# MARKOV BRANCHING DIFFUSIONS: MARTINGALES, GIRSANOV TYPE THEOREMS AND APPLICATIONS TO THE LONG TERM BEHAVIOUR

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**Abstract**

Consider a spatial branching particle process where the underlying motion is a conservative diffusion on $D \subseteq \mathbb{R}^d$ corresponding to the elliptic operator $L$ on $D$, and the branching is strictly binary (dyadic), with spatially varying rate $\beta(x) \geq 0$ (and $\beta \not\equiv 0$) which is assumed to be bounded from above. We prove that, under extremely mild circumstances the process exhibits local extinction if and only if $\lambda_c \leq 0$, where $\lambda_c$ denotes the generalized principal eigenvalue for the operator $L + \beta$ on $D$. (This criterion is analogous to the one obtained by Pinsky (1996) for the local extinction of superdiffusions). Furthermore we show that when the process does not exhibit local extinction, every nonempty open subset is occupied infinitely often with positive probability which can be characterized by a solution bounded in $(0, 1]$ to the semilinear elliptic equation $Lu + \beta(u^2 - u) = 0$ on $D$. Moreover, in this case, there is an exponential rate of growth on sufficiently large compact domains, and this rate can be arbitrarily close to $\lambda_c$. In order to reach these conclusions we first develop some results concerning innerproduct and multiplicative martingales and their relation to the operators $L + \beta$ and $L + \beta\psi$ respectively, where $\psi(x) = x^2 - x$. In the case of the innerproduct martingales we show that for some circumstances they can be used as changes of measure for the law of the branching process in a similar way that Girsanov densities act as changes of measure in the context of diffusions. More specifically, the change of measure induces a drift consistent with a certain Doob's $h$-transform on the path of a randomized ancestral line of descent. These concepts are essentially spatial versions of spine decompositions for Galton-Watson processes given in Lyons *et al.* (1995).

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# 1 Introduction and main results

## 1.1 Markov branching diffusions

Let $D \subseteq \mathbb{R}^d$ be a domain and consider $Y = \{Y(t) : t \geq 0\}$, the diffusion process (with probabilities $\{\mathbb{P}_x,\ x \in D\}$) corresponding to the elliptic operator $L$ on $D$ satisfying

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} \text{ on } D, \tag{1}$$

where the symmetric matrix $a(x) = a_{ij}(x)$ is positive definite for $0 \neq x \in D$ and $a_{ij} \in C^2(D)$ , $b_i \in C^1(D)$. (We assumed these stronger than usual smoothness assumptions for convenience – they guarantee that there will be no problem defining the adjoint operator.) We assume that $Y$ is *conservative*, that is, that $\tau^D := \inf\{t \geq 0 : Y(t) \notin D\}$ satisfies $\mathbb{P}_x(\tau^D < \infty) = 0$ (in other words, $Y$ has an a.s. infinite lifetime). Furthermore let $0 \leq \beta \in C^\alpha(D)$, $\alpha \in (0,1]$ be bounded from above on $D$ and $\beta \not\equiv 0$. [Here $C^\alpha(D)$ denotes the usual Hölder space.] Then, the binary $(L,\beta)$-branching diffusion under the measure $P_x$ is defined (informally) as follows. A single particle starts at position $x \in D$, performs a $Y$-motion on $D$. This particle is killed with spatially dependent rate $\beta$. At the moment and spatial position of its death, the particle produces precisely two offspring. Each of these two individuals proceed independently to perform $Y$-motions killed at rate $\beta$ at which point they reproduce in the same way as their parent and so on. At each time $t > 0$ the branching diffusion consists of a point process $Z_t$ defined on Borel sets with almost surely finite total mass $Z_t(D)$.

In the sequel, the notation $P_x$, $E_x$ and $Z$ will be used for the branching diffusion and the notation $\mathbb{P}_x$, $\mathbb{E}_x$ and $Y$ will be used for the diffusion on $D$ corresponding to $L$. Moreover, we shall use the *Ulam-Harris labelling notation.* That is an individual $u$ is identified by its line of decent from the initial ancestor. More precisely, if $u = (i_1, ..., i_{n-1}, i_n)$ then she is the $i_n$th child of the $i_{n-1}$th child of ....of the $i_1$th child of the initial ancestor. Thus $uv$ refers to the individual who, from $u$'s perspective, has line of descent expressed as $v$. Further, the length $|u|$ is equal to the generation in which individual $u$ lives. We shall use $N_t$ to denote the set of individuals alive at time $t$ and $\{Y_u(t) : u \in N_t\}$ for their positions in $D$. In this way we have for example $Z_t(A) = \text{card}\{u \in N_t : Y_u(t) \in A\}$ where $A$ is any Borel set.

A whole array of questions can be asked about the large time behavior of the process $Z = \{Z_t(\cdot)\}_{t\geq 0}$. Basic questions which address the concepts of 'local extinction' and 'recurrence' (however undefined as yet) focus on whether this process will visit nonempty open sets infinitely often and if so, how can this be quantified. Surprisingly there are few results in this direction in existing literature. In the late fifties and sixties, there were a small cluster of papers which considered simple properties of either general or specific examples of branching diffusions. Specifically we speak of the (former) Soviet and Japanese contributions of Sevast'yanov (1958), Skorohod (1964), Watanabe (1965, 1967) and Ikeda *et al.* (1968a,b 1969).

Some results can be found amongst these references pertaining to the kind of problems we have alluded to above, in particular Watanabe (1967). In more recent times, the number of articles concerning growth and spread of branching diffusions are again largely restricted to special cases; for example branching random walks and branching Brownian motions or simple variations thereof. Notably Biggins (1979, 1992) has produced local limit theorems analogous to the one in Watanabe (1967) which demonstrate that numbers of particles in any Borel set can be appropriately rescaled over time by their average to achieve a 'Law of Large Numbers' type result. Ogura (1983) also showed for branching diffusions where the motion process is a Brownian motion with (a restricted class of) space dependent drift that in a given compact set, the number of particles will become zero and remain zero or a strong law of large numbers can be produced for the number of particles in that set.

By comparison, the recent developments and popularity of measure-valued diffusions (superdiffusions) have given a more thorough treatment of analogous issues concerning concentration and migration of mass. It is almost impossible to give a full account of books and papers on measure-valued diffusions. We therefore restrict ourselves here by mentioning the two basic textbooks Dawson (1993) and Dynkin (1994) and the recent monograph Etheridge (2000) on measure-valued processes in general, and the articles Delmas (1999), Engländer and Pinsky (1999), Iscoe (1988), Pinsky (1995b), Pinsky (1996) and Tribe (1994) regarding the long term behaviour of these processes (when then underlying motion process is a diffusion) in particular.

In this article we have three main goals. The first is to formalize the relationship between the operators $L+\beta$ and $L+\beta\psi$ (where $\psi(x) = x^2 - x$) and two classes of martingales related to the Markov branching diffusion. Secondly to introduce the concept of 'spine' decomposition for Markov branching diffusions. Thirdly we aim to show the functionality of these two by applying them to the fundamental question of *local extinction* versus *recurrence*.

## 1.2 Inner-product and multiplicative martingales

In the mid nineties the article of Lyons et al. (1995) appeared which formalized a new approach to analyzing some fundamental problems in the Galton-Watson process. It was shown that a classical martingale, that has been a popular object of study in earlier years (numbers alive in the $n$-th generation divided by their expectation), can serve as a Radon-Nikodym change of measure on the space of Galton-Watson branching trees. The effect of this change of probability is to pick out a randomized line of decent, called the *spine*, and to size-bias offspring distributions associated with each node along the spine. In the case of Galton-Watson processes, the spine dominates the behavior of the process under the new measure. That is to the extent that simple and intuitively appealing probabilistic proofs of classical theorems can be achieved where the original proofs were more analytical and complicated. In fact, this phenomenon is not particular to Galton-Watson processes. A number of authors have shown that similar constructions can be produced for a variety of non-spatial and discrete time spatial branching processes; see Lyons (1997), Kurtz *et al.* (1997), Olofsson (1998), Athreya (2000), Kyprianou and Rahimsadeh Sani (2001), Biggins and Kyprianou (2001). In each of these cases a naturally occurring martingale is used as the appropriate change of measure. The 'natural occurrence' of these martingales is a consequence of the fact that they are constructed from positive harmonic functions. Take for example the case of a discrete time typed branching process. Roughly speaking for this case, the afore mentioned martingales are of the form $\lambda^{-n}\sum_i h(X_i)$ where the sum is typically taken over individuals alive in the $n$-th generation, $X_i$ is the a type of individual $i$ (which could be for example a birth position) and $h$ is an eigenfunction with respect to the expectation operator with eigenvalue $\lambda$. The last of these properties can be written, using obvious notation, $E_x[\sum_i h(X_i)] = \lambda h(x)$ where $\lambda$ is the eigenvalue. See Athreya and Ney (1972, Chapter VI.4), Athreya (2000) and Biggins and Kyprianou (2001).

In Section 2 we show that the idea of martingales from harmonic functions transfers comfortably into the context of Markov branching diffusions. In this case, the analogous class of martingales takes the form $\exp\{-\lambda t\}\int h(y)\eta_t(dy)$ where $\eta$ is an appropriately evolving point process embedded within the Markov branching process and $h$ is positive and harmonic with respect to $L+\beta-\lambda$ for some appropriate $\lambda \in \mathbb{R}$. These martingales we refer to as innerproduct martingales (for the case of branching Brownian motion, they have also been referred to in the past as 'additive' martingales).

In Section 3 we follow the trend of the previously mentioned literature and show that we can use a specific class of these martingales to serve as changes of measure for the branching diffusion. The effect of the change of measure is to perform a Doob's $h$-transform on the diffusion along a randomized ancestral line of descent, the spine, whilst doubling the rate of fission along this path. It will turn out that this fact is a direct consequence of Girsanov's theorem for diffusions and Poisson processes. It is worth remarking that in the context of superprocesses, there exists a path decomposition similar in spirit to the spine construction when one conditions a supercritical superdiffusion to survive. In this case, the conditioned process can be recovered by taking a particle Markov branching diffusion as a 'backbone' process along which there is a continuum of immigration at each space-time point according to the superdiffusion conditioned on extinction; and finally add a version of this process with a random number of initial particles to the process conditioned on extinction. This is also referred to as the 'immortal particle picture' (see Etheridge (2000), Evans and O'Connell (1994) and Engländer and Pinsky (1999) for an overview). It is anticipated that there is a link between these backbone processes and the spines that we define here. We hope to offer in future work some insight into their true relationship.

There exists another class of martingales in the context of Markov branching diffusions. These are closely related to the non-linear operator $L + \beta\psi$ (recall that $\psi(x) = x^2 - x$). The works of Ikeda *et al.* (1968a,b and 1968), Skorohod (1964) and McKean (1975) all demonstrate clear links between multiplicative martingales and positive bounded solutions to parabolic differential equations of the form $u_t = Lu + \beta(u^2 - u)$. Later, other authors such as Neveu (1988), Champneys *et al.* (1995), Harris (1999) and Kyprianou (2001) used these facts to work with these multiplicative martingales as tools in the context of (typed) branching Brownian motion. Of particular note is their use to date in analyzing travelling wave solutions to the Kolmogorov-Petrovskii-Piskunov (K-P-P) equation or variants of it. In this article, we too shall need multiplicative martingales as tools in our proofs. For this reason, we spend a little time in Section 4 to discuss their relationship with the semi-linear operator $L + \beta\psi$. It will turn out that when evaluating the probability of local extinction in certain circumstances (discussed in the following section) we are essentially considering whether there exist travelling waves solutions to a generalized version of the K-P-P equation.

### 1.3 Local extinction versus local exponential growth; recurrence

The issue of local extinction can be understood in the following context. Given any nonempty open set $B \subset\subset D$ (the notation $B \subset\subset D$ means that the closure of $B$ is a bounded subset of $D$) what are the necessary and sufficient conditions for this set to be visited for arbitrarily large times with positive probability? It turns out that the answer to this question boils down to a simple dichotomy concerning a spectral condition on the linear operator $L + \beta$. That is to say, whether its generalized principal eigenvalue (defined shortly) is positive or not. This fact reflects a similar scenario that has been obtained for superdiffusions in Pinsky (1996) and Engländer and Pinsky (1999). We are also able to provide some weak results pertaining to characterizing the growth in the number of individuals in the given set $B$ when it *is* visited for arbitrarily large times with positive probability. In section 7 we discuss several concrete examples of branching diffusions where the issues surrounding the dichotomy we demonstrate can be clearly seen.

To formulate the question of local extinction more precisely we make the following definition.

**Definition 1 (local extinction)** Fix an $x \in D$. We say that $Z$ under $P_x$ *exhibits local extinction* if for every Borel set $B \subset\subset D$, there exists a random time $\tau_B$ such that

$$P_x(\tau_B < \infty) = 1 \text{ and } P_x(Z_t(B) = 0 \text{ for all } t \geq \tau_B) = 1.$$

**Remark 2** Since $Z$ is a discrete particle system, the above definition of local extinction is tantamount to

$$P_x\left(\lim_{t\uparrow\infty} Z_t(B)=0\right)=1.$$

In the sequel we will use the following notation. We write $C^{2,\alpha}(D)$ to denote the space of twice continuously differentiable functions with all their second order derivatives belonging to $C^{\alpha}(D)$. A $C^{2,\alpha}$-boundary is defined with the help of $C^{2,\alpha}$-maps in the usual way.

