

# Some Hopf Algebras of Trees

Pepijn van der Laan

vdlaan@math.uu.nl

Mathematisch Instituut, Universiteit Utrecht

December 9, 2001

## 1 Introduction

In the literature several Hopf algebras that can be described in terms of trees have been studied. This paper tries to answer the question whether one can understand some of these Hopf algebras in terms of a single mathematical construction.

We recall the Hopf algebra of rooted trees as defined by Connes and Kreimer in [3]. Apart from its physical relevance, it has a universal property in Hochschild cohomology. We generalize the operadic construction by Moerdijk [13] of this Hopf algebra to more general trees (with colored edges), and prove a universal property in coalgebra Hochschild cohomology. For a Hopf operad  $\mathbf{P}$ , the construction is based on the operad  $\mathbf{P}[\lambda_n]$  obtained from  $\mathbf{P}$  by adjoining a free  $n$ -ary operation.

For a specific multi-parameter family of Hopf algebras obtained in this way, we give explicit formulas for the (a priori inductively defined) comultiplication and for the Lie bracket underlying the dual. Some special cases yield natural examples of pre-Lie and dendriform (and thus associative) algebras. We apply these results to construct some known Hopf algebras of trees. Notably, the Loday-Ronco Hopf algebra of planar binary trees [11], and the Brouder-Frabetti pruning Hopf algebra [1].

The author plans to discuss the simplicial algebra structure on the set of initial algebras with free  $n$ -ary operation  $P_n$  for an operad  $\mathbf{P}$  with multiplication in future work, together with some relations of the  $\mathbf{P}[\lambda_n]$  to the algebra Hochschild complex.

The author is grateful to Ieke Moerdijk for suggesting the Hopf  $\mathbf{P}$ -algebras  $P_n$  and their simplicial structure and for motivating as well as illuminating discussions, and to Loïc Foissy and Maria Ronco for pointing out errors in previous versions of this manuscript.

## 2 Preliminaries

This section fixes notation on trees and describes some results of Connes and Kreimer[3], and Chapoton and Livernet [2] that motivated this paper.

**Rooted trees**  $t$  are isomorphism classes of finite partially ordered sets which

- (i). have a minimal element  $r$  ( $\forall x \neq r : r < x$ ); we call  $r$  the **root**, and
- (ii). satisfy the tree condition that  $(y \neq z) \wedge (y < x) \wedge (z < x)$  implies  $(y < z) \vee (z < y)$ .

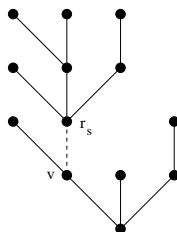
The elements of a tree are called **vertices**. A pair of vertices  $v < w$  is called an **edge** if there is no vertex  $x$  such that  $v < x < w$ . The number of vertices of a tree  $t$  is denoted by  $|t|$ . A **path** from  $x$  to  $y$  in a tree is a sequence  $(x_i)_i$  of elements  $x = x_n > x_{n-1} > \dots > x_1 > x_0 = y$  of maximal length. We will say that  $x$  is above  $y$  in a tree if there is a path from  $x$  to  $y$ . Thus we may depict a rooted tree as a finite directed graph, with one terminal vertex, the root. A vertex is called a **leaf** if there is no other vertex above it. A vertex is an **internal vertex** if it is not a leaf. We draw trees the natural way, with the root downwards. An automorphism of a tree  $t$  is an automorphism of the partially ordered set.

A **forest** is a finite, partially ordered set satisfying only property (ii) of the above. A forest always is a disjoint union of trees, its connected components. A rooted tree can be pictured as a tree with unlabeled vertices and uncolored edges. A forest is a set of these.

In the sequel we need trees with colored edges. That is, there is a function from the set of edges to a fixed set of colors. Vertices can be labeled as well (i.e. there is a function from the set of vertices to a fixed set of labels). Also trees with a linear ordering on the incoming edges at each vertex are used. A colored forest is a linearly ordered set of colored trees. An automorphism of a colored tree is an automorphism of the underlying tree compatible with the coloring of the edges.

We consider two operations on trees: cutting and grafting. Let  $s$  and  $t$  be trees and let  $v \in t$  be a vertex of  $t$ . Define  $s \circ_v t$  to be the tree that as a set is the disjoint union of  $s$  and  $t$  endowed with the ordering obtained by adding the edge  $v < r_s$ . This operation is called **grafting**  $s$  onto  $v$ . When using trees with colored edges, we write  $s \circ_v^i t$  for grafting  $s$  onto  $v$  by an edge of color  $i$ . For trees with labeled vertices, grafting preserves the labels. The operation of grafting  $s$  on  $t$  is visualized below.

A **cut** in a tree  $t$  is a subset of the set of edges of  $t$ . A cut is **admissible** if for each leaf  $m$  of  $t$  the unique path  $m > \dots > x_2 > x_1 > r$  to the root contains

Figure 1: grafting  $s$  on  $t$ 

at most one edge of the cut. Removing the edges in a cut  $c$  yields a forest. We denote by  $R^c(t)$  the connected component of this forest containing the root and by  $P^c(t)$  the complement of  $R^c(t)$ . We denote the set of admissible cuts of  $t$  by  $C(t)$ . For colored and labeled trees, cutting preserves the colors of edges that are not cut and labels of vertices. The left picture below shows an admissible cut, whereas the cut on the right is not admissible.

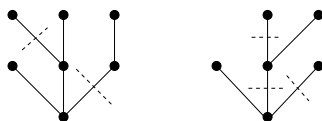


Figure 2: an admissible cut (left) and a non-admissible cut (right)

We describe the **Hopf algebra  $H_R$  of rooted trees** (defined in Connes-Kreimer [3]) as follows. Let  $k$  be a field. The set of rooted trees and the empty set generate a Hopf algebra over  $k$ . As an algebra it is the polynomial algebra in formal variables  $t$ , one for each rooted tree. Thus  $H_R$  is spanned by forests. Multiplication corresponds to taking the direct union. The algebra  $H_R$  is of course commutative. The unit is the empty tree.

Comultiplication  $\Delta$  is defined on rooted trees  $t$  and extended as an algebra homomorphism:

$$\Delta(t) = \sum_{c \in C(t)} P^c(t) \otimes R^c(t),$$

where the sum is over admissible cuts. The counit  $\varepsilon : H_R \rightarrow k$  takes value 1 on the empty tree and 0 on other trees.

The Hopf algebra  $H_R$  is  $\mathbb{Z}$ -graded with respect to the number of vertices in forests. The homogeneous elements of degree  $m$  are products  $t_1 \cdot \dots \cdot t_n$  of trees, such that  $\sum |t_i| = m$ . Both the product and the coproduct preserve the grading.

Denote by  $V^*$  the graded dual of a graded vector space  $V$ :

$$V^* = \bigoplus_n (V_n)^*,$$

the direct sum of the duals of the spaces of homogeneous elements. A basis for  $(H_R)_n^*$  is  $\{D_{t_1 \dots t_k} \mid \sum_i |t_i| = n\}$ , the dual basis to the basis for  $(H_R)_n$  given by products of trees. Of course,  $H_R^*$  is a cocommutative Hopf algebra, with comultiplication

$$\Delta(D_{t_1 \dots t_k}) = \sum_{i=0}^k \sum_{\sigma \in S(i, k-i)} D_{t_{\sigma^{-1}(1)}, \dots, t_{\sigma^{-1}(i)}} \otimes D_{t_{\sigma^{-1}(i+1)}, \dots, t_{\sigma^{-1}(k)}},$$

where  $S(i, k-i)$  is the set of  $(i, k-i)$ -shuffles in  $S_k$ . We sum over these to avoid repetition of terms in the right hand side. The primitive elements are those dual to (single) rooted trees.

As proved by Connes and Kreimer [3], the Lie algebra which has  $H_R^*$  as its universal enveloping algebra is the linear span of rooted trees with the Lie bracket

$$[D_t, D_s] = \sum_u (n(t, s, u) - n(s, t, u)) D_u,$$

where  $n(t, s, u)$  is the number of admissible cuts  $c$  of  $u$  such that  $P^c(u) = t$  and  $R^c(u) = s$ . Chapoton and Livernet [2] show that this is the Lie algebra associated to the free pre-Lie algebra on one generator (cf. section 7 for details).

### 3 Operads

**3.1 CONVENTION** For the rest of this paper, we restrict to the category of vector spaces over a fixed field  $k$ , but the general theory carries over to any symmetric monoidal category with countable coproducts and quotients of actions by finite groups on objects (cf. Moerdijk [13]).

