From certain Lagrangian minimal submanifolds of the 3-dimensional complex projective space to minimal surfaces in the 5-sphere.

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Abstract

We study 3-dimensional Lagrangian submanifolds, without totally geodesic points, of the 3-dimensional complex projective space $\mathbb{C}P^3(4)$ which satisfy Chen's equality. We show how to associate with such a submanifold a minimal surface in $S^5(1)$ of which the ellipse of curvature is a circle. We also give a brief sketch of how this construction may be reversed. Details of the reverse construction will appear in a forthcoming paper.

1 Introduction

A totally real submanifold M^n of the complex projective space $\mathbb{C}P^m(4)$ of constant holomorphic sectional curvature 4 is said to satisfy Chen's equality if

(1)
$$\delta_M(p) = \frac{n^2(n-2)}{2(n-1)} H^2(p) + \frac{1}{2}(n+1)(n-2),$$

for each $p \in M$, where H denotes the length of the mean curvature vector and δ_M is the Riemannian invariant, introduced by Chen in [5], defined by

$$\delta_M(p) = \tau(p) - (\inf K)(p).$$

Here

$$(\inf K)(p) = \inf \left\{ K(\pi) \,|\, \pi \text{ is a 2-dimensional subspace of } T_pM \right\},$$

where $K(\pi)$ is the sectional curvature of π, and $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$ denotes the scalar curvature defined in terms of an orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space T_pM.

In this paper we study Lagrangian submanifolds of $\mathbb{C}P^3(4)$ which satisfy Chen's equality. In particular we show how we can, in a natural way, locally associate with such a

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submanifold a minimal surface in $S^5(1)$ with ellipse of curvature a circle. We also indicate briefly how this construction can be reversed, and details of this will appear in a forthcoming paper [2].

We now give some background material. In [5], Chen proved that for any submanifold M^n of a real space form $R^m(c)$ of constant sectional curvature c,

$$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)}H^2(p) + \frac{1}{2}(n+1)(n-2)c. \tag{2}$$

The class of submanifolds of $R^m(c)$ for which equality in (2) is attained at each point p turns out to be surprisingly large, and these submanifolds have been studied by several authors, for example see [6] and [11].

In [7] it was observed that the inequality (2) also holds for a totally real submanifold M^n of a complex space form of constant holomorphic sectional curvature $4c$, and we shall say that M^n satisfies Chen's equality if equality in (2) is attained at each point $p \in M^n$. If h denotes the second fundamental form of M^n, the *nullity subspace* $\mathcal{D}(p)$ at a point $p \in M^n$ is defined by

$$\mathcal{D}(p) = \{Z \in T_pM \mid h(X,Z) = 0 \text{ for all } X \in T_pM\},$$

and, if M^n is Lagrangian with dimension $n \geq 3$, then M^n satisfies Chen's equality if and only if M^n is minimal and $\mathcal{D}(p)$ has dimension at least $(n-2)$ at each point [8].

Lagrangian submanifolds with constant scalar curvature satisfying Chen's equality were classified in [7] and those with integrable nullity distribution $\mathcal{D}$ in [8].

We remark that, for a Lagrangian submanifold of a complex space form, the cubic form $\langle h(X,Y), JZ\rangle$ is totally symmetric. Following the ideas of Bryant [4], minimal Lagrangian submanifolds can be divided into different classes, depending on the properties of this cubic form. These different classes have been further investigated in [4] for 3-dimensional minimal Lagrangian submanifolds of $\mathbb{C}^3$. In fact, the same distinction can be made for 3-dimensional totally real submanifolds of other complex space forms, and Lagrangian submanifolds of $\mathbb{C}P^3(4)$ satisfying Chen's equality correspond to Case 4 of Proposition 1 of [4].

For the rest of the paper we restrict ourselves to the case of Lagrangian submanifolds of $\mathbb{C}P^3(4)$. This is the lowest non-trivial dimension, since any totally real immersion of a surface in $\mathbb{C}P^n(4)$ clearly satisfies Chen's equality.