Let

$$\lambda_c=\lambda_c(L+\beta,D):=\inf\{\lambda\in\mathbb{R}\ :\ \exists u>0 \text{ satisfying } (L+\beta-\lambda)u=0 \text{ in } D\}$$

denote the *generalized principal eigenvalue* for $L+\beta$ on $D$ (see section 4.4 in Pinsky (1995a) or Appendix A for further elaboration). From a probabilistic point of view, the generalized principle eigenvalue can be equivalently expressed as

$$\lambda_c=\sup_{\{A:\ A\subset\subset D,\ \partial A \text{ is } C^{2,\alpha}\}}\lim_{t\uparrow\infty}\frac{1}{t}\log E_x\left(\exp\left(\int_0^t\beta(Y(s))\,ds\right);\tau^A>t\right),$$

for any $x\in D$, where $\tau^A=\inf\{t\geq 0: Y(t)\notin A\}$. From the above probabilistic representation of $\lambda_c$ it is clear that $\lambda_c<\infty$ since $\beta$ is bounded from above. It is standard theory (see Appendix A) that for any $\lambda\geq\lambda_c$, there exist a function $0<\phi\in C^{2,\alpha}(D)$ such that $(L+\beta)\phi=\lambda\phi$ on $D$.

The main results concerning local extinction/exponential growth are as follows.

**Theorem 3 (local extinction versus local exponential growth)**

*(i) For any Borel set $B\subset\subset D$ and $x\in D$,*

$$P_x\left(\limsup_{t\uparrow\infty} Z_t(B)=0 \text{ or } \infty\right)=1.$$

*(ii) Fix $x\in D$. The branching diffusion $Z$ under $P_x$ exhibits local extinction if and only if there exists a function $h>0$ satisfying $(L+\beta)h=0$ on $D$, that is, if and only if $\lambda_c\leq 0$.*

*(iii) When $\lambda_c>0$, there exists a function $\rho$ in $[0,1)$, such that*

$$P_x\left(\lim_{t\uparrow\infty} Z_t(B)=0\right)=\rho(x) \text{ for all nonempty open } B\subset\subset D, \tag{2}$$

*and furthermore, $\rho$ solves $L\rho+\beta(\rho^2-\rho)=0$ on $D$.*

*(iv) When $\lambda_c > 0$, $Z$ exhibits local exponential growth: for any $\lambda < \lambda_c$ there exists a (large enough) $B_\lambda \subset\subset D$ such that for all $x \in D$,*

$$P_x\left(\lim_{t\uparrow\infty} e^{-\lambda t} Z_t(B_\lambda) = \infty\right) > 0.$$

The next corollary follows from part (ii).

**Corollary 4** *The local extinction property does not depend on $x \in D$; that is, either $Z$ under $P_x$ exhibits local extinction for all $x \in D$ or it does not exhibit local extinction for any $x \in D$.*

The reader will note that once the martingale tools are in place, the proof is reasonably straightforward and does not require a great deal of intricacy thus motivating the martingale theory presented in sections 2, 3 and 4. The elementary nature of the arguments can also be seen when comparing our method with the techniques employed by Ogura (1983) to deduce similar conclusions for a much less general class of branching diffusions.

**Remark 5 (total mass)** In Theorem 3 we were concerned about the *local* behaviour of the population size. When considering the *total mass* process $\|Z\| := \langle 1, Z\rangle$, it is easy to see that the growth rate may actually exceed $\lambda_c$. Indeed, take for example a (transient) diffusion corresponding to $L$ on $D$ with $\lambda_0 := \lambda_c(L, D) < 0$ and let $\beta > 0$ be constant. Then $\lambda_c(L+\beta, D) = \beta + \lambda_0 < \beta$, but – since the branching rate is spatially constant – a classical theorem on Yule's processes tells us that $e^{-\beta t}\|Z_t\|$ tends to a nontrivial random variable as $t \to \infty$, that is, that the growth rate of the total mass is $\beta > \lambda_c$.

It is not clear when the function $\rho$ in Theorem 3 is equal trivial solution 0 and when it is otherwise a non-trivial solution to the equation $Lu + \beta(u^2 - u) = 0$ on $D$. If there is no 'non-trivial' solution to the semi-linear elliptic equation, we obtain a 'zero-one law' concerning the probability that a nonempty set $B \subset\subset D$ becomes eventually vacant.

**Corollary 6 (local extinction versus recurrence)** *Assume that the equation $Lu + \beta(u^2 - u) = 0$ has no solution in $[0, 1]$ except the trivial ones $u = 0$ and $u = 1$. Then either*

$$\limsup_{t\uparrow\infty} Z_t(B) = \infty,\ P_x\text{-a.s. for all } x \in D, \text{ and nonempty open } B \subseteq D,$$

*or*

$$\lim_{t\uparrow\infty} Z_t(B) = 0,\ P_x\text{-a.s. for all } x \in D,\ B \subset\subset D$$

*according to whether $\lambda_c > 0$ or $\lambda_c \le 0$.*

An active example of this Corollary concerning branching Brownian motion will be shown in Section 7. In that case, $Lu+\beta(u^2-u)=0$ takes the form of the travelling wave equation to the K-P-P equation; $(1/2)u''+cu'+\beta(u^2-u)=0$ where $c$ is the wave speed and $\beta$ is a constant.

One can think of $Lu+\beta(u^2-u)=0$ as a generalization of the travelling wave equation associated with the K-P-P equation. Essentially then the question of the uniqueness and non-triviality of $\rho$ would seem to be questions about existence and uniqueness of 'travelling waves'. This provides then another motivation for this work. It emphasizes probabilistic interpretations of travelling waves. In the future we hope to offer further insight into these matters.

Pinsky (1996) and Engländer and Pinsky (1999) consider similar questions for another type of spatial branching process, the superdiffusion corresponding to the semilinear operator $Lu+\beta u-\alpha u^2$ where $\alpha$ and $\beta$ are related to the variance of the offspring distribution and to the 'mass creation', respectively. Their conclusions are proved by considering the relationship between the superdiffusion and solutions to the parabolic partial differential equation $\partial u/\partial t=Lu+\beta u-\alpha u^2$ with appropriate initial and boundary conditions. The behaviour of solutions of this class of parabolic equations together with an understanding of how to express the behaviour of the superdiffusion in terms of these solutions is fundamental to their methodology. Given that the branching diffusion we have described here is associated with the semilinear operator $L+\beta\psi$ one might argue that with some mild adaptations to the case $\alpha=\beta$, the analytical arguments of Engländer and Pinsky (1999) can be re-employed here. Indeed that is the case and in Section 6 we follow this line although the results obtained are weaker than what the probabilistic techniques deliver. In particular, only part (ii) of Theorem 3 is proved using the analytic approach. This again emphasizes the motivation for pursuing probabilistic techniques.

### 1.4 Outline

The rest of this paper is organized as follows. In the second section we give a number of results concerning inner-product martingales and their connection to certain partial differential equations. This section will later be completed by section four which treats similar questions for multiplicative martingales. Between these two sections, we present a key section (Section 3) which contains the "spine-construction". In section five we utilize the preceding three sections and prove the theorems stated in subsection 1.3. An analytical proof of Theorem 3 part (ii) will be shown in section six. Then, in section seven, we complete the theory with several concrete examples. Finally, Appendix A is intended to make this paper easier to read by giving the necessary background material.

## 2 Natural inner-product martingales

As is well known, the theory of diffusion processes lays down a clear relationship between positive harmonic functions and the existence of certain martingales, leading to the stochastic representations of solutions to certain partial differential equations. For an account of this theory one can consult Karatzas and Shreve (1991) for example.

The story is very similar for branching diffusions and that forms part of the motivation for this section. In what follows we shall show that positive functions that are harmonic with respect to the operator $L + \beta - \lambda$ for $\lambda \in \mathbb{R}$, either on the whole domain $D$ or a compact subdomain such as a ball $B$, are intimately linked to certain martingales which can be written as functionals of the $(L, \beta)$-branching diffusion. The results we shall prove are by no means an exhaustive analogy of what can be proved for diffusions, but they suffice for our purposes. The connections we present are not exclusively new. It is clear that other authors have pursuing these connections in the past, see for example Watanabe (1967), Ikeda *et al.* (1968a,b, 1969), Champneys *et al.* (1995) and Harris (1999). Our exposition, if not providing more general results within this context, uses more elementary, probabilisitic proofs. In particular, we will not (unlike in section six) rely on the theory of evolution equations and their connection to the Laplace-functional of $Z$, but rather on arguments only using standard branching decomposition together with Itô's formula for the single $L$-particle. Thus, on bounded domains, for example, one does not have to worry about the appropriate boundary condition for the semigroup, nor about the question, why the solution to the integral equation becomes actually a classical solution.

Before continuing with the results we need to introduce some notation. Let $\mathcal{F}_t$ be the natural filtration generated by $Z_t$. For any Borel set $B \subset\subset D$, let us denote by $\widehat{Z} = \left\{\widehat{Z}_t : t \geq 0\right\}$ the process corresponding to the branching diffusion $Z$ where particles are instantly annihilated when they meet $\partial B$. We let $N_t^B$ consist of those individuals who are first in their line of descent to hit $\partial B$. Recall the stopping time $\tau^B = \inf\{t \geq 0 : Y(t) \notin B\}$. We would like to define similar objects for each $u \in N_t^B$. Henceforth for such a $u$, we denote the finite hitting time of $\partial B$ by $\tau_u^B$. For technical reasons, we also need to define $\tau_u^B$ for $u \notin N_t^B$; for such an individual, $\tau_u^B := \infty$. Finally for each individual $u$, let $\sigma_u$ be their time of death and note that in their particular ancestral line of descent, if $|u| = n$ then $\sigma_u$ is the time of the $n$-th arrival in a Poisson process with inhomogenous intensity $\beta(Y(t))$.

**Theorem 7 (Local inner-product martingales)** *Consider a ball $B \subset\subset D$ (or indeed any other compact set with a $C^{2,\alpha}$-boundary). Let $h \in C^{2,\alpha}(\overline{B})$, $h > 0$ on $B$ and $\lambda \in \mathbb{R}$. Then*

$$W_t^h(B) = \frac{\left\langle h, \widehat{Z}_t \right\rangle}{h(x)} e^{-\lambda t} + \sum_{u \in N_t^B} \frac{h\left(Y_u\left(\tau_u^B\right)\right)}{h(x)} e^{-\lambda \tau_u^B}$$

*is a $P_x$-martingale for all $x \in B$ with respect to $\mathcal{F}_t$ if and only if*

$$(L + \beta - \lambda) h = 0 \text{ in } B.$$

**Proof.** We begin by assuming that $W_t^h(B)$ is a martingale so that necessarily $E_x\left(W_t^h(B)\right) = 1$ for all $x \in \overline{B}$ and $t \geq 0$. By conditioning on the first fission point we have for all $x \in B$

$$\begin{aligned} h(x) &= \mathbb{E}_x\left(h\left(Y\left(t \wedge \tau^B\right)\right) e^{-\int_0^{t\wedge\tau^B} \lambda+\beta(Y(s))ds}\right) \\ &\quad +\mathbb{E}_x\left(\int_0^{t\wedge\tau^B} 2\beta\left(Y(s)\right) e^{-\int_0^s \lambda+\beta(Y(r))dr} h\left(Y(s)\right) ds\right). \end{aligned} \tag{3}$$

It follows that

$$\begin{aligned} M_t := &\quad \exp\left(-\int_0^{t\wedge\tau^B} \lambda + \beta\left(Y(s)\right) ds\right) h\left(Y\left(t \wedge \tau^B\right)\right) \\ &+ \int_0^{t\wedge\tau^B} 2\beta\left(Y(s)\right) e^{-\int_0^s \lambda+\beta(Y(r))dr} h\left(Y(s)\right) ds \end{aligned} \tag{4}$$

is a martingale. If $h \in C^2(B)$, an application of Itô's formula shows that when $M_t$ is written as an Itô diffusion, as the drift component is necessarily zero, we are forced to have $(L - \beta - \lambda)h = -2\beta h$ on $B$. That is $(L + \beta - \lambda)h = 0$ on $B$, finishing the proof in one direction.

For the other direction, let us assume that $(L + \beta - \lambda)h = 0$ on $B$ (and hence necessarily $h \in C^2(B)$). Suppose it can be proved that $E_x W_t^h(B) = 1$ for all $t \geq 0$ and $x \in B$. Then the result follows from the decomposition

$$\begin{aligned} W_{t+s}^h(B) &= \sum_{u\in\widehat{N}_t} e^{-\lambda t}\frac{\left\langle h, \widehat{Z}_s(u)\right\rangle}{h(x)} + \sum_{u\in N_t^B} \frac{h\left(Y_u\left(\tau_u^B\right)\right)}{h(x)} e^{-\lambda\tau_u^B} \\ &\quad + \sum_{u\in\widehat{N}_t} e^{-\lambda t} \sum_{uv\in N_t^B} \frac{h\left(Y_{uv}\left(\tau_{uv}^B\right)\right)}{h(x)} e^{-\lambda\left(\tau_{uv}^B - t\right)} \\ &= \sum_{u\in\widehat{N}_t} e^{-\lambda t} W_t^h(B,u) + \sum_{u\in N_t^B} \frac{h\left(Y_u\left(\tau_u^B\right)\right)}{h(x)} e^{-\lambda\tau_u^B} \end{aligned}$$

where given $\mathcal{F}_t$, $\widehat{Z}_s(u)$ and $W_t^h(B,u)$ are independent copies of $\widehat{Z}_s$ and $W_t^h(B)$ under the law $P_{Y_u(t)}$ respectively.

