# HIGHER ORDER RESONANCE IN TWO DEGREE OF FREEDOM HAMILTONIAN SYSTEM

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This paper reviews higher order resonance in two degrees of freedom Hamiltonian systems. We consider a positive semi-definite Hamiltonian around the origin. Using normal form theory, we give an estimate of the size of the domain where interesting dynamics takes place, which is an improvement of the one previously known. Using a geometric numerical integration approach, we investigate this in the elastic pendulum to find additional evidence that our estimate is sharp. In an extreme case of higher order resonance, we show that phase interaction between the degrees of freedom occurs on a short time-scale, although there is no energy interchange.

## 1 Introduction

Studies on two degrees of freedom Hamiltonian system has a long history: dating back to Euler in 1772 with his description of three body problem. The presence of resonance in a Hamiltonian system of differential equations, strongly affects the dynamics. In this paper, we give a review of studies on higher order resonance in two degrees of freedom Hamiltonian system. An early discussion of this problem, using formal methods, is given by Kevorkian[6].

The analysis in this paper involves asymptotic analysis (perturbation method) and normalization. Using these techniques, we construct an approximation to the original system and then study the dynamics of the approximate system. We also use a numerical method to gain confirmation of our analytical result. The numerical method that we used is based heavily on a geometrical approach to preserve some of the geometric structure of the system.

We start this paper with a mathematical setting of two degrees of freedom Hamiltonian system in $\mathbb{R}^4$. We also give a brief idea of the normalization in this section. For introduction to Hamiltonian system, see Abraham and Marsden[1] or Arnol'd[2], for normalization see Churchil et.al.[4]. In Section 3 we study the resonance domain, which is the main focus of interest in higher order resonance. Using normal form, we show that we can improve the estimate of

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the size of this domain. We check this statement also numerically and it is presented in Section 4. The last section, we present part of an on going research on an extreme type of higher order resonance.

## 2 Two degrees of freedom Hamiltonian systems at higher order resonance

Consider a two degrees of freedom Hamiltonian system, defined in $\mathbb{R}^4$ with coordinate $(\boldsymbol{q},\boldsymbol{p}) = (q_1,q_2,p_1,p_2)$ and a symplectic form $\mathrm{d}\boldsymbol{q}\wedge\mathrm{d}\boldsymbol{p} = \mathrm{d}q_1\wedge\mathrm{d}p_1 + \mathrm{d}q_2\wedge\mathrm{d}p_2$, with Hamiltonian

$$H = \frac{1}{2}\omega_1({q_1}^2+{p_1}^2) + \frac{1}{2}\omega_2({q_2}^2+{p_2}^2) + H_3(\boldsymbol{q},\boldsymbol{p}) + H_4(\boldsymbol{q},\boldsymbol{p}) + \dots, \qquad (1)$$

where $H_k$ is a homogeneous polynomial of degree $k$ and $\omega_1,\omega_2 > 0$. Let $\varepsilon$ be a small parameter ($\varepsilon \ll 1$). We localize in the neighborhood of the origin by rescaling the variables by $q_1 = \varepsilon\overline{q_1}$, $q_2 = \varepsilon\overline{q_2}$, $p_1 = \varepsilon\overline{p_1}$, and $p_2 = \varepsilon\overline{p_2}$. Dividing the Hamiltonian (1) by $\varepsilon^2$, we arrive at the Hamiltonian

$$H = \frac{1}{2}\omega_1({q_1}^2+{p_1}^2) + \frac{1}{2}\omega_2({q_2}^2+{p_2}^2) + \varepsilon H_3(\boldsymbol{q},\boldsymbol{p}) + \varepsilon^2 H_4(\boldsymbol{q},\boldsymbol{p}) + \dots, \qquad (2)$$

where we have dropped the bar.

