$ form a Poisson point process indexed by local time and with parameter $s'(b)/s(b)^{-1}$, it follows that

$$\overline{\mathbb{E}}^\eta\left[\mathrm{e}^{(\eta-\gamma)L(\tau^\Psi_{[b,\infty)})}\right] = \frac{s'(b)}{(\eta-\gamma)s(b)+s'(b)}.$$

Recall from the proof of Lemma 3.3 that under $\overline{\mathbb{P}}^\eta$, $s(b)=(1-\mathrm{e}^{-2b\eta})/2\eta$. Putting this back into (3.15) and (3.14), we recover the result in Theorem 3.2.

4 Perpetual Integral Option

Analogously to what was done with at the begining of the last section, we combine Theorem 1.2 with the Girsanov density $s^{-1}\exp\{-rt\}S_t$ under $\mathbb{P}_x^{(r/\sigma-\sigma/2)}$ and insist that the option holder must exercise in an almost surely finite time to achieve

$$\Pi^{\mathrm{int}}(t,s,\varphi) = S_t \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,\infty}} \mathbb{E}_x^{(r/\sigma+\sigma/2)}\left[\mathrm{e}^{-\lambda\tau}\frac{\int_t^\tau S_u du + (s\varphi + \int_0^t S_u du)}{S_\tau}\middle|\mathcal{F}_t\right].$$

We introduce the new stochastic process $\Phi = \{\Phi_t, t \geq 0\}$ with

$$\Phi_t := \frac{\int_0^t S_u du + s\varphi}{S_t}$$

which can easily be verified to be a Markov process. For convenience let us now assume that the Brownian motion driving the stock has been observed since some time $-M \leq 0$ and we shall interpret the constant φ to be the quantity $s^{-1}\int_{-M}^0 S_u du$ (and assume that this is $\mathcal{F}_0$ measurable). Thus if $\tilde{\mathbb{P}}_\varphi^\gamma$ is the probablilty measure under which W is a $\mathbb{P}_0^\gamma$-Brownian motion but the process Φ has value at time zero equal to φ, then it follows that

$$\Pi^{\mathrm{int}}(t,s,\varphi) = \mathrm{e}^{-\lambda t} S_t \Pi^{\mathrm{I}}(\Phi_t)$$

with

$$\Pi^{\mathrm{I}}(\varphi) = \sup_{\tau\in\mathcal{T}_{0,\infty}} \tilde{\mathbb{E}}_\varphi^{(r/\sigma+\sigma/2)}\left[\mathrm{e}^{-\lambda\tau}\Phi_\tau\right] \tag{4.1}$$

where $\tilde{\mathbb{E}}_\varphi^\gamma$ is expectation with respct to $\tilde{\mathbb{P}}_\varphi^\gamma$ and, in effect, we may now take $\Phi_t := \int_{-M}^t S_u du/S_t$ (which is not a function of s).

As before, we have the following result, which characterizes the optimal stopping time in (4.1) as a hitting time of the process Φ.

Corollary 4.1 *The optimal stopping time in* (4.1) *is a hitting time of the form*

$$\inf\{t \geq 0 : \Phi_t \geq \tilde{\varphi}\}. \tag{4.2}$$

PROOF. The proof is analogous to that of Corollary 3.1. □

The problem of pricing the perpetual integral option, just as in the case of the perpetual Russian option, is reduced to the evaluation of a Laplace transform of a stopping time of a Markov process. The following Theorem essentially gives the analytical structure to the final price given in Theorem 1.6.