It remains then to prove then that $E_x W_t^h(B) = 1$ for all $t \geq 0$ and $x \in B$. First some new notation. Let $\mathcal{L}$ to be those individuals who, during their life time, cross the boundary $\partial B$ before time $t$ for the first time their ancestral history *or* are alive at time $t$ without having an ancestor (including themselves) who has met the boundary $\partial B$. In the terminology of Chauvin (1991), this is a *stopping line.* Let $\mathcal{A}_{\mathcal{L}}(n)$ be those individuals in the $n$-th generation who are neither in $\mathcal{L}$, nor have an ancestor in $\mathcal{L}$. Now define

$$W_t^h(B,n) = \sum_{\substack{u\in\mathcal{L} \\ |u|\leq n}} e^{-\lambda\left(t\wedge\tau_u^B\right)}\frac{h\left(Y_u\left(t\wedge\tau_u^B\right)\right)}{h(x)} + \sum_{u\in\mathcal{A}_{\mathcal{L}}(n)} 2e^{-\lambda\sigma_u}\frac{h\left(Y_u\left(\sigma_u\right)\right)}{h(x)}.$$

We shall now show that this is a mean one martingale with respect to $\mathcal{G}_n$, the natural $\sigma$-algebra generated by the complete life all individuals up to and including the $n$-th generation. Note that for individuals $u$ in the $n$-th generation or less, $\sigma_u$, $\tau_u^B$ and membership of $\mathcal{L}$ are $\mathcal{G}_n$-measurable.

Using the assumption on $h$ and Itô's formula (recall that $h$ has bounded derivatives on $\overline{B}$) , it follows that (4) is a martinagle and thus we have a stochastic representation for $h$ given by (3). This representation can otherwise be written

$$\begin{aligned} h(x) &= \mathbb{E}_x\left(\mathbf{1}_{(\nu>t\wedge\tau^B)}e^{-\lambda\left(t\wedge\tau^B\right)}h\left(Y\left(t\wedge\tau^B\right)\right)\right.\\ &\qquad\left.+\mathbf{1}_{(\nu\le t\wedge\tau^B)}2e^{-\lambda\nu}\beta\left(Y\left(\nu\right)\right)h\left(Y\left(\nu\right)\right)\right) \end{aligned} \tag{5}$$

for all $t\ge 0$ where $\nu$ is the first fission time in our branching process. Now consider the decomposition

$$\begin{aligned} W_t^h(B,n+1) &= \sum_{\substack{u\in\mathcal{L}\\|u|\le n}} e^{-\lambda\left(t\wedge\tau_u^B\right)}\frac{h\left(Y_u\left(t\wedge\tau_u^B\right)\right)}{h(x)}\\ &\quad+\sum_{u\in\mathcal{A}_{\mathcal{L}}(n)}\sum_{i=1}^{2}\left\{\frac{h\left(Y_ui\left(t\wedge\tau_{ui}^B\right)\right)}{h(x)}e^{-\lambda\left(t\wedge\tau_{ui}^B\right)}\mathbf{1}_{\left(\sigma_{ui}>t\wedge\tau_{ui}^B\right)}\right.\\ &\qquad\qquad\left.+2e^{-\lambda\sigma_{ui}}\frac{h\left(Y_ui\left(\sigma_{ui}\right)\right)}{h(x)}\mathbf{1}_{\left(\sigma_{ui}\le t\wedge\tau_{ui}^B\right)}\right\}. \end{aligned}$$

By taking conditional expectations in this decomposition with respect to $\mathcal{G}_n$ and applying (5) the martingale property is proved.

Now note, using (5) again, that

$$\begin{aligned} E_x\left(W_t^h(B,n)\right) &= E_x\left(W_t^h(B,0)\right)\\ &= \mathbb{E}_x\left(\mathbf{1}_{(\nu>t\wedge\tau^B)}e^{-\lambda\left(t\wedge\tau^B\right)}\frac{h\left(Y\left(t\wedge\tau^B\right)\right)}{h(x)}\right.\\ &\qquad\left.+\mathbf{1}_{(\nu\le t\wedge\tau^B)}2e^{-\lambda\nu}\frac{\beta\left(Y\left(\nu\right)\right)h\left(Y\left(\nu\right)\right)}{h(x)}\right)\\ &= 1. \end{aligned}$$

The branching process does not explode in a finite time, and hence $\mathcal{A}_{\mathcal{L}}(n)\to\emptyset$ almost surely. This together with monotonicity implies

$$\begin{aligned} 1 &= \lim_{n\uparrow\infty}E_x\left(W_t^h(B,n)\right)\\ &= E_x\left(W_t^h(B)\right)+\lim_{n\uparrow\infty}E_x\left(\sum_{u\in\mathcal{A}_{\mathcal{L}}(n)}2e^{-\lambda\sigma_u}\frac{h\left(Y_u\left(\sigma_u\right)\right)}{h(x)}\right). \end{aligned}$$

We want to prove that the limit on the right hand side is zero. To this end note that for $u\in\mathcal{A}_{\mathcal{L}}(n)$, $\sigma_u\in[0,t]$, $h$ is bounded on $B$, $\mathcal{A}_{\mathcal{L}}(n)\subseteq\{|u|=n:\sigma_u<t\}$,

and the $n$-th generation contains $2^n$ members. We thus have the upper bound

$$E_x\left(\sum_{u\in\mathcal{A}_{\mathcal{L}}(k)} 2e^{-\lambda\sigma_u}\frac{h\left(Y_u\left(\sigma_u\right)\right)}{h\left(x\right)}\right)\leq Ke^{|\lambda|t}2^k P_x\left(\nu_k<t\right)$$

where $\nu_k$ is the time of the $k$-th arrival in the Poisson process $\{n_t : t\geq 0\}$ whose intensity at time $t$ conditional on $Y$ is $\beta\left(Y\left(t\right)\right)$. Since $\{n_t\geq k\}$ is equivalent to $\{\nu_k<t\}$ it follows that

$$\begin{aligned}\sum_{k\geq 0}E_x\left(\sum_{u\in\mathcal{A}_{\mathcal{L}}(k)} 2e^{-\lambda\sigma_u}\frac{h\left(Y_u\left(\sigma_u\right)\right)}{h\left(x\right)}\right) &\leq Ke^{|\lambda|t}\sum_{k\geq 0}\mathbb{E}_x\left(\frac{\left(2B_t\right)^k}{k!}e^{-B_t}\right)\\ &= Ke^{|\lambda|t}\mathbb{E}_x\left(e^{B_t}\right)\\ &\leq Ke^{\left(|\lambda|+\sup_{x\in B}\beta(x)\right)t}.\end{aligned}$$

where $B_t=\exp\left\{\int_0^t\beta\left(Y\left(s\right)\right)ds\right\}$. The finiteness of this sum implies necessarily that

$$\lim_{n\uparrow\infty}\mathbb{E}_x\left(\sum_{u\in\mathcal{A}_{\mathcal{L}}(n)} 2e^{-\lambda\sigma_u}\frac{h\left(Y\left(\sigma_u\right)\right)}{h\left(x\right)}\right)=0$$

hence $E_x\left(W_t^h\left(B\right)\right)=1$ and the proof of the Theorem is concluded. ■

The following result is an immediate consequence of the previous theorem. Denote by $\lambda$ the principal eigenvalue of $L+\beta$ on $B$ (that is the supremum of the real part of the spectrum). Recall that (see for example Theorem 3.3.1 in Pinsky (1995a)), the corresponding Dirichlet-eigenfunction $\phi$ belongs to $C^{2,\alpha}(\overline{B})$, and $\phi>0$ on $B$ (while $\phi=0$ on $\partial B$).

**Corollary 8 (Dirichlet inner product-martingales)** *The process $M^\phi$ defined by*

$$M_t^\phi=\sum_{u\in\widehat{N}_t}e^{-\lambda t}\frac{\phi\left(Y_u\left(t\right)\right)}{\phi\left(x\right)}=e^{-\lambda t}\frac{\langle\phi,Z_t\rangle}{\phi\left(x\right)}$$

*is a $P_x$-martingale with respect to the filtration generated by the branching process for all $x\in B$.*

Here is another version of Theorem 7 but with a little weaker assumption on $h$. In fact we will not need this version in the rest of the proofs but it shows there are much deeper connections with Dirichlet problems for diffusions.

**Theorem 9** *Let $B\subset\subset D$ be a ball, $h:\overline{B}\to(0,\infty)$ a $C\left(\overline{B}\right)$-function and $\lambda\geq 0$. Then*

$$W_t^h\left(B\right)=\frac{\left\langle h,\widehat{Z}_t\right\rangle}{h\left(x\right)}e^{-\lambda t}+\sum_{u\in N_t^B}\frac{h\left(Y_u\left(\tau_u^B\right)\right)}{h\left(x\right)}e^{-\lambda\tau_u^B}$$

*is a $P_x$-martingale with respect to the filtration generated by the branching process for all $x \in B$ if and only if*

$$(L + \beta - \lambda)\, h = 0 \text{ in } B.$$

**Proof.** Suppose that $W_t^h(B)$ is a martingale. Then necessarily we have (3). Indeed by bounded convergence ($\lambda \geq 0$) we can write

$$\begin{aligned} h(x) &= \mathbb{E}_x\left(h\left(Y\left(\tau^B\right)\right) e^{-\int_0^{\tau^B} \lambda + \beta(Y(s))ds}\right) \\ &\quad + \mathbb{E}_x\left(\int_0^{\tau^B} 2\beta\left(Y(s)\right) e^{-\int_0^s \lambda+\beta(Y(r))dr} h\left(Y(s)\right) ds\right). \end{aligned} \tag{6}$$

The unique solution to the Dirichlet problem (c.f. Karatzas and Shreve (1991))

$$\begin{aligned} (L - (\beta + \lambda))\, u &= -2\beta h \text{ in } B \\ u &= h \text{ on } \partial B. \end{aligned}$$

has stochastic representation equal to the right hand side of the above stochastic representation of $h$ and hence $u = h$. That is, $(L + \beta - \lambda)\, h = 0$ in $B$. Note that we used $\lambda \geq 0$ to guarantee the potential is positive and that $h$ is continuous on $\overline{B}$. These conditions together are sufficient to guarantee that the above Dirichlet problem has a unique solution with a stochastic representation.

Now suppose that $(L + \beta - \lambda)\, h = 0$ in $B$. The function $h$ can be written as a solution to the above Dirichlet problem. By uniqueness and the stochastic representation of the solution this implies that (6) and hence (3) or equivalently (5) holds. The proof is completed in the same way as the previous Theorem. ∎

For both the Theorems above we have used a technique of approximating the expectation of our candidate martingale via a generational decomposition. This technique is essentially based on a method used by Chauvin (1991) who used it to show the existence of multiplicative martingales for branching Brownian motion. Not surprisingly this method will be used again when we look in more detail at multiplicative martingales.

The martingales we have considered so far are 'local' in the sense that they concern the branching diffusion up to containment in a bounded domain $B$. Once this containment is removed, it is not necessarily true that we can make discounted inner products which function as martingales. This reflects a similar situation for diffusions. We finish this section with the global version of the previous results.

**Corollary 10 (Inner-product (super)martingale)** *Suppose that $0 < h \in C^{2,\alpha}(D)$ solves the elliptic equation $(L + \beta - \lambda)\, h = 0$ on $D$. Then*

$$W_t^h = e^{-\lambda t} \frac{\langle h, Z_t \rangle}{h(x)}$$

*is a (nonnegative, right-continuous) $P_x$-supermartingale for all $x \in D$, having an almost sure limit as $t \uparrow \infty$. Conversely if $W_t^h$ is a martingale then $(L + \beta - \lambda)\, h = 0$ on $D$.*

**Proof.** By taking an increasing sequence of balls $B_n$ such that $D \subseteq \bigcup_n B_n$ we produce a sequence of equalities in $n$ of the form

$$E_x\left(W_t^h\left(B_n\right)\right) = 1 \text{ for all } x \in B_n, t \geq 0.$$

(Note that obviously $h \in C^{2,\alpha}(\overline{B_n})$ for all $n$.) Fatou's Lemma together with conservativeness of the underlying diffusion $Y$ implies that for all $x \in D$ (starting the limit from sufficiently large $n$)

$$1 = \lim_{n\uparrow\infty} E_x\left(W_t^h\left(B_n\right)\right) \geq E_x\left(\lim_{n\uparrow\infty} W_t^h\left(B_n\right)\right) = E_x\left(W_t^h\right).$$

The supermartingale property follows from the decomposition

$$W_{t+s}^h = \sum_{u\in N_t} e^{-\lambda t}\frac{h\left(Y_u\left(t\right)\right)}{h\left(x\right)} W_s^h\left(u\right)$$

where given $\mathcal{F}_t$, $W_s^h(u)$ are independent copies of $W_s^h$ under the law $P_{Y_u(t)}$.

For the converse, use conditioning on the first fission time to produce another martingale similar to (4) with $\tau^B \wedge t$ replaced by $t$, then apply Itô's formula as before. ■

**Remark 11** When considering the proof of Theorem 7 one can adopt the methodology in one of the directions there to prove that if $(L+\beta-\lambda)h = 0$ in $D$ then $W_t^h$ is a martingale, providing certain growth conditions hold on $h$. We do not specify general growth conditions, but we will use this fact in the first and last examples of Section 7

# 3 Girsanov-type theorems, $h$-transforms and spines for branching diffusions

The aim here is to give a construction of a change of measure with respect to $P_x$ which has the effect of identifying a randomized distinguished line of descent, the spine, and adjusting the diffusion and rate of reproduction along that line of decent according to a classical Doob's $h$ -transform. This change of measure, discussed in Subsection 3.2, is defined by the Dirichlet innerproduct martingales discussed in Section 2. The structure of these martingales can be decomposed into traditional Girsanov densities for diffusions and jump processes thus rendering them contemporary with the ubiquitous Girsanov Theorem.

