FROM GREEDY TO LAZY EXPANSIONS AND THEIR DRIVING DYNAMICS

KARMA DAJANI AND COR KRAAIKAMP

ABSTRACT. In this paper we study the ergodic properties of non-greedy series expansions to non-integer bases $\beta > 1$. It is shown that the so-called 'lazy' expansion is isomorphic to the 'greedy' expansion. Furthermore, a class of expansions to base $\beta > 1$, $\beta \notin \mathbb{Z}$, 'in between' the lazy and the greedy expansions are introduced and studied. It is shown that these expansions are isomorphic to expansions of the form $Tx = \beta x + \alpha \pmod 1$. Finally, for β equal to the 'Golden Mean', a random expansion to base β is given.

1. INTRODUCTION

As is well-known, it is quite straightforward to develop any $x \in [0,1)$ in a series expansion to any integer base $r > 1$. Almost every[1] $x \in [0,1)$ has a unique series expansion

$$x = \sum_{k=1}^{\infty} \frac{a_k}{r^k}, \quad a_k \in \{0, 1, \dots, r-1\}, \tag{1.1}$$

denoted by $x = .b_1 b_2 \cdots b_n \cdots$. Only rationals p/q with $q = p_1^{\ell_1} \cdots p_m^{\ell_m}$ (where the ℓ_i's are non-negative integers and the p_i's are the prime divisors of r), have two different expansions of the form (1.1), one of them being finite while the other expansion ends in an infinite string of $r-1$'s. Underlying these so-called r-ary expansions of the form (1.1) are maps $T_r : [0,1) \to [0,1)$, given by

$$T_t(x) = rx \pmod 1,$$

and the digits $a_k = a_k(x)$, $k \geq 1$, are given by

$$a_k = \lfloor r T_r^{k-1}(x) \rfloor, \quad k \geq 1,$$

where $\lfloor \xi \rfloor$ denotes the largest integer not exceeding ξ. Clearly T_r is related to the Bernoulli-shift on r symbols, and the Lebesgue measure λ is T_r-invariant.

In case of a non-integer $\beta > 1$ the situation is quite different. Again any number $x \in [0,1)$ can be expanded to base β:

$$x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}, \quad b_k \in \{0, 1, \dots, \lfloor \beta \rfloor\}. \tag{1.2}$$

However, one easily sees that for a given non-integer $\beta > 1$ almost every $x \in [0,1)$ has infinitely many different series expansions of the form (1.2). As in the case of the r-ary expansion, an expansion of $x \in [0,1)$ of the form (1.2) can be obtained by using the map $T_\beta : [0,1) \to [0,1)$, given by

$$T_\beta(x) = \beta x \pmod 1,$$

see also Figure 1. In this case we speak of the *β-expansion of x*. In 1957, A. Rényi

Date: **February 1, 2001.**

1991 *Mathematics Subject Classification.* 11Kxx, 28Dxx.

Key words and phrases. entropy, ergodic, β-expansion.

[1] All *almost all* statements are with respect to Lebesgue measure λ.

FIGURE 1. The *greedy map* T_β (here $\beta = \sqrt{2}$)

[R] introduced these maps T_β, and studied their ergodic properties. Rényi showed that

$$([0,1),\, \mu_\beta,\, T_\beta)$$

forms an ergodic system, where μ_β is a T_β-invariant probability measure equivalent to λ with density h_β, with

$$1 - \frac{1}{\beta} \le h_\beta \le \frac{1}{1 - \frac{1}{\beta}}.$$

Independently, A.O. Gel'fond (in 1959) [G] and W. Parry [P1] (in 1960) showed that

$$h_\beta(x) = \frac{1}{F(\beta)} \sum_{x < T^n(1)} \frac{1}{\beta^n} 1_{[0,1)}(x)\,,$$

where $F(\beta) = \int_0^1 (\sum_{x<T^n(1)} \frac{1}{\beta^n})dx$ is a normalizing constant. After Parry the ergodic properties of T_β were studied by several authors. E.g., F. Hofbauer [H] showed that μ_β is the measure of maximal entropy, and M. Smorodinsky [Sm] "closed the gap" between the ergodic properties of T_β for $\beta \in \mathbb{Z}$ and $\beta \notin \mathbb{Z}$, by showing that for each non-integer $\beta > 1$ the system $([0,1), \mu_\beta, T_\beta)$ is weakly Bernoulli, see also [DKS]. A deep result by N. Friedman and D. S. Ornstein [FO] then yields that the natural extension of $([0,1), \mu_\beta, T_\beta)$ is a Bernoulli automorphism.

The β-expansion of x is also known as the *greedy expansion* of x. The digits b_n, $n \ge 1$, of the greedy expansion of x are recursively given by

$$b_n = k \text{ (with } 0 \le k \le \lfloor\beta\rfloor) \iff \sum_{k=1}^{n-1} \frac{b_k}{\beta^k} + \frac{b}{\beta^n} \le x < \sum_{k=1}^{n-1} \frac{b_k}{\beta^k} + \frac{b+1}{\beta^n}.$$

Clearly this yields a series expansion of x of the form (1.2), and setting

$$t_n = t_n(x) := \beta^n \sum_{k=n+1}^{\infty} \frac{b_k}{\beta^k}, \quad n \ge 0,$$

it is an exercise to show that $t_n = T_\beta^n(x)$ for $n \ge 0$.