The idea of Birkhoff normalization is to transform the Hamiltonian such that it depend only on the so-called *action variables*. In general, this is only possible in the *non-resonance* situation. We proceed by bringing the Hamiltonian (2) to the *Birkhoff-Gustavson* normal form. Consider the *resonance relation*: $n\omega_1 + m\omega_2 = 0$. Let $(n,m)$ integer solution of the resonance relation such that $|n|+|m| = r$ while $m$ and $n$ are relatively prime. Introducing the action variables $\tau_j = ({q_j}^2+{p_j}^2)/2$ and the angle variables $\varphi_j = \arctan(p_j/q_j)$, $j = 1,2$, the Hamiltonian (2) is transformed to the normal form of the form

$$H = \omega_1\tau_1 + \omega_1\tau_1 + \mathcal{P}(\tau_1,\tau_2,\varepsilon) + \varepsilon^{m+n-2}\mathcal{R}(\tau_1,\tau_2,\varphi_1,\varphi_2), \qquad (3)$$

where $\mathcal{P}$ is a polynomial of degree $[\![r/2]\!]$. Moreover, the function $\mathcal{R}$ depend only on the *resonance combination angle*: $n\varphi_1 + m\varphi_2$ (instead of an individual angle).

## 3 The resonance domain

In this paper we are interested on the *higher order resonance* cases: $r \geq 5$. For the lower order resonance ($r = 3$ and $r = 4$), see for instance Nayfeh and Mook[7], or van der Burgh[10]. Sanders[8] is one of the first who described the dynamics of (2) at high order resonance. He found that the phase-space is foliated by invariant tori parameterized by taking the actions $\tau_1$ and $\tau_2$ to be constant. Using the KAM theorem, most of these tori persist under a Hamiltonian perturbation. There exists also the so-called *resonant manifold.* A neighborhood of this manifold, called the *resonance domain*, is the location where interesting dynamics is found. At each energy level, the normal form produces at least one elliptic periodic solution and one saddle type periodic solution; they lie in the resonance manifold which is embedded in the energy manifold.

Writing $\varphi = n\varphi_1 + m\varphi_2$ as the resonance combination angle, the equations of motion derived from the normal form (3) is

$$\begin{aligned} \dot{\tau}_j &= \varepsilon^{m+n-4} \frac{\partial \mathcal{R}}{\partial \varphi_j}, \qquad j = 1, 2 \\ \dot{\varphi} &= n\frac{\partial P_2}{\partial \tau_1} + m\frac{\partial P_2}{\partial \tau_2} + O(\varepsilon), \end{aligned} \tag{4}$$

where we have re-scaled time by $\varepsilon^2 t$, and $P_2$ is the quadratic part of $\mathcal{P}$. Consider a system of two linear equation

$$\begin{cases} m\tau_1 + n\tau_2 &= E_\circ \\ n\frac{\partial P_2}{\partial \tau_1} + m\frac{\partial P_2}{\partial \tau_2} &= 0, \end{cases} \tag{5}$$

for some $E_\circ \in \mathbb{R}^+$. Note that the first equation in (5) represents nothing but the approximate energy manifold $H_2 = E_\circ$. If

$$\Delta = \begin{vmatrix} m & n \\ n\frac{\partial^2 P_2}{\partial {\tau_1}^2} & m\frac{\partial^2 P_2}{\partial {\tau_2}^2} \end{vmatrix} \neq 0,$$

then the location of the resonance domain can be approximated by the solution of (5). Moreover, transfer of energy (or interaction) between the degrees of freedom occurs in the neighborhood of that solution. Sanders[8] also gave an estimate for the size, $d_\varepsilon$, of the resonance domain, which is of $O(\varepsilon^{-(m+n-4)/6})$, and the time-scale of interaction is $\varepsilon^{-(m+n)/2}$.

In van den Broek[9], numerical evidence has been given that the size is actually smaller than the above estimation, so there is room for improvement of the estimate of $d_\varepsilon$. In Tuwankotta and Verhulst[12], we use a geometric approach to give an estimate for $d_\varepsilon$. The idea is to use the normal form theory to construct an approximation of Poincaré section of the flow of (2). Suppose we construct the section in the $q_1$-$p_1$-direction. Each of the periodic solutions (inside the resonance domain) mentioned above will appear as an $n$-periodic point of the map. Obviously, there will be $2n$ of such points in the section (since we have two periodic orbits). The saddle type point will be

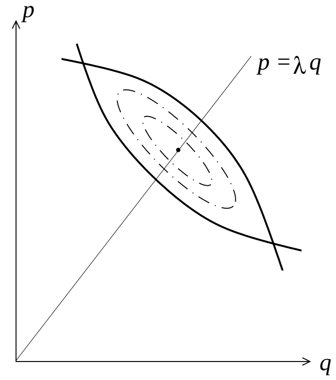


Figure 1. An illustration of calculation of the estimate for the $d_\varepsilon$.

connected to its neighboring saddle point with a heteroclinic cycle. We draw a line connecting the origin and one of the elliptic points. This line will intersect the heteroclinic cycle. The estimate for $d_\varepsilon$ in Tuwankotta and Verhulst[12] is achieved by estimating the distance between the two intersection points. We summarize in the following lemma.