We remind the reader that whilst the results in this section are new, we are not necessarily introducing new technology. We will show here how the fundamental concepts behind Lyons et al (1995) concerning size basing and spine decompositions translate and generalize to the Markov branching diffusion setting.

Let $B$ be a bounded domain with $C^{2,\alpha}$-boundary . Let $\lambda = \lambda_c(L+\beta, B)$ and $\phi$ be as in Corollary 8. Section 4.7 of Pinsky (1995a) concludes that $L+\beta-\lambda$ is a *critical* operator. Let $\widetilde{\phi}$ be the eigenfunction corresponding to the formal adjoint of $L+\beta-\lambda_c$. Since $\phi$ and $\widetilde{\phi}$ are bounded, we have $\int_B \phi(x)\widetilde{\phi}(x)dx < \infty$ and thus by definition, $L+\beta-\lambda$ is a product-critical operator (see the Appendix). Note also that on account of invariance properties under $h$-transforms, the operator

$$(L+\beta-\lambda)^{\phi} = L + a\frac{\nabla\phi}{\phi}\cdot\nabla$$

is also product-critical, which means that the corresponding diffusion is positive recurrent; thus ergodizing with full support on the interior of the domain $B$.

### 3.1 Girsanov theorems for diffusions and Poisson processes

Let $P_x$ be the law of the diffusion $Y$ corresponding to the elliptic operator $L$ on $D$. There are two important changes of measure associated with these processes which we shall now discuss.

Firstly, we quote a special version of Girsanov's theorem for diffusions which concerns Doob's $h$-transforms. Before stating this result we shall define $\{\mathcal{G}_t\}_{t\geq 0}$ a filtration with respect to which the process $Y$ is adapted.

**Proposition 12 (Girsanov's theorem for diffusions)** *Suppose that $B, \phi, \widetilde{\phi}$ and $\lambda$ compose the Dirichlet set-up as above then there exists a probability measure ${\mathbb{P}_x}^{\phi}$ defined by*

$$\left.\frac{d\mathbb{P}_x^{\phi}}{d\mathbb{P}_x}\right|_{\mathcal{G}_t}(Y) = \mathbf{1}_{(t<\tau^B)}\frac{\phi\left(Y\left(t\right)\right)}{\phi\left(x\right)}\exp\left\{-\int_0^t \lambda-\beta\left(Y\left(s\right)\right)ds\right\}$$

*where ${\mathbb{P}_x}^{\phi}$ is the law of a diffusion corresponding to the operator*

$$L + a\frac{\nabla\phi}{\phi}\cdot\nabla.$$

*The new diffusion does not hit the boundary and is positive recurrent in $B$.*

The second change of measure that will be of importance is the Girsanov-type change of measure for the Poisson process $n$ (conditioned on the path of $Y$ ). We do not claim the result is new, it is included merely for completeness as it is easy to prove and not necessarily easy to find in the literature. For this we suppose that $\{\mathcal{H}_t\}_{t\geq 0}$ is a filtration with respect to which $n$ is adapted. Further denote by $\mathbb{L}_Y^{\beta}$ the law conditioned on the process $Y$ of a non-homogeneous Poisson process, $n = \{\nu_i : i = 1, ..., n_t\}_{t\geq 0}$, with instantaneous rate $\beta\left(Y\left(t\right)\right)$.

**Proposition 13 (Girsanov-type theorem for Poisson processes)**

$$\left.\frac{d\mathbb{L}_Y^{2\beta}}{d\mathbb{L}_Y^{\beta}}\right|_{\mathcal{H}_t}(n) = 2^{n_t}\exp\left\{-\int_0^t \beta\left(Y(s)\right)ds\right\}.$$

**Proof.** There are a number of ways that this can be proved. The simplest and quickest is to recall that the (non-homogeneous) Poisson process $n_t$, being a submartingale, can be characterized by its compensator (that is the increasing process that appears in its Doob-Meyer decomposition) and therefore through a martingale representation, see for example Kallenberg (1997). In this context that means that $n$ is a $\mathbb{L}_Y^{2\beta}$-Poisson process with instantaneous rate $2\beta\left(Y(t)\right)$ if and only if $\eta_t = n_t - \int_0^t 2\beta\left(Y(s)\right) ds$ is a martingale with respect to $H_t$. On account of the fact that $n$ has independent increments, it suffices to check that $\eta_t$ has $\mathbb{L}_Y^{2\beta}$ -expectation 1 for all $t > 0$. This is a straightforward computation using the above Radon-Nikodym derivative. ■

## 3.2 Spines on bounded domains

**Theorem 14 (Girsanov-type theorem for branching diffusions)** *Assume that $B, \phi, \widetilde{\phi}$ and $\lambda$ compose the Dirichlet set-up as in Proposition 12; then there exists a probability measure $Q_x$ for $Z$ defined by*

$$\left.\frac{dQ_x}{dP_x}\right|_{\mathcal{F}_t} = M_t^{\phi}.$$

*Further, under this change of measure, the branching diffusion has a randomized line of descent, the spine, that diffuses with corresponding operator*

$$L + a\frac{\nabla\phi}{\phi}\cdot\nabla$$

*which does not hit the boundary $\partial B$ and is positive recurrent.*

**Proof.** The branching diffusion $Z$ can be considered to be defined on the space $(\mathcal{T}, \mathcal{F}, P_x)$ where $(\mathcal{T}, \mathcal{F})$ is the appropriate measurable space of marked trees. Note that the marks are points in the path space of of $Y$ and $n$. (See Chauvin (1991) for a rigorous definition). For any $\Upsilon \in \mathcal{T}$ there exist distinguishable genealogical lines of descent from the initial ancestor. In following such a line of descent we identify its spatial path $\xi = \{\xi_t\}_{t\geq 0}$.

Let $\mathcal{T}^*$ be the enriched space of marked trees in $\mathcal{T}$ with distinguished line of descent and $\mathcal{F}^* = \sigma\left(\mathcal{F}, \mathcal{T}^*\right)$ the sigma algebra it generates. Write $\mathcal{T}_t$ for the subspace of marked trees in $\mathcal{T}$ which are truncated (in the obvious way) at time $t$. Let $\mathcal{F}_t^* = \sigma\left(\mathcal{T}_t^*\right)$ where $\mathcal{T}_t^*$ is the space of marked trees in $\mathcal{T}_t$ with distinguished line of descent.

Given any $(\Upsilon, \xi) \in \mathcal{T}_t^*$, the measure $P_x|_{\mathcal{F}_t}$ for each $x \in B$ can be decomposed according to a particular choice of distinguished line of descent giving the path of the spine so that

$$dP_x\left(\Upsilon\right)|_{\mathcal{F}_t} = \sum_{u\in N_t} \mathbf{1}_{\left(\xi_t = Y_u(t)\right)}\, dP_x^*\left(\Upsilon, \xi\right)|_{\mathcal{F}_t^*} \tag{7}$$

where $P_x^*|_{\mathcal{F}_t^*}$ is a non-probability measure satisfying

$$dP_x^*(\Upsilon,\xi)|_{\mathcal{F}_t^*} = d\mathbb{P}_x(\xi)|_{\mathcal{F}_t^*} \times d\mathbb{L}_\xi^\beta(n)|_{\mathcal{F}_t^*} \times \prod_{i=1}^{n_t} dP_{\xi_{\nu_i}}(\Upsilon_i)|_{\mathcal{F}_{t-\nu_i}}$$

and the tree $\Upsilon$ is decomposed into the distinguished line of descent (with path $\xi$) and $\Upsilon_i$, the marked subtrees growing off it.

We can define a bivariate probability measure $Q_x^*|_{\mathcal{F}_t^*}$ on the measurable space $(\mathcal{T}_t^*, \mathcal{F}_t^*)$ for each $x \in B$ such that for $(\Upsilon,\xi) \in T_t^*$,

$$dQ_x^*(\Upsilon,\xi)|_{\mathcal{F}_t^*} = \mathbf{1}_{(\tau^B>t)} \frac{\phi(\xi_t)}{\phi(x)} e^{-\lambda t}\, dP_x^*(\Upsilon,\xi)|_{\mathcal{F}_t^*} \tag{8}$$

where we understand $\tau^B = \inf\{t \geq 0 : \xi_t \notin B\}$. Decomposition (7) enables us to marginalize $Q_x^*|_{\mathcal{F}_t^*}$ to a probability measure on $(\mathcal{T}_t, \mathcal{F}_t)$, say $Q_x|_{\mathcal{F}_t}$, satisfying

$$\begin{aligned}
dQ_x(\Upsilon)|_{\mathcal{F}_t} &= \sum_{u\in N_t} \mathbf{1}_{(\xi_t = Y_u(t))}\, dQ_x^*(\Upsilon,\xi)|_{\mathcal{F}_t^*} \\
&= e^{-\lambda t} \sum_{u\in \widehat{N}_t} \mathbf{1}_{(\xi_t = Y_u(t))} \frac{\phi(\xi_t)}{\phi(x)}\, dP_x^*(\Upsilon,\xi)|_{\mathcal{F}_t^*} \\
&= e^{-\lambda t} \sum_{u\in \widehat{N}_t} \frac{\phi(Y_u(t))}{\phi(x)}\, dP_x(\Upsilon)|_{\mathcal{F}_t} \\
&= M_t^\phi\, dP_x(\Upsilon)|_{\mathcal{F}_t}.
\end{aligned}$$

The effect of the change of this change of measure on the branching process can be seen though (8). Rewrite this identity as

$$\begin{aligned}
dQ_x^*(\Upsilon,\xi)|_{\mathcal{F}_t^*} &= \mathbf{1}_{(\tau^B>t)} \frac{\phi(\xi_t)}{\phi(x)} e^{-\int_0^t \lambda - \beta(\xi_s)ds}\, d\mathbb{P}_x(\xi)|_{\mathcal{F}_t^*} \\
&\quad \times \frac{1}{2^{n_t}} \times 2^{n_t} e^{-\int_0^t \beta(\xi_s)ds} \times d\mathbb{L}_\xi^\beta(n)|_{\mathcal{F}_t^*} \\
&\quad \times \prod_{i=1}^{n_t} dP_{\xi_{\nu_i}}(\Upsilon_i)|_{\mathcal{F}_{t-\nu_i}} \\
&= \frac{1}{2^{n_t}} \times d\mathbb{P}_x^\phi(\xi)\big|_{\mathcal{F}_t^*} \times d\mathbb{L}_\xi^{2\beta}(n)|_{\mathcal{F}_t^*} \\
&\quad \times \prod_{i=1}^{n_t} dP_{\xi_{\nu_i}}(\Upsilon_i)|_{\mathcal{F}_{t-\nu_i}}.
\end{aligned}$$

With Propositions 12 and 13 in mind we see that this decomposition suggests that under $Q_x$ the branching process evolves as follows:

**(i)** a particle moves as a diffusion corresponding to the operator

$$(L+\beta-\lambda)^\phi = L + a\frac{\nabla\phi}{\phi}\cdot\nabla.$$

**(ii)** at rate $2\beta$ this particle undergoes binary fission,

**(iii)** at the instant of fission one of the particles is chosen with probability $1/2$,

**(iv)** the chosen particle repeats steps (i)–(iii) and

**(v)** the particle which is not chosen initiates an $(L,\beta)$ -branching diffusion.

The randomized line of descent we refer to as the spine. The spatial path along the spine corresponds to the operator $(L+\beta-\lambda)^{\phi}$. Indeed for further confirmation, one can perform the following calculation showing the distributions of the spine position and the number of fissions along the spine. Let $A \subseteq D$ be a Borel set, then

$$
\begin{aligned}
& Q_x^*\left(\xi_t \in A, n_t = k\right) \\
&= \int \sum_{u \in N_t} \mathbf{1}_{(Y_u(t)=\xi_t \in A,\ n_t=k)}\ dQ_x^*\left(\Upsilon,\xi\right)|_{\mathcal{F}_t^*} \\
&= \int \sum_{u \in \widehat{N}_t} \mathbf{1}_{(Y_u(t)=\xi_t \in A,\ n_t=k)} \times \frac{1}{2^k} \\
&\qquad \times\ d\mathbb{P}_x^{\phi}(\xi)\big|_{\mathcal{F}_t^*} \times d\mathbb{L}_{\xi}^{2\beta}\left(n\right)|_{\mathcal{F}_t^*} \times \prod_{i=1}^{k} dP_{\xi_{\nu_i}}\left(\Upsilon_i\right)|_{\mathcal{F}_{t-\nu_i}} \\
&= \frac{1}{2^k}\sum_{i=1}^{2^k} \int \mathbf{1}_{(Y_i(t)=\xi_t \in A,\ n_t=k)} \times d\left[\mathbb{P}_x^{\phi} \times \mathbb{L}_{\xi}^{2\beta}\right]\left(\xi,n\right)|_{\mathcal{F}_t^*} \\
&\qquad\qquad \times \prod_{i=1}^{k} dP_{\xi_{\nu_i}}\left(\Upsilon_i\right)|_{\mathcal{F}_{t-\nu_i}}
\end{aligned}
$$

where the indicator in the integral forces there to be precisely $2^k$ nodes in the marked trees we are considering hence justifying the third equality. Completing the computation we thus have

$$
Q_x^*\left(\xi_t \in A, n_t = k\right) = \left[\mathbb{P}_x^{\phi} \times \mathbb{L}_Y^{2\beta}\right]\left(Y\left(t\right) \in A, n_t = k\right)
$$

indicating that the behaviour of the spine is that of the specified $h$-transformed diffusion with the instantaneous rate reproduction doubled. ■

**Remark 15** In many cases (see the later sections containing Examples) the inner-product martingales which were shown in Corollary 10 can be shown to be martingales. In these instances, using a more general version of the Girsanov Theorem for diffusions, it is possible to construct a spine which is not necessarily confined to a compact domain by following the same procedure as above.