Theorem 4.2 *For Borel sets B let*

$$\tau_B^\Phi = \inf\{s \geq 0 : \Phi \in B\}.$$

For $\varphi \in [0,b)$ we have

$$\tilde{\mathbb{E}}_\varphi^{(r/\sigma+\sigma/2)}\left[\mathrm{e}^{-\lambda\tau_{[b,\infty)}^\Phi}\right] = \frac{u(\varphi)}{u(b)} \tag{4.3}$$

where the function u is given by

$$u(x) = \int_0^\infty \mathrm{e}^{-2z/\sigma^2} z^{-(y_1+1)}(1+zx)^{y_2}\mathrm{d}z.$$

with $y_1 < y_2$ the roots of $y^2 - (1+2r/\sigma^2)y - (2\lambda/\sigma^2) = 0$.

We shall shortly prove this Theorem but let us proceed by showing that the price of the integral option can now be quickly obtained.

PROOF OF THEOREM 1.6. The proof is given along the same lines as the proof of Theorem 1.5. Begin by noting that Corollary 4.1 implies that the relation (4.1) can be rewritten as

$$\Pi^{\mathrm{I}}(\varphi) = \sup_{m\geq 0} V_m^{(4)}(\varphi)$$

where

$$V_m^{(4)}(\varphi) = \begin{cases} m\tilde{\mathbb{E}}_\varphi^{(r/\sigma+\sigma/2)}\left[\mathrm{e}^{-\lambda\tau_{[m,\infty)}^\Phi}\right] & 0 \leq \varphi \leq m \\ \varphi & \varphi > m. \end{cases}$$

¿From Theorem 4.2 we find for the value function

$$V_m^{(4)}(\varphi) = \begin{cases} m \cdot \frac{u(\varphi)}{u(m)} & 0 \leq \varphi \leq m \\ \varphi & \varphi > m. \end{cases}$$

The function $f(m) := m/u(m)$ is positive and differentiable such that $f(0) = 0$ and $f(m)$ decreases to 0 as $m \to \infty$. Since u is increasing and strictly convex it thus follows that there is a unique point in $[0, \infty)$ satisfying $f'(m) = 0$ or equivalently $u(m) = mu'(m)$. The theorem is proved. □

We now conclude this section by proving the main result, Theorem 4.2. The main idea behind the proof is to take advantage of Lamberti's relation, namely that for a standard Brownian motion with drift γ,

$$\mathrm{e}^{\sigma(W_t+\gamma t)} = \left[R^{(2\gamma/\sigma)} \left(\frac{\sigma^2}{4} A_t^{(\gamma,\sigma)} \right) \right]^2. \tag{4.4}$$

where $R^\gamma(t)$ is a Bessel process of index γ satisfying the stochastic differential equation $dR^\gamma(t) = dW_t + (\gamma - 1)dt/2R^\gamma(t)$ with $R^\gamma(0) = 1$ and

$$A_t^{(\gamma,\sigma)} = \int_0^t \mathrm{e}^{\sigma W_s^\gamma} ds = \int_0^t \mathrm{e}^{\sigma(W_s+\gamma s)} ds. \tag{4.5}$$

Thus $\tau_{[b,\infty)}^\Phi$ may be considered to be of the form

$$\tau_{[b,\infty)}^\Phi = \inf \left\{ t \geq 0 : R^{(2\gamma/\sigma)} \left(\frac{\sigma^2}{4} A_t^{(\gamma,\sigma)} \right) \leq \sqrt{\frac{4}{b\sigma^2} \left(\frac{\sigma^2}{4} A_t^{(\gamma,\sigma)} \right) + \frac{\varphi}{b}} \right\}$$

where $\gamma = r/\sigma + \sigma/2$. One can now see that the necessary fluctuation theory we need concerns Bessel processes. Unlike the case of the Russian option the necessary fluctuation results we shall apply are quite deep and specific. We summarise them in the following two Lemmas whose proofs can be found in [23] and [22] respectively. The first Lemma is not too difficult to recover from the Girsanov Theorem, but the second needs considerably more work to prove.