Let  $kS_n$  denote the group algebra of the permutation group  $S_n$  on  $n$  objects. An **operad**  $\mathbf{P}$  in this paper will mean an operad with unit. Thus an operad consists of a collection of right  $kS_n$ -modules  $\mathbf{P}(n)$ , together with an associative,  $kS_n$ -equivariant composition

$$\gamma : \mathbf{P}(n) \otimes \mathbf{P}(m_1) \otimes \dots \otimes \mathbf{P}(m_n) \longrightarrow \mathbf{P}(m_1 + \dots + m_n),$$

such that there exists an identity  $\text{id} : k \rightarrow \mathbf{P}(1)$  with the obvious property and a unit  $u : k \rightarrow \mathbf{P}(0)$ , which need not be an isomorphism. The existence of the unit map is the only change with respect to the usual definition of an operad (consult Kriz and May [8], or Ginzburg and Kapranov [7], or Getzler and Jones [6]). Following Getzler and Jones [6] we define an **operad with multiplication** as an operad together with an associative element  $\mu \in \mathbf{P}(2)$ .

The category of **pointed vector spaces** has as objects vector spaces  $V$  with a base point  $u : k \rightarrow V$ . Morphisms are base point preserving  $k$ -linear maps. We use notation  $1 := u(1)$ . There are some canonical functors: The forgetful functor to vector spaces will be used implicitly. The free associative algebra functor  $T$  on a pointed vector space  $V$  is left adjoint to the forgetful functor from unital associative algebras to pointed vector spaces.

For any pointed vector space  $V$ , there is an operad  $\mathbf{End}_V$  such that  $\mathbf{End}_V(n) = \text{Hom}_k(V^{\otimes n}, V)$ . The pointed structure of  $V$  is only used to to define  $u : k \rightarrow \mathbf{End}(V)(0)$ , the elements of  $\text{Hom}_k(V^{\otimes n}, V)$  need not preserve the base point. A **P-algebra** structure on  $V$  is a map of operads (i.e. preserving relevant structure) from  $\mathbf{P}$  to  $\mathbf{End}_V$ . Equivalently, a **P-algebra** is a vector space  $V$  together with linear maps

$$\gamma_V : \mathbf{P}(n) \otimes_{kS_n} (V^{\otimes m_1} \otimes \dots \otimes V^{\otimes m_n}) \longrightarrow V^{\otimes m_1 + \dots + m_n},$$

compatible with composition in  $\mathbf{P}$  and unit in the natural sense.

**3.2 EXAMPLE** Let  $\mathbf{k}$  be the operad, with  $\mathbf{k}(0) = k = \mathbf{k}(1)$  and  $\mathbf{k}(n) = 0$ , otherwise (with  $u = \text{id}$ ). Algebras for  $\mathbf{k}$  are pointed vector spaces. The operad  $\mathbf{k}$  is the initial operad in our setting.

We denote by  $\mathbf{Com}$  the operad with as algebras unital commutative algebras. This operad satisfies  $\mathbf{Com}(n) = k$  for all  $n$  (where  $u : k \rightarrow k$  is the identity map).

Likewise  $\mathbf{Ass}$  is the operad satisfying  $\mathbf{Ass}(n) = kS_n$ , the group algebra of  $S_n$ , as a right  $S_n$ -module. The  $\mathbf{Ass}$ -algebras are unital associative algebras. An operad with multiplication is an operad  $\mathbf{P}$  together with a map of operads from  $\mathbf{Ass}$  to  $\mathbf{P}$ .

A map of operads  $\varphi : \mathbf{P} \rightarrow \mathbf{Q}$  induces an obvious functor  $\varphi^* : \mathbf{Q}\text{-Alg} \rightarrow \mathbf{P}\text{-Alg}$ . This functor has a left adjoint  $\varphi_! : \mathbf{P}\text{-Alg} \rightarrow \mathbf{Q}\text{-Alg}$ . Let  $\mathbf{P}$  be any operad and let  $i : k \rightarrow \mathbf{P}$  be the unique inclusion of operads. Then  $i_!$  is the unitary free  $\mathbf{P}$ -algebra functor.

Let  $\mathbf{P}$  be an operad. We can form the operad  $\mathbf{P}[\lambda_n]$  by adjoining a free  $n$ -ary operation to  $\mathbf{P}(n)$ . Algebras for  $\mathbf{P}[\lambda_n]$  are just  $\mathbf{P}$ -algebras  $A$  endowed with a linear map  $\alpha : A^{\otimes n} \rightarrow A$ . The reader familiar with the collections (cf. Getzler and Jones [6], or Ginzburg and Kapranov [7]), will recognize  $\mathbf{P}[\lambda_n]$  as the coproduct of operads

$$\mathbf{P}[\lambda_n] = \mathbf{P} \oplus_k FE_n$$

of  $\mathbf{P}$  and the free operad  $FE_n$  on the collection  $E_n$  defined by  $E_n(n) = kS_n$  and  $E_n(m) = 0$  for  $m \neq n$ .

Denote the initial  $\mathbf{P}[\lambda_n]$ -algebra  $\mathbf{P}[\lambda_n](0)$  by  $P_n$ . For any  $\mathbf{P}[\lambda_n]$ -algebra  $(A, \alpha)$ , there is by definition a unique  $\mathbf{P}$ -algebra morphism  $\gamma$  such that the following diagram commutes.

$$\begin{array}{ccc} P_n^{\otimes n} & \xrightarrow{\lambda_n} & P_n \\ \gamma^{\otimes n} \downarrow & & \downarrow \gamma \\ A^{\otimes n} & \xrightarrow{\alpha} & A \end{array}$$

The operad  $\mathbf{P}[\lambda_n]$  can be described in terms of planar trees. A planar tree is understood to have a linear ordering on the incoming edges at each vertex and might have external edges. There is a linear ordering on all external edges. An element of  $\mathbf{P}[\lambda_n](m)$  can be represented by a tree with  $m$  external edges and each vertex  $v$  with  $n$  incoming edges labeled either by an element of  $\mathbf{P}(n)$  or by  $\lambda_n$ . We identify two such representing trees if they can be reduced to the same tree using edge contractions implied by composition in  $\mathbf{P}$ , equivariance with respect to the  $kS_n$ -action at each internal vertex  $v$  and  $kS_m$ -equivariance at the external edges. The initial  $\mathbf{P}[\lambda_n]$ -algebra  $\mathbf{P}[\lambda_n](0)$  is described by such trees without external edges.

**3.3 EXAMPLE** Let  $\mathbf{P} = \mathbf{Com}$ . The initial algebra  $C_n = \mathbf{Com}[\lambda_n](0)$  can be described as the free commutative algebra on trees with edges colored by  $\{1, \dots, n\}$  (and no ordering on the edges). A bijection  $T$  is given by induction on the number of applications of  $\lambda$  as follows.  $T(1) = 1 = \emptyset$ , and  $T(\lambda(1, \dots, 1)) = r$  (the one vertex tree), and  $T(\lambda(x_1, \dots, x_n))$  is the tree obtained from the forest  $x_1 \cdot \dots \cdot x_n$  by adjoining a new root and connecting the roots of each tree in  $x_i$  to the new root by an edge of color  $i$ . Figure 3 shows (twice) the tree  $T(\lambda(x, \lambda(xx, 1)\lambda(x, 1)))$  in  $C_2$ , where  $x = \lambda(1, 1)$ .



Figure 3: trees not equal in  $A_2$ , but equal in  $C_2$

**3.4 EXAMPLE** Let  $\mathbf{P} = \mathbf{Ass}$ . The initial algebra  $A_n = \mathbf{Ass}[\lambda_n](0)$  can be described as the free associative algebra on trees with edges colored by  $\{1, \dots, n\}$  and at each vertex the incoming edges of each color endowed with a linear ordering. Equivalently, we could say, a linear ordering on all incoming edges extending the ordering on the colors. A bijection is given by the map  $T$  with the same inductive definition as in the previous example, but using associative multiplication. In the associative

case, the two trees in figure 3 are not identified. The tree to the right in figure 3 is  $T(\lambda(x, \lambda(x, 1)\lambda(xx, 1)))$  in  $A_2$ , while the tree to the left is  $T(\lambda(x, \lambda(xx, 1)\lambda(x, 1)))$  (where  $x = \lambda(1, 1)$ ).

3.5 EXAMPLE Let  $\mathbf{P} = \mathbf{k}$ . Then  $k_n = \mathbf{k}[\lambda_n](0)$  can be identified with the free vector space on trees (not forests, but possibly empty) with edges labeled by  $\{1, \dots, n\}$  and at most one incoming edge of each color at each vertex. For the relation to  $n$ -ary trees, see section 8.

