Parametric and autoparametric resonance

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1 Introduction

Parametric and autoparametric resonance play an important part in many applications while posing interesting mathematical challenges. The linear dynamics is already nontrivial whereas the nonlinear dynamics of such systems is extremely rich and largely unexplored. The role of symmetries is essential, both in the linear and in the nonlinear analysis.

1.1 Parametric excitation

The classical example of autoparametric excitation is the swinging pendulum with oscillating support. The equation of motion describing the model is

$$\ddot{x} + (\alpha^2 + p(t)) \sin x = 0 \quad , \tag{1}$$

where $p(t)$ is a periodic function. Upon linearization - replacing $\sin x$ by x - we obtain Hill's equation (section 3.1).
It is well-known that special tuning of the frequency α and the period of excitation (of $p(t)$) produces interesting instability phenomena (resonance).
More generally we may study nonlinear parametric equations of the form

$$\ddot{x} + k\dot{x} + (\alpha^2 + p(t))F(x) = 0 \quad , \tag{2}$$

where $k > 0$ is the damping coefficient, $F(x) = x + bx^2 + cx^3 + \cdots$ and time is scaled so that:

$$p(t) = \sum_{l \in Z} a_{2l} e^{2ilt} \quad , \quad a_0 = 0 \quad , \quad a_{-2l} = \bar{a}_{2l} \quad , \tag{3}$$

is a π-periodic function with zero average. We may also take for $p(t)$ a quasi-periodic or almost-periodic function.
The books by Yakubovich and Starzhinskii (1975) cover most of the classical theory. There are many open questions for equations (1) and (2); in section 3 we shall discuss some recent results.

Note that in parametric excitation we have an oscillator with an independent source of energy. Often the oscillator can be described by a one degree of freedom system but of course more degrees of freedom may play a part; see for instance in section 4 the case of coupled Mathieu-equations as studied by Ruijgrok, Tondl and Verhulst (1993).
In autoparametric systems we have interaction of at least two modes.

1.2 Autoparametric excitation

The classical example of an autoparametric system is the elastic pendulum, which consists of a spring fixed at one end. The spring may swing in a plane like a pendulum and oscillate at the same time. An interesting phenomenon arises if the ratio of the linear frequencies in the longitudinal and transversal directions is 2 : 1. In that case, if we start with an oscillation of the spring in the (near) vertical direction, this normal mode motion is unstable and energy is transferred gradually to the swinging motion and back; for a detailed description see van der Burgh (1968) or Nayfeh and Mook (1979). This phenomenon of destabilisation of a normal mode by a special tuning of the frequencies is called autoparametric resonance or autoparametric instability. In the example of the elastic pendulum, it takes place in a conservative setting. Such a situation may arise if we have a two degrees of freedom Hamiltonian sytem described by $H(p_1, p_2, q_1, q_2)$ which is discrete symmetric in one of the degrees of freedom. For instance $H(p_1, -p_2, q_1, -q_2) = H(p_1, p_2, q_1, q_2)$. Stability of the (p_1, q_1) normal mode has been studied extensively by normal form techniques, see Verhulst(1998) for a survey.
Here we will study autoparametric resonance in a more structurally stable

context with models which usually involve damping and an external source of energy. An example is given by a damped system of a forced, elastically mounted mass with a pendulum attached. This system can be seen as an extension of the classical parametrically excited pendulum.
We shall give a general characterization of autoparametric systems.

Following Tondl, Ruijgrok, Verhulst and Nabergoj (2000) we characterize autoparametric systems as vibrating systems which consist of at least two constituing subsystems. One is a Primary System which will generally be in a vibrating state. This Primary System can be externally forced, self-excited, parametrically excited or a combination of these, it has N degrees of freedom and will be described by the coordinates $x_i, \dot{x}_i,\ i = 1, 2, ..., N$.
The second constituing subsystem is called the Secondary System. The Secondary System is coupled to the Primary System in a nonlinear way, but such that the Secondary System can be at rest while the Primary System is vibrating. In engineering this state is called the *semi-trivial solution*, in physics it is called a *normal mode*. In fact there can exist an infinite number of semi-trivial solutions, for instance all the transient states to a periodic solution of the Primary System, but in discussing semi-trivial solutions we shall ignore transient states. In general, the Secondary System has n degrees of freedom and will be described by the coordinates $y_j,\ \dot{y}_j, j = 1, 2, ..., n$.

In the instability (parameter) intervals of the semi-trivial solution we have *autoparametric resonance*. The vibrations of the Primary System act as parametric excitation of the Secondary System which will no longer remain at rest. In this context this is called autoparametric excitation.
It is clear that in studying autoparametric systems, the determination of stability and instability conditions of the semi-trivial solution or normal mode is always the first step. After this it is of interest to look for other periodic solutions, bifurcations and classical or chaotic limit sets.

1.3 Applications of autoparametric resonance

In actual engineering problems, the loss of stability of the semi-trivial response of the Primary System depends on frequency tuning of the various components of the system, and on the interaction (the coupling) between the Primary System and the Secondary System. Autoparametric vibrations

occur only in a limited region of the system parameters. This property is of course of great importance for engineering purposes.
In a large number of problems we wish to diminish the vibration amplitudes of the Primary System; sometimes this is called 'quenching' of vibrations'. In other cases we have a coupled Secondary System which we would like to keep at rest.
An example of the last situation occurs in airplanes where the engines are mounted under the wings by elastic suspenders. The side deflection of the engine due to the elasticity of the suspenders can be described by the simple model of a mass on a leaf spring. Vertical vibrations of the wing (the Primary System) can, under certain conditions, initiate the swinging motion of the suspended engines (the Secondary System). Autoparametric excitation, with the motion of the wings supplying the energy, can lead to violent vibrations of the engines, resulting in a fatal failure of the suspenders.
Some mechanical systems wich are externally excited can be turned into autoparametric systems by using additional subsystems, for example by adding a pendulum. This has the effect of transferring the excitation energy to the added subsystem, thereby diminishing or quenching the vibration of the original basic system. Such a pendulum, or a similarily acting additional subsystem, has the same purpose as a tuned absorber commonly used in externally excited systems and self-excited systems. There are certain structures, machines and devices excited by different types of self-excitation (relative dry friction, a flowing medium, etc.) where these tools could be applied. The additional subsystem (the Secondary System) can be designed in various ways. The applications are usually in mechanics. It would be instructive and useful to look for examples in other fields, for example in electronic or biological systems which are governed by the same type of differential equations. These systems could lead to new possibilities for the application of autoparametric resonance.

The monograph by Tondl et al. (2000) contains a survey of the literature; see also Schmidt and Tondl (1986) and Cartmell (1990).

2 Normal form aspects

The term "normalisation" is used whenever an expression or quantity is put in a simpler, standardized form. For instance, a $n \times n$-matrix with constant coefficients can be put in Jordan normal form by a suitable transformation. When all eigenvalues are different, this is a diagonal matrix.
Introductions to normalisation can be found in Arnold (1983), Golubitsky and Schaeffer (1985), Golubitsky et al. (1988), Kuznetsov (1998) and Cicogna and Gaeta (1999). For the relation between averaging and normalisation the reader is referred to Sanders and Verhulst (1985) and Verhulst (1996).

2.1 The autonomous case

For autonomous, nonlinear differential equations we can arrive at normalisation of a vectorfield, in the neighbourhood of a fixed point, as follows. Assume that $x = 0$ is the fixed point, and write the system of differential equations as:

$$\dot{x} = Ax + f(x) \quad , \tag{4}$$

with $x \in \mathbb{R}^n$, A a constant $n \times n$-matrix; $f(x)$ can be expanded in homogeneous vector polynomials starting with quadratic terms:

$$f(x) = f_2(x) + f_3(x) + \cdots \quad . \tag{5}$$

Here the vector polynomial $f_m(x)$ is homogeneous of degree $m > 2$.
Normalisation of equation (4) means that by successive transformation we remove as many terms of equation (4) as possible. It would be ideal if we could remove all the nonlinear terms, i.e. linearize equation (4) by transformation. In general, however, some nonlinearities will be left.
To normalise equation (4) near $x = 0$ we introduce a near-identity transformation of the form:

$$x = y + h_2(y) + h_3(y) + \cdots \quad , \tag{6}$$

where $h_m(y)$ are homogeneous vector polynomials of degree m. The unknown polynomials $h_m(y)$ will be determined successively by substituting the near-identity transformation into equation (4).

The eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix A are resonant if for some $i \in \{1, 2, \ldots, n\}$ one has:

$$\sum_{j=1}^{n} m_j \lambda_j = \lambda_i \quad , \tag{7}$$

with $m_j \geq 0$ integers and $m_1 + m_2 + \cdots + m_n \geq 2$.
If the eigenvalues of A are non-resonant, we can remove all the nonlinear terms and so linearise the system. However, this is less useful then it appears, as in general the sequence of successive transformations to perform this will be divergent.
The usefulness of normalisation lies in removing nonresonant terms to a certain degree to simplify the analysis.

2.2 Normalisation of time-dependent vectorfields

In many problems we have time-dependent systems such as equations involving the Mathieu equation. Details of proofs and methods to compute the normal-form coefficients in such cases can be found in Arnold (1983), Iooss and Adelmeyer (1992).
Consider the following parameter and time dependent equation:

$$\dot{x} \;=\; F(x, \mu, t) \quad , \tag{8}$$

with $x \in \mathbb{R}^m$ and the parameters $\mu \in \mathbb{R}^p$.Here $F(x, \mu, t) : \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^m$ is C^∞ in x and μ and T-periodic in the t-variable. We assume that $x = 0$ is a solution, so $F(0, \mu, t) = 0$ and, moreover assume that the linear part of the vectorfield $D_x F(0, 0, t)$ is time independent for all $t \in \mathbb{R}$. We will write $L_0 = D_x F(0, 0, t)$.
Expanding $F(x, \mu, t)$ in a Taylor series with respect to x and μ yields the equation:

$$\dot{x} \;=\; L_0 x + \sum_{n=2}^{k} F_n(x, \mu, t) + \mathcal{O}(|(x, \mu)|^{k+1}) \quad , \tag{9}$$

where the $F_n(x, \mu, t)$ are homogeneous polynomials in x and μ of degree n with T-periodic coefficients.