Lemma 4.3 *Let $\hat{P}_x^\gamma$ be the law of a Bessel process with parameter γ started from $x > 0$ and $\hat{E}_x^\gamma$ expectation with respect to this measure. For any stopping time T, define $I_T = \int_0^T [R(s)]^{-2} ds$ where $\{R(t) : t \geq 0\}$ is a Bessel process. Suppose that T is $\hat{P}_x^\gamma$-almost surely finite, then for $\lambda > 0$*

$$\hat{E}_x^\gamma \left[\mathrm{e}^{-\lambda I_T} \right] = \hat{E}_x^\nu \left[\left(\frac{x}{R(T)} \right)^{(\nu-\lambda)} \right]$$

where $\nu = \sqrt{2\lambda + \gamma^2}$.

Lemma 4.4 *Define for Bessel processes $\{R(t) : t \geq 0\}$ stopping times of the form*

$$T(b) = \inf\{t \geq 0 : R(t) \leq b\sqrt{1+t}\}.$$

For any $\gamma \geq -1/2, x > b, m \geq 0$ we have

$$\hat{E}_x^\gamma \left[\left(\frac{1}{1+T(b)} \right)^m \right] = \frac{U(m, \gamma+1, x^2/2)}{U(m, \gamma+1, b^2/2)}, \tag{4.6}$$

where U is the confluent hypergeometric Kummer's function of the second kind. That is to say that for real valued a, b, z,

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-zt} t^{a-1} (1+t)^{b-a-1} \mathrm{d}t.$$

(See [13] for a description of this class of functions).

Proof of Theorem 4.2. The first thing to note is that the time change $A_t^{(\gamma,\sigma)}$ satisfies the inverse relation (see [17])

$$\frac{4}{\sigma^2} \int_0^{\sigma^2 A_t^{(\gamma,\sigma)}/4} \left[R^{(2\gamma/\sigma)}(s) \right]^{-2} \mathrm{d}s = t \qquad t \geq 0. \tag{4.7}$$

Thus we can rewrite $\tau^{\Phi}_{[b,\infty)}$ in the form

$$\tau^{\Phi}_{[b,\infty)} = \frac{4}{\sigma^2}\int_0^{\tilde{T}} [R(s)]^{-2}\,\mathrm{d}s \tag{4.8}$$

under $\hat{P}^{(2\gamma/\sigma)}_1$ where

$$\tilde{T} = \inf\left\{t : R(t) \le \sqrt{\frac{4}{b\sigma^2}t + \frac{\varphi}{b}}\right\}.$$

Bessel processes have a scaling property that can be considered to be inherited from Brownian motion. Namely that if R is a Bessel with index γ with $R(0) = 1$, then for any constant $c > 0$, $R' := \{c^{-1/2}R(ct), t \ge 0\}$ is also an Bessel process with index γ but starting from $c^{-1/2}$. It thus follows after a brief calculation that $\tilde{T}$ is equal in $\hat{P}^{(2\gamma/\sigma)}_1$-law to $(\varphi\sigma^2/4)\cdot T(\sqrt{\varphi/b})$ under $\hat{P}^{(2\gamma/\sigma)}_z$ where $z = (\varphi\sigma^2/4)^{1/2}$. Combining this observation with Lemma 4.3, one can check that

$$\tilde{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\varphi}\left[\mathrm{e}^{-\lambda\tau^{\Phi}_{[b,\infty)}}\right] = \left(\frac{b}{\varphi}\right)^{-y_1}\hat{E}^{(y_2-y_1)}_{\sqrt{4/\sigma^2\varphi}}\left[\left(\frac{1}{1+T(\sqrt{4/\sigma^2 b})}\right)^{-y_1}\right]$$

Applying Lemma 4.4 one finds, after some algebra, the stated expression. □

5 Canadization

From a financial point of view, perpetual options may be considered as rather exotic objects, since in the real world options never have an infinite time of expiration. As we will show below, perpetual-type options can be linked to American type options of finite expiration.