From this construction of a spine using the martingale $M_t^{\phi}$ we are able to find conditions under which the martingale itself will converge in mean. The importance of this, as will be seen in the next Theorem, is that the measure $Q_x$

becomes absolutely continuous with respect to $P_x$. This will eventually enable statements about the process under $Q_x$, where the behaviour of the spine is quite specific, to be transferred into statements under $P_x$.

**Theorem 16** *Suppose that $B, \lambda$ and $\phi$ are the same as in Proposition 12. If $\lambda > 0$ then the martingale $M_t^\phi$ is $L^1(P_x)$ convergent for all $x \in B$ and hence $Q_x \ll P_x$.*

**Proof.** First recall that $M^\phi = \lim_{t\uparrow\infty} M_t^\phi$ exists since we are dealing with a positive martingale. Note that the result is trivial when $x \in \partial B$ and hence we only consider the case that $x$ is in the interior of $B$. To prove this theorem we make use of a standard element of measure theory. For the case at hand, it says that

$$Q_x \ll P_x \Longleftrightarrow \limsup_{t\uparrow\infty} M_t^\phi < \infty \ Q_x\text{-a.s.} \Longleftrightarrow E_x\left(M^\phi\right) = 1.$$

Suppose that $\mathcal{S} = \sigma\left((\xi_t, n_t) : t \geq 0\right)$ is the sigma algebra generated by the movement and reproduction along the spine. Let $E_x^{Q^*}$ be the expectation operator associated with $Q_x^*$. Taking expectations under $Q_x^*$ conditional on $S$ we have

$$\begin{aligned}
E_x^{Q^*}\left(M_t^\phi \middle| \mathcal{S}\right) &= E_x^{Q^*}\left(\sum_{i=1}^{n_t}\sum_{iu\in\hat{N}_t} e^{-\lambda t}\frac{\phi\left(Y_{ui}\left(t\right)\right)}{\phi\left(x\right)}\middle|\mathcal{S}\right) + e^{-\lambda t}\frac{\phi\left(\xi_t\right)}{\phi\left(x\right)} \\
&= \sum_{i=1}^{n_t} e^{-\lambda\nu_i}\frac{\phi\left(\xi_{\nu_i}\right)}{\phi\left(x\right)} E_{\xi_{\nu_i}}^{Q^*}\left(M_{t-\nu_i}^\phi\middle|\mathcal{S}\right) + e^{-\lambda t}\frac{\phi\left(\xi_t\right)}{\phi\left(x\right)} \\
&= \sum_{i=1}^{n_t} e^{-\lambda\nu_i}\frac{\phi\left(\xi_{\nu_i}\right)}{\phi\left(x\right)} + e^{-\lambda t}\frac{\phi\left(\xi_t\right)}{\phi\left(x\right)} \\
&\leq \left(\sum_{i=1}^{n_t} e^{-\lambda\nu_i} + e^{-\lambda t}\right)\times const. \qquad (9)
\end{aligned}$$

where in the last inequality we have used the fact that $\phi$ is bounded from above on $B$. Now note since $\beta$ is bounded from above, the process $n$ is stochastically bounded above by the Poisson process with constant rate $\infty > \beta' \geq \sup_{x\in D}\beta\left(x\right)$. Since for this 'upper-bounding' homogeneous Poisson process, the equivalent version of the final sum in is almost surely convergent (one can apply the Law of Large Numbers, or, alternatively, check that the right hand side has finite expectation), then so is (9). We have thus proved that

$$\limsup_{t\uparrow\infty} E_x^{Q^*}\left(M_t^\phi\middle|\mathcal{S}\right) < \infty \ Q_x^*\text{-a.s.}$$

It now follows from Fatou's Lemma that

$$E_x^{Q^*}\left(\liminf_{t\uparrow\infty} M_t^\phi\middle|\mathcal{S}\right) < \infty \ Q_x^*\text{-a.s.}$$

and therefore $\liminf_{t\uparrow\infty} M_t^{\phi} < \infty$ $Q_x^*$-almost surely. Since $\left(M_t^{\phi}\right)^{-1}$ is a $Q_x$-martingale it has a limit $Q_x^*$-almost surely and hence we have proved that $\limsup_{t\uparrow\infty} M_t^{\phi} < \infty$ $Q_x$ -almost surely and $L^1(P_x)$ convergence follows. ■

# 4 Natural multiplicative martingales

Browsing existing literature one will again get the feeling that the results we present in this section are already embedded within existing knowledge. The fundamental issue is the link between solutions to the non-linear elliptic equation $Lf + \beta(f^2 - f) = 0$ (on both compact domains $B$ as well as $D$) and certain martingales which take the form of a multiplicative structure. The reader is referred to Skorohod (1964), Ikeda et al. (1968a,b, 1969), McKean (1975), Neveu (1988) and Harris (1999). Consistent with our earlier remarks about inner product martingales we claim that the results here are to some extent more general in this context than the current literature necessarily offers. Further, our proofs are again probabilistic relying on similar techniques used for the inner product martingales. Just like we did it for inner-product martingales, we start with a local version. Again the reader will note the use of stochastic representations of solutions to differential equations and generational decompositions.

**Theorem 17 (Local multiplicative martingales)** *Let $B \subset\subset D$ be a ball and let $f : \overline{B} \to (0,1]$ and $f \in C\left(\overline{B}\right)$. Then*

$$\pi_t^f(B) = \prod_{u \in \hat{N}_t} \frac{f(Y_u(t))}{f(x)} \prod_{u \in N_t^B} \frac{f\left(Y_u\left(\tau_u^B\right)\right)}{f(x)}$$

*is an $P_x$-martingale for all $x \in B$ if and only if*

$$Lf + \beta\left(f^2 - f\right) = 0 \text{ on } B.$$

**Proof.** First assume that $\pi_t^f(B)$ is a martingale. Necessarily $E_x\left(\pi_t^f(B)\right) = 1$. By conditioning on the first fission time we obtain for all $t \geq 0$,

$$f(x) = \mathbb{E}_x\left(f\left(Y\left(t \wedge \tau^B\right)\right) e^{-\int_0^{t\wedge\tau^B} \beta(Y(s))ds} + \int_0^{t\wedge\tau^B} \beta(Y(s)) f^2(Y(s)) e^{-\int_0^s \beta(Y(s))ds} ds\right) \quad (10)$$

showing that

$$f\left(Y\left(t \wedge \tau^B\right)\right) e^{-\int_0^{t\wedge\tau^B} \beta(Y(s))ds} + \int_0^{t\wedge\tau^B} \beta(Y(s)) f^2(Y(s)) e^{-\int_0^s \beta(Y(s))ds} ds \quad (11)$$

is a martingale. By dominated convergence, it can even be seen to be a uniformly integrable martingale and $L^1(P_x)$ convergence implies

$$
\begin{aligned}
f(x) &= \mathbb{E}_x\left(f\left(Y\left(\tau^B\right)\right)e^{-\int_0^{\tau^B}\beta(Y(s))ds}\right)\\
&\qquad +\mathbb{E}_x\left(\int_0^{\tau^B}\beta\left(Y\left(s\right)\right)f^2\left(Y\left(s\right)\right)e^{-\int_0^s\beta(Y(s))ds}ds\right) \qquad (12)
\end{aligned}
$$

Now consider the Dirichlet problem

$$
\begin{aligned}
(L-\beta)\,u &= -\beta f^2 \text{ in } B\\
u &= f \text{ on } \partial B
\end{aligned}
$$

The unique solution to this problem (c.f. Karazas and Shreve (1991)) has stochastic representation equal to the right hand side of (12) and hence $u=f$. That is to say, $Lf+\beta\left(f^2-f\right)=0$ in $B$.

Now suppose that $Lf+\beta\left(f^2-f\right)=0$ in $B$. Considering $f$ as the unique solution to the above Dirichlet problem, we have (12) and hence (10). This latter can otherwise be written as

$$
f(x)=\mathbb{E}_x\left(\mathbf{1}_{(\nu>t\wedge\tau^B)}f\left(Y\left(t\wedge\tau^B\right)\right)+\mathbf{1}_{(\nu\leq t\wedge\tau^B)}f^2\left(Y\left(\nu\right)\right)\right).
$$

(Recall that $\nu$ denotes the first fission time). Using the same notation as before, let

$$
\pi_t^f\left(B,n\right)=\prod_{\substack{u\in\mathcal{L}\\|u|\leq n}}\frac{f\left(Y_u\left(t\wedge\tau_u^B\right)\right)}{f(x)}\prod_{u\in\mathcal{A}_{\mathcal{L}}(n)}\frac{f^2\left(Y_u\left(\sigma_u\right)\right)}{f(x)}. \qquad (13)
$$

We claim that $\pi_t^f\left(B,n\right)$ is a $\mathcal{G}_n$-martingale. To see this note that

$$
\begin{aligned}
\pi_t^f\left(B,n+1\right) &= \prod_{\substack{u\in\mathcal{L}\\|u|\leq n}}\frac{f\left(Y_u\left(t\wedge\tau_u^B\right)\right)}{f(x)}\\
&\qquad\times\prod_{u\in\mathcal{A}_{\mathcal{L}}(n)}\prod_{i=1}^{2}\left\{\mathbf{1}_{\left(\sigma_{ui}>t\wedge\tau_{ui}^B\right)}\frac{f\left(Y_{ui}\left(t\wedge\tau_{ui}^B\right)\right)}{f(x)}\right.\\
&\qquad\qquad\left.+\,\mathbf{1}_{\left(\sigma_{ui}\leq t\wedge\tau_{ui}^B\right)}\frac{f^2\left(Y_{ui}\left(\sigma_{ui}\right)\right)}{f(x)}\right\}
\end{aligned}
$$

and apply (13) to achieve the martingale property. Since $f$ is bounded in $(0,1]$ it follows that $\pi_t^f\left(B,n\right)$ is both an almost surely and $L^1(P_x)$ convergent martingale. The branching process is non-explosive which means that

$\lim_{n\uparrow\infty}\mathcal{A}_{\mathcal{L}}(n)=\emptyset$. These previous two facts together imply that

$$\begin{aligned}
E_x\left(\pi_t^f(B,0)\right) &= E_x\left(\pi_t^f(B)\right)\\
&= \mathbb{E}_x\left(\mathbf{1}_{(\nu>t\wedge\tau^B)}\frac{f\left(Y\left(t\wedge\tau^B\right)\right)}{f(x)}+\mathbf{1}_{(\nu\leq t\wedge\tau^B)}\frac{f^2\left(Y(\nu)\right)}{f(x)}\right)\\
&= 1 \qquad (14)
\end{aligned}$$

for all $t\geq 0$. Observing the decomposition

$$\pi_{t+s}^f(B)=\prod_{u\in\widehat{N}_t}\frac{f\left(Y_u(t)\right)}{f(x)}\pi_s^f(B,u)\prod_{u\in N_t^B}\frac{f\left(Y_u\left(\tau_u^B\right)\right)}{f(x)} \qquad (15)$$

where give $\mathcal{F}_t$, $\pi_s^f(B,u)$ are independent copies of $\pi_s^f(B)$ under the law $P_{Y_u(s)}$, the expectation (14) together with the strong Markov branching property shows that $\pi_t^f(B)$ is a martingale. ■

An obvious consequence is the following result.

**Corollary 18** *Let $f:\overline{B}\to(0,1]$ be a member of $C(\overline{B})$ and furthermore, let $f=1$ on $\partial B$. The product*

$$\prod_{u\in\widehat{N}_t}\frac{f\left(Y_u(t)\right)}{f(x)}$$

*is a martingale if and only if $f$ solves $Lf+\beta\left(f^2-f\right)=0$ on $B$.*

Finally, we have the following result concerning the whole domain.

**Corollary 19 (Multiplicative martingale)** *Let $f:D\to(0,1]$ be continuous. The product*

$$\pi_t^f=\prod_{u\in N_t}\frac{f\left(Y_u(t)\right)}{f(x)}$$

*is a martingale if and only if $Lf+\beta\left(f^2-f\right)=0$ on $D$.*

**Proof.** If $\pi_t^f$ is a martingale then by conditioning on the first fission time we obtain for all $t\geq 0$,

$$f(x)=\mathbb{E}_x\left(f\left(Y(t)\right)e^{-\int_0^t\beta(Y(s))ds}+\int_0^t\beta\left(Y(s)\right)f^2\left(Y(s)\right)e^{-\int_0^s\beta(Y(s))ds}ds\right).$$

The Feynman-Kac formula (c.f. Karatzas and Shreve (1991)) now tells us that the right hand side is the unique solution to $\partial u/\partial t+(L-\beta)u=-\beta f^2$ in $D$ with $u(x,0)=f(x)$. Hence $u=f$ and $Lf+\beta\left(f^2-f\right)=0$ in $D$.