Theorem
Let $k \in \mathbb{N}$. There exists a (parameter-and-time dependent) transformation $x = \hat{x} + \sum_{n=2}^{k} P_n(\hat{x}, \mu, t)$, where $P_n(\hat{x}, \mu, t)$ are homogeneous polynomials in x and μ of degree n with T-periodic coefficients, such that equation (9) takes the form (dropping the hat):

$$\begin{aligned} \dot{x} &= L_0 x + \sum_{n=2}^{k} \tilde{F}_n(x, \mu, t) + \mathcal{O}(|(x, \mu)|^{k+1}) \quad , \\ \dot{\mu} &= 0 \quad . \end{aligned} \tag{10}$$

The truncated vectorfield:

$$\dot{x} = L_0 x + \sum_{n=2}^{k} \tilde{F}_n(x, \mu, t) = \tilde{F}(x, \mu, t) \quad , \tag{11}$$

which will be called the normal form of (8), has the following properties:

a) $\frac{d}{dt} e^{L_0^* t} \tilde{F}(e^{-L_0^* t} x, \mu, t) = 0$, for all $(x, \mu) \in \mathbf{R}^{m+p}, t \in \mathbb{R}$.

b) If equation (8) is invariant under an involution (i.e. $SF(x, \mu, t) = F(Sx, \mu, t)$ with S an invertible linear operator such that $S^2 = I$) then the truncated normal form (11) is also invariant under S. Similarly, if equation (8) is reversible under an involution R (i.e. $RF(x, \mu, t) = -F(Rx, \mu, t)$), then the truncated normal form (11) is also reversible under R.

For a proof, see Iooss and Adelmeyer (1992).

The theorem will be applied to situations where L_0 is semi-simple and has only purely imaginary eigenvalues. We take $L_0 = diag\{i\lambda_1, \dots, i\lambda_m)\}$. In our applications, $m = 2l$ is even and $\lambda_{l+j} = -\lambda_j$ for $j = 1, \dots, l$. The variable x is then often written as $x = (z_1, \dots, z_l, \bar{z}_1, \dots, \bar{z}_l)$. We have:
Assume $L_0 = diag\{i\lambda_1, \dots, i\lambda_m)\}$ then:

- A term $x_1^{\gamma_1} \dots x_m^{\gamma_m} e^{i \frac{2\pi}{T} kt}$ is in the j-th component of the Taylor-Fourier series of $\tilde{F}(x, \mu, t)$ if:

$$-\lambda_j + \frac{2\pi}{T} k + \gamma_1 \lambda_1 + \dots + \gamma_m \lambda_m = 0 \quad . \tag{12}$$

  This is known as the resonance condition.

- Transforming the normal form through $x = e^{L_0 t} w$ leads to an autonomous equation for w:

$$\dot{w} = \sum_{n=2}^{k} \tilde{F}_n(w, \mu, 0) \quad . \tag{13}$$

- An important result is this:
If equation (8) is invariant (respectively reversible) under an involution S, then this also holds for equation (13).

- The autonomous normal form (13) is invariant under the action of the group $\mathcal{G} = \{g \mid gx = e^{jL_0 T} x \ , \ j \in \mathbb{Z}\}$, generated by $e^{L_0 T}$. Note that this group is discrete if the ratios of the λ_i are rational and continuous otherwise.

For a proof of the last two statements see Ruijgrok (1994).

By property b) we can make the system autonomous. This is very effective as the autonomous normal form (13) can be used to prove the existence of periodic solutions and invariant tori of equation (8) near $x = 0$. We have:

Theorem
Let $\varepsilon > 0$, sufficiently small, be given. Scale $w = \varepsilon \hat{w}$.
a) If $\hat{w}_0$ is hyperbolic fixed point of the (scaled) equation (13), then (8) has a hyperbolic periodic solution $x(t) = \varepsilon \hat{w}_0 + \mathcal{O}(\varepsilon^{k+1})$.
b) If the scaled equation (13) has a hyperbolic closed orbit, then (8) has a hyperbolic invariant torus.

These results are related to earlier theorems by Bogoliubov and Mitropolsky (1961). See Guckenheimer and Holmes (1983) and Hale (1969) for proofs in the context of averaging. The proofs are easily adapted to the present situation.

2.3 Remarks on bifurcations

Fortunately there are many good introductions to the basic ideas of bifurcation theory in nonlinear dynamics; we mention Guckenheimer and Holmes

(1983), Thompson and Stewart (1986), Moon (1987), Wiggins (1988), Verhulst (1996), Kuznetsov (1998), also the beautiful and readable introduction by Peitgen, Jürgens and Saupe (1993).
Consider a mechanical system described by the n-dimensional system:

$$\dot{x} = f(x, t, \mu) \quad . \tag{14}$$

The parameter μ, which is m-dimensional, represents the natural control parameters of the system, such as friction-coefficients, the amplitude of forcing functions or coupling constants. Such a description of a mechanical system is actually a description of a class of systems as we will never know the exact values of the parameters μ. Also we are usually interested in what happens for various values of the parameters.
A *bifurcation* is a qualitative change in the dynamics of the system as the parameter μ crosses a critical value μ_c. In autonomous systems this is reflected by a change of the topology of the phase space. For instance a saddle with two neighbouring attracting nodes can merge into one attracting node. Bifurcation theory aims at establishing the critical values μ_c of a system and at describing what happens at the changes. At the value $\mu = \mu_c$ the mechanical system is called *structurally unstable*, any small change of μ will produce a qualitative change. Sometimes such a change is quantitatively a small-scale phenomenon, sometimes it is very dramatic.

Local bifurcations
The simplest bifurcations to study are the so-called *codimension one* bifurcations of equilibria with a one-dimensional parameter. In each case we consider a normal form of the equation near equilibrium, a reason why these are also called local bifurcations. We list them briefly.

- The *saddle-node bifurcation* described by $\dot{x} = \mu - x^2$. If $\mu < 0$ there is no equilibrium solution, at $\mu = 0$ two equilibrium solutions branch off, of which one is stable and one is unstable.
- The *transcritical bifurcation* described by $\dot{x} = \mu x - x^2$. There are two equilibrium solutions, $x = 0$ and $x = \mu$, with an exchange of stability when μ crosses zero.
- The *pitchfork bifurcation* described by $\dot{x} = \mu x - x^3$. If $\mu < 0$ there is one equilibrium solution, $x = 0$, which is stable. If $\mu > 0$, there

are three equilibrium solutions, $x = 0$, $x = \pm\sqrt{\mu}$, of which $x = 0$ is unstable and the other two stable.

- Bifurcation of periodic solutions (*Hopf* bifurcation) can occur when the linearisation of the vectorfield near an equilibrium has two purely imaginary eigenvalues for a certain value of μ, whereas the other eigenvalues all have a non-vanishing real part. This situation is exemplified by the famous van der Pol-equation.

Equations with dimension two and higher, containing also more than one parameter, may display more complicated bifurcation patterns. We may have for instance the double-zero (two eigenvalues zero) bifurcation or *Bogdanov-Takens* bifurcation; at dimension three the case of two purely imaginary eigenvalues and one zero and at dimension four the case of four purely imaginary eigenvalues (*Hopf-Hopf*-bifurcation).
Codimension two bifurcations are discussed by Kuznetsov (1998).

Global bifurcations
The bifurcations studied above have the advantage that they can be studied by considering a neighbourhood of an equilibrium. Normal form calculation or averaging is effective here. More difficult are the global bifurcations which again take place at certain parameter values, but which are not local in the sense described earlier.
Global bifurcations may cause sudden, large-scale changes in the dynamical system. This is reflected in some of the terminology: *catastrophic-explosive* or *crisis*. We mention some important concepts:

- *Neimark-Sacker bifurcation*
  Suppose that we are studying solutions in 4-space as in the case of a coupled autoparametric system with two modes; we assume there exists a periodic orbit which is characterized by four eigenvalues, at least two of which are complex. On varying a parameter, these two eigenvalues cross the imaginary axis producing the Neimark-Sacker bifurcation. This bifurcation usually results in a torus with an associated periodic orbit.

- *Šilnikov bifurcation*
  Let X be a sufficiently smooth vector field on $\mathbb{R}^3$ with equilibrium $O = (0, 0, 0)$ such that:

1. the eigenvalues of O are $\alpha \pm i\beta, \lambda$ with $|\alpha| < |\lambda|$ and $\beta \neq 0$;
2. O has a homoclinic orbit,

then there exists a small perturbation $X + \mu Y$ of the vectorfield X such that the perturbed field contains a horseshoe map which implies the presence of an infinite number of unstable periodic orbits and chaos. Of course the difficulty in establishing a Šilnikov bifurcation lies in demonstrating the existence of an homoclinic orbit.

- *Heteroclinic cycles*
  Consider solutions in 3- or 4-space. Certain (hidden) symmetries may produce robust heteroclinic cycles connecting two or more saddle-sink structures. A nice discussion is given by Krupa (1997).

2.4 Remarks on limit sets

In studying a dynamical system the behaviour of the solutions is for a large part determined by the limit sets of the system. The classical limit sets are equilibria and periodic orbits.
Even when restricting to autonomous equations of dimension three, we have no classification of possible limit sets and we do not expect to have one in the near future. This makes the recognition and description of non-classical limit sets important.
In autoparametric systems which have at least four dimensions the following limit sets, apart from the classical ones, are of interest:

- Chaotic attractors. Various scenarios were found; in Ruijgrok (1995), Ruijgrok and Verhulst (1996), Tondl, Ruijgrok, Verhulst and Nabergoj (2000) flow-induced vibrations are studied which lead to Šilnikov bifurcation and chaotic behaviour. In section 5 also period-doubling and other scenarios play a part.

- Strange attractors without chaos, see for instance Pikovsky and Feudel (1995). Until now such phenomena have not been observed in autoparametric systems but the presence of various forcing periods in such systems make their occurrence quite plausible.

- Attracting tori. These limit sets are not difficult to find; they arise for instance as a consequence of a Neimark-Sacker bifurcation of a periodic solution.
- Attracting heteroclinic cycles, mentioned earlier under bifurcations. For an application see section 6.

3 Parametric excitation

As we have seen in section 1.1 parametric excitation leads to the study of second order equations with periodic coefficients. More in general such equations arise from linearization near T-periodic solutions of T-periodic equations of the form $\dot{y} = f(t, y)$. Suppose $y = \phi(t)$ is a T-periodic solution; putting $y = \phi(t) + x$ produces upon linearization the T-periodic equation

$$\dot{x} = f_x(t, \phi(t))x \tag{15}$$

In autoparametric systems this equation often takes the form

$$\dot{x} = Ax + \varepsilon B(t)x \quad , \tag{16}$$

in which $x \in \mathbb{R}^m$; A is a constant $m \times m$-matrix, $B(t)$ is a continuous, T-periodic $m \times m$-matrix, ε is a small parameter.
For elementary studies of such an equation, the Poincaré-Lindstedt method or continuation method is quite efficient. The method applies to nonlinear equations of arbitrary dimension, but we shall demonstrate its use for equations of Mathieu type.
Many books and papers have appeared on the subject of differential equations with periodic coefficients. We mentioned already Yakubovich and Starzhinskii (1975); see also Meixner and Schäfke (1954), Levy and Keller (1963), Weinstein and Keller (1985, 1987), Arnold (1977, 1983).