2 Preliminaries

Throughout this paper we assume that M^n is a Lagrangian submanifold of $\mathbb{C}P^m(4)$. That is to say, if J is the complex structure of $CP^m(4)$, then J maps the tangent space of M onto the normal space so, in particular $m = n$. If $\tilde{\nabla}$ denotes the standard connection on $\mathbb{C}P^n(4)$, and ∇, $\nabla^\perp$ the induced connections on M and the normal bundle of M in $\mathbb{C}P^n(4)$, then the formulae of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$$

for vector fields X and Y tangential to M and for normal vector fields ξ. Since J is a parallel complex structure, it follows from the above formulae that [9]

$$\nabla^\perp_X JY = J\nabla_X Y,$$
$$A_{JY} X = -Jh(X, Y).$$

These formulae imply that β, defined by

$$\beta(X, Y, Z) = \langle h(X, Y), JZ \rangle,$$

is totally symmetric. Let $\overline{\nabla} h$ be defined by

$$(\overline{\nabla} h)(X, Y, Z) = \nabla^\perp_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The Codazzi equation says that $\overline{\nabla} h$ is also totally symmetric, whereas the Gauss curvature equation states that the curvature R of ∇ is given by

$$R(X, Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y + [A_{JX}, A_{JY}]Z.$$

We now recall the following theorem from [7] and [8].

Theorem 1 *Let M^n be a Lagrangian submanifold of $\mathbb{C}P^n(4)$ of constant holomorphic sectional curvature 4. Then M^n satisfies Chen's equality if and only if around each non totally geodesic point there exist a positive function λ and a local orthonormal moving tangent frame $\{E_1, \ldots, E_n\}$ such that*

$$h(E_1, E_1) = \lambda J E_1,$$
$$h(E_1, E_2) = -\lambda J E_2,$$
$$h(E_2, E_2) = -\lambda J E_1,$$
$$h(E_i, E_j) = 0, \quad j > 2.$$

In this case λ is the positive function given by

$$4\lambda^2 = n(n-1) - 2\tau.$$

For construction purposes, it is often useful to work with a horizontal lift under the Hopf fibration of a Lagrangian immersion. For that purpose we conclude this section with the following correspondence theorem of Reckziegel [13].

Theorem 2 *Let $p : S^{2n+1}(1) \to \mathbb{C}P^n(4)$ be the Hopf fibration. If $E : M^n \to S^{2n+1}(1)$ is a horizontal immersion, then $f = p \circ E : M^n \to \mathbb{C}P^n(4)$ is a Lagrangian immersion.*

Conversely, let M^n be simply connected and let $f : M^n \to \mathbb{C}P^n(4)$ be a Lagrangian immersion. Then there exists a 1-parameter family of horizontal lifts $E : M^n \to S^{2n+1}(1)$ such that $f = p \circ E$. Any two such lifts E_1 and E_2 are related by $E_2 = e^{i\theta} E_1$, where θ is a constant.

3 Lagrangian submanifolds of $\mathbb{C}P^3(4)$ satisfying Chen's equality

¿From now on we assume that M is a 3-dimensional Lagrangian submanifold of $\mathbb{C}P^3(4)$ satisfying Chen's equality. We will also assume that M is orientable, connected and simply connected and does not have any totally geodesic points. In this case, it follows from Theorem 1 that there exist globally defined vector fields E_1, E_2, E_3 on M satisfying

$$\begin{aligned} h(E_1,E_1) &= \lambda JE_1, & h(E_1,E_3) &= 0,\\ h(E_1,E_2) &= -\lambda JE_2, & h(E_2,E_3) &= 0,\\ h(E_2,E_2) &= -\lambda JE_1, & h(E_3,E_3) &= 0, \end{aligned}$$

where λ is a strictly positive function determined by the scalar curvature. It is now easy to check that, as is also indicated in Proposition 1 of [4], E_3 is determined up to sign whereas E_1 is determined up to rotations about E_3 by $2\pi/3$. However, as we will see later, the possible choices of E_1, E_2 and E_3 give exactly two minimal surfaces in $S^5(1)$ and these are closely related to each other.

Theorem 2 shows the existence of a globally defined horizontal lift $E_0 : M \to S^7(1)$, and we will identify a point of M with its image under E_0 in $S^7(1)$. Similarly, we identify E_1, E_2, E_3 with their images under the derivative dE_0 of E_0.