## 4 Hopf Operads

A **Hopf operad** (cf. Getzler and Jones [6], and also Moerdijk [13]) is an operad in the category of coalgebras over  $k$ . Thus, a **Hopf operad**  $\mathbf{P}$  is a ( $k$  linear) operad together with a morphisms

$$k \xleftarrow{\varepsilon} \mathbf{P}(n) \xrightarrow{\Delta} \mathbf{P}(n) \otimes \mathbf{P}(n),$$

such that  $\Delta$  and  $\varepsilon$  satisfy the usual axioms of a coalgebra:  $\Delta$  is coassociative and  $\varepsilon$  is a counit for  $\Delta$  (cf. Sweedler [15]), and such that the composition  $\gamma$  of the operad is a coalgebra morphism. Moreover,  $\Delta$  should be compatible with the  $kS_n$ -action, where  $\mathbf{P}(n) \otimes \mathbf{P}(n)$  is a  $kS_n$ -module via the diagonal coproduct. For any Hopf operad  $\mathbf{P}$ , the tensor product of two  $\mathbf{P}$ -algebras is a  $\mathbf{P}$ -algebra again.

A Hopf  $\mathbf{P}$ -algebra  $A$  is a  $\mathbf{P}$ -algebra  $A$  in the category of (counital!) coalgebras. Note that we can not use the description of algebras in terms of the endomorphism operad in the category of coalgebras, since coalgebra homomorphisms do not form a linear space. In this generality it is not natural to consider antipodes. A Hopf Ass-algebra is just a bialgebra.

Let  $\varphi : \mathbf{Q} \rightarrow \mathbf{P}$  be a map of Hopf operads. The map  $\varphi$  induces functors

$$\varphi^* : \mathbf{P}\text{-Alg} \rightarrow \mathbf{Q}\text{-Alg} \quad \text{and} \quad \bar{\varphi}^* : \mathbf{P}\text{-HopfAlg} \rightarrow \mathbf{Q}\text{-HopfAlg}.$$

The map  $\varphi^*$  has a left adjoint  $\varphi_!$  (we work with  $k$ -vector spaces). Note that  $\varphi^*(k) = k$  and for  $\mathbf{P}$ -algebras  $A$  and  $B$  we have  $\varphi^*(A \otimes B) = \varphi^*(A) \otimes \varphi^*(B)$  (the map  $\varphi$  is compatible with  $\Delta$ ). Using this observation, conclude that the adjunction induces algebra maps  $\varphi_!(k) = \varphi_!\varphi_*(k) \rightarrow k$  and  $\varphi_!(A \otimes B) \rightarrow \varphi_!(\varphi^*\varphi_!A \otimes \varphi^*\varphi_!B) \rightarrow \varphi_!(A) \otimes \varphi_!(B)$ . These maps serve to show that  $\varphi_!$  lifts to a left adjoint

$$\bar{\varphi}_! : \mathbf{Q}\text{-HopfAlg} \longrightarrow \mathbf{P}\text{-HopfAlg}$$

of  $\bar{\varphi}^*$ .

Let  $\mathbf{P}$  be a Hopf operad. The aim of this section is to state a general result on Hopf operad structures on  $\mathbf{P}[\lambda_n]$ . We use the notations  $\lambda = \lambda_n$  and  $P_n = \mathbf{P}[\lambda_n](0)$  when no confusion can arise.

Let  $(A, \alpha)$  be a  $\mathbf{P}[\lambda_n]$ -algebra and let  $\sigma_1, \sigma_2 : A^{\otimes n} \rightarrow A$  be a pair of linear maps and define for each such pair a map  $(\sigma_1, \sigma_2) : (A \otimes A)^{\otimes n} \rightarrow A \otimes A$  by

$$(\sigma_1, \sigma_2) = (\sigma_1 \otimes \alpha_n + \alpha_n \otimes \sigma_2) \circ \tau,$$

where  $\tau$  is the  $n$ -fold twist map which identifies  $(A \otimes A)^{\otimes n}$  with  $A^{\otimes n} \otimes A^{\otimes n}$  by the symmetry of the tensor product. For vector spaces  $V_1, \dots, V_{2n}$ , the map  $\tau$  is the natural map

$$\tau : V_1 \otimes \dots \otimes V_{2n} \mapsto V_1 \otimes \dots \otimes V_{2n-1} \otimes V_2 \otimes V_4 \dots \otimes V_{2n}$$

induced by the symmetry (in the dg case this involves some signs).

We now list some results for linear maps  $\sigma_i$  that define a Hopf algebra structure on  $P_n$  but do not extend to  $\mathbf{P}[\lambda_n]$ . The results in this section are obtained along the same lines as the results in Moerdijk [13]. Our approach is more general in the sense that Moerdijk considered only the case  $n = 1$ .

Let  $(P_n, \lambda_n)$  be the initial  $\mathbf{P}[\lambda_n]$ -algebra. For any pair of  $n$ -ary linear maps  $\sigma_1, \sigma_2$  there is a unique  $\mathbf{P}[\lambda_n]$ -algebra morphism  $(P_n, \lambda_n) \rightarrow (P_n \otimes P_n, (\sigma_1, \sigma_2))$ . That is, a unique  $P$ -algebra morphism such that

$$\begin{array}{ccc} P_n^{\otimes n} & \xrightarrow{\lambda} & P_n \\ \downarrow \Delta^{\otimes n} & & \downarrow \Delta \\ (P_n \otimes P_n)^{\otimes n} & \xrightarrow{(\sigma_1, \sigma_2)} & P_n \otimes P_n \end{array}$$

commutes. Define  $\varepsilon : P_n \rightarrow k$  as the unique  $\mathbf{P}[\lambda]$ -algebra morphism  $(P_n, \lambda_n) \rightarrow (k, 0)$ , which extends  $\varepsilon : P(0) \rightarrow k$ . The following proposition is a generalization of Moerdijks construction [13]. The proof is completely analogous.

**4.1 PROPOSITION** *Let  $n \in \mathbb{N}$ ,  $\lambda_n$  and  $P_n$  be defined as above. Let  $\sigma_i : P_n^{\otimes n} \rightarrow P_n$  for  $i = 1, 2$  be linear maps. If  $\sigma_i$  satisfies*

$$\begin{aligned} \varepsilon \circ \sigma_i &= \varepsilon^{\otimes n}, \quad \text{and} \\ (\sigma_i \otimes \sigma_i) \circ \tau \circ \Delta^{\otimes n} &= \Delta \circ \sigma_i; \end{aligned}$$

*then there exists a unique Hopf- $P$  algebra structure on  $P_n$  extending the Hopf  $P$ -algebra structure on  $P(0)$  such that  $\Delta \circ \lambda = (\sigma_1, \sigma_2) \circ \tau \circ \Delta^{\otimes n}$  and  $\varepsilon \circ \lambda = 0$ .*

In the following examples  $|t|$  denotes the number of applications of  $\lambda_n$  in an element  $t \in \mathbf{P}[\lambda_n](m)$ , for any  $m$ . In the case  $\mathbf{P} = \text{Ass}$ , or  $\mathbf{P} = \text{Com}$ , or  $\mathbf{P} = \mathbf{k}$  this corresponds to counting the number of vertices in a tree.



4.2 EXAMPLE Let  $\mathbf{P}$  be a Hopf operad. We consider the case  $n = 1$ .

i) For any  $q_1, q_2 \in k$ , the endomorphisms  $\sigma_i(t) = q_i^{|t|}t$  define a coassociative Hopf  $\mathbf{P}$ -algebra structure on  $P_1$  (cf. Moerdijk [13] for the initial algebra).

ii) Suppose that  $\mathbf{P}$  is a Hopf operad with multiplication. For any  $q_1, q_2 \in k$ , the endomorphisms  $\sigma_i(t) = q_i^{|t|}r^{|t|}$  define a Hopf  $\mathbf{P}$ -algebra structure on  $P_1$ . (Here  $r$  is  $\lambda(1)$ , and powers are with respect to the multiplication in  $\mathbf{P}$ .)

4.3 EXAMPLE Let  $\mathbf{P}$  be a Hopf operad with multiplication. For any choice of  $q_{ij} \in k$  for all  $j \leq n$ , the maps

$$\sigma_i(t_1, \dots, t_n) = q_{i1}^{|t_1|} \cdot \dots \cdot q_{in}^{|t_n|} t_1 \cdot \dots \cdot t_n$$

(where  $i = 1, 2$ ) give a Hopf  $\mathbf{P}$ -algebra structure on  $P_n$ .

## 5 Cohomology

In the previous section we constructed Hopf operad structures on  $\mathbf{P}[\lambda_n]$ . This section aims to describe the relation of these Hopf operads and Hochschild cohomology for coalgebras.