Let us consider an American type option with finite expiration T and system of pay-off functions $\{\pi_t : 0 \le t \le T\}$. The holder of the option has the right to exercise it at any time *before* T. If the holder does not exercise before this finite time then he receives a payment π_T at expiry. By considering Theorems 1.2 and 1.3 for the sequence of payments $\{\pi_{t\wedge T} : t \ge 0\}$ we have the arbitrage free price of this an American type

$$\Pi_T = \sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}^{(r/\sigma-\sigma/2)}_x\left[\mathrm{e}^{-r\tau}\pi_\tau\right]$$

with optimal stopping time

$$\tau^* = \inf\{0 \le t \le T : \Pi_T(t) \le \pi_t\}$$

where the hedging capital, as in section 1, is given by

$$\Pi_T(t) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}^{(r/\sigma-\sigma/2)}_x\left[\mathrm{e}^{-r\tau}\pi_\tau\,|\mathcal{F}_t\right].$$

Unlike the perpetual case, the optimal stopping time is (in general) not a hitting time of a level, but will now be the crossing time of a non-flat space time boundary. For this optimal exercise boundary no explicit formula's are known. For an account of the American put with finite time of expiration see for example [15]. Since no explicit solution is known for this problem, we consider instead a reasonable approximation. We follow the lead of [5]. The idea is to randomize T in a sensible way. That is, to replace T by an independent random variable.

Let $T_1, T_2, \ldots$ be a sequence of independent exponential variables with mean T, which are also independent of $\mathbf{F}$ and denote their probability measures and expectation respectively by P and E. An n-step approximation is understood to mean replacing the claim process $\pi_{t\wedge T}$ by $\pi_{t\wedge T^{(n)}}$ where $T^{(n)} = n^{-1}\sum_1^n T_i$, which has a Gamma$(n, n/T)$-distribution. Note by the strong law of large numbers $T^{(n)} \to T$ almost surely as n tends to infinity. The next result shows this approximation procedure makes sense.

Proposition 5.1 *Let π_t be continuous and suppose there are $\epsilon, C > 0$ such that the family $\{\mathrm{e}^{-r\tau}\pi_\tau : \tau \in \mathcal{T}_{0,T+\epsilon}\}$ is uniformly integrable with respect to $\mathbb{P}^{r/\sigma-\sigma/2}$ and $\sup_{\tau\in\mathcal{T}_{0,\infty}, u>T+\epsilon}\mathbb{E}^{(r/\sigma-\sigma/2)}[\mathrm{e}^{-r(\tau\wedge u)}\pi_{\tau\wedge u}] \le C$. Then the sequence $\{\Pi^{(n)} : n \ge 1\}$ given by*

$$\Pi^{(n)} = \sup_{\tau\in\mathcal{T}_{0,\infty}} E\left[\mathbb{E}^{(r/\sigma-\sigma/2)}_x\left[\mathrm{e}^{-r(\tau\wedge T^{(n)})}\pi_{\tau\wedge T^{(n)}}\right]\right]$$

converges for each x to Π_T as n tends to infinity.

PROOF. For simplicity, write $g_t = e^{-rt}\pi_t$, $\gamma = (r/\sigma - \sigma/2)$ and $P^\gamma = P \times \mathbb{P}^{r/\sigma-\sigma/2}$. By an extension of Theorem 1.3 to the finite expiration case, we know there exists an optimal stopping time $\tau^* \in \mathcal{T}_{0,T}$ such that $\Pi_T = \mathbb{E}^\gamma[g_{\tau^*}]$. Note that $\tau^* \in \mathcal{T}_{0,\infty}$ and hence $\Pi^{(n)} \geq E^\gamma[g_{\tau^*\wedge T^{(n)}}]$