For reasons which will become apparent in Sections 2 and 3 we expand the domain of T_β from $[0,1)$ to $\Lambda_\beta := [0, \lfloor\beta\rfloor/(\beta-1))$. Now let $T_\beta : \Lambda_\beta \to \Lambda_\beta$ be defined by

$$T_\beta(x) = \begin{cases} \beta x \pmod 1, & 0 \le x < 1, \\ \\ \beta x - \lfloor\beta\rfloor, & 1 \le x < \lfloor\beta\rfloor/(\beta-1). \end{cases}$$

Notice that for each $x \in \Lambda_\beta$ there exists a unique integer $n_0 = n_0(x)$ such that for all $n \ge n_0$ one has that $T_\beta^n(x) \in [0,1)$. In view of this we let h_β be as before on [0,1), and define $\mu_\beta\left([1, \lfloor\beta\rfloor/(\beta-1))\right) = 0$. Due to this, the system

$$\left(\Lambda_\beta := \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right), \mu_\beta, T_\beta\right),$$

is weak-Bernoulli, since the "original" system on [0,1) is.

In the last decade an interest in expansions to non-integer bases $\beta > 1$ other than the greedy expansion has developed. In particular in papers by P. Erdös, M. and I. Joo, V. Komornik, P. Loreti, F. Schnitzer and others, the so-called *lazy expansion* to base $\beta \in (1,2)$ has been studied, see e.g. [EJK], [KL1], [KL2], and (the references in) [JS]. In particular in these (and other) papers the lazy-expansion of 1, and its relation to the greedy-expansion of 1 has been thoroughly investigated.

In general, for a non-integer $\beta > 1$, the digits $(\tilde{b}_k)_{k\ge1}$ of the lazy-expansion of $x \in \Lambda_\beta$ are recursively given by

$$\tilde{b}_n = 0 \iff \sum_{k=1}^{n-1} \frac{\tilde{b}_k}{\beta^k} + \frac{\lfloor\beta\rfloor}{\beta^{n+1}} + \frac{\lfloor\beta\rfloor}{\beta^{n+2}} + \cdots \ge x \tag{1.3}$$

and $\tilde{b}_n = b$ (with $1 \le b \le \lfloor\beta\rfloor$) if and only if both

$$\sum_{k=1}^{n-1} \frac{\tilde{b}_k}{\beta^k} + \frac{b-1}{\beta^n} + \frac{\lfloor\beta\rfloor}{\beta^{n+1}} + \frac{\lfloor\beta\rfloor}{\beta^{n+2}} + \cdots < x \tag{1.4}$$

and

$$\sum_{k=1}^{n-1} \frac{\tilde{b}_k}{\beta^k} + \frac{b}{\beta^n} + \frac{\lfloor\beta\rfloor}{\beta^{n+1}} + \frac{\lfloor\beta\rfloor}{\beta^{n+2}} + \cdots \ge x \tag{1.5}$$

are satisfied. By induction we always have that for $n \in \mathbb{N}$

$$\sum_{k=1}^{n} \frac{\tilde{b}_k}{\beta^k} \le x \le \sum_{k=1}^{n} \frac{\tilde{b}_k}{\beta^k} + \frac{\lfloor\beta\rfloor}{\beta^{n+1}} \sum_{k=0}^{\infty} \frac{1}{\beta^k}.$$

Since

$$\lim_{n\to\infty} \frac{\lfloor\beta\rfloor}{\beta^{n+1}} \sum_{k=0}^{\infty} \frac{1}{\beta^k} = \lim_{n\to\infty} \frac{\lfloor\beta\rfloor}{\beta^n} \frac{1}{\beta-1} = 0,$$

it follows that the series expansion $\sum_{k=1}^{\infty} \tilde{b}_k/\beta^k$ of x converges to x.

In Section 2 we show that there is an ergodic map S_β underlying the lazy-expansion, which is isomorphic to (our extended version of) T_β. From this, and the fact that the isomorphism can be given explicitly, several conclusions will be drawn.

In Section 3 we will introduce a new class of transformations $S_{\beta,\alpha}$, each of them yielding a series-expansion (1.2) of any $x \in \Lambda_\beta$ "in-between" the lazy-expansion and the greedy-expansion of x. We will see that each $S_{\beta,\alpha}$ is essentially isomorphic to

$$T_{\beta,\alpha^*}(x) = \beta x + \alpha^* \pmod 1,$$

where $\alpha^* = \lfloor\beta\rfloor 1 - (\alpha+1)(\beta-1)$. The maps $T_{\beta,\alpha}$ were previously studied by Parry in 1964 [P2] and by R. Palmer in 1979 [Pa], see also [FL]. In Section 4 an

FIGURE 2. The *lazy map* S_β (here $\beta = \pi$)

example of a "random expansion" to base β (where β equals the *golden ratio* G, i.e., $\beta = G = (1+\sqrt{5})/2$) will be discussed. Loosely speaking, this "random expansion" will be a random mix of the greedy and lazy expansions.

2. Lazy expansions

Let $\beta > 1$, $\beta \notin \mathbb{Z}$, fixed. Setting for $x \in \Lambda_\beta = [0, \lfloor\beta\rfloor/(\beta-1))$ and $n \in \mathbb{N}$:

$$\tilde{t}_{n-1} \ = \ \tilde{t}_{n-1}(x) \ = \ \beta^{n-1} \sum_{k=n}^{\infty} \frac{\tilde{b}_k}{\beta^k},$$

where $\tilde{b}_k$ is for $k \geq 1$ defined as in Section 1. Since

$$x \ = \ \sum_{k=1}^{n-1} \frac{\tilde{b}_k}{\beta^k} \ + \ \sum_{k=n}^{\infty} \frac{\tilde{b}_k}{\beta^k} \ = \ \sum_{k=1}^{n-1} \frac{\tilde{b}_k}{\beta^k} \ + \ \frac{1}{\beta^{n-1}} \, \tilde{t}_{n-1},$$

it follows from (1.3), (1.4) and (1.5) that

$$\tilde{b}_n = 0 \quad \Longleftrightarrow \quad \tilde{t}_{n-1} \leq \frac{\lfloor\beta\rfloor}{\beta(\beta-1)},$$

and, if $d \in \{1, 2, \ldots, \lfloor\beta\rfloor\}$

$$\tilde{b}_n = d \quad \Longleftrightarrow \quad \frac{(d-1)\beta + \lfloor\beta\rfloor - (d-1)}{\beta(\beta-1)} \ < \ \tilde{t}_{n-1} \ \leq \ \frac{d\beta + \lfloor\beta\rfloor - d}{\beta(\beta-1)}.$$