Let  $A$  be a Hopf Ass-algebra, and let  $C_{(p)}^*$  be the graded vector space, which in degree  $q$  is  $\text{Hom}_k(A^{\otimes p}, A^{\otimes q})$ . For  $\varphi \in \text{Hom}_k(A^{\otimes p}, A^{\otimes q})$  we define a differential by the formula

$$d\varphi = (\mu^{(p)} \otimes \varphi) \circ \tau \circ \Delta^{\otimes p} + \sum_{i=1}^q (-1)^i \Delta_{(i)} \circ \varphi + (-1)^{q+1} (\varphi \otimes \mu^{(p)}) \circ \tau \circ \Delta^{\otimes p},$$

where  $\Delta_{(i)}(x_1 \otimes \dots \otimes x_q) = (x_1 \otimes \dots \otimes \Delta(x_i) \otimes \dots \otimes x_q)$ . This is the coalgebra-Hochschild complex with respect to the left and right coaction of  $A$  on  $A^{\otimes p}$  given by

$$(\mu^{(p)} \otimes \text{id}) \circ \tau \circ \Delta : A^{\otimes p} \longrightarrow A \otimes A^{\otimes p}$$

(or with  $(\text{id} \otimes \mu^{(p)})$ , where  $\mu^{(p)}$  is the unique  $p$ -fold application of  $\mu$ . This complex (for  $p, q > 0$ ) is the  $p$ -th column in the bicomplex used by Lazarev and Movshev [9] to compute what they call the cohomology of a Hop algebra.

Let  $\mathbf{P}$  be a Hopf operad, and let  $A$  be a Hopf  $\mathbf{P}$ -algebra, let  $p \geq 1$  and  $\sigma_1, \sigma_2 : A^{\otimes p} \rightarrow A$  be coalgebra morphisms. Then

$$(\sigma_1 \otimes \text{id}) \circ \tau \circ \Delta^{\otimes p} : A^{\otimes p} \rightarrow A \otimes A^{\otimes p} \quad \text{and}$$

$$(\text{id} \otimes \sigma_2) \circ \tau \circ \Delta^{\otimes p} : A^{\otimes p} \rightarrow A^{\otimes p} \otimes A$$

define a left and a right coaction of  $A$  on  $A^{\otimes p}$  and we can define the complex  $C_{\sigma_1 \sigma_2}^q$  with the boundary  $d_{\sigma_1 \sigma_2}$ . We can write the differential explicitly as

$$d\varphi = (\sigma_1 \otimes \varphi) \circ \tau \circ \Delta^{\otimes p} + \sum_{i=1}^q (-1)^i \Delta_{(i)} \circ \varphi + (-1)^{q+1} (\varphi \otimes \sigma_2) \circ \tau \circ \Delta^{\otimes p}.$$

This is the Hochschild boundary with respect to these coactions. The cohomology of this complex will be denoted  $H_{\sigma_1\sigma_2}^*(A)$ .

Let  $\mathbf{P}$  be a Hopf operad. A **natural  $n$ -twisting function**  $\varphi$  is a map from Hopf  $\mathbf{P}$ -algebras  $B$  to coalgebra maps  $\varphi^{(B)} : B^{\otimes n} \rightarrow B$ , such that the map  $\varphi$  commutes with augmented  $\mathbf{P}$ -algebra morphisms  $f : A \rightarrow B$  (i.e.  $f \circ \sigma_i^{(A)} = \sigma_i^{(B)} \circ f^{\otimes n}$  for  $i = 1, 2$ ).

If  $\sigma_1$  and  $\sigma_2$  are natural  $p$ -twisting functions such that  $\sigma_i^{(B)}$  satisfies the conditions from 4.1 for any Hopf  $\mathbf{P}$  algebra  $B$ ; then  $H_{\sigma_1\sigma_2}^*(B)$  is defined for any Hopf- $\mathbf{P}$  algebra  $B$ , and natural in  $B$ .

Let  $(B, \beta)$  be a Hopf  $\mathbf{P}$ -algebra. Then  $\beta$  is a 1-cocycle in the  $(\sigma_1, \sigma_2)$ -complex iff

$$(\sigma_1 \otimes \beta)\Delta - \Delta\beta + (\beta \otimes \sigma_1)\Delta = 0.$$

**5.1 EXAMPLE** Let  $\mathbf{P}$  be a Hopf operad. Then  $\sigma = p \in \mathbf{P}(n)$  defines a natural  $n$ -twisting function. More generally, if  $\sigma = (\varepsilon^{\otimes k} \otimes p)\tau$  for some  $p \in \mathbf{P}(n-k)$  and some  $\tau \in S_n$ , then  $\sigma$  defines a natural  $n$ -twisting function.

We now characterize the universal property of the Hopf  $\mathbf{P}$ -algebra  $P_n$ . Again this is a direct generalization of Moerdijk [13].

**5.2 THEOREM** *Let  $(\sigma_1, \sigma_2)$  be a pair of natural twisting functions of Hopf  $\mathbf{P}$ -algebras, satisfying the conditions of Theorem 4.1 and endow  $P_n$  with the induced Hopf  $\mathbf{P}$ -algebra structure. For every Hopf  $\mathbf{P}$ -algebra  $B$ , there is a one-one correspondence between*

(i). *Hopf  $\mathbf{P}$ -algebra maps  $c_\beta : P_n \rightarrow B$  that lift to a map  $(P_n, \lambda) \rightarrow (B, \beta)$  of  $\mathbf{P}[\lambda_n]$ -algebras and*

(ii). *cocycles  $\beta$  in the  $(\sigma_1, \sigma_2)$ -twisted complex of  $B$ .*

**PROOF** The previous paragraph shows that  $\lambda$  is a cocycle in the  $(\sigma_1, \sigma_2)$ -complex of  $P_n$  with respect to the  $(\sigma_1, \sigma_2)$ -coproduct on  $P_n$ .

First suppose that  $c_\beta$  has such a lift  $c_\beta : (P_n, \lambda) \rightarrow (B, \beta)$  (which is obviously unique). Since  $\sigma_1$  and  $\sigma_2$  are natural  $n$ -twisting functions and  $c_\beta$  is a map of augmented  $\mathbf{P}$ -algebras, we have the following commutative diagram of  $\mathbf{P}[\lambda_n]$ -algebras:

$$\begin{array}{ccc} (P_n, \lambda) & \xrightarrow{\Delta} & (P_n \otimes P_n, (\sigma_1, \sigma_2)) \\ c \downarrow & & \downarrow c \otimes c \\ (B, \beta) & \xrightarrow{\Delta} & (B \otimes B, (\sigma_1, \sigma_2)). \end{array}$$

This shows that  $\Delta \circ \beta = (\sigma_1, \sigma_2) \circ \Delta$ , which is the cocycle relation for  $\beta$  in the  $(\sigma_1, \sigma_2)$ -twisted complex.

Let  $\beta$  be a cocycle in the  $(\sigma_1, \sigma_2)$ -twisted complex of  $B$ . Then (by the universal property of  $P_n$ ) there is a unique  $\mathbb{P}[\lambda_n]$ -algebra map  $c : (P_n, \lambda) \rightarrow (B, \beta)$ . We are done as soon as we have verified that the diagram above is a commutative diagram of  $\mathbb{P}[\lambda_n]$ -algebras. To show commutativity of the diagram, note that the universal property of  $P_n$  shows that  $\varepsilon \circ c = \varepsilon$  is the unique morphism of  $(P_n, \lambda)$  to  $(k, 0)$ . The intertwining relation for natural  $n$ -twisting functions with respect to augmented  $\mathbb{P}$  algebra morphisms now implies that  $c \otimes c$  is a morphism of  $\mathbb{P}[\lambda_n]$ -algebras. Since both  $c$  and  $\Delta$  are maps of  $\mathbb{P}[\lambda_n]$ -algebras, the universal property of  $P_n$  implies that both composites in the diagram are the unique  $\mathbb{P}[\lambda_n]$ -algebra morphism from  $(P_n, \lambda)$  to  $(B \otimes B, (\sigma_1, \sigma_2))$ . QED

The result above tells us that  $\mathbb{P}[\lambda_n]$  has the following universal property in twisted coalgebra Hochschild cohomology, which we state once more in  $\mathbb{P}[\lambda_n]$ -free language.

**5.3 COROLLARY** *Let  $(B, \beta)$  be a pair of a Hopf  $P$ -algebra together with a cocycle in the  $(\sigma_1, \sigma_2)$ -twisted complex. Then there is a unique Hopf  $P$ -algebra morphism  $c : P_n \rightarrow B$  such that the following diagram commutes:*

$$\begin{array}{ccc} P_n^{\otimes n} & \xrightarrow{\lambda} & P_n \\ \downarrow c^{\otimes n} & & \downarrow c \\ B^{\otimes n} & \xrightarrow{\beta} & B. \end{array}$$

This section concludes with some functorial properties of the construction. The proof of the following proposition is completely analogous the the proof for the case  $n = 1$  in Moerdijk [13].