3.1 Elementary theory

Floquet theory tells us that the solutions of (16) can be written as:

$$x(t) = \Phi(t)e^{Ct} \quad , \tag{17}$$

with $\Phi(t)$ a T-periodic $m \times m$-matrix, C a constant $m \times m$-matrix. The determination of C provides us with the stability behaviour of the solutions. A particular case of equation (16) is Hill's equation:

$$\ddot{x} + b(t)x = 0 \quad , \tag{18}$$

which is of second order; $b(t)$ is a scalar T-periodic function. A particular case of equation (18) which arises frequently in applications is the Mathieu equation:

$$\ddot{x} + (a + \varepsilon \cos 2t)x = 0, \qquad a > 0, \tag{19}$$

which is reversible. A typical question is: for which values of a and ε in (a, ε)-parameter space is the trivial solution $x = \dot{x} = 0$ stable?
Solutions of (19) can be written in the Floquet form (17), where in this case $\Phi(t)$ will be π-periodic. The eigenvalues λ_1, λ_2 of C, which are called characteristic exponents, determine the stability of the trivial solution. For the characteristic exponents of equation (16) we have:

$$\sum_{i=1}^{n} \lambda_i = \frac{1}{T} \int_0^T Tr(A + \varepsilon B(t))dt \quad , \tag{20}$$

see theorem 6.6 in Verhulst (1996).
So in the case of equation (19) we have:

$$\lambda_1 + \lambda_2 = 0 \quad . \tag{21}$$

The exponents are functions of ε, $\lambda_1 = \lambda_1(\varepsilon)$, $\lambda_2 = \lambda_2(\varepsilon)$ and clearly $\lambda_1(0) = i\sqrt{a}$, $\lambda_2(0) = -i\sqrt{a}$. As $\lambda_1 = -\lambda_2$, the characteristic exponents, which are complex conjugate, are purely imaginary or real. The implication is that if $a \neq n^2$, $n = 1, 2, \ldots$ the characteristic exponents are purely imaginary and $x = 0$ is stable near $\varepsilon = 0$. If $a = n^2$ for some $n \in \mathbb{N}$, however, the imaginary part of $\exp(Ct)$ can be absorbed into $\Phi(t)$ and the characteristic exponents may be real. We assume now that $a = n^2$ for some $n \in \mathbb{N}$, or near this value, and we shall look for periodic solutions of $x(t)$ of equation (19) as these solutions bound the stable from the unstable solutions. We put:

$$a = n^2 - \varepsilon\beta \quad , \tag{22}$$

with β a constant, and we apply the Poincaré-Lindstedt method to find the periodic solutions; see Verhulst (1996), appendix 2.
We find that periodic solutions exist for $n = 1$ if:

$$a = 1 \pm \frac{1}{2}\varepsilon + \mathcal{O}(\varepsilon^2) \quad .$$

In the case $n = 2$ periodic solutions exist if ;

$$\begin{aligned} a &= 1 - \frac{1}{48}\varepsilon^2 + \mathcal{O}(\varepsilon^4) \quad , \\ a &= 1 + \frac{5}{48}\varepsilon^2 + \mathcal{O}(\varepsilon^4) \quad . \end{aligned} \tag{23}$$

The corresponding instability domains are called Floquet tongues.
On considering higher values of n, we have to calculate to a higher order of ε. At $n = 1$ the boundary curves are intersecting at positive angles at $\varepsilon = 0$, at $n = 2$ $(a = 4)$ they are tangent; the order of tangency increases as $n - 1$ (contact of order n), making instability domains more and more narrow with increasing resonance.

The Mathieu equation with viscous damping
In real-life applications there is always the presence of damping. We shall consider the effect of its simplest form, small viscous damping. Equation (19) is extended by adding a linear damping term:

$$\ddot{x} + \kappa\dot{x} + (a + \varepsilon\cos 2t)x = 0 \quad , \quad a, \kappa > 0 \quad . \tag{24}$$

We assume that the damping coefficient is small, $\kappa = \varepsilon\kappa_0$, and again we put $a = n^2 - \varepsilon\beta$ to apply the Poincaré-Lindstedt method.
We find periodic solutions in the case $n = 1$ if:

$$a = 1 \pm \sqrt{\frac{1}{4}\varepsilon^2 - \kappa^2} \quad . \tag{25}$$

Relation (25) corresponds with the curve of periodic solutions, which in (a, ε)-parameter space separates stable and unstable solutions, see figure 1. We observe the following phenomena. If $0 < \kappa < \frac{1}{2}\varepsilon$, we have an instability domain which by damping has been lifted from the a-axis; also *the width has shrunk*. If $\kappa > \frac{1}{2}\varepsilon$ the instability domain has vanished.

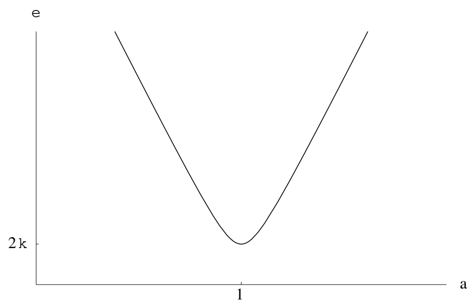


Figure 1: In the Mathieu equation with damping (24) the instability domains are lifted from the a-axis.

Repeating the calculations for $n \geq 2$ we find no instability domains at all; damping of $\mathcal{O}(\varepsilon)$ stabilizes the system for ε small. To find an instability domain we have to decrease the damping, for instance if $n = 2$ we have to take $\kappa = \varepsilon^2 \kappa_0$.

3.2 Geometric aspects of resonance tongues

Consider again Hill's equation in the form

$$\ddot{x} + (a^2 + bp(t))x = 0 \quad , \tag{26}$$

with periodic $p(t)$ and parameters a, b. Most of the results are restricted to the reversible case $p(t) = p(-t)$ of which the classical Mathieu-equation is an example.

Broer and Levi (1995) note that the Mathieu-equation is special in the sense that for the general reversible equation resonance tongues may intersect thereby creating instability pockets in the bifurcation diagram. They study the properties of Hill's map which relates (a, b) to the Poincaré matrix of equation (26). This leads to observations on the sharpness of resonance tongues, the presence of instability pockets and the consequences of reversible perturbations like $p(t) = \cos(t) + \varepsilon \cos(2t)$.

This geometric approach is continued by Broer and Simó (2000) who explore again reversible perturbations of Mathieu's equation. They conclude that at a certain order n of resonance, perturbations by lower order harmonics introduces instability pockets and any order of tangency of the tongues.

It will be no surprise that many more complication arise in Broer and Simó

(1998) where $p(t)$ is allowed to be quasi-periodic with two rationally independent frequencies.

3.3 Bifurcations in a nonlinear Mathieu-equation

It is important to have insight in the consequences of adding nonlinear terms to the Mathieu-equation. Consider the following equation:

$$\ddot{x} + \kappa \dot{x} + (\alpha^2 + p(t))F(x) = 0 \quad , \tag{27}$$

where $\kappa > 0$ is the damping coefficient, $F(x) = x + bx^2 + cx^3 + \cdots$, and time is scaled so that:

$$p(t) = \sum_{l \in Z} a_{2l} e^{2ilt} \quad , \quad a_0 = 0 \quad , \quad a_{-2l} = \bar{a}_{2l} \quad , \tag{28}$$

is a π-periodic function with zero average. As we have seen in section 3.1, the trivial solution $x = 0$ is unstable when $\kappa = 0$ and $\alpha^2 = n^2$, for all $n \in \mathbb{N}$. Fix a specific $n \in \mathbb{N}$ and assume that α^2 is close to n^2. We will study the bifurcations from the solution $x = 0$ in the case of primary resonance, which by definition occurs when the Fourier expansion of $p(t)$ contains nonzero terms $a_{2n}e^{2int}$ and $a_{-2n}e^{-2int}$. The bifurcation parameters in this problem are the detuning $\sigma = \alpha^2 - n^2$, the damping coefficient κ and the Fourier coefficients of $p(t)$, in particular a_{2n}. They are assumed to be small and of equal order of magnitude.

In Broer and Vegter (1992) the conservative case (or Hamiltonian) $\kappa = 0$ was studied, here we consider the dissipative case $\kappa > 0$. The analysis is based on Ruijgrok (1995) and can also be found in Ruijgrok and Verhulst (1996), Tondl et al. (2000).

Normal form equations

To find the time-periodic normal form of (27), we put $x = x_1$, $\dot{x} = x_2$ and write

$$\begin{aligned} \dot{x}_1 &= x_2 \quad , \\ \dot{x}_2 &= -\kappa x_2 - (n^2 + \sigma + p(t))F(x_1) \quad . \end{aligned} \tag{29}$$

Equation (29) can be written in complex form, using $z = nx_1 - ix_2$ and expanding $F(x_1)$:

$$\dot{z} = inz - \frac{1}{2}\kappa(z - \bar{z}) + \frac{1}{2n} i(\sigma + p(t))(z + \bar{z}) + \cdots \quad . \tag{30}$$

The equation for $\bar{z}$ has been omitted.
To equation (30) we apply the time-periodic normal form procedure as described in section 2.2. The righthand side of (30) is expanded in powers of z, $\bar{z}$ and the parameters, which will be indicated by $\mu = (\sigma, \kappa, a_2, a_4, \dots)$. A long calculation yields the time-dependent normal form of (30), up to second order:

$$\begin{aligned}\dot{z} &= inz + (-\frac{1}{2}\kappa + \frac{1}{2n}i\sigma)z + \frac{1}{2n}ia_{2n}e^{i2nt}\bar{z} + \Lambda(z,\bar{z},\mu,t) + K(z,\bar{z},\mu,t) + \\ &\quad igz|z|^2 + \mathcal{O}(|(z,\bar{z},\mu)|^4) \quad , \end{aligned} \tag{31}$$

where:

$$\begin{aligned} g &= (\frac{3}{4}c - \frac{10}{3}b^2) \quad , \qquad (32) \\ K(z,\bar{z},\mu,t) &= -\frac{ib}{6n^2}(a_{-n}e^{-int}z^2 + 2a_ne^{int}|z|^2 - 3a_{3n}e^{3int}\bar{z}^2) \quad , \end{aligned}$$

and $\Lambda(z,\bar{z},\mu,t)$ contains terms which are linear in z and quadratic in the parameters. It can be assumed that after a suitable time translation, a_{2n} is real and positive. From this point on, it will be assumed that $g \neq 0$. This condition is satisfied by "almost all" choices of $F(x)$, and can therefore be called generic.
The normal form (31) can be made autonomous through the transformation $z = we^{int}$. After some rescaling of the parameters and also scaling $w = \varepsilon^{1/2}\hat{w}$ (following Broer and Vegter (1992)), equation (31) becomes (dropping the hats and time scaling $\tau = \varepsilon t$):

$$w' = (-\kappa + i\sigma)w + ia_{2n}\bar{w} + \varepsilon^{1/2}K(w,\bar{w},\mu) + igw|w|^2 + \mathcal{O}(\varepsilon) \quad , \tag{33}$$

with

$$K(w,\bar{w},\mu) = -\frac{ib}{3n}(a_{-n}w^2 + 2a_n|w|^2 - 3a_{3n}\bar{w}^2) \quad , \tag{34}$$

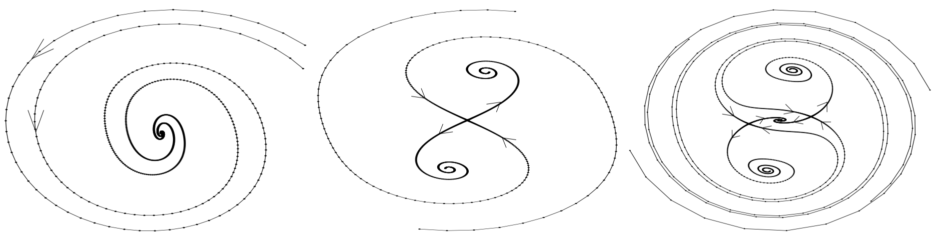
Equation (33) is invariant under $(w,\bar{w}) \to -(w,\bar{w})$ (up to $\mathcal{O}(\varepsilon)$ terms) and can be treated as a perturbation of a symmetric system.