In view of this we define the *lazy map* $S_\beta : \Lambda_\beta \to \Lambda_\beta$ by

$$S_\beta(x) \ = \ \beta x - d, \qquad \text{for } x \in \Delta(d), \tag{2.1}$$

where

$$\Delta(0) \ = \ \left[0, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}\right]$$

and

$$\Delta(d) \ = \ \left(\frac{(d-1)\beta + \lfloor\beta\rfloor - (d-1)}{\beta(\beta-1)}, \frac{d\beta - d + \lfloor\beta\rfloor}{\beta(\beta-1)}\right], \quad d \in \{1, 2, \ldots, \lfloor\beta\rfloor\},$$

i.e., to get the time 0 partition one starts from $\lfloor\beta\rfloor/(\beta-1)$ by taking $\lfloor\beta\rfloor$ intervals of length $1/\beta$ from right to left. The last interval with endpoints 0 and $(\lfloor\beta\rfloor + 1 - \beta)/\beta(\beta-1)$, corresponding to the lazy digit 0, is longer than the rest, see also Figure 2. As in the greedy case it is an easy exercise to show that

$$\tilde{t}_n(x) = S_\beta^n(x), \quad \text{for } n \geq 0.$$

Notice that from the dynamics of S_β it follows that for every $x \in (0, \lfloor\beta\rfloor/\beta(\beta-1))$ there exists a unique $n_0 = n_0(x) \in \mathbb{N}$ such that

$$S_\beta^n(x) \notin \left[0, \frac{\lfloor\beta\rfloor + 1 - \beta}{\beta - 1}\right), \quad \text{for all } n \geq n_0,$$

i.e., the interval $A_\beta = [(\lfloor\beta\rfloor + 1 - \beta)/(\beta-1), \lfloor\beta\rfloor/(\beta-1))$ is an 'attractor' of the map S_β (of length 1). Now let $\psi : \Lambda_\beta \to \Lambda_\beta$ be given by

$$\psi(x) = \frac{\lfloor\beta\rfloor}{\beta - 1} - x,$$

then $\psi([0,1)) = A_\beta$, and for $x \in [0,1)$ one has

$$T_\beta(x) = \beta x - d, \quad \text{for } x \in \psi^{-1}\left(\Delta(\lfloor\beta\rfloor - d)\right).$$

We have the following result.

Theorem 1. *The map $\psi : \Lambda_\beta \to \Lambda_\beta$ is measurable and $\psi T_\beta = S_\beta \psi$. Furthermore, the system*

$$\left(\left(0, \frac{\lfloor\beta\rfloor}{\beta-1}\right], \rho_\beta, S_\beta\right) \quad \textit{is weak Bernoulli,}$$

where ρ_β is a probability measure on Λ_β, given by

$$\rho_\beta(A) = \mu_\beta(\psi^{-1}(A)), \quad \textit{for any Lebesgue set } A \subset \Lambda_\beta.$$

Proof. Clearly ψ is measurable since it takes cylinders to cylinders and if $x \in C(d) = [(d-1)/\beta, d/\beta)$, where $d \in \{1, 2, \dots, \lfloor\beta\rfloor\}$, then $T_\beta(x) = \beta x - (d-1)$, and we find that

$$\psi\left(T_\beta(x)\right) = \frac{\lfloor\beta\rfloor}{\beta-1} - \beta x + d - 1. \tag{2.2}$$

We also have that

$$\psi(x) \in \left(\psi(\frac{d}{\beta}), \psi(\frac{d-1}{\beta})\right] = \left(\frac{(\lfloor\beta\rfloor - d)\beta + d}{\beta(\beta-1)}, \frac{(\lfloor\beta\rfloor - (d-1))\beta + d - 1}{\beta(\beta-1)}\right],$$

for $d \in \{1, 2, \dots, \lfloor\beta\rfloor\}$. Thus

$$S_\beta\left(\psi(x)\right) = \frac{\lfloor\beta\rfloor}{\beta-1} - \beta x + d - 1 = \psi(T_\beta(x)).$$

A similar reasoning works for the case that x is in the interval $[\lfloor\beta\rfloor/\beta, \lfloor\beta\rfloor/(\beta-1)]$.

Since $\psi : \Lambda_\beta \to \Lambda_\beta$ is a bijection, it follows by construction of ρ_β that ψ is a measure theoretical isomorphism. Hence $(\Lambda_\beta, \rho_\beta, S_\beta)$ inherits the mixing properties of $(\Lambda_\beta, \mu_\beta, T_\beta)$ and is therefore weak Bernoulli. ☐

Remarks 1 1. It was already noticed in [EJK] in the case $1 < \beta < 2$ that if $x \in [0,1)$ has a greedy expansion $x = .b_1 b_2 \dots b_n \dots$, then $\psi(x)$ has as lazy expansion $.(1-b_1)(1-b_2)\dots(1-b_n)\dots$, i.e., $\tilde{b}_n = 1 - b_n$, for $n \in \mathbb{N}$. Clearly a similar relation holds in general. I.e., if $\beta > 1$ is non-integer, and if $x \in \Lambda_\beta$ has as greedy expansion $x = .b_1 b_2 \dots b_n \dots$, then then $\psi(x)$ has as lazy expansion $.(\lfloor\beta\rfloor - b_1)(\lfloor\beta\rfloor - b_2)\dots(\lfloor\beta\rfloor - b_n)\dots$, i.e., $\tilde{b}_n = \lfloor\beta\rfloor - b_n$, for $n \in \mathbb{N}$.