**5.4 PROPOSITION** *Let  $\varphi : Q \rightarrow P$  be a map of Hopf operads, and let  $\sigma_1, \sigma_2$  satisfy the conditions of 4.1 on  $Q_n$  such that  $\sigma_1 \sigma_2$  extend to natural  $n$ -twisting functions of  $Q$ -algebras; then*

- (i). *The  $\sigma_i$  induce natural twisting functions  $\sigma_i$  of  $P$ -algebras, satisfying the conditions of 4.1 on  $H$ .*
- (ii). *The unique map of  $Q[\lambda_n]$ -algebras  $j_0 : Q_n \rightarrow \varphi^*(P_n)$  is a map of Hopf  $Q$ -algebras.*
- (iii). *The unique map of  $\mathbb{P}[\lambda_n]$ -algebras  $j : \varphi_!(Q_n) \rightarrow P_n$  is a map of Hopf  $P$ -algebras.*

5.5 EXAMPLE Let  $\sigma = (\varepsilon^{\otimes m} \otimes p)\tau$  for some  $m \leq n$ , some  $p \in \mathbf{P}(n - m)$  and some  $\tau \in S_n$ . If  $p$  is group-like a group-like element in the sense that  $\Delta(p) = p \otimes p$ , then  $\sigma$  satisfies the conditions of theorems 4.1 and 5.3. In this case maps of operads  $\psi : \mathbf{Q} \rightarrow \mathbf{P}$  and  $\varphi : \mathbf{P} \rightarrow \mathbf{Q}$  induce functors  $\mathbf{P}\text{-HopfAlg} \rightarrow \mathbf{Q}\text{-HopfAlg}$  that map the natural twisting function  $\sigma$  to a natural twisting functions  $\bar{\psi}^*\sigma$  and  $\bar{\varphi}_!\sigma$  of Hopf  $\mathbf{Q}$ -algebras.

## 6 A Deformation

Moerdijk [13] gives an inductive formula for a 2-parameter deformation of  $\Delta$  on  $P_1$  for any Hopf operad  $\mathbf{P}$ . Regarding  $P_n$ , we have seen in example 4.3 that there is a  $2n$ -parameter deformation of the coproduct given by  $\sigma_1 = \sigma_2 = u \circ \varepsilon^{\otimes n}$  (provided that  $\mathbf{P}$  is an Hopf operad with multiplication). This section obtains an explicit formula for the  $2n$ -parameter deformation of the coproduct in  $C_n$  and the Lie algebra structure underlying the dual.

6.1 CONVENTION In this section we fix  $\mathbf{P} = \mathbf{Com}$ , and  $n \in \mathbb{N}$ . We use notation  $C_n = \mathbf{P} = \mathbf{Com}[\lambda_n](0)$ . For  $i = 1, 2$  and  $1 \leq j \leq n$ , let  $q_{ij} \in k$ , and define

$$\sigma_i(t_1, \dots, t_n) = t_1 \cdot \dots \cdot t_n \cdot \prod_j q_{ij}^{|t_j|}.$$

According to example 4.3, these  $\sigma_i$  make  $C_n$  a commutative Hopf algebra.

A subforest  $s$  of a rooted tree  $t$  is a subset of the partially ordered set  $t$  with the induced partial ordering. For colored trees, the color of the edge connecting  $v > w$  in  $s$  is the color of edge connecting  $w$  to its direct predecessor in the unique path from  $v$  to  $w$  in  $t$ . For  $v \in s$  we denote by  $p_k(v, s, t)$  the number of edges of color  $k$  in the path in  $t$  from  $v$  to the root of  $t$  that have their lower vertex in  $s^c$ . For forests  $t$  we define  $p_k(v, s, t)$  as  $p_k(v, s \cap t', t')$ , where  $t'$  is the connected component of  $t$  containing  $v$ . There is an easy but useful lemma on the calculus of the  $p_k$ .

6.2 LEMMA *Let  $t$  and  $s$  be subforests of a forest  $u$ . Let  $v \in s$  and set  $t' = t \cup v$ ,  $s' = s \cap t'$ ,  $t'' = t^c \cup v$  and  $s'' = s \cap t''$ . Then*

$$p_k(v, s, u) = p_k(v, s', t') + p_k(v, s'', t''),$$

where  $t'$ ,  $t''$ ,  $s'$  and  $s''$  are interpreted as subforests of  $u$ .

PROOF The lemma follows at once when we observe that a vertex in the path from  $v$  to the root in  $u$  that is not in  $s$  is either in  $t'$  or in  $t''$ . QED

Define

$$q(s, t) := \prod_j q_{1j}^{\sum_{v \in s} p_j(v, s, t)} \cdot \prod_j q_{2j}^{\sum_{v \in s^c} p_j(v, s^c, t)}.$$

Intuitively,  $q(s, t)$  counts for  $v \in s$  the number of edges of color  $j$  are in the path from  $v$  to the root that have their lower vertex in  $s^c$  and adds a factor  $q_{1j}$  for each of these, and  $q(s, t)$  counts for  $v \in s^c$  the number of edges of color  $j$  are in the path from  $v$  to the root that have their lower vertex in  $s$  and adds a factor  $q_{2,j}$  for each of these.

6.3 THEOREM *Under the assumptions of 6.1, we have for a forest  $t \in C_n$  the formula*

$$\Delta(t) = \sum_{s \subset t} q(s, t) s \otimes s^c,$$

where the sum is over all subforests  $s$  of  $t$ .

PROOF We use induction with respect to the number of applications of  $\lambda$ . The formula is trivial for the empty tree.

Let  $t = \lambda(x_1, \dots, x_n)$  be a tree and suppose that the formula holds for all forests with less than  $|t|$  vertices. Subforests of  $t$  are either of the form  $s = \cup_i s_i$ , a (disjoint) union of subforests of the  $x_i$ , or of the form  $s = r \cup (\cup_i s_i)$ , a (disjoint) union of subforests of the  $x_i$  together with the root. By definition,

$$\begin{aligned} \Delta(t) &= \sum_{s_i \subset x_i} s_1 \cdot \dots \cdot s_n \otimes \lambda(s_1^c, \dots, s_n^c) \prod_i q_{1i}^{|s_i|} q(s_i, x_i) \\ &\quad + \sum_{s_i \subset x_i} \lambda(s_1, \dots, s_n) \otimes s_1^c \cdot \dots \cdot s_n^c \prod_i q_{2i}^{|s_i^c|} q(s_i, x_i). \end{aligned}$$

But by the lemma above,

$$\prod_i q_{1i}^{|s_i|} q(s_i, x_i) = \prod_j q_{1j}^{\sum_{v \in s} p_j(v, s, t)} \cdot \prod_j q_{2j}^{\sum_{v \in s^c} p_j(v, s^c, t)}$$

for  $s = \cup_i s_i = s_1 \cdot \dots \cdot s_n$  and  $s^c = r \cup (\cup_i s_i^c) = \lambda(s_1^c, \dots, s_n^c)$ ; and

$$\prod_i q_{2i}^{|s_i^c|} q(s_i, x_i) = \prod_j q_{1j}^{\sum_{v \in s} p_j(v, s, t)} \cdot \prod_j q_{2j}^{\sum_{v \in s^c} p_j(v, s^c, t)}$$

for  $s = r \cup (\cup_i s_i) = \lambda(s_1, \dots, s_n)$  and  $s^c = \cup_i s_i = s_1^c \cdot \dots \cdot s_n^c$ . Putting these together yields the formula. QED

An **antipode** for a Hopf Ass-algebra  $A$  is an inverse for the identity map  $id$  with respect to the convolution product

$$f * g = \mu \circ (f \otimes g) \circ \Delta$$

on the  $k$ -linear maps  $\text{Hom}_k(A, A)$ .

In the situation of theorem 6.3 it makes sense to ask if there exists an antipode for arbitrary choice of the coefficients  $q_{ij}$ . An explicit description of  $S$  follows by a general argument. A Hopf Ass-algebra  $A$  is called **connected** if it is  $\mathbb{Z}$ -graded, concentrated in non-negative degree, and satisfies  $A_0 = k \cdot 1$ . The **augmentation ideal** of a connected Hopf algebra  $A$  is the ideal  $\bigoplus_{n \geq 1} A_n$ .

6.4 LEMMA (MILNOR AND MOORE [12]) *Let  $A$  be a connected Hopf Ass-algebra. Then there exists an antipode on  $A$  which for  $x$  in the augmentation ideal is given by*

$$S(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \mu^{(k)} \circ \bar{\Delta}^{(k)}(x),$$

where  $\bar{\Delta} = \Delta - (id \otimes 1 + 1 \otimes id)$ , and  $\mu^{(0)} = id = \bar{\Delta}^{(0)}$ , and  $\mu^{(k)} : A^{\otimes k+1} \rightarrow A$  and  $\bar{\Delta}^{(k)} : A \rightarrow A^{\otimes k+1}$  are defined using (co)associativity for  $k > 0$  and  $\mu^{(0)} = id = \bar{\Delta}^{(0)}$ .