Dynamics and bifurcations of the symmetric system
The symmetry given by $(w,\bar{w}) \to -(w,\bar{w})$ implies, amongst other things,

that all fixed points come in pairs, and that bifurcations of the origin will be symmetric (such as pitchfork bifurcations). We observe that the normal form equation is symmetric when either $F(x)$ in equation (27) is odd in x or when n is odd. This is reflected by equation (33), which is invariant under $(x, y) \to -(x, y)$ only if the quadratic terms vanish. From (34) it is easy to see that this indeed is the case when $F(x)$ is odd, since then $b = 0$. Similarly, when n is odd, $p(t) = \sum_{l \in \mathbf{Z}} a_{2l} e^{2ilt}$ does not contain terms a_{-n}, a_n or a_{3n} and all the coefficients in (34) equal zero.
The symmetric equation, truncated at $\mathcal{O}(\varepsilon)$, is given by:

$$\dot{w} = (-\kappa + i\sigma)w + i a_{2n}\bar{w} + igw|w|^2 \quad . \tag{35}$$

It is not difficult to show that, for sufficiently large R, the disc $|w| < R$ is invariant under the flow of (35), and that the only attractors in this region are fixed points. The dynamics of (35) is summarized in figure 2.



(i) $(\sigma, a_{2n}) \in I$ (ii) $(\sigma, a_{2n}) \in II$ (iii) $(\sigma, a_{2n}) \in III$

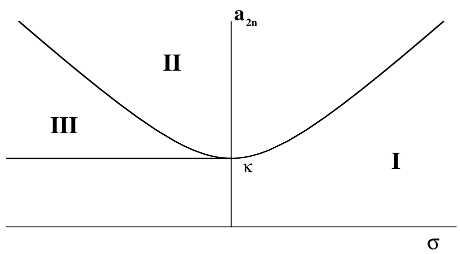


Figure 2: Bifurcation diagram in the (σ, a_{2n})-plane and phase-portraits of equation (35).

Outside the hyperbola $\kappa^2 + \sigma^2 = a_{2n}^2$ (that is, outside region II) the trivial solution is stable. On the hyperbola a pitchfork bifurcation occurs, which is supercritical if $\sigma > 0$ and subcritical if $\sigma < 0$. On the half line $a_{2n} = \kappa$, $\sigma < 0$ there occurs a double saddle-node bifurcation, i.e. two simultaneous saddle-nodes.

Bifurcations in the general case
As was remarked earlier, the general equation (33) can be seen as a non-symmetric $\mathcal{O}(\varepsilon^{1/2})$ perturbation of the symmetric case. The analysis involves all codimension one bifurcations and a centre manifold computation; see Tondl et al. (2000).

4 Rotor dynamics

The following formulation is based on Tondl (1991) and Ruijgrok, Tondl and Verhulst (1993). Consider a rigid rotor consisting of a heavy disk of mass M which is rotating around an axis. The axis of rotation is elastically mounted on a foundation; the connections which are holding the rotor in an upright position are also elastic. To describe the position of the rotor we have the axial displacement u in the vertical direction (positive upwards), the angle of the axis of rotation *with respect to* the z-axis and *around* the z-axis . Instead of these two angles we will use the projection of the centre of gravity motion on the horizontal (x, y)-plane. The equations of motion become:

$$
\begin{aligned}
\ddot{x} + 2\alpha\dot{y} + x &= \frac{MR}{I_1}\ddot{u}x \quad , \\
\ddot{y} - 2\alpha\dot{x} + y &= \frac{MR}{I_1}\ddot{u}y \quad , \\
\ddot{u} + 4\eta^2 u &= \frac{1}{R}(\dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y}) - \frac{g}{\Omega^2} \quad .
\end{aligned} \tag{36}
$$

Here $\alpha, \eta, M, R, I_1, g$ and Ω are positive, physical parameters.
In the following, we will consider a rotor system which is externally excited through the support. Thus, a special solution to equations (36) is the semi-trivial solution:

$$
\begin{gathered}
x_0(t) = 0, y_0(t) = 0, \\
u_0(t) = a\cos 2\eta t - \frac{g}{4\eta^2\Omega^2} = a\cos 2\eta t - \frac{Mg}{2k_0},
\end{gathered} \tag{37}
$$

corresponding to oscillations in the upright position.
Without loss of generality we have taken the phase of the oscillation equal to zero. The amplitude a depends on the initial conditions.
We will consider the situation where $a << 1$, and study the stability of the semi-trivial solution by assuming asymptotic expansions for x, y and u:

$$\begin{aligned} x = \varepsilon x_1 + \varepsilon^2 x_2 + ..., y = \varepsilon y_1 + \varepsilon^2 y_2 + ..., \\ u = -\frac{Mg}{2k_0} + \frac{I_1}{MR}\varepsilon \cos 2\eta t + \varepsilon^2 u_2 + ..., \end{aligned} \tag{38}$$

with $\varepsilon = a\frac{MR}{I_1}$ a small positive parameter. Inserting these expressions into (36) yields up to first order in ε:

$$\begin{aligned} \ddot{x}_1 + 2\alpha\dot{y}_1 + x_1 &= -4\varepsilon\eta^2 x_1 \cos 2\eta t \quad , \\ \ddot{y}_1 - 2\alpha\dot{x}_1 + y_1 &= -4\varepsilon\eta^2 y_1 \cos 2\eta t \quad . \end{aligned} \tag{39}$$

In the equation for u all terms to order ε vanish. Approximation of $O(\varepsilon)$ means neglection of all nonlinear terms in a neighbourhood of the trivial equilibrium solution $(x, \dot{x}, y, \dot{y}) = (0, 0, 0, 0)$ of equations (39).

4.1 A sum-resonance in the linear system

Replacing (x_1, y_1) by (x, y) in equations (39), we can write:

$$\begin{aligned} \ddot{x} + 2\alpha\dot{y} + (1 + 4\varepsilon\eta^2 \cos 2\eta t)x &= 0 \quad , \\ \ddot{y} - 2\alpha\dot{x} + (1 + 4\varepsilon\eta^2 \cos 2\eta t)y &= 0 \quad . \end{aligned} \tag{40}$$

System (40) constitutes a system of Mathieu-like equations, where we have neglected the effects of damping (see the next sections). The natural frequencies of the unperturbed system (40), $\varepsilon = 0$, are $\omega_1 = \sqrt{\alpha^2 + 1} + \alpha$ and $\omega_2 = \sqrt{\alpha^2 + 1} - \alpha$. It is well-known that when the frequency of the autoparametric excitation, 2η, satisfies a resonance condition with the eigenfrequencies of the unperturbed system ($\varepsilon = 0$), then the trivial solution of (40), i.e. the semi-trivial solution of (36), can become unstable. We shall determine the instability domains for ε small.
By putting $z = x + iy$, system (40) can be written as:

$$\ddot{z} - 2\alpha i\dot{z} + (1 + 4\varepsilon\eta^2 \cos 2\eta t)z = 0 \quad . \tag{41}$$

Introducing the new variable:

$$v = e^{-i\alpha t} z \quad , \tag{42}$$

and assuming $\eta t = \tau$, we obtain:

$$v'' + \left(\frac{1+\alpha^2}{\eta^2} + 4\varepsilon \cos 2\tau \right) v = 0 \quad , \tag{43}$$

where the prime denotes differentiation with respect to τ. By writing down the real and imaginary parts of this equation, we get two identical Mathieu equations.

We conclude that the trivial solution is stable for ε small enough, providing that $\sqrt{1+\alpha^2}$ is not close to $n\eta$, for some $n = 1, 2, 3, \ldots$. The first-order interval of instability, $n = 1$, arises if:

$$\sqrt{1+\alpha^2} \approx \eta \quad . \tag{44}$$

If (44) is satisfied the trivial solution of equation (43) is unstable. Therefore, the trivial solution of system (40) is also unstable. Note that this instability arises when:

$$\omega_1 + \omega_2 = 2\eta \quad ,$$

i.e. when the sum of the eigenfrequencies of the unperturbed system equals the autoparametric excitation frequency 2η. This is known as a combination sum-resonance of first order.The domain of instability can be calculated as in section 3.1; we find for the boundaries:

$$\eta_b = \sqrt{1+\alpha^2}\ (1 \pm \varepsilon) + O(\varepsilon^2) \quad . \tag{45}$$

The second-order interval of instability of equation (43), $n = 2$, arises when:

$$\sqrt{1+\alpha^2} \approx 2\eta \quad , \tag{46}$$

i.e. $\omega_1 + \omega_2 \approx \eta$. This is known as a combination sum-resonance of second order. As above, we find the boundaries of the domains of instability:

$$\begin{aligned} 2\eta &= \sqrt{1+\alpha^2} \quad \left(1 + \tfrac{1}{24}\varepsilon^2\right) + O(\varepsilon^4) \quad , \\ 2\eta &= \sqrt{1+\alpha^2} \quad \left(1 - \tfrac{5}{24}\varepsilon^2\right) + O(\varepsilon^4) \quad . \end{aligned} \tag{47}$$

Higher order combination resonances can be studied in the same way; the domains of instability in parameter space continue to narrow as n increases. It must be kept in mind that the parameter α is proportional to the rotating frequency of the disk and to the ratio of the moments of inertia.