2. By definition of the 'lazy measure' ρ_β one has that the density d_β of ρ_β equals

$$d_\beta(x) = h_\beta\left(\psi^{-1}(x)\right), \quad \text{for } x \in A_\beta,$$

and $d_\beta = 0$ for $x \notin A_\beta$.

3. Let $\beta > 1$, $\beta \notin \mathbb{Z}$ and let $x \in \Lambda_\beta$. For $n \in \mathbb{N}$, let $b_i \in \{0, 1, \dots, \lfloor\beta\rfloor\}$, $1 \le i \le n$. Then we define the *asymptotic density* $\mathcal{D}(b_1, b_2, \dots, b_n; x)$ of the block $b_1, b_2, \dots, b_n$ by

$$\mathcal{D}(b_1, \dots, b_n; x) := \lim_{N\to\infty} \frac{1}{N} \#\{0 \le i \le N-1;\ b_{i+1}(x) = b_1, \dots, b_{i+n}(x) = b_n\}.$$

Similarly the asymptotic density $\tilde{\mathcal{D}}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n; x)$ is defined for the lazy expansion. For instance, Rènyi [R1] showed that in case $\beta = G$ for almost all x one has that $\mathcal{D}(1; x) = \frac{G^2}{G^2+1} = .7236\dots$. In this case one also has that $\mathcal{D}(11; x) = \tilde{\mathcal{D}}(00; x) = 0$.

Corollary 1. *Let $\beta > 1$, $\beta \notin \mathbb{Z}$ and let $n \in \mathbb{N}$. Furthermore, let $b_i \in \{0, 1, \dots, \lfloor\beta\rfloor\}$ for $1 \le i \le n$. Then for almost all $x \in \Lambda_\beta$ one has that*

$$\mathcal{D}(b_1, b_2, \dots, b_n; x) = \tilde{\mathcal{D}}(\lfloor\beta\rfloor - b_1, \lfloor\beta\rfloor - b_2, \dots, \lfloor\beta\rfloor - b_n; x)$$

For $x \in \Lambda_\beta$, we define the greedy resp. lazy convergents $C_n = C_n(x)$ resp. $\tilde{C}_n = \tilde{C}_n(x)$, $n \ge 1$, of x by

$$C_n := \sum_{k=1}^{n} \frac{b_k}{\beta^k}, \quad \text{resp.} \quad \tilde{C}_n := \sum_{k=1}^{n} \frac{\tilde{b}_k}{\beta^k}, \quad n \ge 1.$$

From the definitions of the greedy and lazy maps one might be tempted to think that one always has that

$$x - C_n \le x - \tilde{C}_n, \quad \text{for } n \ge 1.$$

However, this is incorrect, as the following example shows. Let $\beta = 1.618$, $x = 0.619$, then using MAPLE one finds that the greedy expansion of x equals

$$.1000000000000010001\dots,$$

and the lazy expansion of x is

$$.0101010101010111101 0110\dots,$$

and that $C_n = C_1 = .6180469716\dots$ for $n = 2, \dots, 14$, $C_n = .6187803401\dots$ for $n = 15, \dots, 18$ and $C_{19} = .6188873461\dots$. Furthermore, $\tilde{C}_n = .6188093591\dots$ for $n = 17, 18$, and $C_{19} = .6189163651\dots$. Thus we see that $\tilde{C}_n > C_n$ for $n = 17, 18, 19$. Notice that

$$C_3^* := \frac{1}{\beta^2} + \frac{1}{\beta^3} = .6180649139\dots,$$

so there exist expansions of x to base β which are neither lazy nor greedy for which the convergents (sometimes) perform better than the greedy convergents.

In order to compare the quality of approximation of the two algorithms we define *approximation coefficients* $\theta_n = \theta_n(x)$ resp. $\tilde{\theta}_n = \tilde{\theta}_n(x)$ by

$$\theta_n = \theta_n(x) := \beta^n(x - C_n), \quad \tilde{\theta}_n = \tilde{\theta}_n(x) := \beta^n(x - \tilde{C}_n), \quad \text{for } n \ge 1.$$

Clearly $T_\beta^n(x) = \theta_n$ and $S_\beta^n(x) = \tilde{\theta}_n$ for $n \ge 0$. But then it follows from the ergodic theorem that the limits

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \theta_k(x) \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \tilde{\theta}_k(x) \quad \text{exist}$$

and equal $M_{greedy} := \int_0^{\frac{\lfloor\beta\rfloor}{\beta-1}} x\,d\mu_\beta$ resp. $M_{lazy} := \int_0^{\frac{\lfloor\beta\rfloor}{\beta-1}} x\,d\rho_\beta$, for almost all x.

We have the following result, which states that on average for almost all x the greedy convergents approximate x ‘better’ than the lazy convergents of x.

Proposition 1. *Let $\beta > 1$, $\beta \notin \mathbb{Z}$, then*

$$M_{greedy} + M_{lazy} = \frac{\lfloor\beta\rfloor}{\beta - 1} \quad \text{and } M_{greedy} < M_{lazy}.$$

Proof. The first statement follows directly from the relation between h_β and d_β, viz.,

$$\begin{aligned} M_{lazy} &= \int_{\Lambda_\beta} x \, d\rho_\beta(x) = \int_0^{\frac{\lfloor\beta\rfloor}{\beta-1}} x h_\beta\left(\frac{\lfloor\beta\rfloor}{\beta-1} - x\right) dx \\ &= \int_0^{\frac{\lfloor\beta\rfloor}{\beta-1}} \left(\frac{\lfloor\beta\rfloor}{\beta-1} - y\right) h_\beta(y) \, dy = \frac{\lfloor\beta\rfloor}{\beta-1} - M_{greedy}. \end{aligned}$$

For the second statement, notice that by definition of M_{greedy} one has that

$$M_{greedy} = \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \int_0^{T_\beta^n(1)} \frac{x}{\beta^n} \, dx = \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \frac{(T_\beta^n(1))^2}{2\beta^n}.$$