**Lemma 3.1** *In two degrees of freedom Hamiltonian systems at higher order resonance $m:n$ with $m$ and $n$ natural numbers satisfying $m+n \geq 5$, the size $d_\varepsilon$ of the resonance domain is*

$$d_\varepsilon = O(\varepsilon^{\frac{m+n-4}{2}}), \tag{6}$$

*with a time-scale of interaction $O(\varepsilon^{-(m+n)/2})$.*

The presence of an appropriate discrete symmetry completely changes the hierarchy of resonances in the system. As demonstrated in Tuwankotta and Verhulst[12], the 1 : 2-resonance for instance, has to be viewed as a 2 : 4-resonance in the presence of mirror symmetry in the second degree of freedom. As a consequence, it becomes a higher order resonance and thus the lemma above holds. The number of periodic solutions in the resonance manifold embedded in the energy manifold is then doubled.

## 4 Geometric Integration based on a splitting method

As mentioned previously, the first indication that the estimate of $d_\varepsilon$ can be improved, is found numerically. The next thing for us to do is then to check the estimate (6) numerically. For this purpose, we choose the Elastic Pendulum, which is a classical mechanical problem with discrete symmetry. This system serves as a model of a lot of applications, see the references in Tuwankotta and Verhulst[12].

The elastic pendulum is a mathematical pendulum in which the rod is replaced by a linear spring. The Hamiltonian of the elastic pendulum is

$$H = \frac{1}{2ml^2}\left({p_z}^2 + \frac{{p_\varphi}^2}{(z+1)^2}\right) + \frac{sl^2}{2}\left(z + \frac{l-l_\circ}{l}\right)^2 - mgl(z+1)\cos(\varphi), \qquad (7)$$

where $\varphi$ is the deviation from the vertical position, $z$ is the radial oscillations, $s$ is the spring constant, $m$ is the mass of the pendulum, and $l$ is the after load length of the pendulum. See Nayfeh[7],van der Burgh[10],Tuwankotta and Verhulst[12],Tuwankotta and Quispel[11] for details.

Due to the size of the domain which is relatively small, and the time-scale of interaction which is relatively long, it is not easy to get the numerical confirmation of the estimate (6). We need a method of time integration that is accurate enough after relatively long integration time as well as being fast in the real time computation.

By mean of an example, Tuwankotta and Quispel [11] demonstrated how the *Baker-Campbell-Hausdorf* (BCH) formula can be used to construct an approximation of the flow of (7). The idea is to split the Hamiltonian (7) into parts which are individually integrable. By composing the exact flow of each part, and using the BCH formula, the numerical integration scheme is then constructed. Using this method, an independent confirmation that our estimate is sharp, is achieved. See Tuwankotta and Quispel[11] for details, Table 1 for numerical result and also Figure (2).

| Resonance | Resonant part | Analytic $\log_\varepsilon(d_\varepsilon)$ | Numerical $\log_\varepsilon(d_\varepsilon)$ | Error |
|---|---|---|---|---|
| $4:1$ | $H_5$ | $1/2$ | 0.5091568 | 0.01 |
| $6:1$ | $H_7$ | $3/2$ | 1.5079998 | 0.05 |
| $4:3$ | $H_7$ | $3/2$ | 1.4478968 | 0.09 |
| $3:1$ | $H_8$ | 2 | 2.0898136 | 0.35 |

Table 1. Comparison between the analytic estimate and the numerical computation of the size of the resonance domain of four of the most prominent higher order resonances of the elastic pendulum. The second column of this table indicates the part of the expanded Hamiltonian in which the lowest order resonant terms are found.

Apart from this confirmation, we note that this splitting method preserves some geometric properties of the system such as, linear symmetry, time-reversal symmetry, the symplectic form and also the linear resonance.