We now introduce functions a, b, c, d by

$$\begin{aligned} a &= \langle \nabla_{E_1} E_1, E_3\rangle,\\ b &= \langle \nabla_{E_1} E_2, E_3\rangle,\\ c &= \langle \nabla_{E_1} E_1, E_2\rangle,\\ d &= \langle \nabla_{E_2} E_1, E_2\rangle. \end{aligned}$$

Then, Lemma 4.1 of [8] implies that the connection ∇ is given by

$$\begin{aligned} \nabla_{E_1}E_1 &= cE_2 + aE_3, & \nabla_{E_2}E_1 &= dE_2 - bE_3, & \nabla_{E_3}E_1 &= -(1/3)bE_2,\\ \nabla_{E_1}E_2 &= -cE_1 + bE_3, & \nabla_{E_2}E_2 &= -dE_1 + aE_3, & \nabla_{E_3}E_2 &= (1/3)bE_1,\\ \nabla_{E_1}E_3 &= -aE_1 - bE_2, & \nabla_{E_2}E_3 &= bE_1 - aE_2, & \nabla_{E_3}E_3 &= 0, \end{aligned}$$

and that the function λ satisfies the following system of differential equations:

$$E_1(\lambda) = -3\lambda d, \quad E_2(\lambda) = 3\lambda c, \quad E_3(\lambda) = \lambda a.$$

We now use the above constructions to define a map $\mathcal{E}$ from M to the unitary group $U(4)$ given by

$$\mathcal{E} = (E_0, E_1, E_2, E_3). \tag{3}$$

The equations for the second fundamental form h and the connection ∇ written down earlier in this section may be written in terms of the $\mathbb{C}^4$-valued functions $E_0, \ldots, E_3$ as follows:

$$
\begin{aligned}
&(4) \qquad dE_0 = \omega^1 E_1 + \omega^2 E_2 + \omega^3 E_3, \\
&(5) \qquad dE_1 = -\omega^1 E_0 + i\lambda\omega^1 E_1 + (c\omega^1 + (d - i\lambda)\omega^2 - \tfrac{1}{3}b\omega^3)E_2 + (a\omega^1 - b\omega^2)E_3, \\
&(6) \qquad dE_2 = -\omega^2 E_0 - (c\omega^1 + (d + i\lambda)\omega^2 - \tfrac{1}{3}b\omega^3)E_1 - i\lambda\omega^1 E_2 + (b\omega^1 + a\omega^2)E_3, \\
&(7) \qquad dE_3 = -\omega^3 E_0 - (a\omega^1 - b\omega^2)E_1 - (b\omega^1 + a\omega^2)E_2,
\end{aligned}
$$

where $\{\omega^1, \omega^2, \omega^3\}$ is the dual basis to $\{E_1, E_2, E_3\}$.

Note that taking a different horizontal lift would imply that we rotate E_0 (and thus also E_1, E_2 and E_3) by a factor $e^{i\theta}$, where θ is a constant. Thus we may choose a lift E_0 for which $\mathcal{E}$ lies in $SU(4)$ at some point. It then follows from (4), (5), (6) and (7) that $\mathcal{E}$ always lies in $SU(4)$ so, by choosing a suitable horizontal lift E_0, we may assume that

$$
\mathcal{E} : M \to SU(4). \tag{8}
$$

4 An associated minimal surface in $S^5(1)$

We begin this section with a brief description of the use of the Hodge star operator to give the standard double cover of $SO(6)$ by $SU(4)$.

Recall that the standard Hermitian inner product on $\mathbb{C}^4$ extends to a Hermitian inner product $\langle\ ,\ \rangle$ on the second exterior power $\Lambda^2\mathbb{C}^4$ of $\mathbb{C}^4$. Then the Hodge star operator $* : \Lambda^2\mathbb{C}^4 \to \Lambda^2\mathbb{C}^4$ is the sesquilinear map defined by

$$
\langle \beta, *\alpha \rangle \omega = \alpha \wedge \beta, \quad \alpha, \beta \in \Lambda^2\mathbb{C}^4,
$$

where $\omega = e_0 \wedge e_1 \wedge e_2 \wedge e_3$ with $e_0, \ldots, e_3$ the standard unitary basis of $\mathbb{C}^4$. Then $\Lambda^2\mathbb{C}^4 = V \oplus iV$, where V is the real subspace of $\Lambda^2\mathbb{C}^4$ which is the $(+1)$-eigenspace of $*$. The star operator commutes with the natural action of $SU(4)$ on $\Lambda^2\mathbb{C}^4$, so that $SU(4)$ acts on V as a group of orthogonal transformations. If we now choose an orthonormal basis of V, we obtain the double cover of $SO(6)$ by $SU(4)$. It turns out that the most convenient basis for our purposes is that given by