Furthermore, by definition of d_β one has

$$\begin{aligned} M_{lazy} &= \int_{A_\beta} x d_\beta(x) \, dx = \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \int_{\psi(T_\beta^n(1))}^{\frac{\lfloor\beta\rfloor}{\beta-1}} \frac{x}{\beta^n} \, dx \\ &= \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \frac{\left(\frac{\lfloor\beta\rfloor}{\beta-1}\right)^2 - \left(\psi(T_\beta^n(1))\right)^2}{2\beta^n}. \end{aligned}$$

The first result follows from the observation that for every $n \geq 0$ one has that

$$\left(T_\beta^n(1)\right)^2 < \left(\frac{\lfloor\beta\rfloor}{\beta-1}\right)^2 - \left(\psi(T_\beta^n(1))\right)^2,$$

a statement equivalent to $T_\beta^n(1) < \lfloor\beta\rfloor/(\beta-1)$, which is obviously correct for every $n \geq 0$. □

As an example, we consider here $\beta = G = (1+\sqrt{5})/2$. in this case Rényi [R1] already showed that

$$h_G(x) = \frac{G^3}{G^2+1} 1_{[0,g)}(x) + \frac{G^2}{G^2+1} 1_{[g,1)}(x),$$

where $g = 1/G$. But then one finds that

$$M_{greedy} = \frac{1}{\sqrt{5}} = .4472\ldots \quad \text{and} \quad M_{greedy} = G - \frac{1}{\sqrt{5}} = 1.17082\ldots.$$

3. (β, α) EXPANSIONS

In this section we will discuss a new class of series expansions to any non-integer base $\beta > 1$. Notice that both the greedy map T_β and the lazy map S_β have 'attractors' of length 1. For each

$$\alpha \in \left[0, \frac{\lfloor\beta\rfloor}{\beta-1} - 1\right]$$

we will define a map $N_{\beta,\alpha}$ on Λ_β, which has as attractor the interval $[\alpha, \alpha+1)$. Just as the greedy map T_β and the lazy map S_β the map $N_{\beta,\alpha}$ generates a series expansion (1.2) to base β. Let the partition points $d_1, \ldots, d_{\lfloor\beta\rfloor}$ be given by:

$$d_i := \frac{\alpha + i}{\beta}, \quad i = 1, \ldots \lfloor\beta\rfloor,$$

FIGURE 3. $N_{\beta,\alpha}$ for $\beta = \sqrt{5}$ and $\alpha = g^2$.

see also Figure 3, then $N_{\beta,\alpha} : \Lambda_\beta \to \Lambda_\beta$ is defined by

$$N_{\beta,\alpha}(x) := \begin{cases} \beta x, & x \in [0, d_1), \\ \beta x - i, & x \in [d_i, d_{i+1}),\ 1 \le i < \lfloor \beta \rfloor, \\ \beta x - \lfloor \beta \rfloor, & x \in [d_{\lfloor \beta \rfloor}, \frac{\lfloor \beta \rfloor}{\beta - 1}). \end{cases}$$

In order to understand the dynamical properties of $N_{\beta,\alpha}$ we consider consider the map $\psi^* : [\alpha, \alpha+1) \to [0,1]$, given by $\psi^*(x) := \alpha + 1 - x$. Setting

$$T^*(x) = \psi^*(N_{\beta,\alpha}(\psi^{*-1}(x))).$$

We have the following lemma.

Lemma 1. *Let $\beta > 1$, $\beta \notin \mathbb{Z}$, and let $\alpha \in [0, \frac{\lfloor \beta \rfloor}{\beta - 1} - 1)$. Then*

$$T^*(x) = \beta x + \alpha^* \ (\text{mod } 1),$$

where $\alpha^* = \lfloor \beta \rfloor - (\alpha + 1)(\beta - 1)$.

Proof. The proof is essentially the same as the first part of Theorem 1, and is therefore omitted. □

Remarks 2 Maps $T(x) = \beta x + \alpha$ (mod 1) were first introduced and studied by Parry [P2]. Parry showed that T is ergodic with respect to the Lebesgue measure λ, and that there exists a unique T-invariant probability measure τ $(=\tau_{\beta,\alpha}) \ll \lambda$, with density

$$h_\tau(x) = K \left(\sum_{x < T^n(1)} \frac{1}{\beta^n} - \sum_{x < T^n(0)} \frac{1}{\beta^n} \right) 1_{[0,1)}(x),$$

where $K = K_{\beta,\alpha}$ is a normalizing constant. In [Pa], R. Palmer extended results by R. Bowen [B], Parry [P2] and Smorodinsky [Sm] by giving the exact regions in the (β, α)-plane in which T is weakly Bernoulli (WB). Palmer also determined the eigenvalues of all those transformations T which are not WB. Since Palmer's thesis [Pa] was never published, we will recall here some of her results, see also [FL].

Theorem 2. (Palmer, 1979) *Let $\beta > 1$, $0 \le \alpha < 1$. Then ....*

From Lemma 1 and Palmer's theorem we at once have the following corollary.

Corollary 2. *Let $\beta > 1$, $0 \leq \alpha < 1$. Then .....*

4. Random expansions to base $\beta = (1+\sqrt{5})/2$

Let $\beta = G = (1+\sqrt{5})/2$ be the *golden mean.* In this section we consider the greedy map T_G on $[0,1)$, and the lazy map S_G on $[g, G)$, where $g = \beta - 1 = 1/G = (\sqrt{5}-1)/2$. Let $L = [0, g)$, $M = [g, 1)$ and $R = [1, G)$. For any $x \in [0, G)$ we will use the following "random" algorithm to generate expansions of x to base G which are neither greedy nor lazy nor an (β, α) expansion as described in the previous section. Note that the maps T_G and S_G overlap on the interval M. We will use M as a "switch region", where one is allowed to replace a digit 1 generated by the greedy algorithm to a 0 by switching the map to the corresponding lazy algorithm, and conversely. In the previous section this was done in a deterministic way, we now will do it in a random way. The digits are obtained as follows.