4.2 Instability by damping

To examine the stability of the semi-trivial solution of (36), we add a small linear damping to system (40), with positive damping parameter $\mu = 2\varepsilon\kappa$. This leads to the equation:

$$\ddot{z} - 2\alpha i\dot{z} + \left(1 + 4\varepsilon\eta^2 \cos 2\eta t\right) z + 2\varepsilon\kappa\dot{z} = 0 \quad . \tag{48}$$

Because of the damping term, we can no longer reduce the complex equation (48) to two identical second order real equations, as we did in the previous section.
In the sum-resonance of the first order, $\omega_1 + \omega_2 \approx 2\eta$ and the solution of the unperturbed ($\varepsilon = 0$) equation can be written as:

$$z(t) = z_1 e^{i\omega_1 t} + z_2 e^{-i\omega_2 t} \quad , \; z_1, z_2 \in \tag{49}$$

with $\omega_1 = \sqrt{\alpha^2+1} + \alpha,\ \omega_2 = \sqrt{\alpha^2+1} - \alpha$.
Applying variation of constants leads to equations for z_1 and z_2:

$$\begin{aligned}
\dot{z}_1 &= \frac{i\varepsilon}{\omega_1+\omega_2}(2\kappa(i\omega_1 z_1 - i\omega_2 z_2 e^{-i(\omega_1+\omega_2)t}) + \\
&\qquad 4\eta^2 \cos 2\eta t(z_1 + z_2 e^{-i(\omega_1+\omega_2)t})) \quad , \\
\dot{z}_2 &= \frac{-i\varepsilon}{\omega_1+\omega_2}(2\kappa(i\omega_1 z_1 e^{i(\omega_1+\omega_2)t} - i\omega_2 z_2) + \\
&\qquad 4\eta^2 \cos 2\eta t(z_1 e^{i(\omega_1+\omega_2)t} + z_2)) \quad .
\end{aligned} \tag{50}$$

To calculate the instability interval around the value $\eta_0 = \frac{1}{2}(\omega_1+\omega_2) = \sqrt{\alpha^2+1}$, we put:

$$\eta = \eta_0 + \varepsilon\sigma \quad , \tag{51}$$

where σ is a parameter, independent of ε, which indicates the detuning from exact resonance. In the following we want to obtain the values of σ for which the trivial solution of (50) becomes unstable.
Inserting (51) into (50) and after transforming:

$$z_1 = v_1 e^{i\varepsilon\sigma t} \quad , \; z_2 = v_2 e^{-i\varepsilon\sigma t} \quad , \tag{52}$$

we obtain upon averaging for v_1 and v_2:

$$\begin{aligned}
\dot{v}_1 &= \frac{\varepsilon}{\eta_0}\left(-(\omega_1\kappa + i\sigma\eta_0)v_1 + i\eta_0^2 v_2\right) \quad , \\
\dot{v}_2 &= \frac{\varepsilon}{\eta_0}\left(-i\eta_0^2 v_1 - (\omega_2\kappa - i\sigma\eta_0)v_2\right) \quad .
\end{aligned} \tag{53}$$

The stability of the trivial solution of (53) is determined by the real parts of the corresponding eigenvalues.
Let λ' be an eigenvalue of (53). Defining λ by $\lambda' = \frac{\varepsilon}{\eta_0}\lambda$, the eigenvalue equation for (53) becomes:

$$\lambda^2 + 2\eta_0\kappa\lambda + \kappa^2 - 2i\alpha\kappa\sigma\eta_0 + \sigma^2\eta_0^2 - \eta_0^4 = 0 \quad , \tag{54}$$

which has the roots:

$$\lambda^{+,-} = -\eta_0\kappa \pm \sqrt{(\alpha\kappa + i\eta_0\sigma)^2 + \eta_0^4} \quad . \tag{55}$$

For $\kappa = 0$ (no damping), we find that $\lambda^+ = \sqrt{\eta_0^4 - \sigma^2\eta_0^2}$, and the trivial solution is unstable if $|\sigma| < \eta_0 = \sqrt{\alpha^2+1}$. This result is consistent with the stability-boundary found earlier. However, if $\kappa > 0$ we find after some calculations that $\mathrm{Re}(\lambda^+) > 0$ if:

$$|\sigma| < \sqrt{\eta_0^4 - \kappa^2} \quad , \tag{56}$$

and $\eta_0^4 - \kappa^2 > 0$ otherwise there is stability for all values of σ.
If $\kappa = 0$ equation (45) applies; if $\kappa > 0$ we find, using equations (51) and (56), that to first order:

$$\begin{aligned}
\eta_b &= \sqrt{1+\alpha^2}\left(1 \pm \varepsilon\sqrt{1+\alpha^2 - \frac{\kappa^2}{\eta_0^2}} + ...\right) \quad , \\
&= \sqrt{1+\alpha^2}\left(1 \pm \sqrt{(1+\alpha^2)\varepsilon^2 - \left(\frac{\mu}{\eta_0}\right)^2} + ...\right) \quad .
\end{aligned} \tag{57}$$

It follows that the domain of instability actually becomes larger when damping is introduced. This phenomenon can occur only for combination intervals of instability.
The most unusual aspect of the above expression for the instability interval,

however, is that there is a discontinuity at $\kappa = 0$. If $\kappa \to 0$, then the boundaries of the instability domain tend to the limits $\eta_b \to \sqrt{1+\alpha^2}(1 \pm \varepsilon\sqrt{1+\alpha^2})$ which differs from the result we found when $\kappa = 0$: $\eta_b = \sqrt{1+\alpha^2}(1 \pm \varepsilon)$. We will give a summary of the mathematical analysis of this remarkable aspect in subsection 4.4.
In mechanical terms, the broadening of the instability-domain is caused by the coupling between the two degrees of freedom of the rotor in lateral directions which arises in the presence of damping. Such phenomena are typical for gyroscopic systems and have been noted earlier in the literature; see Banichuk et al. (1989), Bolotin (1963) or Bratus (1990). The explanation of the discontinuity, however, seems to be new.

4.3 Hysteresis and phase-locking

Consider non-trivial solutions of (36) and the possibility of hysteresis and phase-locking. To obtain bounded solutions, the original equation of motion must be slightly modified, for example by assuming progressive damping. Thus we will take as damping function: $f(z, \dot{z}) = \kappa\dot{z} + \delta|z|^2\dot{z}$, with κ, $\delta > 0$. After scaling of z by a factor $\left(\frac{k}{\delta}\right)^{\frac{1}{2}}$, equation (48) becomes:

$$\ddot{z} - 2i\alpha\dot{z} + \left(1 + 4\varepsilon\eta^2\cos 2\eta t\right) z + \varepsilon\kappa\dot{z}(1 + |z|^2) = 0. \tag{58}$$

After some rescaling, putting $\eta_0 = \sqrt{1+\alpha^2}$ and normalisation we find in (real) polar coordinates:

$$\begin{aligned}
r_1' &= \varepsilon\left(r_2\sin(\varphi_1 - \varphi_2) - \frac{\kappa\omega_1}{2\eta_0^2}r_1(1 + r_1^2) - \frac{\kappa\alpha}{\eta_0^2}r_1 r_2^2\right) \quad , \\
\varphi_1' &= \varepsilon\left(\frac{r_2}{r_1}\cos(\varphi_1 - \varphi_2) - \sigma\right) \quad , \\
r_2' &= \varepsilon\left(r_1\sin(\varphi_1 - \varphi_2) - \frac{\kappa\omega_2}{2\eta_0^2}r_2(1 + r_2^2) + \frac{\kappa\alpha}{\eta_0^2}r_2 r_1^2\right) \quad , \\
\varphi_2' &= \varepsilon\left(\frac{r_1}{r_2}\cos(\varphi_1 - \varphi_2) - \sigma\right) \quad .
\end{aligned} \tag{59}$$

From the right-hand-side of (59), it is seen that only the phase difference $\psi = \varphi_1 - \varphi_2$ is relevant, so we can study the reduced system:

$$\begin{aligned}
r_1' &= \varepsilon\left(r_2 \sin\psi - \frac{\kappa\omega_1}{2\eta_0^2} r_1(1+r_1^2) - \frac{\kappa\alpha}{\eta_0^2} r_1 r_2^2\right) \quad , \\
r_2' &= \varepsilon\left(r_1 \sin\psi - \frac{\kappa\omega_2}{2\eta_0^2} r_2(1+r_2^2) + \frac{\kappa\alpha}{\eta_0^2} r_2 r_1^2\right) \quad , \\
\psi' &= \varepsilon\left(\frac{(r_1^2+r_2^2)}{r_1 r_2}\cos\psi - 2\sigma\right) \quad .
\end{aligned} \tag{60}$$

Equations (60) have been investigated with the help of the software package AUTO (see Doedel (1981)). This program is able to track the fixed points of a system as a parameter is changed (in our case σ), calculate the stability of the fixed point and, most importantly, detect bifurcations. For system (60) the results are as follows: the zero solution $r_1 = r_2 = 0$ is unstable iff $|\sigma| \leq \sigma_1 = \sqrt{1+\alpha^2 - \frac{1}{4}\kappa^2}$, as we already know from linear analysis; for $0 \leq |\sigma| \leq \sigma_1$ there is also an asymptotically stable fixed point. For $|\sigma| > \sigma_2$ only the zero-solution is stable, and there are no other fixed points. For $\sigma_1 < |\sigma| < \sigma_2$ we have two non-zero solutions, one asymptotically stable and one unstable.
It follows that there is hysteresis in the system. When $|\sigma| < \sigma_1$, the solutions of system (60) will tend to the non-zero fixed point. As $|\sigma|$ is increased, this fixed point will remain an attractor until σ_2 is reached, after which this fixed point disappears and the solution will suddenly "jump" to the zero solution.

System (59) also exhibits phase-locking. When the reduced system (60) tends to a non-trivial fixed point, this implies that the phase difference $\psi = \varphi_1 - \varphi_2$ will converge to a fixed (and in general non-zero) value ψ_0. It then follows from (59) that for the asymptotically fixed point we can write:

$$\varphi_1(t) = \varepsilon\nu t, \;\; \varphi_2(t) = \varepsilon\nu t - \psi_0 \quad ,$$

with $\nu = \frac{r_1}{r_2}\cos\psi_0 - \sigma$. We can now reconstruct the solution of the original equation (58) by inverting the various transformations, and we find that for $|\sigma| \leq \sigma_2$ there exists a stable solution of (58) given by

$$z(t) = r_1 e^{i(\omega_1 t + \varepsilon\nu_1 t)} + r_2 e^{i(-\omega_2 t - \psi_0 + \varepsilon\nu t)} + \mathcal{O}(\varepsilon) \quad , \tag{61}$$

on time-scale $1/\varepsilon$, with $\omega_1 = \sqrt{\alpha^2+1}+\alpha$, $\omega_2 = \sqrt{\alpha^2+1}-\alpha$ and r_1, r_2 and ψ_0 (the fixed points of (60)) depending only on σ. Solution (61) consists of two dominant vibration components, one with forward precession frequency ω_1, the second with backward precession frequency $-\omega_2$. The motion at this resonance is generally non-periodic and the trajectory of the centre of gravity in the horizontal plane is not closed (Tondl (1995)).
For $|\sigma| \geq \sigma_2$, this solution suddenly disappears and only $z(t) = 0$ remains as an asymptotically stable solution.