$$
\begin{aligned}
u_0 &= (1/\sqrt{2})(e_0 \wedge e_3 + e_1 \wedge e_2), & u_5 &= (i/\sqrt{2})(e_0 \wedge e_3 - e_1 \wedge e_2), \\
u_1 &= (1/\sqrt{2})(e_0 \wedge e_1 + e_2 \wedge e_3), & u_4 &= (i/\sqrt{2})(e_0 \wedge e_1 - e_2 \wedge e_3), \\
u_2 &= (1/\sqrt{2})(e_0 \wedge e_2 + e_3 \wedge e_1), & u_3 &= (i/\sqrt{2})(e_0 \wedge e_2 - e_3 \wedge e_1),
\end{aligned}
$$

and we will use the orthonormal basis $\{u_0, \ldots, u_5\}$ to identify V with $\mathbb{R}^6$ in the usual way. We now let $\mathcal{U} = (U_0, \ldots, U_5) : M \to SO(6)$ be the composite of this double cover with the map $\mathcal{E} : M \to SU(4)$ constructed in Section 3 from a suitable horizontal lift E_0. Then the relation between $U_0, \dots, U_5$ and $E_0, \dots, E_3$ is obtained by replacing lower case letters by upper case letters in the equations above. It then follows from (4), (5), (6) and (7) that

(9) $dU_0 = (-a\omega^1 + (1+b)\omega^2)U_1 - ((1+b)\omega^1 + a\omega^2)U_2,$

(10) $dU_1 = -(-a\omega^1 + (1+b)\omega^2)U_0 + (c\omega^1 + d\omega^2 + (1-b/3)\omega^3)U_2 + \lambda\omega^1 U_3 - \lambda\omega^2 U_4,$

(11) $dU_2 = ((1+b)\omega^1 + a\omega^2)U_0 - (c\omega^1 + d\omega^2 + (1-b/3)\omega^3)U_1 - \lambda\omega^2 U_3 - \lambda\omega^1 U_4,$

(12) $dU_3 = -\lambda\omega^1 U_1 + \lambda\omega^2 U_2 + (c\omega^1 + d\omega^2 - (1+b/3)\omega^3)U_4 + (a\omega^1 + (1-b)\omega^2)U_5,$

(13) $dU_4 = \lambda\omega^2 U_1 + \lambda\omega^1 U_2 - (c\omega^1 + d\omega^2 - (1+b/3)\omega^3)U_3 - ((1-b)\omega^1 - a\omega^2)U_5,$

(14) $dU_5 = -(a\omega^1 + (1-b)\omega^2)U_3 + ((1-b)\omega^1 - a\omega^2)U_4.$

Taking now the projection of $\mathcal{U}$ onto the first factor, i.e. the map $p \mapsto U_0(p)$, we obtain a map from M into the 5-dimensional sphere $S^5(1)$ in $\mathbb{R}^6$. Notice that the definition of U_0 implies that U_0 is invariant under rotations of E_1 and E_2, which shows that the possible choices of E_1, E_2, E_3 referred to in Section 3 lead only to two antipodally opposite pairs of maps U_0 into $S^5(1)$, one pair obtained from the other by changing the sign of E_2 or E_3. As we shall see in the proof of the following theorem, this essentially corresponds to replacing the corresponding minimal surface in $S^5(1)$ by its polar [12], [1].

Theorem 3 *Let $p \in M$ and assume that $a^2 + (1+b)^2 \neq 0$ in a neighborhood X of the point p. Then the image of the restriction of U_0 to X is a minimal surface S in $S^5(1)$ with ellipse of curvature a circle. Moreover, provided that $a^2 + (1-b)^2$ is not identically zero on X, S is linearly full in $S^5(1)$. In that case, the polar immersion of this minimal surface is congruent to the minimal surface obtained via this construction when E_3 is replaced by $-E_3$.*

Proof. Let $p \in M$. Since $dU_0(E_3) = 0$, it follows that U_0 remains constant along the integral curves of E_3. Therefore, in order to describe the image of U_0 around p, it is sufficient to consider through p a surface N^2 which is transversal to E_3. Then there exist local functions α and β such that the tangent space of N^2 is spanned by $V_1 = E_1 + \alpha E_3$ and $V_2 = E_2 + \beta E_3$. We denote the restriction of U_0 to N^2 by g. It then follows from (9) that