Start with a point $x \in [0, G)$,

* if $x \in L$, then set $d_1 = d_1(x) = 0$ and let $K(x) = T_G(x) = Gx$;
* if $x \in R$, then set $d_1 = d_1(x) = 1$ and let $K(x) = S_G(x) = Gx - 1$;
* if $x \in M$, then flip a coin with $P(\text{HEADS}) = p$, where $0 \leq p \leq 1$. If the coin flip is HEADS, then set $d_1 = d_1(x) = 1$ and let $K(x) = T_G(x) = Gx - 1$. If the coin toss is TAILS, then set $d_1 = d_1(x) = 0$ and let $K(x) = S_G(x) = Gx$.

Summarizing,

$$d_1 = d_1(x) = \begin{cases} 0 & \text{if } x \in L \text{ or } x \in M \text{ and TAILS}, \\ 1 & \text{if } x \in R \text{ or } x \in M \text{ and HEADS}. \end{cases} \tag{4.1}$$

For $n \in \mathbb{N}$, let $d_n = d_n(x) = d_1(K^{n-1}x)$.

Proposition 2. *Given $x \in [0, G)$, the digits d_n given by the above procedure satisfy*

$$x = \sum_{k=1}^{\infty} \frac{d_k}{\beta^k}.$$

Proof. Notice that $K(x) = Gx - d_1(x) = Gx - d_1$. Hence,

$$x = \frac{d_1(x)}{G} + \frac{K(x)}{G},$$

and iterating this n-times yields

$$\begin{aligned} x &= \frac{d_1(x)}{G} + \frac{1}{G}\left(\frac{d_1(K(x))}{G} + \frac{K^2(x)}{G}\right) \\ &= \frac{d_1(x)}{G} + \frac{d_2(x)}{G^2} + \frac{K^2(x)}{G^2} \\ &= \cdots \\ &= \frac{d_1(x)}{G} + \frac{d_2(x)}{G^2} + \ldots + \frac{d_n(x)}{G^n} + \frac{K^n(x)}{G^n}, \end{aligned}$$

from which it follows that

$$\left| x - \sum_{k=1}^{n} \frac{d_k(x)}{G^k} \right| = \frac{1}{G^n} |K^n(x)| \leq \frac{G}{G^n} = g^{n-1} \to 0 \quad \text{as } n \to \infty.$$

□

It is well-known that the dynamical system underlying the β-transformation or greedy algorithm on $[0,1)$ with $\beta = G$, can be symbolically described by the ergodic Markov chain on 2 symbols 0 and 1 and transition matrix given by

$$\begin{pmatrix} g & g^2 \\ 1 & 0 \end{pmatrix},$$

see e.g. [R1], or [R2]. The stationary distribution corresponds to the Parry measure. Analogously, the lazy transformation S_G on $[g, G)$ can be described by the ergodic Markov chain on 2 symbols 0 and 1, and transition matrix

$$\begin{pmatrix} 0 & 1 \\ g^2 & g \end{pmatrix}.$$

We will now describe how our random digits d_n given by (4.1) are generated by an ergodic Markov chain on 3 symbols ℓ, m and r, and transition matrix

$$\mathbf{P} = \begin{pmatrix} g & g^2 & 0 \\ p & 0 & 1-p \\ 0 & g^2 & g \end{pmatrix}.$$

This Markov chain has stationary distribution $\pi = (\pi_\ell, \pi_m, \pi_r)$, given by

$$\pi_\ell = \frac{pG^2}{G^2+1}, \quad \pi_m = \frac{1}{G^2+1} \quad \text{and} \quad \pi_r = \frac{(1-p)G^2}{G^2+1}.$$

Notice that if $p = 1$ one gets the Parry measure and if $p = 0$ one gets the lazy measure. Any sequence of ℓ's, m's and r's that is generated by this Markov chain corresponds to a random expansion to base $\beta = G$ as described in the beginning of this section. To see this, let

$$X_1,\, X_2,\, X_3,\, \ldots$$

be a sequence generated by this Markov chain. Define a sequence $(d_n)_{n\in\mathbb{N}}$ of 0's and 1's as follows:

$$d_n = \begin{cases} 0 & \text{if } X_n = \ell \quad \text{or} \quad (X_n = m \text{ and } X_{n+1} = r), \\ \\ 1 & \text{if } X_n = r \quad \text{or} \quad (X_n = m \text{ and } X_{n+1} = \ell). \end{cases} \tag{4.2}$$

Setting $x = \sum_{k=1}^{\infty} d_k/G^k$, we now show that $(d_1, d_2, \ldots)$ can also be generated by the initial procedure (4.1) (here we know apriori the flip times and the results of the coin flips!). In other words, given the flip times and outcomes of the coin tosses, we want to show for $n \in \mathbb{N}$ that

$$K^{n-1}(x) \in \begin{cases} L & \Longleftrightarrow \quad X_n = \ell, \\ M & \Longleftrightarrow \quad X_n = m, \\ R & \Longleftrightarrow \quad X_n = r. \end{cases}$$

For this it is enough to show that if $n_0 \in \mathbb{N}$ is the **first** index n for which $X_n = m$, then $K^{n_0-1}(x) \in M$ and either $x, K(x), \ldots, K^{n_0-2}(x) \in L$ or $x, K(x), \ldots, K^{n_0-2}(x) \in R$. If this is shown, we begin again with the point $K^{n_0}(x)$ and $X_{n_0+1}, X_{n_0+2}, \ldots$.