4.4 Parametrically forced oscillators in sum resonance

In section 4.2 we came upon an unexpected result: when adding linear damping to system (40) there is a striking discontinuity in the bifurcational behaviour. Phenomena like this have already been observed and described by Yakubovich and Starzhinskii (1975) and Szemplinska-Stupnicka (1990). Hoveijn and Ruijgrok (1992) have presented a geometrical explanation using 'all' the parameters as unfolding parameters. It will turn out that four parameters are needed to give a complete description. Fortunately three suffice to visualize the situation.
The problem of section 4.2 is analyzed in a slightly more general setting. Consider the following type of differential equation

$$\dot{z} = A_0 z + \varepsilon f(z, \omega_0 t; \lambda),\ z \in \mathbb{R}^4,\ \lambda \in \mathbb{R}^p, \tag{62}$$

which describes a system of two parametrically forced coupled oscillators. Here A_0 is a 4×4 matrix with purely imaginary eigenvalues $\pm i\omega_1$ and $\pm i\omega_2$. The vector valued function f is 2π-periodic in $\omega_0 t$ and $f(0, \omega_0 t; \lambda) = 0$ for all t and λ. Equation (62) can be resonant in many different ways. Because of our rotor-problem, we consider the resonance $\omega_1 + \omega_2 = \omega_0$ where the system exhibits instability. The parameter λ is used to control detuning from resonance and damping.
We summarize the analysis; for technical details see Hoveijn and Ruijgrok (1992).

- The first step is to put equation (62) into normal form by normalization or averaging. In section 2.2 we stated a normal form theorem tailored to our situation. We construct the normal form and identify the detuning and damping parameters. In the normalized equation the time

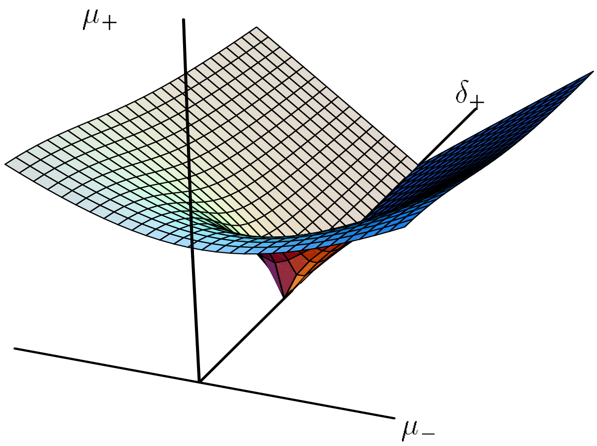


Figure 3: The critical surface in (μ_+, μ_-, δ_+) space. $\mu_+ = \mu_1 + \mu_2$, $\mu_- = \mu_1 - \mu_2$, $\delta_+ = \delta_1 + \delta_2$. Only the part $\mu_+ > 0$ and $\delta_+ > 0$ is shown. The parameters δ_1, δ_2 control the detuning of the frequencies, the parameters μ_1, μ_2 the damping of the oscillators.

dependence appears only in the high order terms. But the autonomous part of this equation contains enough information to determine the stability regions of the origin.
The linear part of the normal form is $\dot{z} = A(\delta, \mu)z$ with

$$A(\delta, \mu) = \begin{pmatrix} B(\delta, \mu) & 0 \\ 0 & \overline{B}(\delta, \mu) \end{pmatrix} \quad , \tag{63}$$

and

$$B(\delta, \mu) = \begin{pmatrix} i\delta_1 - \mu_1 & \alpha_1 \\ \overline{\alpha}_2 & -i\delta_2 - \mu_2 \end{pmatrix} \quad . \tag{64}$$

Since $A(\delta, \mu)$ is the complexification of a real matrix, it commutes with complex conjugation. Furthermore, according to the normal form theorem and using the fact that ω_1 and ω_2 are independent over the integers, the normal form of (62) has the continuous symmetry group:

$$\mathcal{G} = \{g \mid gz = ({}^{is}x_1, {}^{is}x_2, {}^{-is}y_1, {}^{-is}y_2,), s \in \mathbb{R}\} \quad .$$

- The second step is to test the linear part $A(\delta, \mu)$ of the normalized equation for structural stability. This family of matrices is parametrized by the detunings δ of ω_1 and ω_2 and the damping parameters μ. We first identify the most degenerate member N of this family and then show that $A(\delta, \mu)$ is its versal unfolding in the sense of Arnold (1983). The family $A(\delta, \mu)$ is equivalent to a versal unfolding $U(\lambda)$ of N in $\mathbf{u}$ where:

$$N = \begin{pmatrix} N_1 & 0 \\ 0 & N_1 \end{pmatrix} \quad \text{and} \quad N_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad .$$

- Put differently, the family $A(\delta, \mu)$ is structurally stable, whereas $A(\delta, 0)$ is not. Thus we find that the stability diagram actually 'lives' in a four dimensional space. In this space, the stability regions of the origin are separated by a *critical surface* which is the hypersurface where $A(\delta, \mu)$ has at least one pair of purely imaginary complex conjugate eigenvalues. This critical surface is diffeomorphic to the *Whitney umbrella*, see figure 3. It is the singularity of the Whitney umbrella that causes the discontinuous behaviour of the planar stability diagram. The structural stability argument guarantees that our results are 'universally valid', i.e. they qualitatively hold for every system in sum resonance.

We conclude that the discontinuity in the bifurcation-diagram, found in section 4.2 for the rotor system, is not due to a special choice of coefficients or to the approximation procedure. The discontinuity is a fundamental structural instability in linear gyroscopic systems with at least two degrees of freedom and with linear damping.

5 Autoparametric resonance of a parametric oscillator

The following results can be found in Fatimah and Ruijgrok (2001).
We consider an autoparametric system which consists of an oscillator, coupled with a parametrically-excited subsystem, of the form:

$$\begin{aligned} x'' + x + \varepsilon(k_1 x' + \sigma_1 x + a\cos 2\tau x + \frac{4}{3}x^3 + c_1 y^2 x) = 0 \\ y'' + y + \varepsilon(k_2 y' + \sigma_2 y + c_2 x^2 y + \frac{4}{3}y^3) = 0 \end{aligned} \tag{65}$$

where σ_1 and σ_2 are the detunings from the $1:1$-resonance of the oscillators. The system (65) is invariant under $(x,y) \to (x,-y)$, $(x,y) \to (-x,y)$, and $(x,y) \to (-x,-y)$.
Using the method of averaging as a normalization procedure we investigate the stability of solutions of system (65). Introduce the transformation:

$$x = u_1 \cos\tau + v_1 \sin\tau \quad ; \quad x' = -u_1 \sin\tau + v_1 \cos\tau \tag{66}$$

$$y = u_2 \cos\tau + v_2 \sin\tau \quad ; \quad y' = -u_2 \sin\tau + v_2 \cos\tau \tag{67}$$

After rescaling $\tau = \frac{\varepsilon}{2}\tilde{\tau}$ the averaged system of (65) becomes:

$$\begin{aligned} u_1' &= -k_1 u_1 + (\sigma_1 - \frac{1}{2}a)v_1 + v_1(u_1^2 + v_1^2) + \frac{1}{4}c_1 u_2^2 v_1 + \frac{3}{4}c_1 v_2^2 v_1 + \frac{1}{2}c_1 u_2 v_2 u_1 \\ v_1' &= -k_1 v_1 - (\sigma_1 + \frac{1}{2}a)u_1 - u_1(u_1^2 + v_1^2) - \frac{3}{4}c_1 u_2^2 u_1 - \frac{1}{4}c_1 v_2^2 u_1 - \frac{1}{2}c_1 u_2 v_2 v_1 \\ u_2' &= -k_2 u_2 + \sigma_2 v_2 + v_2(u_2^2 + v_2^2) + \frac{1}{4}c_2 u_1^2 v_2 + \frac{3}{4}c_2 v_1^2 v_2 + \frac{1}{2}c_2 u_1 v_1 u_2 \\ v_2' &= -k_2 v_2 - \sigma_2 u_2 - u_2(u_2^2 + v_2^2) - \frac{3}{4}c_2 u_1^2 u_2 - \frac{1}{4}c_2 v_1^2 u_2 - \frac{1}{2}c_2 u_1 v_1 v_2. \end{aligned} \tag{68}$$

At the solution $X_\circ = (\pm u_\circ, \pm v_\circ, 0, 0)$, corresponding to the semi-trivial solution of the system, linearization produces the matrix

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$$

5.1 Bifurcation and Stability of the solution

The stability boundary of the semi-trivial solution $X_\circ$ graphically corresponds to a surface in the 3-dimensional parameter space (σ_1, a, σ_2), this surface is shown in figure 4.

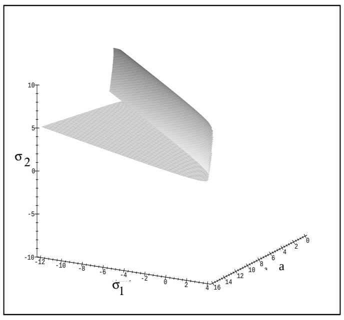
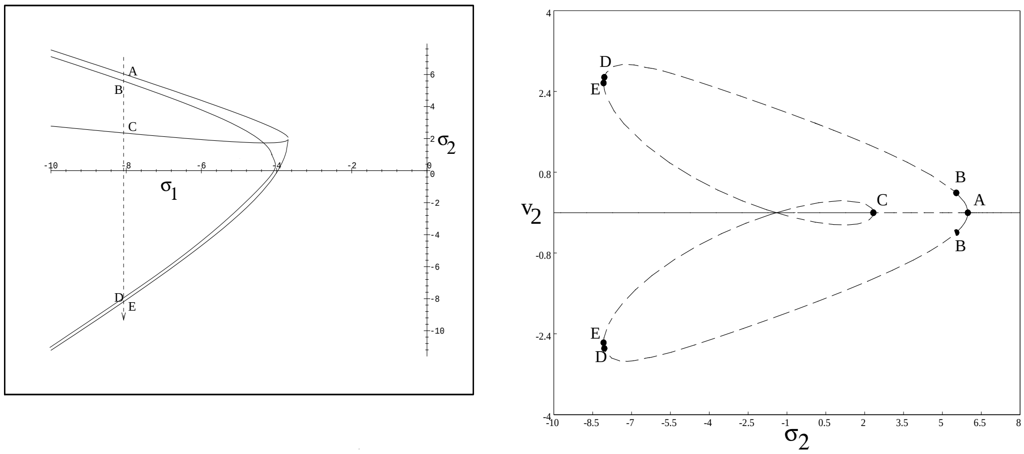


Figure 4: The parameter-space of the stability boundary of the semi-trivial solution of system (68) in the (a, σ_1, σ_2)-space for fixed $k_1 = 1$, $k_2 = 1$, $c_1 = 1$ $c_2 = -1$.



(a) (b)

Figure 5: (a) The parameter diagram of system (68) in the (σ_1, σ_2)-plane. (b) The stability diagram of system (68) in the $(\sigma_2, v2)$-plane. Points A and C indicate branching points of the semi-trivial solution. Points B and D indicate Hopf points and point E indicates a saddle-node point. In figure (b), A solid line means the solution is stable and the dashed line that it is unstable.