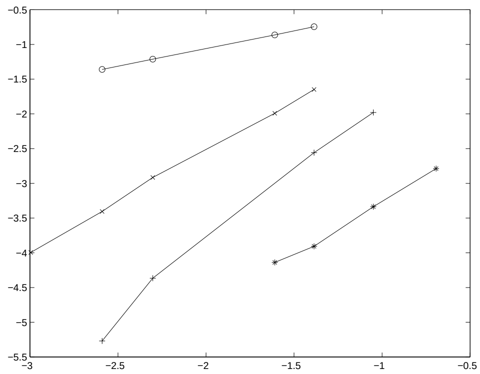


Figure 2. Plots of $\log(d_\varepsilon)$ against $\log(\varepsilon)$ for various resonances. The 4 : 1-resonance is plotted using '$-\circ$', the 3 : 1-resonance is using '$-+$', the 4 : 3-resonance is using '$-\times$' and the 6 : 1 resonance is using '$-*$'.

## 5 Systems with widely separated frequencies

Studies on coupled oscillator systems have a long history. They serve as models in many applications, such as vibrating mechanical structures. Most of the concern in these studies is to see how energy interchanges between the oscillators. In this frame work, our preceding sections suggest that in the higher order resonance case, the energy exchanged between the oscillators is small. This is in agreement with the traditional knowledge in this field. However, in 1990, a lot of studies (see Haller[5]) have been devoted to an extreme type of higher order resonance, i.e. systems with widely separated frequencies.

In the Hamiltonian case, Broer et.al.[3] gave a description of the unfolding of the origin of this type of system. Using normal form and singularity theory, they found that the codimension of the origin is 1 for the non semi-simple case and 3 for the semi-simple case. As a supplement to this study, we study also the dynamics in time of the semi-simple type of this system and some degeneracies due to symmetry (see Tuwankotta and Verhulst[13]).

The Hamiltonian that we consider is

$$H = \frac{1}{2}({q_1}^2 + {p_1}^2) + \frac{1}{2}\varepsilon({q_2}^2 + {p_2}^2) + H_3(\boldsymbol{q}) + H_4(\boldsymbol{q}) + \dots . \qquad (8)$$

We re-scale the variables in the usual way to arrive at the perturbative Hamiltonian. The unperturbed Hamiltonian is $I_\circ = \frac{1}{2}({q_1}^2+{p_1}^2)$. We then normalize (8) with respect to the $S^1$-action defined by the flow of the Hamiltonian vector

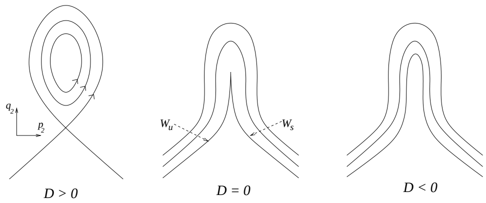


Figure 3. Phase portrait of the Poincaré section on $(q_2, p_2)$-plane for fixed value of $I_\circ \neq 0$ and $\beta \neq 0$. The geometry of phase-space of the system (9) is achieved by taking the cross product of them with $S^1$

field $X_{I_\circ}$. The normalized Hamiltonian is

$$H = I_\circ + \varepsilon \left( \frac{1}{2} \left( {q_2}^2 + {p_2}^2 \right) - \alpha I_\circ q_2 - \frac{1}{3} \beta {q_2}^3 \right), \tag{9}$$

where $\alpha$ and $\beta$ are parameters. If $\beta = 0$, the system degenerates to a linear system. Apart from $I_\circ$, there is one other parameter $D = 1 - 4\alpha\beta I_\circ$ which is important for the bifurcation scenario in this system. See Figure (3).

The conclusion of this section is that in Hamiltonian system with widely separated frequencies, there are no energy interchanges. Nevertheless, the phase interaction between the oscillators is stronger than the the one we found in the previous section. For the degenerate case and details in the study on this type of systems, see Tuwankotta and Verhulst[13].

## 6 Discussion

We have presented a review on higher order resonance in two degrees of freedom Hamiltonian systems. The estimate of the size of the resonance domain where the interesting dynamics takes place, has been improved. We also show numerical evidence that our new estimate is sharp. In dealing with the higher order resonance numerically, the geometric integration provides relatively cheap computation time in getting accurate results. The study of Hamiltonian systems with widely separated frequencies will be completed in the near future.

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