By continuity of g and Fatou's lemma we find that

$$\begin{aligned}\liminf_{n\to\infty}\Pi^{(n)} &\geq \liminf_{n\to\infty} E^\gamma[g_{\tau^*\wedge T^{(n)}}]\\ &\geq E^\gamma[\liminf_{n\to\infty} g_{\tau^*\wedge T^{(n)}}]\\ &= \mathbb{E}^\gamma[g_{\tau^*\wedge T}] = \Pi_T.\end{aligned}$$

To finish the proof we thus have to proof that

$$\limsup_{n\to\infty}\Pi^{(n)} = \limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}_{0,\infty}} E^\gamma[g_{\tau\wedge T^{(n)}}] \leq \Pi_T.$$

Using the bound on $E^\gamma[g^{\tau\wedge T^{(n)}}|T^{(n)}]$, we find that

$$\begin{aligned}\Pi^{(n)} &= \sup_{\tau\in\mathcal{T}_{0,\infty}} E^\gamma[g_{\tau\wedge T^{(n)}}\mathbf{1}_{\{T^{(n)}\leq T+\epsilon\}}] + \sup_{\tau\in\mathcal{T}_{0,\infty}} E^\gamma[g_{\tau\wedge T^{(n)}}\mathbf{1}_{\{T^{(n)}> T+\epsilon\}}]\\ &\leq \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}^\gamma[g_{\tau\wedge(T+\epsilon)}] + C\cdot P(T^{(n)} > T+\epsilon)\end{aligned}$$

which after taking the limsup for $n\to\infty$ converges to $\Pi_{T+\epsilon}$, by virtue of the fact that $T^{(n)}$ converges to T a.s.. The proof is completed by showing that $\Pi_{T+\epsilon}$ tends to Π_T as ϵ tends to zero. To do so, note that

$$\begin{aligned}\left|\sup_{\tau\in\mathcal{T}_{0,\infty}}\mathbb{E}^\gamma[g_{\tau\wedge(T+\epsilon)}] - \sup_{\tau\in\mathcal{T}_{0,\infty}}\mathbb{E}^\gamma[g_{\tau\wedge T}]\right| &\leq \sup_{\tau\in\mathcal{T}_{0,\infty}}\mathbb{E}^\gamma[|g_{\tau\wedge(T+\epsilon)} - g_{\tau\wedge T}|]\\ &= \sup_{\tau\in\mathcal{T}_{0,\infty}}\mathbb{E}^\gamma[|g_\tau - g_T|\mathbf{1}_{\{T<\tau\leq T+\epsilon\}}]\\ &= \mathbb{E}^\gamma[|g_{\tau^*_\epsilon} - g_T|\mathbf{1}_{\{T<\tau^*_\epsilon\leq T+\epsilon\}}]\end{aligned}$$

where τ^*_ϵ is given by Theorem 1.3. The expectation in the previous line converges to zero by uniform integrability. Hence it follows that $\Pi_{T+\epsilon}$ can be made arbitrarily close to Π_T by making ϵ sufficiently small. $\square$

The Canadization of an American-type option is the 1-step approximation as described above. That is to say the expiration date is randomized by an independent exponential distribution with parameter $\alpha = T^{-1}$. In all the cases we are interested in, American calls and puts, Russians and integrals, their Canadized price are of the form

$$\begin{aligned}\widehat{\Pi}(\gamma) &= \sup_{\tau\in\mathcal{T}_{0,\infty}} E_\gamma\left[\mathrm{e}^{-r(\tau\wedge T_1)}f(\Gamma_{\tau\wedge T_1})\right]\\ &= \sup_{\tau\in\mathcal{T}_{0,\infty}} E_\gamma\left[\mathrm{e}^{-(r+\alpha)\tau}f(\Gamma_\tau) + \alpha\int_0^\tau \mathrm{e}^{-(r+\alpha)s}f(\Gamma_s)\mathrm{d}s\right]\end{aligned}$$

where $\Gamma = \{\Gamma_t : t\geq 0\}$ is a continous Markov process starting from γ under some measure whose expectation operator is E_γ and f is a non-negative, monotone increasing, convex function. It can be easily checked using Theorem 1.3 that the optimal stopping time is of the form

$$\tau^* = \inf\{t\geq 0 : \widehat{\Pi}(\Gamma_t) \leq f(\Gamma_t)\}$$

Hence on account of the properties of f, we can reason as in the previous sections to conclude that τ^* is hitting time of the Markov process Γ.