The bifurcation continuation program CONTENT - see Kuznetsov (1998) or Khibnik et al. (1992) - is used to study the non-trivial solutions branching from these bifurcation points. The results are illustrated in figure 5. For fixed $a = 2.1$, a supercritical Hopf bifurcation occurs at point B for $\sigma_2 = 5.5371$ and at point D for $\sigma_2 = -8.051$. A saddle-node bifurcation occurs at point E for $\sigma_2 = -8.0797$.We find a stable periodic orbit for all values of σ_2 in the interval $5.4119 < \sigma_2 < 5.5371$. As σ_2 is decreased, period doubling of the stable periodic solution is observed. There is an infinite number of such period doubling bifurcations, until the value $\sigma_2^* = 5.2505$ is reached. There are strong indications for the presence of Šilnikov bifurcation.

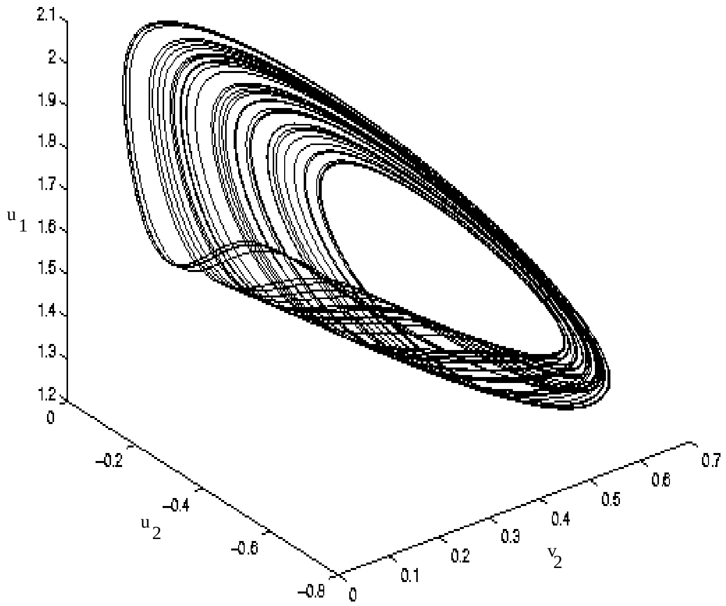


Figure 6: The strange attractor of the averaged system (68). The phase-portraits in the (u_2, v_2, u_1)-space for $c_2 < 0$ at the value $\sigma_2 = 5.3$. The Kaplan-Yorke dimension for $\sigma_2 = 5.3$ is 2.29.

5.2 Global Bifurcations and chaotic solutions

We perform a further analysis of system (65) based on Fatimah and Ruijgrok (2001). The autoparametric system is of the form:

$$\begin{aligned} x'' + k_1 x' + q_1^2 x + ap(\tau)x + f(x,y) &= 0 \\ y'' + k_2 y' + q_2^2 y + g(x,y) &= 0 \end{aligned} \tag{69}$$

where $f(x,y) = c_1 xy^2 + \frac{4}{3}bx^3$, $g(x,y) = \frac{4}{3}by^3 + c_2x^2y$, and $p(\tau) = \cos 2\tau$.

The aim is to show the existence of non-trivial solutions which are periodic, quasi-periodic or chaotic in a more rigorous, analytical way. To this end we combine the analysis of a codimension 2 bifurcation with the application of a generalized Melnikov method to yield a full picture of the dynamics of (65). The results of this theoretical analysis, in particular concerning the

existence of chaotic solutions, show a remarkable degree of agreement with the numerical results.
A bifurcation analysis of the averaged system will show the existence of periodic and quasi-periodic solutions of (65) as well as solutions that are homoclinic to a periodic solution.
Let X_0 be a fixed point of the averaged equation. The translation $X = X_\circ + Z$, leads to

$$Z' = F(X_\circ + Z) = G(Z). \tag{70}$$

In section (5.1) we noted that the linear part of $G(Z)$ has the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$.
The bifurcation we want to study occurs when both A_1 and A_2 have one zero eigenvalue. This double zero eigenvalue bifurcation is not equivalent with the standard Bogdanov-Takens bifurcation, because the system (70) has the symmetry $S = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. It is not difficult to write down the equations in the 2-dimensional center manifold. We note that the symmetry S carries over to the bifurcation-equations. The bifurcation equations are:

$$\begin{aligned} x' &= -\mu - rx^2 + y^2 \\ y' &= \lambda y - xy + y^3 \end{aligned} \tag{71}$$

where μ and λ are bifurcation parameters, and r constant with $r \neq 0$. We

assume $-1 < r < 0$.
The results of the analysis can be stated as follows: There is a neighborhood U of $(x, y) = (0, 0)$ and a neighborhood V of $(\mu, \lambda) = (0, 0)$ such that V is divided into regions as shown in figure 7.

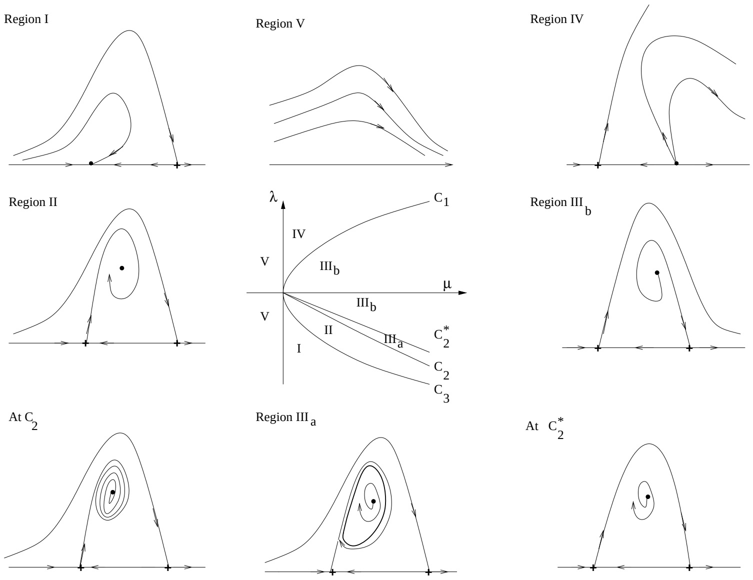


Figure 7: The phase-portraits of equation (71) in the (x, y)-plane for a specific value (μ, λ) in each region.

Transforming to the original parameters (σ_1, σ_2) to reconstruct the parameter diagram of the full system (70), we obtain the parameter diagram as shown in figure 8.
In numerical simulations, we found that there is at most one stable periodic solution in the narrow area between C_5 and C_6. This stable periodic solution then undergoes a series of period doubling bifurcations, leading to a strange attractor. This attractor has positive Lyapunov exponents (section 5.1).

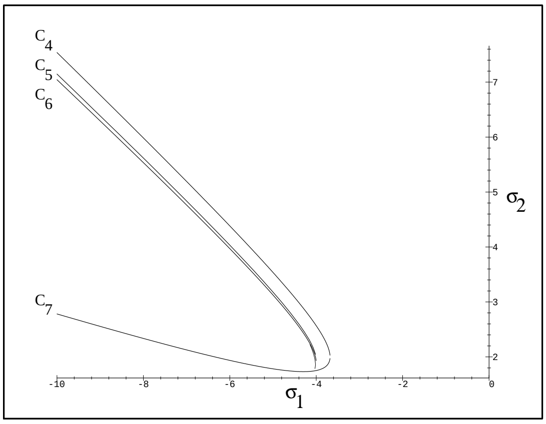


Figure 8: Parameter diagram of system (65) in the (σ_1, σ_2)-plane, for fixed values $a = 2.1$, $k_1 = k_2 = c_1 = 1$, and $c_2 = -1$. C_4 and C_7 represent branching curves of the semi-trivial solution $X_\circ$. C_5 represents the Hopf curve of the non-trivial solution and C_6 represents the saddle connection curve.

5.3 Analytical study of the chaotic solution, using a generalized Melnikov method

The generalized Melnikov method developed by Kovacic and Wiggins (1992) consists of studying a system in which the unperturbed problem is an integrable Hamiltonian system having a normally hyperbolic invariant set whose stable and unstable manifold intersect nontransversally. The structure of the unperturbed system is that of two uncoupled one-degree of freedom Hamiltonian systems.

The global geometry associated with the integrable structure is used to develop coordinates which are then used in determining if any of the homoclinic orbits to the normally hyperbolic invariant set survive under perturbation. To prove this involves an application of higher dimensional Melnikov theory developed in Wiggins (1988). A rather technical analysis shows the existence of a Šilnikov orbit in the averaged equation, which implies chaotic dynamics, also for the original system.

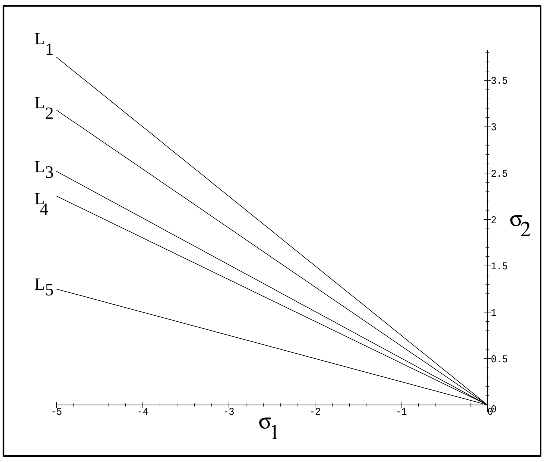


Figure 9: Parameter diagram of system (65) in the (σ_1, σ_2)-plane for values $k_1 = k_2 = 1$, $c_1 = 1$, $c_2 = -1$, and $a = 2.1$

The area in parameter-space where a Šilnikov orbit exits is plotted in figure 9, and is bordered by the lines L_3 and L_4. In that figure, we have also indicated the lines where the semi-trivial solution bifurcates to a non-trivial solution (lines L_1 and L_5), and where the Hopf-bifurcation of the non-trivial solution occurs (L_2). These last curves are found from the previous analysis by a suitable rescaling. As was mentioned, these curves correspond to a high degree with the numerical results obtained earlier. For instance, for the parameter-values $\bar{a} = 2.1$, $\bar{k_1} = \bar{k_2} = 1$, $c_1 = 1$, $c_2 = -1$, and $\sigma_1 = -8$ we numerically found a strange attractor for $5.2505 < \sigma_2 < 5.3195$, whereas the the values as predicted by the curves L_3 and L_4 yield $5.26 < \sigma_2 < 5.33$.