Suppose now that $Lf + \beta\left(f^2 - f\right) = 0$ on $D$. Let $B_n$ be an increasing sequence of balls such that $D \subseteq \bigcup_n B_n$. Since $E_x\left(\pi_t^f\left(B_n\right)\right) = 1$ for all $x \in B_n$, bounded convergence and the conservativeness of the underlying diffusion implies that for all $x \in D$

$$1 = \lim_{n\uparrow\infty} E_x\left(\pi_t^f\left(B_n\right)\right) = E_x\left(\lim_{n\uparrow\infty} \pi_t^f\left(B_n\right)\right) = E_x\left(\pi_t^f\right).$$

The martingale property now follows by a decomposition similar to (15). ■

In contrast to the equivalent version of this Corollary for inner-product martinagales, note that the method of inflating domains preserves the martingale equality on account of boundedness.

**Remark 20** Using the Maximum Principle given in Proposition 3 of Pinsky (1996) and/or Proposition 7.1 of Engländer and Pinsky (1999), there is at most one non-trivial (that is not identically one) solution to $Lf + \beta(f^2 - f) = 0$ on $B$ with boundary condition 1. Also any function $g \geq 0$ which is not identically 1 solving the same equation on $B$ is smaller or equal than $f$. Note that in the above two references for the maximum principle the term $u^2 - u$ is replaced by $u - u^2$. One recovers the relevant form here by taking $1 - u$ in place of $u$.

Now assume that $\lambda_c > 0$. There exists a domain $B_\epsilon \subset\subset D$ for which $\lambda = \lambda_c\left(L + \beta, B_\epsilon\right)$, the generalized principal eigenvalue of $L + \beta$ on $B_\epsilon$, satisfies $0 < \lambda_c - \epsilon \leq \lambda \leq \lambda_c$. Note that once we know we can find such $B_\epsilon$ then any $B'_\epsilon$ with a smooth boundary satisfying $B_\epsilon \subset\subset B'_\epsilon \subset\subset D$ also has this property. Fix $B_\epsilon$ and define the branching diffusion $\widehat{Z}$ obtained from $Z$ by killing the particles on meeting the boundary $B_\epsilon$ which we can assume is $C^{2,\alpha}$-smooth. As previously remarked, the operator $L + \beta - \lambda$ is critical on $B_\epsilon$. Let $\phi$ denote the corresponding Dirichlet-eigenfunction. In Section 2 it is shown how to construct an innerproduct martingale of the form

$$M_t^\phi := e^{-\lambda t}\frac{\left\langle \phi, \widehat{Z}_t\right\rangle}{\phi\left(x\right)}$$

where $x \in B_\epsilon$. The next Theorem uses the notation of this paragraph.

**Theorem 21** *When $\lambda_c > 0$ there exists a unique non-trivial solution to $Lu + \beta\left(u^2 - u\right) = 0$ such that $u \in (0,1)$ when in $B_\epsilon$ and $u = 1$ on $\partial B_\epsilon$, which can be characterized as either $u(x) = p_\epsilon\left(x\right) = P_x\left(\widehat{Z}\text{ becomes extinct}\right)$ or $u(x) = q_\epsilon\left(x\right) = P_x\left(\lim_{t\uparrow\infty} M_t^\phi = 0\right)$.*

**Proof.** Note that it is automatic from their definition that on the boundary of $B_\epsilon$ both probabilities are one. It suffices then to prove that both $p_\epsilon$ and $q_\epsilon$ can be used to construct product martingales and are non-trivial. Uniqueness is guaranteed by the previous Theorem and its following Remark.

Theorem 16 shows that $M_t^\phi$ is $L^1(P_x)$ convergent for all $x \in B_\epsilon$. This implies necessarily that the limit $M^\phi$ is strictly positive with positive probability and on the complement of the extinction of $\widehat{Z}$ there can be no mass in the limit. Consequently for $x \in B_\epsilon$ we have

$$0 < p_\epsilon(x) \leq q_\epsilon(x) < 1 \text{ with } p_\epsilon(x) = q_\epsilon(x) = 1 \text{ on } \partial B_\epsilon.$$

This shows that the two proposed probabilities are non-trivial.

For the case of $p_\epsilon$, note that by a classical branching decomposition, for all $t, s \geq 0$

$$\left\{\widehat{Z}_{t+s}\left(B_\epsilon\right) \ = 0\right\} = \left\{\widehat{Z}_s\left(u, B_\epsilon\right) \ = 0 \text{ for all } u \in \widehat{N}_t\right\}$$

where given $\mathcal{F}_t$, $\widehat{Z}_s\left(u, B_\epsilon\right)$ are independent copies of $\widehat{Z}_s\left(B_\epsilon\right)$ under the measure $P_{Y_u(t)}$. It follows that

$$P_x\left(\widehat{Z}_{t+s}\left(B_\epsilon\right) \ = 0\right) = E_x\left(\prod_{u \in \widehat{N}_t} P_{Y_u(t)}\left(\widehat{Z}_s\left(B_\epsilon\right) \ = 0\right)\right).$$

Taking the limit as $s \uparrow \infty$, the Dominated Convergence Theorem implies that for all $t \geq 0$

$$p_\epsilon\left(x\right) = E_x\left(\prod_{u \in \widehat{N}_t} p_\epsilon\left(Y_u\left(t\right)\right)\right).$$

A standard branching decomposition again shows that this last identity guarantees that $\prod_{u \in \widehat{N}_t} p_\epsilon\left(Y_u\left(t\right)\right)$ is a product martingale.

Now turning to $q_\epsilon$, we note that, again appealing to a standard branching decomposition, the innerproduct martingale $M_t^\phi$ can be written as follows for $t, s \geq 0$

$$M_{t+s}^\phi = e^{-\lambda t} \sum_{u \in \widehat{N}_t} \frac{\phi\left(Y_u\left(t\right)\right)}{\phi\left(x\right)} M_s^\phi\left(u\right)$$

where given $F_t$, $M_s^\phi\left(u\right)$ are independent copies of $M_s^\phi$ under the measure $P_{Y_u(t)}$. Taking limits as $s \uparrow \infty$, it is again clear from this equality that

$$q_\epsilon\left(x\right) = E_x\left(\prod_{u \in \widehat{N}_t} q_\epsilon\left(Y_u\left(t\right)\right)\right)$$

and hence $\prod_{u \in \widehat{N}_t} q_\epsilon\left(Y_u\left(t\right)\right)$ is a martingale. ■

# 5 Local extinction versus local exponential growth and recurrence: probabilistic arguments

## 5.1 Proof of Theorem 3 (i)

Let $\Omega_0 := \{\omega \in \Omega : \limsup_{t\uparrow\infty} Z_t(B) > 0\}$. First, note that it is sufficient to prove that for all $K \in \mathbb{R}^+$,

$$P_x\left(\left\{\limsup_{t\uparrow\infty} Z_t(B) < K\right\} \cap \Omega_0\right) = 0. \tag{16}$$

Indeed, from (16) it follows that

$$P_x\left(\left\{\limsup_{t\uparrow\infty} Z_t(B) < \infty\right\} \cap \Omega_0\right) = 0.$$

Therefore we now prove (16). Recall that $\beta$ is continuous and $\beta \not\equiv 0$, that is, $\beta$ is bounded away from zero on some ball. Using this along with well-known positivity and continuity properties of the transition kernel $p(t,x,y)$, it is straightforward to prove that

$$\varepsilon(K,B) := \inf_{x\in\bar{B}} P_x(Z_1(B) \geq K) > 0. \tag{17}$$

Although (17) is intuitively clear, the precise formulation of its proof is a bit tedious, and therefore we skip here the technicalities.

Then, by the strong Markov-property,

$$P_x\left(\left\{\limsup_{t\uparrow\infty} Z_t(B) < K\right\} \cap \Omega_0\right) \leq \lim_{n\uparrow\infty} (1-\varepsilon(K,B))^n = 0,$$

finishing the proof.

## 5.2 Proof of Theorem 3 (ii)

Assume that $\lambda_c \leq 0$. This means that $L+\beta$ is critical or subcritical which in turn means by definition (see Appendix A) that one can pick an $h > 0$ with $(L+\beta)h = 0$ on $D$. According to Corollary 10 Section 2, the innerproduct $\langle h, Z_t\rangle$ is a supermartingale, bounded above in expectation by $h(x)$. Since $h$ is bounded below on compact domains, it follows that for all $B \subset\subset D$, $Z_t(B) \leq$ const.$\times \langle h, Z_t\rangle$ and hence by comparison, when $x \in B$

$$\limsup_{t\uparrow\infty} Z_t(B) < \infty \ P_x\text{-a.s.}$$

It now follows from part (i) that

$$\limsup_{t\uparrow\infty} Z_t(B) = \lim_{t\uparrow\infty} Z_t(B) = 0 \ P_x\text{-a.s.}$$

Now assume that $\lambda_c > 0$ and let $B_\epsilon$, $\lambda$, $\widehat{Z}$, $\phi$, $M_t^\phi$, $p_\epsilon$ and $q_\epsilon$ be as in Theorem 21. Theorem 14 shows that $M_t^\phi$ can be used as a change of measure for the law of the branching diffusion,

$$\left.\frac{dQ_x}{dP_x}\right|_{\mathcal{F}_t} := M_t^\phi.$$

Further, under $Q_x$ there is a randomized ancestral line of descent whose path is the diffusion $\xi = \{\xi_t : t \geq 0\}$ corresponding to the operator

$$L + a\frac{\nabla\phi}{\phi}\cdot\nabla,$$

on $B_\epsilon$, so that $\xi$ lives on $B_\epsilon$, and is positive recurrent.

It was shown in Theorem 21 that

$$p_\epsilon(x) = P_x\left(\widehat{Z} \text{ becomes extinct}\right) = P_x\left(M^\phi = 0\right) = q_\epsilon(x)$$

and $p_\epsilon(x) = q_\epsilon(x) = 1$ on $\partial B_\epsilon$. Clearly $Q_x \ll P_x$ and on $\{M^\phi > 0\}$ the two measures are equivalent (with probability $1 - q_\epsilon = 1 - p_\epsilon$). As we already know that under $Q_x$ there is one particle (whose path is that of the spine) which ergodizes on $B_\epsilon$ with full support on the interior of $B_\epsilon$, then we can say that for any Borel $B \subseteq B_\epsilon$ and $x \in B_\epsilon$,

$$Q_x\left(\liminf_{t\uparrow\infty} Z_t(B) > 0\right) = 1.$$

Since $dQ_x = M^\phi dP_x$, it follows that

$$\begin{aligned}
P_x\left(\limsup_{t\uparrow\infty} Z_t(B) = \infty\right) &= P_x\left(\liminf_{t\uparrow\infty} Z_t(B) > 0\right) \\
&\geq 1 - q_\epsilon(x) \\
&= 1 - p_\epsilon(x) > 0
\end{aligned}$$

Note we have used again the fact that with probability 1, the $\limsup_{t\uparrow\infty} Z_t(B)$ is either zero or infinity from part (i). Since we can arbitrarily inflate $B_\epsilon$ to enclose any $x \in D$, we have proved that non-local extinction occurs with strictly positive probability for all Borel subsets of $D$. Note, generally speaking the larger $B_\epsilon$ is chosen, the higher the value of the lower bound. Part (ii) of the Theorem is now proved.

## 5.3 Proof of Theorem 3 (iii)

For each $x \in D$ we can take a sequence of domains with smooth boundaries $\{B_n\}_{n\geq 1}$, satisfying $x \in B_n \subset\subset B_{n+1}$, $\bigcup B_n = D$ (and $\lambda_c(L+\beta, B_n) \uparrow \lambda_c$.) It follows by monotonicity that in a pointwise sense $p_n(x) \downarrow p(x)$ where $p_n(x) = p_\epsilon(x)$ if we would take $B_\epsilon = B_n$. Note that in the limit $p(x) \in [0,1)$. Further,

$$Lp + \beta\left(p^2 - p\right) = 0 \text{ on } D. \tag{18}$$

Remark 20 shows that in fact, $p$ is the maximal solution to (18) in $[0,1)$. Referring to the last paragraph of the previous part of the proof, we now have that $\rho(x) := P_x\left(\limsup_{t\uparrow\infty} Z_t(B) = 0\right) \leq p(x)$ for all $x \in D$. The function $\rho(x)$ is also a solution to the equation (18). To see this, note again from the branching property that for $t, s \geq 0$

$$\left\{\limsup_{s\uparrow\infty} Z_{t+s}(B) \ = 0\right\} = \left\{\limsup_{s\uparrow\infty} Z_s(u,B) \ = 0 \text{ for all } u \in N_t\right\}$$

where given $\mathcal{F}_t$, $Z_s(u,B)$ are independent copies of $Z_s(B)$. In a similar way to previous analysis, it follows that $\prod_{u\in N_t} \rho(Y_u(t))$ is a martingale and hence by Theorem 19, $\rho$ also solves (18).

### 5.4 Proof of Theorem 3 (iv)

Denote $\epsilon := \lambda_c - \lambda > 0$. Let $B_\epsilon$, $\widehat{Z}$ and $\phi$ be as in the second part of the proof. Obviously, it is enough to show that there exists a $B_\epsilon \subset\subset D$ such that

$$P_x\left(\liminf_{t\to\infty} e^{-\lambda t} Z_t(B_\epsilon) > 0\right) > 0, \tag{19}$$

because then the original statement follows by replacing $B_\epsilon$ with $B_{\epsilon/2}$. We now therefore prove (19). Since $\phi$ is bounded from above , we have

$$\begin{aligned} P_x(\liminf_{t\uparrow\infty} e^{-\lambda t} Z_t(B_\epsilon) = 0) &\leq P_x(\liminf_{t\uparrow\infty} e^{-\lambda t} \widehat{Z}_t(B_\epsilon) = 0) \\ &\leq P_x(\lim_{t\uparrow\infty} e^{-\lambda t}\langle \phi, \widehat{Z}_t\rangle = 0) \\ &= q_\epsilon(x) \end{aligned}$$

The last probability in the estimate is smaller than 1, giving the required statement.