In the following examples, note that it is no longer necessary that the parameter λ is positive in order to guarentee the existence of a solution. A finite (allbeit) random expiry date removes this necessity.

Example 5.2 (1-Step American Put Approximation, $\lambda = 0$) The first approximation $\Pi^{\mathrm{put}}_{T_1}(s)$ to the price of a American put with expiration T

$$\Pi^{\mathrm{put}}_T(s) = \sup_{\tau\in\mathcal{T}_{0,T}}\mathbb{E}^{(r/\sigma-\sigma/2)}_x[\mathrm{e}^{-r\tau}(K-S_\tau)^+]$$

is equal to the supremum over all $l>0$ of

$$\mathbb{E}^{(r/\sigma-\sigma/2)}_x[\mathrm{e}^{-(r+\alpha)\tau^W_{(-\infty,l]}}(K-S_{\tau^W_{(-\infty,l]}})^+] + \alpha\mathbb{E}^{(r/\sigma-\sigma/2)}_x\left[\int_0^{\tau^W_{(-\infty,l]}}\mathrm{e}^{-(r+\alpha)t}(K-S_t)^+\mathrm{d}t\right]. \quad (5.1)$$

After a tedious computation we find $\Pi^{\mathrm{put}}_{T_1}(s)$ to be equal to

$$\begin{cases} \left(\frac{s}{K}\right)^{-x_1}\left(K\frac{x_2}{x_2-x_1}\frac{\alpha}{\alpha+r} - K\frac{1+x_2}{x_2-x_1}\right) + \left(\frac{s}{l_*}\right)^{-x_1} K\frac{x_2}{x_2-x_1}\frac{r}{r+\alpha} & \text{if } s \geq K \\ K\frac{\alpha}{r+\alpha} - s + \left(\frac{s}{K}\right)^{x_2}\left(K\left(\frac{1-x_1}{x_2-x_1}\right) - K\frac{-x_1}{x_2-x_1}\frac{\alpha}{\alpha+r}\right) + \left(\frac{s}{l_*}\right)^{-x_1} K\frac{x_2}{x_2-x_1}\frac{r}{r+\alpha} & \text{if } s \in (l_*, K) \\ K - s & \text{if } s \leq l_*. \end{cases}$$

where $x_1 < x_2$ are the roots of $x^2 - (1-2r/\sigma^2)x - 2(\alpha+r)/\sigma^2 = 0$. The optimal exercise level is given by

$$l_* = K\left(\frac{-rx_1}{r+\alpha-rx_1}\right)^{\frac{1}{x_2}}.$$

Example 5.3 (1-Step Russian Option Approximation, $\lambda = 0$) According to the preceding a first approximation to the price of a Russian option with expiry T

$$\Pi^{\mathrm{russ}}_T(s,\psi) = s \cdot \Pi^{\mathrm{R}}_T(\psi) = s \cdot \sup_{\tau\in\mathcal{T}_{0,T}} \overline{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\psi}[\Psi_\tau]$$

is equal to $\Pi^{\mathrm{russ}}_{T_1}(s,\psi) = s \cdot \Pi^{\mathrm{R}}_{T_1}(\psi)$ where $\Pi^{\mathrm{R}}_{T_1}(\psi)$ is equal to the supremum over all $b > 0$ of

$$b\overline{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\psi}\left[\mathrm{e}^{-\alpha\tau^{\Psi}_{[b,\infty)}}\right] + \alpha\overline{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\psi}\left[\int_0^{\tau^{\Psi}_{[b,\infty)}} \mathrm{e}^{-\alpha t}\Psi_t \mathrm{d}t\right]. \tag{5.2}$$