6 Autoparametric resonance of a self-excited oscillator

We consider a self-excited auto-parametric system of Rayleigh type in the non-dimensional form:

$$\begin{aligned} x'' - \beta(1 - x'^2)x' + x + \gamma_1 y^2 &= 0 \\ y'' + \kappa y' + q^2 y + \gamma_2 xy &= 0. \end{aligned} \tag{72}$$

where $\beta > 0$, is the self-excitation coefficient, $\kappa > 0$ is the damping coefficient of the Excited System, γ_1 and γ_2 are the nonlinear coupling coefficients; q is the tuning coefficient expressing the ratio of natural frequencies of the undamped linearized subsystems, where the frequency of the x-mode is normalized to 1. We review results by Abadi (2001) who restricts the discussion to the important resonance $q = \frac{1}{2}$ and nearby (detuned) values.
Assuming that all the parameters in system (72) are small and in order to apply the averaging method (see Sanders and Verhulst (1985)), we rescale the parameters as follows. Let $\beta = \varepsilon\bar{\beta}$, $\kappa = \varepsilon\bar{\kappa}$, $\gamma_1 = \varepsilon\bar{\gamma}_1$, $\gamma_2 = \varepsilon\bar{\gamma}_2$, and take $q^2 = \frac{1}{4} + \varepsilon\sigma$. Then, we have the following standard form

$$\begin{aligned} x'' + x &= \varepsilon(\beta(1 - x'^2)x' - \gamma_1 y^2) \\ y'' + \frac{1}{4}y &= -\varepsilon(\kappa y' + \sigma y + \gamma_2 xy). \end{aligned} \tag{73}$$

6.1 The semi-trivial solution and its stability

The semi-trivial solution is defined as the solution of the system (73) by putting $y = 0$. Thus, we have the well-known Rayleigh equation

$$x'' + x = \varepsilon\beta(1 - x'^2)x', \; y = 0. \tag{74}$$

By using the averaging method we analyze (74) and put $x_0(\tau) = R_0\cos(\tau + \psi_0) = \sqrt{\frac{4}{3}}\cos\tau$ which is an approximation to the (stable) periodic solution of (74) up to $\mathcal{O}(\varepsilon)$.
To investigate the stability of the semi-trivial solution $x_0(\tau)$ in the full system (73), we apply a small perturbation to the solution. Thus we obtain the stability boundary which is given by

$$q^2 = \frac{1}{4} + \varepsilon\sqrt{\frac{\gamma_2^2 R_0^2 - \kappa^2}{4}}, \tag{75}$$

which exists for $\kappa \leq \gamma_2\sqrt{\frac{4}{3}}$ corresponding to a *bifurcation value* where the semi-trivial solution changes its properties and a non-trivial solution is initiated.

6.2 Analysis of periodic solutions

The non-trivial solutions of (73) can be written in the following form:

$$x = R_1 \cos(\tau + \psi_1) \text{ and } y = R_2 \cos(\frac{1}{2}\tau + \psi_2). \tag{76}$$

We substitute (76) into (73), then we apply the averaging method. The averaged system obtained can be transformed (with the transformation: $\rho = R_2^2,\ u = R_1 \cos\phi,\ v = R_1 \sin\phi$) into the following system:

$$\begin{aligned}
\rho' &= -\kappa\rho - \gamma_2 \rho v \\
u' &= \frac{1}{2}\beta(1 - \frac{3}{4}R_1^2)u + \gamma_2 uv + 2\sigma v \\
v' &= \frac{1}{2}\beta(1 - \frac{3}{4}R_1^2)v + \frac{1}{4}\gamma_1\rho - \gamma_2 u^2 - 2\sigma u,
\end{aligned} \tag{77}$$

where $R_1^2 = u^2 + v^2$.

Note that system (77) is invariant under $(\rho, u, v) \rightarrow (\rho, -u, v)$. $\rho = 0$ is an invariant manifold of the system. In the analysis below we will make use of the existence of the invariant manifolds $\rho = 0$ and $u = 0$.

We study the fixed points of system (77) which correspond to periodic solutions in the exact resonance case ($\sigma = 0$) and for near-resonance ($\sigma \neq 0$).

6.3 Symmetry at exact resonance

Putting $\sigma = 0$ and assuming that $\beta > 0,\ \gamma_1 > 0$, and $\gamma_2 > 0$, we solve $f_1(\rho, u, v) = 0, f_2(\rho, u, v) = 0, f_3(\rho, u, v) = 0$, where $f_1,\ f_2,\ f_3$ are the right hand sides of (77), to obtain the following fixed points. $\mathbf{x_{00}} = (\rho_0, u_0, v_0) = (0, 0, 0)$ (the trivial solution), $\mathbf{x_{10}} = (0, 0, \sqrt{\frac{4}{3}})$, and $\mathbf{x_{20}} = (0, 0, -\sqrt{\frac{4}{3}})$ (the semi-trivial solution) and the following fixed points corresponding with non-trivial periodic solutions

$$\mathbf{X_1} = \left(\frac{2\beta\kappa}{\gamma_1\gamma_2}(1 - \frac{3}{4}\frac{\kappa^2}{\gamma_2^2}), 0, -\frac{\kappa}{\gamma_2}\right), \tag{78}$$

and

$$\mathbf{X_2} = \left(\frac{16}{3}\frac{\gamma_2}{\gamma_1}(1 - 2\frac{\kappa}{\beta}), \pm\sqrt{\frac{4}{3}(1 - \frac{3}{4}\frac{\kappa^2}{\gamma_2^2}) - \frac{8}{3}\frac{\kappa}{\beta}}, -\frac{\kappa}{\gamma_2}\right). \tag{79}$$

Using linear analysis applied to each of the critical points, we can determine the stabilities of them. Also, by varying the values of the damping coefficient κ of the Excited System, we find a rich pattern of different bifurcations. The bifurcations can be visualized by the following diagram, obtained by using the CONTENT numerical continuation package.

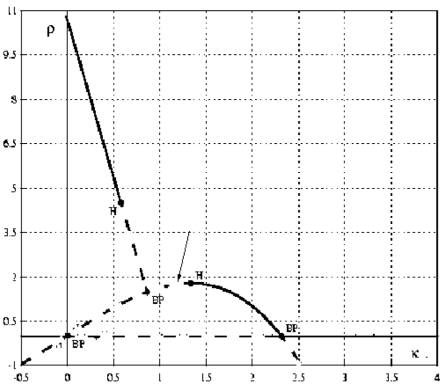


Figure 10: **Exact resonance**. Bifurcation diagram of system (77) projected on the $\kappa-\rho$ plane, for $\beta = 2,\ \gamma_1 = 1,\ \gamma_2 = 2$. BP stands for branching point and H stands for Hopf point.

An important question concerns the stability of the solution in the parameter interval where both semi-trivial and non-trivial solutions are unstable (pointed by an arrow in figure 10). Due to the super-critical Hopf bifurcation, a stable limit cycle appears (for a short interval of κ). Then, the limit cycle breaks up and a *robust heteroclinic cycle*, which is stable, emerges; see Krupa (1997). Figure 11 gives a clear illustration of the cycle in the 3-dimensional phase-space $\rho - u - v$.

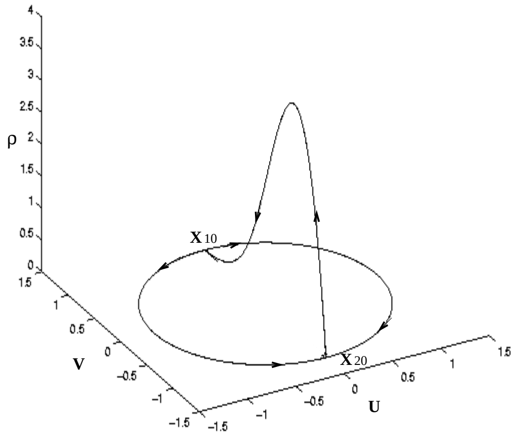


Figure 11: The robust heteroclinic cycle connecting the saddle points $\mathbf{x_{10}}$ and $\mathbf{x_{20}}$.

6.4 Breaking of symmetry and long-periodic solutions

Applying a small detuning ($\sigma \neq 0$) to our system gives more interesting phenomena, especially with respect to the non-trivial solution if we take σ closer to 0. We slightly perturb equation (77) such that the symmetry under $u = 0$ is broken; a *forced symmetry breaking* takes place; see again Krupa (1997) for a general discussion.
Figure 12 illustrates the phenomenon.

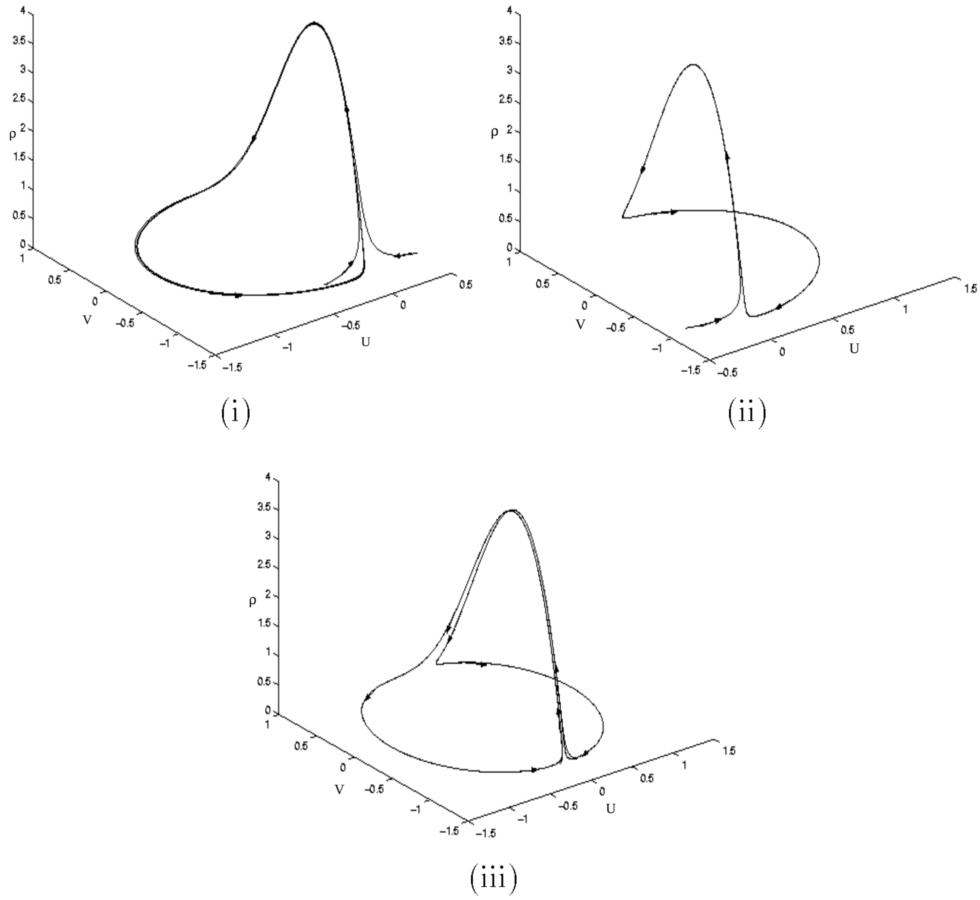


Figure 12: **Forced symmetry breaking.** (i) a long-periodic orbit for $\sigma = 0.01$, (ii) a long-periodic orbit for $\sigma = -0.01$, (iii) the combination of (i) and (ii).

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