## 6 Local extinction criterion: analytical arguments

In this section we present an *analytical* proof of the local extinction criterion. Although we have already given a probabilistic proof for the result in the previous section, we feel that this paper becomes more complete by showing here how the result can be derived from an analogous one which can be found in the superprocess literature. This analogous result for superdiffusions (which we will utilize in the proof) has been proved recently by Pinsky (see Pinsky (1996, Theorem 6) and the remark afterwards) using quite a bit of heavy analytical machinery. As far as the proof of the condition for local extinction is concerned, we will show how to derive this from Pinsky's result using a comparison argument between branching diffusions and superdiffusions. Our proof of the condition for *no* local extinction will be essentially the same as his proof for superdiffusions.

Regarding the comparison mentioned above, it is likely that the deeper reason for it is compounded in the fact that one can decompose a superdiffusion using "immigration" and an underlying supercritical (strictly) binary branching diffusion (see Evans and O'Connell (1994) and Engländer and Pinsky (1999, Section 6). Intuitively, if the underlying branching process visits a compact again and again (no local extinction), then so does the superdiffusion as well. For the rigorous proof we will utilize a result on the "weighted occupation time" for branching particle systems obtained by Evans and O'Connell (1994) (also used for proving the immigration picture in the same paper).

**Proof of the criterion on local extinction.**

(i) Assume that $\lambda_c \leq 0$. Let $(x,s) \mapsto \phi(s,x)$ be jointly measurable in $(x,s)$ and let $\phi(s) = \phi(s,\cdot)$ be nonnegative and bounded for each $s \geq 0$. By Evans and O'Connell (1994, Theorem 2.2), $E_x\left[\exp\left(-\int_0^t \langle \phi(s), Z_s\rangle ds\right)\right] = u(t,x)$, where $u$ is the so-called *mild solution* of the evolution equation

$$\begin{aligned}&\frac{\partial u}{\partial s}(s) = Lu(s) - \beta u(s) + \beta u^2(s) - \phi(t-s)u(s),\ 0 < s \leq t,\\ &\lim_{s\downarrow 0} u(s) = 1.\end{aligned} \tag{20}$$

[Here we used the notation $u(s) = u(s,\cdot)$]. Pick a $\phi \in C_c^+(D)$ satisfying $\phi(x) > 0$, for $x \in B$ and $\phi(x) = 0$, for $x \in D \setminus B$. Let $u = u_{t,\theta}^{(T)}$ be the mild solution of the evolution equation

$$\begin{aligned}&\frac{\partial u}{\partial s}(s) = Lu(s) - \beta u(s) + \beta u^2(s) - \theta\phi 1_{[t,\infty)}(T-s)u(T-s),\ 0 < s \leq T,\\ &\lim_{s\downarrow 0} u(s) = 1.\end{aligned} \tag{21}$$

For the rest of the proof of part (i), let the starting point $x \in D$ be fixed. Using the argument given in Iscoe (1988, p.207), we have that $Z$ exhibits local extinction if and only if

$$\lim_{t\uparrow\infty}\lim_{\theta\uparrow\infty}\lim_{T\uparrow\infty} u_{t,\theta}^{(T)}(T,x) = 1. \tag{22}$$

Consider now $X$, the $(L,\beta,\beta,D)$-superdiffusion (that is, the superdiffusion corresponding to the operator $Lu + \beta u - \beta u^2$ on $D$ – see Engländer and Pinsky (1999) for the definition) and let $U = U_{t,\theta}^{(T)}$ be the mild solution of the evolution equation

$$\begin{aligned}&\frac{\partial U}{\partial s}(s) = LU(s) + \beta U(s) - \beta U^2(s) + \theta\phi 1_{[t,\infty)}(T-s),\ 0 < s \leq T,\\ &\lim_{s\downarrow 0} U(s) = 0.\end{aligned} \tag{23}$$

Again, the argument given in Iscoe (1988,p. 207) shows that the support of $X$ exhibits local extinction (that, is the property in Definition 1 holds with $X$ in place of $Z$) if and only if

$$\lim_{t\uparrow\infty}\lim_{\theta\uparrow\infty}\lim_{T\uparrow\infty} U_{t,\theta}^{(T)}(T,x) = 0. \tag{24}$$

Using (24), Pinsky (1996) has shown (Theorem 6 and the remark afterwards) that the assumption $\lambda_c \leq 0$ is equivalent to the local extinction of the support of $X$. Thus, (24) follows from $\lambda_c \leq 0$. We now show that (24) implies (22), which will complete the proof of this part. Making the substitution $v := 1 - u$, we have that v is the mild solution of the evolution equation

$$\begin{aligned}
&\frac{\partial v}{\partial s}(s) = \\
&Lv(s) + \beta v(T-s) - \beta v^2(s) + \theta\phi 1_{[t,\infty)}(T-s)(1 - v(T-s)),\ 0 < s \leq T, \\
&\lim_{s\downarrow 0} v(s) = 0.
\end{aligned} \tag{25}$$

By Iscoe (1988, pp. 204), $U$ and $v$ (with $t, \theta$ fixed) have the following probabilistic representations:

$$\begin{aligned}
U(T,x) &= E_x \exp\left(\int_0^T \mathrm{d}s \, \langle \theta\phi 1_{[t,\infty)}(s), X_s\rangle\right), \\
v(T,x) &= E_x \exp\left(\int_0^T \mathrm{d}s \, \langle \theta\phi 1_{[t,\infty)}(s)(1 - v(s)), X_s\rangle\right).
\end{aligned} \tag{26}$$

¿From these equations it is clear that $v \leq U$. Hence

$$\lim_{t\uparrow\infty}\lim_{\theta\uparrow\infty}\lim_{T\uparrow\infty} v_{t,\theta}^{(T)}(T,x) = 0. \tag{27}$$

(ii) Assume now that $\lambda_c > 0$. The proof of this part is almost the same as the proof of the analogous statement for superdiffusions Pinsky (1996, p.262-263). In that proof it is shown that the assumption $\lambda_c > 0$ guarantees the existence of a (large) subdomain $D_0 \subset\subset D$, and a function $v \geq 0$ defined on $D_0$ which is not identically zero and which satisfies

$$\begin{aligned}
&Lv + \beta v - \beta v^2 = 0 \text{ in } D_0 \\
&\lim_{x\to\partial D_0} v(x) = 0, \\
&v > 0 \text{ in } D_0.
\end{aligned} \tag{28}$$

Since $f \equiv 1$ also solves $Lf + \beta f - \beta f^2 = 0$ *in* $D_0$, the elliptic maximum principle (see Pinsky (1996, Proposition 3) and Engländer and Pinsky (1999, Proposition 7.1)) implies that $v \leq 1$. Let $w := 1 - v$. Then $w \geq 0$ and furthermore $w$ satisfies

$$\begin{aligned}
&Lw - \beta w + \beta w^2 = 0 \text{ in } D_0 \\
&\lim_{x\to\partial D_0} w(x) = 1, \\
&w < 1 \text{ in } D_0.
\end{aligned} \tag{29}$$

Let $\hat{P}$ denote the probability for the branching diffusion $\hat{Z}$ obtained from $Z$ by killing the particles upon exiting $\partial D_0$. Obviously $\hat{P}_x(\hat{Z}$ survives$) \leq P_x(Z(t, D_0) > 0$ for arbitrary large $t'$s), and thus, it is enough to show that

$$0 < \hat{P}_x(\hat{Z} \text{ survives}). \tag{30}$$

We now need the fact that $w > 0$ on $D_0$. This follows from the equation

$$(L - \beta(1-w))w = 0 \text{ in } D_0$$

and the strong maximum principle (Theorem 3.2.6 in Pinsky (1995)) applied to the linear operator $L - \beta(1-w)$.

Now an argument similar to the one in the proof of Lemma 17 shows that

$$\hat{E}_x e^{\langle \log w, \hat{Z}(t) \rangle} = w(x), \; t \geq 0. \tag{31}$$

Suppose that (30) is not true. Then the left-hand side of (31) converges to 1 as $t \uparrow \infty$. On the other hand, the right-hand side of (31) is independent of $t$ and is smaller than 1, which is a contradiction. Consequently, (30) is true. $\square$

# 7 Examples

In this section we will present five examples for branching diffusions which will illustrate the general results of this paper.

## 7.1 Branching Brownian motion (with drift).

This is one of the most simple branching diffusion models one can consider, particularly if $D = \mathbb{R}$ and $\beta$ is a positive constant. In this case $L = 1/2 \left(\mathrm{d}^2/\mathrm{d}x^2\right)$. It is well know that the positive harmonic functions of $L + \beta$ are $\exp\{-\theta x\}$ for $\theta \in R$ giving $\lambda_c = \beta > 0$ (corresponding to the case $\theta = 0$). Here one notes that we have the luxury of innerproduct martingales defined on the whole domain. Indeed the martingale with growth rate $\lambda_c$ is the classical martingale $N_t e^{-\beta t}$ which converges almost surely and in mean to its limit $W$.

It is easy to reason without the technology we have presented in this article that the process visits any Borel set infinitely often with probability one. This follows by virtue of the fact that every line of descent diffuses as a standard Brownian motion which exhibits these properties.

Watanabe (1967) has proved an even stronger result than this conclusion. He has shown that over and above local extinction with probability zero, a strong law of large numbers exists:

$$\lim_{t \uparrow \infty} \frac{Z_t(B)}{e^{\beta t} t^{-1/2}} = (2\pi)^{-1/2} |B| \cdot W \;\; P_x\text{-a.s.}$$

(In fact Watanabe proved versions of this result for higher dimentional Brownian motion too as well as some other special branching diffusions related to

Brownian motion). Suppose now we consider a branching Brownian motion in $\mathbb{R}$ with a constant drift $\varepsilon \in \mathbb{R}$. One could reason in this case that despite the fact that particles exhibit transient motion, for a small enough $\varepsilon$ the comparative reproduction ensures that space fills up everywhere with particles. The necessary comparison is of course captured in the spectral condition on $\lambda_c$ in Theorem 3. In this case we have $L = 1/2\left(\mathrm{d}^2/\mathrm{d}x^2\right) + \varepsilon\left(\mathrm{d}/\mathrm{d}x\right)$ and hence, again referring to well know facts, the positive harmonic functions are again of the form $\exp\{-\theta x\}$ for $\theta \in R$. This time $\lambda_c = \beta - (1/2)\,\varepsilon^2$ (achieved at $\theta = \epsilon$) which is then positive if $|\varepsilon| < \sqrt{2\beta}$.

In Theorem 3 we should still be concerned about the probabilities $\rho(x)$. It is not yet clear whether they will be identically one or not. The travelling wave solutions to the K-P-P equation come to our rescue. According to Theorem 2(ii)-(iii), $\rho \in [0,1]$ satisfies

$$\frac{1}{2}\frac{d^2\rho}{dx^2} + \varepsilon\frac{d\rho}{dx} + \beta\left(\rho^2 - \rho\right) = 0.$$

However, Kolmogorov et al. (1937) proved that there are no non-trivial solutions bounded in $[0,1]$ to this, the travelling wave K-P-P equation, for $|\varepsilon| < \sqrt{2\beta}$ and otherwise there is a unique non-trivial solution. Consistently with Corollary 6 we see that there is local extinction with probability zero or one (that is, $\rho = 0$ or $\rho \equiv 1$) according to whether $\lambda_c > 0$ or $\lambda_c \leq 0$ respectively.

In the case of non-zero drift, it can be easily checked, using methods similar to those in the proof of Theorem 7, that inner-product supermartingale associated with $\lambda_c$ is also a martingale. Further, following the discussion in Section 3, it is possible to produce a spine from this martingale along which there is a doubled rate of reproduction and the associated diffusion operator is simply that of standard Brownian motion (so the $\varepsilon$-drift has been removed).

## 7.2 Transient $L$ and compactly supported $\beta$

Consider the case when $L$ corresponds to a transient (but conservative) diffusion on $D \subseteq \mathbb{R}^d$ and $\beta$ is a smooth nonnegative compactly supported function. Since the generalized principal eigenvalue coincides with the classical principal eigenvalue for smooth bounded domains, it follows that for any nonempty bounded $B \subset\subset D$ one can pick such a $\beta$ with $B = \mathrm{supp}(\beta)$ and so that $L + \beta$ is supercritical on $D$, that is $\lambda_c > 0$ (all one has to do is to ensure that the infimum of $\beta$ on a somewhat smaller smooth domain $B' \subset\subset B$ is larger then the absolute value of the principal eigenvalue on $B'$. Then, a fortiori, $L + \beta$ is supercritical on $D$ as well). On the other hand, by the transience assumption, it is clear that the initial $L$-particle wanders out to infinity with positive probability without ever visiting $B$ (and thus without ever branching), when starting from $x \notin B$.

In light of Corollary 5, this now shows that *there exists a non-trivial travelling wave solution* to $Lu + \beta\left(u^2 - u\right) = 0$ for such an $L$ and $\beta$. To the best of our knowledge, this is a new result concerning generalized K-P-P equations.