By an application of Itô's lemma to the process $\exp(-\alpha t)\Psi_t$, we find that

$$\begin{aligned} \overline{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\psi}\left[\int_0^{\tau^{\Psi}_{[b,\infty)}} \mathrm{e}^{-\alpha t}\Psi_t \mathrm{d}t\right] = -\frac{1}{r+\alpha}\Bigg(b\overline{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\psi}\left[\mathrm{e}^{-\alpha\tau^{\Psi}_{[b,\infty)}}\right] \\ - \psi - \overline{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\psi}\left[\int_0^{\tau^{\Psi}_{[b,\infty)}} \mathrm{e}^{-\alpha t} S_t^{-1}\mathrm{d}M_t\right]\Bigg). \end{aligned}$$

where $M_t = \overline{S}_t$. In order to evaluate the second expecation on the right hand side above, we note that M_t is a local time process satisfying $M_\infty = \infty$. Its inverse local time process M_t^{-1} is the first hitting time of the set $[t,\infty)$ for the process S. Infact $M_t^{-1} = L(\sigma^{-1}\log t)$. By substituting inverse local time of M_t in place of t we thus achieve after a little manipulation involving a change of integral and expectation

$$\begin{aligned} &\overline{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\psi}\left[\int_0^{\tau^{\Psi}_{[b,\infty)}} \mathrm{e}^{-\alpha t} S_t^{-1}\mathrm{d}M_t\right] \\ &= \int_{\frac{\log\psi}{\sigma}}^{\infty} \overline{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\psi}\left[\sigma\mathrm{e}^{-\alpha L^{-1}(s)}\mathbf{1}_{\{\sup_{0\leq u\leq L^{-1}(s)}(\overline{W}_u - W_u)<b/\sigma\}}\right]\mathrm{d}s \end{aligned}$$

where $0 \leq \log\psi \leq b$. A similar calculation to (3.12) yields

$$\overline{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\psi}\left[\int_0^{\tau^{\Psi}_{[b,\infty)}} \mathrm{e}^{-\alpha t} S_t^{-1}\mathrm{d}M_t\right] = \sigma\frac{b^{y_2}\psi^{y_1} - b^{y_1}\psi^{y_2}}{y_2 b^{y_1} - y_1 b^{y_2}}$$

where $y_1 < y_2$ are the roots of $y^2 - (1+2r/\sigma^2)y - 2\alpha/\sigma^2 = 0$. Thus, the first approximation is given by

$$\Pi^{\mathrm{russ}}_{T_1}(s,\psi) = s\cdot\left\{\frac{r}{r+\alpha}b_*\cdot\frac{y_2\psi^{y_1} - y_1\psi^{y_2}}{y_2 b_*^{y_1} - y_1 b_*^{y_2}} + \frac{\alpha}{r+\alpha}\left(\psi + \sigma\frac{b_*^{y_2}\psi^{y_1} - b_*^{y_1}\psi^{y_2}}{y_2 b_*^{y_1} - y_1 b_*^{y_2}}\right)\right\}$$

where y_1, y_2 are as before and the optimal exercise level b_* is the unique solution of

$$r(y_2(1-y_1)b^{y_1} + y_1(y_2-1)b^{y_2}) + \alpha\sigma(y_2-y_1)b^{y_1+y_2-1} = 0. \tag{5.3}$$

Note that uniqueness follows since the function of b in (5.3) is concave and differentiable with a positive derivative at 1.