We condider two cases:

- if $n_0 = 1$, then either $(X_1 = m$ and $X_2 = r)$ (hence $d_1 = 0$, $d_2 = 1$) or $(X_1 = m$ and $X_2 = \ell)$ (hence $d_1 = 1$, $d_2 = 0$). In the first case (using that $G^2 = G + 1$)

$$\frac{1}{G} = \frac{1}{G^2} + \frac{1}{G^3} + \sum_{k=4}^{\infty} \frac{0}{G^k} \le x = \frac{1}{G^2} + \sum_{k=3}^{\infty} \frac{d_k}{G^k} \le \sum_{k=2}^{\infty} \frac{1}{G^k} = 1.$$

  In the second case (i.e., the case $(X_1 = m,\, X_2 = \ell)$),

$$\frac{1}{G} \le x = \frac{1}{G} + \sum_{k=3}^{\infty} \frac{d_k}{G^k} \le \frac{1}{G} + \sum_{k=4}^{\infty} \frac{1}{G^k} = \frac{1}{G} + \frac{1}{G^3(G-1)} = 1.$$

- if $n_0 > 1$, then either
  (a) $X_1 = \ell = X_2 = \ldots = X_{n_0-1}$ and $(X_{n_0} = m$ and $X_{n_0+1} = r)$, which implies that $d_1 = \ldots = d_{n_0-1} = 0$, $d_{n_0} = 0$, $d_{n_0+1} = 1$,
  (b) $X_1 = \ell = X_2 = \ldots = X_{n_0-1}$ and $(X_{n_0} = m$ and $X_{n_0+1} = \ell)$, which implies that $d_1 = \ldots = d_{n_0-1} = 0$, $d_{n_0} = 1$, $d_{n_0+1} = 0$,

(c) $X_1 = r = X_2 = \ldots = X_{n_0-1}$ and ($X_{n_0} = m$ and $X_{n_0+1} = r$), which implies that $d_1 = \ldots = d_{n_0-1} = 1$, $d_{n_0} = 0$, $d_{n_0+1} = 1$,

(d) $X_1 = r = X_2 = \ldots = X_{n_0-1}$ and ($X_{n_0} = m$ and $X_{n_0+1} = \ell$), which implies that $d_1 = \ldots = d_{n_0-1} = 1$, $d_{n_0} = 1$, $d_{n_0+1} = 0$.

In case (a), we get $x, K(x) = Gx, \ldots, K^{n_0-2}(x) = G^{n_0-2}x \in L$ and

$$\frac{1}{G} = \frac{1}{G^2} + \frac{1}{G^3} \le K^{n_0-1}(x) = G^{n_0-1}x = \frac{1}{G^2} + \sum_{k=3}^{\infty} \frac{d_{k+n_0-1}}{G^k} \le 1,$$

which yields that $K^{n_0-1}(x) \in M$.

In case (b), we get $x, K(x) = Gx, \ldots, K^{n_0-2}(x) = G^{n_0-2}x \in L$ and

$$\frac{1}{G} \le K^{n_0-1}(x) = G^{n_0-1}x = \frac{1}{G} + \frac{0}{G^2} + \sum_{k=3}^{\infty} \frac{d_{k+n_0-1}}{G^k} \le 1,$$

which yields that $K^{n_0-1}(x) \in M$.

In case (c), $x, K(x) = Gx - 1, \ldots, K^{n_0-2}(x) = G(K^{n_0-3}(x)) - 1 \in R$. Since

$$x = \frac{1}{G} + \ldots + \frac{1}{G^{n_0-1}} + \frac{0}{G^{n_0}} + \frac{1}{G^{n_0+1}} + \frac{d_{n_0+2}}{G^{n_0+2}} + \ldots$$

it follows that

$$K(x) = \frac{1}{G} + \ldots + \frac{1}{G^{n_0-2}} + \frac{0}{G^{n_0-1}} + \frac{1}{G^{n_0}} + \frac{d_{n_0+2}}{G^{n_0+1}} + \ldots$$

$$\vdots$$

$$K^{n_0-2}(x) = \frac{1}{G} + \frac{0}{G^2} + \frac{1}{G^3} + \frac{d_{n_0+2}}{G^4} + \ldots \in R.$$

Therefore,

$$\frac{1}{G} \le K^{n_0-1}(x) = \frac{0}{G} + \frac{1}{G^2} + \sum_{k=3}^{\infty} \frac{d_{k+n_0-1}}{G^k} \le 1,$$

and we see that $K^{n_0-1}(x) \in M$. Finally, case (d) follows in a similar way.

Notice that if we are given the sequence of digits $(d_n)_{n \in \mathbb{N}}$ one is able to recover the original sequence of ℓ's, m's and r's in a unique way. Let $n \in \mathbb{N}$ be the first index for which

$$d_1 = \cdots = d_n \quad \text{and} \quad d_n \ne d_{n+1}.$$

Mark the block $d_n d_{n+1}$ and start again with d_{n+2}: find the first $m \ge 0$ such that

$$d_{n+2} = \cdots = d_{n+m} \quad \text{and} \quad d_{n+m} \ne d_{n+m+1},$$

mark the block $d_{n+m} d_{n+m+1}$ and repeat this procedure beginning with d_{n+m+2}. Once this blocking at 'switch times' is done, one is able to retrieve the original sequence.

For indices n that are not blocked, use the following correspondence:

$$d_n = 0 \quad \longleftrightarrow \quad X_n = \ell,$$
$$d_n = 1 \quad \longleftrightarrow \quad X_n = r.$$

For blocked indices $d_n d_{n+1}$ use the following correspondence:

$$01 \quad \longleftrightarrow \quad mr,$$
$$10 \quad \longleftrightarrow \quad m\ell.$$

Here is an example:

$$\cdot 1 \quad \mathbf{1\,0} \quad \mathbf{0\,1} \quad \mathbf{1\,0} \quad \mathbf{1\,0} \quad \mathbf{0\,1} \quad 1 \quad 1 \quad \mathbf{0\,1} \quad 1 \quad \mathbf{1\,0} \quad \mathbf{0\,1} \cdots.$$

$$\cdot\, r\; m\; \ell\; m\; r\; m\; \ell\; m\; \ell\; m\; r\; r\; r\; m\; r\; r\; m\; \ell\; m\; r\; \cdots$$

Let (X, σ_X) be the shift space consisting of all possible realizations of the above Markov chain on the symbols $\{\ell, m, r\}$, and let μ be the shift-invariant measure corresponding to the stationary distribution π and the transition matrix $\mathbf{P}$.