6.5 COROLLARY *Under the assumptions of convention 6.1, there exists an antipode on the Hopf algebra of theorem 6.3, given by the formula*

$$S(t) = \sum_{k=1}^{|t|} \sum_{\cup_i s_i = t} (-1)^k s_1 \cdots s_k \prod_{1 \leq j < k} q(s_j, s_{j+1} \cup \dots \cup s_k, s_j \cup \dots \cup s_k),$$

where we only sum over partitions  $t = s_1 \cup \dots \cup s_k$  of the forest  $t$  with all forests  $s_i$  non-empty.

PROOF The grading with respect to  $|t|$  gives a grading that makes the Lie algebra connected, since  $\text{Com}(0) = k$ . QED

6.6 EXAMPLE In the case  $n = 1$  with  $q_1 = 1$  and  $q_2 = 0$  we get the formula of Connes and Kreimer [3] for the antipode on the Hopf algebra  $H_R$  of section 2. Note that according to example 5.5 the Connes-Kreimer Hopf algebra is the initial Hopf algebra for the Hopf operad  $\text{Com}[\lambda_1]$ .

We now state a corollary to the formula for the coproduct in theorem 6.3, that gives an explicit formula for the Lie bracket on the primitive elements of the cocommutative Hopf algebra  $C_n^*$ . We know by the Milnor-Moore theorem [12] that  $C_n^*$  is the universal enveloping algebra of the Lie algebra of its primitive elements.

6.7 COROLLARY *Let  $P = \text{Com}$ , and  $t \in C_n$ , and  $\Delta = \Delta_{q_1 \dots q_{2n}}$ ; then the graded dual  $C_n^*$  is the universal enveloping algebra of the Lie algebra which*

as a vector space is spanned by elements  $D_t$ , where  $t$  is a rooted tree in  $C_n$ . The bracket is given by  $[D_s, D_t] = D_s * D_t - D_t * D_s$ , where

$$D_s * D_t = \sum_{w=s \cup t} q(s, w) D_w,$$

the sum ranges over all rooted trees in  $C_n$  that have  $s$  and  $t$  as complementary colored subforests.

PROOF For any cocommutative Hopf algebra we can define an operation  $*$  on the primitive elements, such that its commutator is the Lie bracket on primitive elements: Simply define  $*$  as the truncation of the product at degree  $> 1$ , with respect to the primitive filtration  $F$ . In this case,  $F_m C_n^*$  is spanned by the elements  $D_u$  dual to forests  $u$  consisting of at most  $m$  trees. The product in  $C_n^*$  is determined by the coproduct in  $C_n$ . Explicitly, for forest  $u$ , we can write the multiplication in  $C_n^*$  as

$$\begin{aligned} D_s D_t(u) &= (D_s \otimes D_t) \Delta(u) \\ &= \sum_{u=w \cup w^c} q(w, u) D_s(w) D_t(w^c). \end{aligned}$$

When we then restrict to the primitive part, we conclude that  $D_s * D_t$  is given by the desired formula. QED

6.8 REMARK With some minor changes, there is an analogue of the above for the case  $\mathbf{P} = \text{Ass}$ . The only change is that we have to remember the ordering of up-going edges at each vertex. To be very explicit, for the coproduct on  $A_n = \text{Ass}[\lambda_n](0)$  we have the formula

$$\Delta(t) = \sum_{s \subset t} \prod_j q_{1j}^{\sum_{v \in s} p_j(v, s, t)} \cdot \prod_j q_{2j}^{\sum_{v \in t} p_j(v, s^c, t)} s \otimes s^c,$$

where the product of trees is associative, non-commutative. The order of multiplication is given by the linear order on the roots of the trees defined by the linear on the incoming edges at each vertex and the partial order on vertices. Dually, the primitive elements of the dual are again spanned by elements dual to single rooted trees (with linear ordering on incoming edges in each vertex). The truncated product  $*$  is given by

$$D_s * D_t = \sum_{w=s \cup t} \prod_j q_{1j}^{\sum_{v \in s} p_j(v, s, w)} \cdot \prod_j q_{2j}^{\sum_{v \in t} p_j(v, t, w)} D_w,$$

where  $w$ ,  $s$  and  $t$  are trees with a linear ordering on the incoming edges of the same color at each vertex and the inclusions of  $s$  and  $t$  in  $w$  have to respect these orderings.

6.9 EXAMPLE Independently, Foissy [4] has found this formula for the Lie bracket in the case where  $n = 1$ ,  $q_1 = 1$  and  $q_2 = 0$  (and **Ass** is the underlying operad), and established the universal property of section 5, which in this case also follows from Moerdijk [13]. He uses this formula to give an explicit isomorphism between the Hopf algebras  $A_1$  and  $A_1^*$  in this case. Moreover, he does the same for planar trees with labeled vertices. The labeled case corresponds to iterated addition of unitary operations, using the same  $q_1$  and  $q_2$  at each stage.

6.10 REMARK For certain values of the parameters  $q_{ij}$ , the formula of theorem 6.3 defines a coproduct on  $k[\lambda_n](0)$ . If for fixed  $i = 1, 2$  there is at most one  $j$  such that  $q_{ij} \neq 0$ , then the subspace spanned by single rooted trees which do not involve multiplication is preserved under  $\Delta$ . Since this subspace is isomorphic to  $k[\lambda_n](0)$ , this implies that in this case the formulas above define a coproduct on  $k[\lambda_n](0)$ . Consequently, this defines an operation  $*$  on the dual, since each element in the dual is primitive. A formula for this is given by the formula for **Com**, modulo trees involving multiplication (their span is denoted  $\mu$ ): the quotient  $\text{Com}[\lambda_n](0)^*/(\mu)$  is canonically isomorphic to  $k[\lambda_n](0)^*$ .

## 7 Pre-Lie algebras

A **(right) pre-Lie algebra** (cf. Gerstenhaber [5], and Chapoton and Livernet [2]) is a  $k$ -vector space  $L$  together with a bilinear map  $- * - : L \times L \rightarrow L$  which satisfies

$$(x * y) * z - x * (y * z) = (x * z) * y - x * (z * y),$$

for each  $x$ ,  $y$  and  $z$  in  $L$ . Note that the bracket  $[x, y] = x * y - y * x$  defines a Lie algebra structure on  $L$ . This Lie algebra is the **associated Lie Algebra** of the pre-Lie algebra  $(L, *)$ .

Remember from [2] that the free pre-Lie algebra  $L_p$  on  $p$  generators is given by the vector space spanned by rooted trees with vertices labeled with elements of  $\mathbf{p} = \{1, 2, \dots, p\}$ . The pre-Lie algebra product is given by grafting trees. That is,

$$t * s = \sum_{v \in t} s \circ_v t$$

for trees  $s$  and  $t$ , where the color of the vertices is preserved. For any tree  $t \in L_p$  define  $r_i(t)$  to be the tree  $t$  with the color of the root changed to  $i$ . Thus  $r_i$  defines a linear endomorphism of the vector space  $L_p$ .



7.1 CONVENTION Let  $\mathbf{P} = \mathbf{Com}$ , choose  $\Delta$  on  $C_n = \mathbf{Com}[\lambda_n](0)$  defined by  $\sigma_i$  as in the deformation used in convention 6.1 and use the notation  $\chi_S$  to denote the characteristic function of a subset  $S \subset X$  which has value 1 on  $S$  and value 0 on  $X - S$ . Denote the primitive elements of a coalgebra  $C$  by  $P(C)$ .

Let  $t \in C_n$  be a tree. Then we call the  $\mathbf{p}$ -connected component of  $t$  the subtree consisting of vertices connected to the root by a path with edges colored exclusively by elements of  $\mathbf{p}$ .

7.2 PROPOSITION Let  $\mathbf{p} \subset \{1, \dots, n\}$  and define  $q_{1j} = \chi_{\mathbf{p}}(j)$  and  $q_{2j} = 0$ . Then

- (i). The product  $*$  of corollary 6.7 defines a pre-Lie algebra structure on the primitive elements  $P(C_n^*)$  of  $C_n^*$ .
- (ii). If  $\mathbf{p} = \{1, \dots, n\}$ , then there is a natural inclusion of this pre-Lie algebra into the free pre-Lie algebra on  $n$  generators. The image in the free pre-Lie algebra is spanned by all sums  $\sum_{i \in \mathbf{p}} t_i$ , of trees that only differ in that the color of the root of  $t_i$  is  $i$ .

PROOF First note that with this choice of  $\sigma_i$ , the formula for the coproduct simplifies to

$$\Delta(t) = \sum_{c \in C(t)} P^c(t) \otimes R^c(t),$$

where we sum over admissible cuts as described in section 2, provided that the cuts are at edges in the  $\mathbf{p}$ -connected component of  $t$ . To see this, consider the formula of corollary 6.7 and note that the coefficient of  $s \otimes s^c$  is 0 iff  $s^c$  has a connected component with respect to edges of colors in  $\mathbf{p}$  which does not contain the root.