Example 5.4 (1-step Integral Option Approximation, $\lambda = 0$) We now show how to find an approximation to the price of the integral option with expiry T, that is, we approximate

$$\Pi^{\mathrm{int}}_T(s,\varphi) = s\cdot\Pi^{\mathrm{I}}_T(\varphi) = s\cdot\sup_{\tau\in\mathcal{T}_{0,T}}\tilde{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\varphi}[\Phi_\tau].$$

The first approximation $\Pi^{\mathrm{I}}_{T_1}$ to the price Π^{I}_{T} is given by the supremum over all $b > 0$ of

$$b\tilde{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\varphi}\left[\mathrm{e}^{-\alpha\tau^{\Phi}_{[b,\infty)}}\right] + \alpha\tilde{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\varphi}\left[\int_0^{\tau^{\Phi}_{[b,\infty)}} \mathrm{e}^{-\alpha t}\Phi_t \mathrm{d}t\right].$$

An application of Itô's lemma to $\exp(-\alpha t)\Phi_t$ shows

$$\tilde{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\varphi}\left[\int_0^{\tau^{\Phi}_{[b,\infty)}} \mathrm{e}^{-\alpha t}\Phi_t \mathrm{d}t\right] = \frac{1}{\alpha+r}\left(1+\varphi-(1+b)\tilde{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\varphi}[\mathrm{e}^{-\alpha\tau^{\Phi}_{[b,\infty)}}]\right)$$

Recalling formula (4.3) we find,

$$\Pi^{\mathrm{int}}_{T_1}(s,\varphi) = s\cdot\left\{\alpha\left(\frac{1+\varphi}{\alpha+r}\right)+\frac{m_* r-\alpha}{\alpha+r}\cdot\frac{u(\varphi)}{u(m_*)}\right\}.$$

where, following the line of reasoning of the proof of Theorem 1.6, $m_* > 0$ is uniquely determined by $u'(m_*)(rm_*-\alpha) = u(m_*)r$,

On a final note we consider how one would evaluate an n-step approximation by using a dynamic programming algorithm with the Russian option.

Example 5.5 (n-Step Russian Option Approximation, $\lambda = 0$) Let $\alpha_n = \alpha/n$ and write $e_i = n^{-1}T_i$ for $i = 1, ..., n$. Define the subsequent stages $h_n, \dots, h_1$ by

$$\begin{aligned} h_n(\psi) &= \sup_{\tau\in\mathcal{T}_{0,\infty}} \overline{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\psi}[\Psi_{\tau\wedge e_n}] \\ &= \overline{\mathbb{E}}^{(\rho/\sigma+\sigma/2)}_{\psi}\left[\mathrm{e}^{-\alpha_n\tau}\Psi_\tau + \int_0^\tau \mathrm{e}^{-\alpha_n t}\Psi_t\mathrm{d}t\right] \end{aligned}$$

$$\begin{aligned} h_m(\psi) &= \sup_{\tau\in\mathcal{T}_{0,\infty}} \overline{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\psi}[\Psi_{t\wedge\sum_m^n e_i}] \\ &= \overline{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\psi}\left[\mathrm{e}^{-\alpha_n\tau}\Psi_\tau + \int_0^\tau \mathrm{e}^{-\alpha_n t}h_{m+1}(\Psi_t)\mathrm{d}t\right] \end{aligned}$$

Using the Markov property it can be checked that the price $\Pi^{(n)}(x)$ of the n-approximation is equal to $h_1(\psi)$, the final outcome of the above dynamic programming algorithm, for all possible starting values ψ of the Markov process. Note each step in the dynamic programming algorithm requires solution of a problem of the form

$$\sup_{\tau\in\mathcal{T}_{0,\infty}} E_\gamma\left[\mathrm{e}^{-\alpha_n\tau}f(\Gamma_\tau)+\alpha\int_0^\tau \mathrm{e}^{-\alpha_n s}g(\Gamma_s)\mathrm{d}s\right]$$

where g is another non-negative, convex, monotone increasing function. It can be reasoned similarly to previously using Theorem 1.3 that for each stage of the algorithm, the optimal stopping time is still a hitting time.

The American and integral options can be dealt with similarly.

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