Similarly, let (Y, σ_Y) be the shift space of all possible sequences of digits $(d_1, d_2, \cdots)$ obtained by using the above 'random map' $K(x)$. Due to the above discussion we that there exists a 2-block factor map $\phi : X \to Y$ given by

$$(\phi(x))_i = \begin{cases} 0 & \text{if } X_i = \ell \text{ or } X_i X_{i+1} = mr, \\ \\ 1 & \text{if } X_i = r \text{ or } X_i X_{i+1} = m\ell, \end{cases}$$

see also (4.1) and (4.2). Then clearly $\psi \circ \sigma_X = \sigma_Y \circ \psi$, and the measure ρ defined on Y by $\rho(A) = \mu\left(\psi^{-1}(A)\right)$ is σ_Y invariant. In other words, ρ is K invariant and ergodic.

Given this correspondence between sequences generated by the Markov chain and the random expansions to base $\beta = G$, we are now able to describe the asymptotic as well as generic behaviour of these sequences.

As an example we give a number of stationary probabilities.

$$\begin{aligned} P(d_n = 0) &= P(X_n = \ell) + P(X_n = m, X_{n+1} = r) \\ &= \pi_\ell + P(X_{n+1} = r \mid X_n = m)\pi_m \\ &= \frac{pG^2}{G^2+1} + (1-p)\frac{1}{G^2+1} \\ &= \frac{pG+1}{G^2+1}, \end{aligned}$$

Note that if $p = 1/2$ one finds that $P(d_n = 0) = 1/2$, as one might expect beforehand due to symmetry.

$$\begin{aligned} P(d_n = 0, d_{n+1} = 1) &= P(X_n = \ell, X_{n+1} = m) + P(X_n = m, X_{n+1} = r) \\ &= P(X_{n+1} = m \mid X_n = \ell)P(X_n = \ell) \\ &\qquad + P(X_{n+1} = r \mid X_n = m)P(X_n = m) \\ &= g^2\,\frac{pG^2}{G^2+1} + (1-p)\,\frac{1}{G^2+1} \\ &= \frac{1}{G^2+1}. \end{aligned}$$

We can also calculate the expected return time to the region M, i.e., the expected time between two flips of symbols $= 1/\pi_m = G^2 + 1$, which is the same for all choices of p.

Acknowledgements We would like to thank Jeff Lagarias for pointing out Palmer's thesis [Pa], and sending us a photo copy of its relevant section.

References

[B] Bowen, R.

[DKS] Dajani, K., Kraaikamp, C. and Solomyak, B. – *The natural extension of the β-transformation*, Acta Math. Hungar. **73** (1996), no. 1-2, 97–109. MR 99d:28029

[EJK] Erdös, P., Joó, I. and Komornik, V. – *Characterization of the unique expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$ and related problems*, Bull. Soc. Math. France **118** (1990), no. 3, 377–390. MR 91j:11006

[FL] Flatto, L. and Lagarias, J. –

[FO] Friedman, N. A. and D. S. Ornstein – *On isomorphisms of weak Bernoulli transformations,* Adv. in Math., **5** (1970), 365-390. MR 43 #478c

[H] Hofbauer, F. – *Maximal measures for piecewise monotonically increasing transformations on* $[0,1]$, Ergodic theory (Proc. Conf., Math. Forschungsinst., Oberwolfach, 1978), 66–77, Lecture Notes in Math., **729**, Springer, Berlin, 1979. MR 81a:28023

[JS] Joó, I., Schnitzer, F.J. – *Expansions with respect to non-integer bases*, Grazer Mathematische Berichte, **329**. Karl-Franzens-Universität Graz, Graz, 1996. MR 98e:11090

[KL1] Komornik, V. and Loreti, P. – *Unique developments in non-integer bases*, Amer. Math. Monthly **105** (1998), 636-639; MR 99k:11017.

[KL2] Komornik, V. and Loreti, P. – *On the expansions in non-integer bases*, Rend. Mat. Appl. (7) 19 (1999), no. 4, 615–634.

[G] Gel'fond, A.O. – *A common property of number systems*, Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959) 809–814. MR 22 #702

[P1] Parry, W. – *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416. MR 26 #288

[P2] Parry, W. – *Representations for real numbers*, Acta Math. Acad. Sci. Hungar. **15** (1964), 95–105.MR 29 #3609

[Pa] Palmer, R.

[R1] Rényi, A. - *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hung. **8** (1957), 472-493. MR 20 #3843

[R2] Rényi, A. - *On algorithms for the generation of real numbers*, Magyar Tud. Akad. Mat. Fiz. Oszt. Kzl. **7** (1957), 265–293. MR 20 #4113

[S] Schweiger, F. - *Ergodic Theory of Fibred Systems and Metric Number Theory*, Oxford University Press, Oxford, 1995.

[Sm] Smorodinsky, M. – *β-automorphisms are Bernoulli shifts*, Acta Math. Acad. Sci. Hungar. **24** (1973), 273–278. MR 49 #10859

Mathematics Department, Utrecht University, 3508TA Utrecht, The Netherlands
E-mail address: `dajani@math.uu.nl`

TU Delft, Department of Information Technology and Systems, CROSS, Mekelweg 4, 2628 CD Delft, The Netherlands
E-mail address: `c.kraaikamp@its.tudelft.nl`