The operation  $*$  of theorem 6.7 is then given by

$$D_s * D_t = \sum_u \sum_{p \in \mathbf{p}} n_p(s, t, u) D_u,$$

where  $u$  ranges over trees, and  $n_p(s, t, u)$  denotes the number of admissible cuts  $c$  of  $u$  that only cut  $p$ -colored edges in the  $\mathbf{p}$ -connected component of  $u$  and satisfy  $P^c(u) = t$  and  $R^c(u) = s$ .

To check the pre-Lie algebra identity, note that  $D_u * (D_t * D_s) - (D_u * D_t) * D_s$  is exactly  $\sum_w n(u, t, s, w) D_w$ , where  $n(u, t, s, w)$  is the number of admissible cuts  $c$  of  $w$ , such that  $P^c(w) = u \cdot t$  and  $R^c(w) = s$ . Since  $n(t, u, s, w) = n(u, t, s, w)$  part (i) follows.

For a tree  $t$ , we denote the cardinality of the automorphism group of  $t$  by  $a_t$ . If  $D_t \in P(C_n^*)$  and  $v$  a vertex of  $t$ , denote by  $a_t^v$  the cardinality of the group of automorphisms of  $t$  that have  $v$  as a fixed point. Define  $m(v, t)$  the cardinality of the orbit of  $v$  under the action by the automorphisms of  $t$ . Let  $s, t \in C_n$  be trees and define  $u = s \circ_v^p t$ . We now see that

$$a_u = \sum_p a_s \cdot a_t^v \cdot n_p(s, t, u).$$

Let  $\mathbf{p} = \{1, \dots, n\}$  and consider the pre-Lie algebra of (i). We claim that there is an isomorphism of pre-Lie algebras from this pre-Lie algebra to the pre Lie algebra on the same vector space  $C_n$  with multiplication given by

$$D_s *' D_t = \sum_{v \in t} \sum_p D_{s \circ_v^p t}.$$

Define a linear automorphism of  $P(C_n^*)$  by  $\varphi(D_t) = a_t D_t$ . We have to show that  $\varphi(D_s * D_t) = \varphi(D_s) *' \varphi(D_t)$ . Write

$$\begin{aligned} D_s *' D_t &= \sum_{v \in t} \sum_p D_{s \circ_v^p t} \\ &= \sum_p \sum_{u = s \circ_v^p t} D_u m(v, t) \end{aligned}$$

where the sum in the second line is over all  $u$  that can be written as  $s \circ_v^p t$  for some  $v$  and where for each such  $u$  we fix a decomposition of each such  $u$  as  $u = s \circ_v^p t$ . To show that  $\varphi$  is a pre-Lie algebra morphism, we have to show that for any such  $u$ ,

$$\sum_p a_s \cdot a_t \cdot n_p(s, t, u) = a_u m(v, t).$$

This follows from the above since  $a_t = a_t^v \cdot m(v, t)$ .

A map from the pre-Lie algebra  $(P(C_n^*), *')$  into the free pre-Lie algebra on  $n$  generators (trees with  $n$ -colored vertices) is given by assigning the color of an edge to the vertex on top and summing over all colors in  $\mathbf{p}$  for the root. That this is an inclusion of pre-Lie algebras and that the image can be described as in (ii) is a direct consequence of the definitions. QED

**7.3 EXAMPLE** In the case  $n = 1$ ,  $\mathbf{p} = \{1\}$ , this yields the identification of  $L_1$  with the primitive elements of  $H_R^*$  with the pre-Lie product  $*$  as in Chapoton and Livernet [2].

## 8 The Pruning coproduct

Brouder and Frabetti [1] use a pruning operator on planar binary trees to define a coproduct on the  $k$ -vector space spanned by planar binary trees. We will see that this coproduct is related to the coproducts described in section 6. We first give a family of coproducts on planar  $n$ -ary trees. In this context the pruning Hopf algebra of Brouder and Frabetti is a direct corollary.

A **planar binary tree** is a non-empty oriented planar graph in which each vertex which is not a leaf or the root has exactly two direct predecessors and one direct successor. Let  $T$  be a planar binary tree. We call each vertex which is not a leaf a **internal vertex** of  $T$ . If  $T$  has  $n + 1$  leaves, it has  $n$  internal vertices. Following Loday and Ronco, we denote the  $k$ -linear span of the set of planar binary trees  $k[Y_\infty]$ . On planar binary trees we have the binary operation  $\lambda$  adding a new root (smallest element) to the union of planar binary trees  $T_1$  and  $T_2$ .

Likewise, we define a **planar  $n$ -ary tree** as a non-empty oriented graph in which each vertex which is not a leaf has exactly  $n$  direct predecessors and one direct successor. The  $n$ -ary operation  $\lambda$  adds a new root to an  $n$ -tuple of planar  $n$ -ary trees. Denote the vector space spanned by the set of  $n$ -ary planar trees by  $k[Y_\infty^{(n)}]$ . The basepoint in  $k[Y_\infty^{(n)}]$  is the tree with one leaf.

8.1 PROPOSITION *Let  $n \geq 0$ .*

- (i). *The pointed vector space  $k[Y_\infty^{(n)}]$  is isomorphic to  $k[\lambda_n](0)$ .*
- (ii). *The unitary free associative algebra on the pointed vector space spanned by planar  $n$ -ary trees has a family of natural Hopf algebra structures parameterized by  $i \in \{1, \dots, n\}$ .*

PROOF Define a bijection  $\xi : k[Y_\infty^{(n)}] \rightarrow k[\lambda_n](0)$  by mapping the planar  $n$ -ary tree with one leaf to the empty tree and

$$\xi(\lambda(T_1, \dots, T_n)) = \lambda_n(\xi(T_1), \dots, \xi(T_n)),$$

for planar  $n$ -ary trees  $T_i$ . Each vertex in a tree in  $k[\lambda_n](0)$  corresponds to an internal vertex in the corresponding  $n$ -ary tree. It is easy to check that this is a basepoint preserving bijection of bases. This proves (i).

Let  $i \in \{1, \dots, n\}$ . The coproducts induced by  $q_{1j} = \chi_{\{i\}}(j)$  and  $q_{2j} = 0$  are well defined on  $k[\lambda_n](0)$  by remark 6.10. This shows that we have a family of coproducts on  $k[\lambda_n](0)$ . The result (ii) is now a direct consequence of the deformation in remark 6.8 and functorial properties. The functor

$\bar{i}_i : k[\lambda_n]\text{-HopfAlg} \rightarrow \text{Ass}[\lambda_n]\text{-HopfAlg}$  induced by the inclusion of Hopf operads  $i : k \rightarrow \text{Ass}$  is the unitary free algebra functor and the result follows by functoriality. QED

8.2 COROLLARY (BROUDER AND FRABETTI [1]) *Let the pruning operator  $P : k[Y_\infty] \rightarrow k[Y_\infty] \otimes k[Y_\infty]$  be defined inductively as*

$$\begin{aligned} P(\lambda(T, 1)) &= 0 \\ P(\lambda(T, S)) &= \sum_i \lambda(T, S'_i) \otimes S''_i + \lambda(T, 1) \otimes S, \end{aligned}$$

if  $P(S) = \sum_i S'_i \otimes S''_i$ . Then the formulas

$$\begin{aligned} \Delta^P(1) &= 1 \otimes 1 \quad \text{and} \\ \Delta^P(T) &= 1 \otimes t + P(T) + T \otimes 1, \end{aligned}$$

define a coassociative coproduct on  $k[Y_\infty]$ . Thus the unitary free associative algebra  $T(k[Y_\infty])$  is a Hopf algebra with the coproduct described on generators by  $\Delta^P$ .

PROOF Let  $n = 2$ , and  $i = 2$ . Apply proposition 8.1. QED

## 9 Associative and dendriform algebras

A **dendriform algebra** (cf. Loday [10], and Loday and Ronco [11]) is a vector space together with two bilinear (non-associative) products  $\prec$  and  $\succ$ , satisfying the identities

$$\begin{aligned} (a \prec b) \prec c &= a \prec (b \prec c) + a \prec (b \succ c) \\ a \succ (b \prec c) &= (a \succ b) \prec c \\ a \succ (b \succ c) &= (a \prec b) \succ c + (a \succ b) \succ c. \end{aligned}$$

A dendriform algebra  $D$  defines an associative algebra on the same vector space with associative product  $*$  defined by

$$a * b = a \prec b + a \succ b.$$

A **dendriform Hopf algebra** (Ronco [14]) is a dendriform algebra together with a coproduct that satisfies

$$\begin{aligned} \Delta(x \prec y) &= \sum x' * y' \otimes x'' \prec y'' + x' * y \otimes x'' + y' \otimes x \prec y'' + y \otimes x \\ \Delta(x \succ y) &= \sum x' * y' \otimes x'' \succ y'' + x * y' \otimes y'' + y' \otimes x \succ y'' + x \otimes y. \end{aligned}$$

9.1 REMARK One motivation for studying dendriform algebras is the classification of Hopf algebras analogous to the Milnor-Moore theorem (cf. Milnor and Moore [12]). The Milnor-Moore theorem gives an equivalence of categories between the category of cocommutative Hopf algebras and the category of Lie algebras (in characteristic 0). A Milnor-Moore theorem for dendriform Hopf algebras can be found in Ronco [14]. This theorem states the equivalence of dendriform Hopf algebras and brace algebras.