(15) $$dg(V_1) = -aU_1 - (1+b)U_2,$$

(16) $$dg(V_2) = (1+b)U_1 - aU_2.$$

Since $a^2 + (1+b)^2 \neq 0$ in X, this implies that g is an immersion of N^2 into $S^5(1)$ and that V_1 and V_2 form an isothermal frame. Moreover, it is clear that U_1 and U_2 span the tangent space of the image of g. Using (10) and (11), we find that the second fundamental form h of g is given by

(17) $$h(V_1, V_1) = -a\lambda U_3 + (1+b)\lambda U_4,$$

(18) $$h(V_1, V_2) = h(V_2, V_1) = (1+b)\lambda U_3 + a\lambda U_4,$$

(19) $$h(V_2, V_2) = a\lambda U_3 - (1+b)\lambda U_4.$$

Hence g is a minimal immersion and, since $h(V_1, V_1)$ and $h(V_1, V_2)$ are orthogonal and have the same length, the ellipse of curvature is a circle. Further, g is linearly full in $S^5(1)$ if and only if U_5 is not a constant vector, i.e. provided $a^2 + (1-b)^2$ is not identically zero. We also note that $(U_1, \ldots, U_5)$ is a strongly adapted orthonormal frame over S in that U_1 and U_2 span the tangent space while U_3 and U_4 span the image of h. In particular, it follows that U_5 is the polar of g.

On the other hand, if we denote the frame obtained by replacing E_3 by $-E_3$ by adding a $\tilde{}$, we see that $\tilde{U}_0 = iU_5$, so that the corresponding minimal surface is congruent to the polar of g. ∎

Note that if $a \equiv 0$ and $b^2 \equiv 1$ on an open set, then, depending on our choice of E_3, one of the maps U_0 or U_5 becomes constant. For simplicity, we assume that U_5 is constant so that the minimal surface determined by U_0 is contained in a totally geodesic $S^4(1)$. Our correspondence is then equivalent to that obtained by Bryant [3] between superminimal immersions in $S^4(1)$ and holomorphic horizontal curves in $\mathbb{C}P^3(4)$.

An example of the above is provided by the exotic immersion of S^3 in $\mathbb{C}P^3(4)$ described in [7]. In fact, it follows from Lemma 4.5 of [7] that for this example we have $b = 1$, $a = 0$ and $\lambda = \frac{2}{\sqrt{3}}$. Thus the corresponding minimal surface S is contained in a totally geodesic $S^4(1)$, and it follows from the Gauss equation that the Gaussian curvature K of S is given by

$$\begin{aligned} K &= 1 + \tfrac{1}{16}(< h(V_1, V_1), h(V_2, V_2) > - < h(V_1, V_2), h(V_1, V_2) >) \\ &= 1 - \tfrac{1}{2}\lambda^2 = \tfrac{1}{3}. \end{aligned}$$

This implies that S is the Veronese surface in $S^4(1)$.

5 Final Remarks

The construction discussed in this paper locally associates to each Lagrangian submanifold M without totally geodesic points of $\mathbb{C}P^3(4)$ satisfying Chen's equality a three-dimensional subbundle of the bundle of strongly adapted orthonormal frames over a minimal surface S in $S^5(1)$ with ellipse of curvature a circle. In a forthcoming paper [2] we discuss the reverse process. In fact, if S is a minimal surface without totally geodesic points in $S^5(1)$ with ellipse of curvature a circle, we construct an essentially unique three-dimensional subbundle of the bundle of strongly adapted orthonormal frames over S which arises via our construction from a Lagrangian submanifold of $\mathbb{C}P^3(4)$ satistfying Chen's equality. This will give an essentially one-to-one local correspondence between Lagrangian submanifolds without totally geodesic points of $\mathbb{C}P^3(4)$ satistfying Chen's equality and minimal surfaces S without totally geodesic points in $S^5(1)$ with ellipse of curvature a circle.

A direct computational approach to this correspondence is also possible. In fact, using special coordinates on the Lagrangian submanifold, we may use the Gauss curvature equation to obtain a system of differential equations for the functions a, b, c, d. The solution of this system leads to a system of partial differential equations for two functions in two variables. This latter system of differential equations turns out to describe minimal surfaces in $S^5(1)$ with ellipse of curvature a circle. This computational approach is followed in [10] when dealing with the similar question in the complex hyperbolic space. In that case the geometric approach described in the present paper breaks down completely.

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