## 7.3 The Harris-Williams branching diffusion

This is a specialized example of a branching diffusion studied for aspects of its prototypical behaviour (and in many ways has served as a source of inspiration for this paper) by Harris and Williams (1996) and Harris (2000). We consider here a variant of their model. The diffusion operator on $\mathbb{R}^2$ is

$$L_\varepsilon = \frac{1}{2}ay^2\frac{\partial^2}{\partial x^2} + \varepsilon\frac{\partial}{\partial x} + \frac{\theta}{2}\left(\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y}\right),$$

where $\theta \in \mathbb{R}$. That is, the operator corresponding to a time changed Brownian motion with drift where the time change is controlled by an independent Ornstein-Uhlenbeck process operating in the $y$-direction. (Since we work with positive definite operators, we should in fact replace the first coefficient $ay^2$ by $af(y)$ where $f > 0$ is obtained by modifying the function $y^2$ on a compact. This, however will not affect the rest of the argument.) In the original version of this model, $\varepsilon = 0$. The branching rate is dependent purely on the $y$ variable and takes the form

$$\beta(x, y) = ry^2 + \rho$$

where $\rho > 0$ and $0 < r < \theta/8$. Note that $\beta$ is not bounded. However this is not a problem. Since, as is clear from the model, the process does not explode and, as far as the parameter values of interest are concerned, the generalized principle eigenvalue is finite.

Further, note that the operator $L_0$ is *symmetric* with respect to the reference density $g(x, y) = \exp(-y^2/2)$. Using a comparison principle for symmetric operators (see Corollary 6.4.2 in Pinsky (1995a)), it follows that $L_0$ corresponds to a recurrent diffusion on $\mathbb{R}^2$. Consequently, the generalized principal eiganvalue of this operator is zero (see Appendix); and therefore

$$\lambda_c(L_0 + \beta, \mathbb{R}^2 L) \geq \rho > 0.$$

For the case that $\varepsilon \neq 0$, an application of an $h$-transform shows that

$$\lambda_c(L_\varepsilon + \beta, \mathbb{R}^2) = \lambda_c(L_0 + \beta, \mathbb{R}^2 L) - \frac{\varepsilon^2}{2} \geq \rho - \frac{\varepsilon^2}{2}.$$

Therefore, for sufficiently small $\varepsilon$ ($|\varepsilon| < \sqrt{2\rho}$) the positivity of the generalized principal eigenvalue is preserved. The conclusion is that for a small enough drift, there is no local extinction. This is again consistent with intuition; that is to say the branching rate 'wins' against transience.

Like the previous example, this is another branching diffusion from which we could *in principle* compare the transition of local extinction to non-local extinction against the event of the existence of travelling waves to the equation

$$\frac{1}{2}ay^2\frac{\partial^2 u}{\partial x^2} + \varepsilon\frac{\partial u}{\partial x} + \frac{\theta}{2}\left(\frac{\partial^2 u}{\partial y^2} - y\frac{\partial u}{\partial y}\right) + \left(ry^2 + \rho\right)\left(u^2 - u\right) = 0. \qquad (32)$$

(Again, one should slightly modify the first coefficient.) Harris and Williams (1996) have shown there exists an asymptotic wave speed, say $\tilde{\varepsilon}$, for the left most particle in the $x$-direction of this branching diffusion when $\varepsilon = 0$. Given the elegant probabilistic arguments in Harris (1999) linking the asymptotic speed of the left most particle in a standard branching Brownian motion and the minimal speed at which travelling waves to the K-P-P equation exist and then the relationship with local extinction expressed in the previous example, one might be inclined to believe a similar relationship holds here. That is to say: is it true that the transition in $\varepsilon$ from local extinction to non-local extinction, seen through the positivity of $\lambda_c(L_\varepsilon + \beta, \mathbb{R}^2)$, coincides precisely with the transition from existence to no existence of travelling waves to (32)? And further that this transition takes place at precisely $\varepsilon = \tilde{\varepsilon}$ ?

Personal communication with Dr. Harris confirms that minimal wave speed for existence in (32) is indeed $\tilde{\varepsilon}$. For the rest, some work awaits.

## 7.4 Branching Ornstein-Uhlenbeck process and more general recurrent motions

Let $L = \frac{1}{2}\Delta - kx \cdot \nabla$ on $\mathbb{R}^d$, $d \geq 1$, where $k > 0$. Then $L$ corresponds to the $d$-dimensional Ornstein-Uhlenbeck process with drift parameter $k$. Note that it is a (positive) recurrent process. Furthermore let $\beta$ be a positive constant. Consider now the $(L, \beta)$-branching diffusion $Z$ (on $\mathbb{R}^d$). We call $Z$ a *branching Ornstein-Uhlenbeck process*. By recurrence it follows that $L$ is a critical operator, and thus $\lambda_c = \lambda_c(L, \mathbb{R}^d) = 0$. Consequently $\lambda_c(L + \beta, \mathbb{R}^d) = \beta$. By Theorem 3(ii) and (iv), $Z$ does not exhibit local extinction, and exhibits local exponential growth (with rate arbitrarily close to $\beta$) on large compact domains.

Of course, non-local extinction with positive probability would immediately follow from the obvious comparison with a single recurrent $L$-particle. It follows that non local extinction and local exponential growth hold even when $L$ is replaced by an arbitrary $L$ on a domain $D$ as far as $L$ corresponds to a recurrent diffusion on $D$. In fact, as Theorem 4.6.3(i) in Pinsky (1996) shows, $\lambda_c(L + \beta, D) > 0$, whenever $L$ corresponds to a recurrent diffusion on $D$ and the branching rate $\beta \geq 0$ is not identically zero. Therefore, *$Z$ does not exhibit local extinction, and exhibits local exponential growth on large compacts for any recurrent motion and any not identically vanishing branching rate.*

## 7.5 Branching outward Ornstein-Uhlenbeck process

Let $L = \frac{1}{2}\Delta + kx \cdot \nabla$ on $\mathbb{R}^d$, $d \geq 1$, where $k > 0$. Then $L$ corresponds to the $d$-dimensional "outward" Ornstein-Uhlenbeck process with drift parameter $k$. This process is transient. Furthermore let $\beta$ be a positive constant, and consider the $(L, \beta)$-branching diffusion $Z$. Following Example 2 in Pinsky (1996), we have that $\lambda_c(L + \beta, \mathbb{R}^d) = \beta - kd$. From Theorem 3(ii) we conclude that if $\beta > kd$ then $Z$ does not exhibit local extinction, and exhibits local exponential growth (with rate arbitrarily close to $\beta - kd$) on large compact domains. However if $\beta \leq kd$ then $Z$ exhibits local extinction.

For this case we have the luxury of having an exact form for the one dimensional space of harmonic functions satisfying $(L+\beta-\lambda_c)h=0$. Indeed, it is easy to see that $h(x)=\exp\{-k|x|^2\}$ satisfies $(L+\beta-\lambda_c)h=0$, and that making an $h$-transform with this $h$, $L+\beta-\lambda_c$ transform into

$$(L+\beta-\lambda)^h=\frac{1}{2}\Delta-kx\cdot\nabla. \tag{33}$$

This operator corresponds to an (inward) Ornstein-Uhlenbeck process which is (positive) recurrent. Thus the operator in (33) is critical, and therefore it has a one-dimensional space of positive harmonic functions; the positive constants. Using the correspondence between the positive solutions for the two operators (invariance under $h$-transforms), we conclude, that in fact, $h(x)=\exp\{-k|x|^2\}$ is the *only* (up to constant multiples) positive harmonic function for $L+\beta-\lambda_c$.

Using the associated inner-product martingale (which can again be shown to be a martingale using methods found in Theorem 7), we can follow the arguments of Section 3 to produce a spine with a doubled rate of reproduction. This spine is precisely the Ornstein-Uhlenbeck process corresponding to the operator (33).

Now, irrespective of the value of $\beta-kd$ it would seem possible to change measure in such a way that there is a positive recurrent spine. This would seem to suggest that there is a strong case for non-local extinction without a condition on $\beta-kd$. However, in order to transfer statements of local survival back to the process under the original measure, we would need mean convergence of the inner-product martingale and thus the condition $\beta-kd>0$ appears.

# 8 Appendix A: A review on criticality theory

Let $L$ be as in (1). We do not assume in this general setting that $L$ corresponds to a conservative diffusion. There exists however a corresponding diffusion process $Y$ on $D$ that solves the *generalized martingale problem* for $L$ on $D$ (see Chapter 1 in Pinsky (1995a)). The process lives on $D\cup\Delta$ with $\Delta$ playing the role of a cemetery state. Let $\beta\in C^{\alpha}(D)$ (again, we do not assume anything further about $\beta$). We denote by $P_x$ and $E_x$ the corresponding probabilities and expectations, and define the *transition measure* $p(t,x,dy)$ for $L+\beta$ by

$$p(t,x,B)=\mathbb{E}_x\left(\exp\left(\int_0^t\beta(Y_s)\,\mathrm{d}s\right);Y_t\in B\right),$$

for measurable $B\subseteq D$.

**Definition 22** If

$$\int_0^\infty p(t,x,B)\,\mathrm{d}t=\mathbb{E}_x\int_0^\infty\exp\left(\int_0^t\beta(Y_s)\,\mathrm{d}s\right)1_B(Y_t)\,\mathrm{d}t<\infty,$$

for all $x \in D$ and all bounded $B \subset D$, then

$$G(x, \mathrm{d}y) = \int_0^\infty p(t, x, \mathrm{d}y)\, \mathrm{d}t$$

is called the *Green's measure* for $L + \beta$ on $D$. If the above condition fails, then the Green's measure for $L + \beta$ on $D$ is said not to exist.

In the former case, $G(x, dy)$ possesses a density, $G(x, dy) = G(x, y)dy$, which is called the *Green's function* for $L + \beta$ on $D$.

For $\lambda \in$ define

$$C_{L+\beta-\lambda} \;=\; \{\; u \in C^2 \;:\; (L+\beta-\lambda)u = 0 \text{ and } u > 0 \text{ in } D \;\}.$$

The operator $L + \beta - \lambda$ on $D$ is called *subcritical* if the Green's function exists for $L+\beta-\lambda$ on $D$; in this case $C_{L+\beta-\lambda} \neq \emptyset$. If the Green's function does not exist for $L+\beta-\lambda$ on $D$, but $C_{L+\beta-\lambda} \neq \emptyset$, then the operator $L+\beta-\lambda$ on $D$ is called *critical.* In this case $C_{L+\beta-\lambda}$ is one-dimensional. The unique function (up to a constant multiple) in $C_{L+\beta-\lambda}$ is called the *ground state* of $L + \beta$ on $D$. The formal adjoint of the operator $L + \beta - \lambda$ on $D$ is also critical with ground state $\tilde{\phi}$. If furthermore $\phi\tilde{\phi} \in L^1(D)$, we call $L + \beta - \lambda$ on $D$ *product $L^1$ critical*, or *product-critical* in short. (For $\phi = \tilde{\phi}$, product-criticality means that $\phi$ is an $L^2$-eigenfunction.) Finally, if $C_{L+\beta-\lambda} = \emptyset$, then $L + \beta - \lambda$ on $D$ is called *supercritical.*

If $\beta \equiv 0$, then $L+\beta$ is not supercritical on $D$ since the function $f \equiv 1$ satisfies $Lf = 0$ on $D$. In this case $L + \beta = L$ is subcritical or critical on $D$ according to whether the corresponding diffusion process, $Y$, is transient or recurrent on $D$ . Product-criticality in this case is equivalent to positive recurrence for $Y$. If $\beta \leq 0$ and $\beta \not\equiv 0$, then $L + \beta$ is subcritical on $D$.

In terms of the solvability of inhomogeneous Dirichlet problems, subcriticality guarantees that the equation $(L + \beta)u = -f$ in $D$ has a positive solution $u$ for every $0 \lneq f \in C_c^\eta$. (Here $C_c^\eta = C_c \cap C^\eta$.) If subcriticality does not hold, then there are no positive solutions for any $0 \lneq f \in C_c^\eta$.

One of the two following possibilities holds :

1) There exists a number $\lambda_c \in$ such that $L-\lambda$ on $D$ is subcritical for $\lambda > \lambda_c$, supercritical for $\lambda < \lambda_c$, and either subcritical or critical for $\lambda = \lambda_c$.

2) $L - \lambda$ on $D$ is supercritical for all $\lambda \in$, in which case we define $\lambda_c = \infty$.

**Definition 23** The number $\lambda_c \in (-\infty, \infty]$ is called the *generalized principal eigenvalue* for $L$ on $D$ .

Note that $\lambda_c \;=\; \inf\,\{\lambda \in :\; C_{L+\beta-\lambda} \neq \emptyset\}$. Also, if $\beta$ is bounded from above, then case 1) holds.

The generalized principal eigenvalue coincides with the classical principal eigenvalue (that is, with the supremum of the real part of the spectrum) if $D$ is bounded with a smooth boundary and the coefficients of $L$ are smooth up to $\partial D$. Also, if $L + \beta$ is symmetric with respect to a reference measure $\rho\, dx$, then $\lambda_c$ equals the supremum of the spectrum of the self-adjoint operator on $L^2(D, \rho\, dx)$ obtained from $L+\beta$ via the Friedrichs' extension theorem.

Let $h \in C^{2,\eta}$ satisfy $h > 0$ in $D$. The operator $(L+\beta)^h$ defined by

$$(L+\beta)^h f = \frac{1}{h}(L+\beta)(hf)$$

is called the *h-transform* of the operator $L+\beta$. Written out explicitly, one has

$$(L+\beta)^h f = L + a\frac{\nabla h}{h} \cdot \nabla + \frac{Lh}{h} + \beta.$$

All the properties defined above are invariant under $h$-transforms.

For further elaboration and proofs see Chapter 4 in Pinsky (1995a).

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