9.2 CONVENTION Let  $k, l \in \mathbb{N}$ . Use  $\mathbf{P} = k$ , and take  $q_{1j} = \delta_{kj}$  and  $q_{2j} = \delta_{lj}$  to define  $\sigma_1$  and  $\sigma_2$  in the deformation of 6.3. According to remark 6.10 this choice of parameters indeed defines a coproduct on  $k[\lambda_p](0)$ .

We use the augmentation of  $u^* : k[\lambda](0)^* \rightarrow k$ .

9.3 PROPOSITION *Consider the graded vector space  $(k[\lambda_p](0))^*$ , graded by the number of applications of  $\lambda_p$  in trees. Then the augmentation ideal of  $(k[\lambda_p](0))^*$  is a graded dendriform algebra with associated multiplication  $*$ .*

PROOF The most convenient way to check the statement is to give an inductive characterization of the operations  $\prec$  and  $\succ$ . To avoid awesome notation, we identify trees  $t$  with their dual  $D_t$ .

The product is characterized by its unit  $1 = \emptyset (= D_\emptyset)$ , and the identity

$$s * t = \lambda(s_1, \dots, s_k * t, \dots, s_n) + \lambda(t_1, \dots, s * t_l, \dots, t_n),$$

as is readily checked. Now define operations

$$\begin{aligned} s \prec t &= \lambda(s_1, \dots, s_k * t, \dots, s_n) \quad \text{and} \\ s \succ t &= \lambda(t_1, \dots, s * t_l, \dots, t_n). \end{aligned}$$

Let  $t = \lambda(t_1, \dots, t_n)$ ,  $u = \lambda(u_1, \dots, u_n)$  and  $s = \lambda(s_1, \dots, s_n)$  and suppose that  $\prec$  and  $\succ$  satisfy the dendriform identities on all trees with in total  $< |t| + |s| + |u|$  vertices. Then we have

$$\begin{aligned} (s \prec t) \prec u &= \lambda(s_1, \dots, (s_k * t) * u, \dots, s_n) \\ &= \lambda(s_1, \dots, s_k * (t * u), \dots, s_n) \\ &= s \prec (t * u) \\ (s \succ t) \prec u &= \lambda(t_1, \dots, s * t_l, \dots, t_k * u, \dots, t_n) \\ &= s \succ (t \prec u). \end{aligned}$$

The first identity follows by the definition of  $*$  and associativity on smaller trees. The second is clear, but uses associativity on smaller trees in case  $k = l$ . The proof of the third dendriform identity can be copied from the

first almost verbatim. It is now left to show that the dendriform operations preserve the grading in the sense that for  $s$  of degree  $m$  and  $t$  of degree  $n$ , the products  $s \prec t$  and  $s \succ t$  are of degree  $m + n$ . If the dendriform operations are graded in total degree smaller than  $m + n$ , assume  $s$  and  $t$  of degree  $m$  and  $n$  respectively. We now know that  $*$  is graded in total degree smaller than  $m + n$ . Since the definition of  $\prec$  and  $\succ$  only uses multiplication of smaller trees, one checks that the grading is preserved. QED

## 10 The Loday-Ronco construction

We still assume convention 9.2. There is an isomorphism of graded vector spaces between  $k[\lambda_n](0)$  and  $(k[\lambda_n](0))^*$  that maps  $t$  to  $D_t$ . Proposition 9.3 shows that we have an associative multiplication  $*$  on  $(k[\lambda_n](0))^*$ . The isomorphism gives an associative multiplication on  $k[\lambda_n](0)$ , which we will also denote by  $*$ .

Denote by  $*_{(n)} : k[\lambda_n](0)^{\otimes n} \rightarrow k[\lambda_n](0)$  the map induced by the associative multiplication, and by  $\varepsilon_{(n)} : k[\lambda_n](0)^{\otimes n} \rightarrow k[\lambda_n](0)$  the map  $u \circ \varepsilon^{\otimes n}$ .

**10.1 PROPOSITION** *Use convention 9.2 and let the product  $*$  of proposition 9.3. Then  $\sigma_1 = *_{(n)}$  and  $\sigma_2 = \varepsilon_{(n)}$  define a Hopf algebra structure on  $k[\lambda_n](0)$  iff  $k = n$  and  $l = 1$ .*

*Moreover if  $k = n$  and  $l = 1$ , the augmentation ideal of  $k[\lambda_n](0)$  is a dendriform Hopf algebra.*

**PROOF** We check the requirements of proposition 4.1. The requirement for the counit is fulfilled for any element of  $(k[\lambda_p](0))^*$ . We prove the claim for coassociativity inductively. Let  $\sigma_i$  as above and assume that the requirement for  $\Delta$  holds up to degree  $n$ . Then coassociativity of  $\Delta$  holds up to degree  $n$ . Compute for  $k = n$  and  $l = 1$  (using Sweedler notation)

$$\begin{aligned} \Delta(a * b) &= \Delta(\lambda(a_1, \dots, a_n) * \lambda(b_1, \dots, b_n)) \\ &= \Delta(\lambda(a_1, \dots, a_n * b) + \lambda(a * b_1, \dots, b_n)) \\ &= \sum a' * \dots * (a_n * b)' \otimes \lambda(a''_1, \dots, (a_n * b)'') \\ &\quad + \sum (a * b_1)' * \dots * b'_n \otimes \lambda((a * b_1)'', \dots, b''_n) \\ &\quad + (a * b) \otimes 1 \end{aligned}$$

for  $a = \lambda(a_1, \dots, a_n)$  and  $b = \lambda(b_1, \dots, b_n)$  such that the sum of their degrees is  $m$ . Then use the induction hypothesis for degrees smaller than  $m$  (in

particular  $\Delta(a_n * b) = \Delta(a_n) * \Delta(b)$  and  $\Delta(a * b_1) = \Delta(a) * \Delta(b_1)$ ). Deduce

$$\begin{aligned}\Delta(a \prec b) &= \sum a' * b' \otimes a'' \prec b'' & \text{and} \\ \Delta(a \succ b) &= a' * b' \otimes a'' \succ b'',\end{aligned}$$

and conclude that  $(k[\lambda_p](0))^*$  is a Hopf algebra if  $k = n$  and  $l = 1$ . The formulas also show that for other choices of  $k$  and  $l$  the induction fails and coassociativity does not hold since  $*$  is not commutative.

The dendriform Hopf identities for  $\bar{\Delta} = \Delta - \text{id} \otimes 1 - 1 \otimes \text{id}$  follow from the above. QED

Loday and Ronco [11] construct a Hopf algebra of planar binary trees. Remember the definition of the vector space  $k[Y_\infty]$  as the vector space spanned by binary trees.

10.2 COROLLARY (LODAY AND RONCO [11]) *Let  $T, T' \in k[Y_\infty]$ . If  $T = \lambda(T_1, T_2)$  and  $S = \lambda(S_1, S_2)$ , then*

$$T * S = \lambda(T_1, T_2 * S) + \lambda(T * S_1, S_2)$$

*defines an associative product on  $k[Y_\infty]$  with as unit the tree with one leaf. Moreover there is a coproduct making  $k[Y_\infty]$  a Hopf algebra. This coproduct is defined inductively by*

$$\Delta(T) = \sum (T'_1 * T'_2) \otimes \lambda(T''_1, T''_2) + T \otimes 1,$$

*where  $\Delta(T) = \sum_{(T)} T' \otimes T''$  is the Sweedler notation of the coproduct. The counit  $\varepsilon$  is the obvious augmentation of the product by projection on the span of the unit.*

*Finally, the augmentation ideal is a dendriform Hopf algebra.*

PROOF Recognize the star product on  $k[\lambda_2](0)$  given by  $q_{12} = 1 = q_{21}$  and  $q_{11} = 0 = q_{22}$ , and the coproduct associated to  $*$ . Apply proposition 10.1.

QED

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Pepijn van der Laan  
Mathematisch Instituut  
Universiteit Utrecht

P.O.Box 80.010  
3508 TA Utrecht  
The Netherlands