On equilibria for discontinuous games: Nash approximation schemes

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ABSTRACT: For discontinuous games Simon and Zame (1990) introduced a new approach to the existence of equilibria. To obtain the required continuity, they convert the original discontinuous payoff function into an upper semicontinuous *payoff correspondence*; graphically, this corresponds to a "vertical interpolation" to close the discontinuity gap at the discontinuity points. The resulting payoff indeterminacy, in the form of *endogenous sharing rules* (i.e., measurabe selections of the payoff correspondence), is an essential feature of their model. The mixed equilibrium existence result obtained by Simon and Zame (1990) generalizes Glicksberg's (1952) existence result for Nash equilibria. This paper proposes to view Simon and Zame's "vertical interpolation" as the limit of a sequence of standard (nonvertical) continuous interpolations across the discontinuity. In other words, we propose to approximate the upper semicontinuous payoff correspondence directly by means of a sequence of continuous payoff functions. To each of these Glicksberg's existence result applies, which yields a sequence of mixed Nash equilibria. The weak limit of this sequence is the equilibrium of Simon and Zame (1990). However, our approach goes beyond existence, because the approximate Nash equilibria can often be easily computed in actual examples (most notably, with the aid of purification methods). This does not only provide a new interpretation of the endogenous sharing rule as a certain conditional expectation of the payoff vector, but the precise information gathered about it in terms of the approximate Nash equilibria and their payoff values is, as we show, of considerable help in the actual computations.

KEYWORDS: Nash equilibrium, discontinuous games, weak convergence of probability measures, approximate continuous selections, endogenous sharing rule, Kuratowski limes superior, supports.

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1. Introduction

This paper introduces a new method to find equilibria with endogenous sharing rules for discontinuous games. Such equilibria were introduced by Simon and Zame (1990) and will be explained below. First, we wish to demonstrate in a concrete example how the method works. This example is the California-Oregon psychologists' game from Simon and Zame (1990); it is a location game in the spirit of Hotelling (1929). This game (call it Γ) takes place along a section of interstate highway 5, which is represented by the interval $[0,4]$. The action space of player 1, the psychologist from California, is $S_1 := [0,3]$ (i.e., the Californian highway stretch), and for player 2, the psychologist from Oregon, the action space is $S_2 := [3,4]$, which stands for the Oregon part of the highway. The payoff function of player 1 is

$$q_1(s_1,s_2) := \begin{cases} \frac{s_1+s_2}{2} & \text{if } s_1 < s_2 \\ 2 & \text{if } s_1 = s_2 = 3 \end{cases}$$

Player 2 has payoff $q_2(s_1,s_2) := 4 - q_1(s_1,s_2)$. Clearly, these functions are discontinuous at the point $(s_1,s_2) = (3,3)$, which corresponds to the California-Oregon border.

Consider the following simple approximation of Γ by means of a sequence $\{\Gamma^{(n)}\}_n$ of continuous games. For $n \in \mathbb{N}$ we define $\Gamma^{(n)}$ as the game with the same two action spaces as Γ and with continuous payoff

$$q_1^{(n)}(s_1,s_2) := \begin{cases} (\frac{1}{2}-n)s_1 + \frac{1}{2} + 3n & \text{if } 3-\frac{1}{n} \leq s_1 \leq 3 \text{ and } s_2 = 3 \\ \frac{s_1+s_2}{2} & \text{otherwise} \end{cases}$$

for player 1 and $q_2^{(n)}(s_1,s_2) := 4 - q_1^{(n)}(s_1,s_2)$ for player 2. Observe that the coefficients $v := \frac{1}{2} - n$ and $w := \frac{1}{2} + 3n$ are chosen in such a way that $v(3-\frac{1}{n}) + w = q_1(3-\frac{1}{n},3)$ and $v3 + w = q_1(3,3)$. This is a standard linear interpolation of $q := (q_1,q_2)$ by $q^{(n)} := (q_1^{(n)}, q_2^{(n)})$. It is naturally connected, in a way to be made more precise below, to a "vertical interpolation" proposed by Simon and Zame (1990). Such vertical interpolation defines a *payoff correspondence* Q_q according to the formula (3) below; in the present example this gives

$$Q_q(s_1,s_2) := \begin{cases} \{(z_1, 4-z_1) : z_1 \in [2,3]\} & \text{if } s_1 = s_2 = 3 \\ \{(q_1(s_1,s_2), q_2(s_1,s_2))\} & \text{otherwise} \end{cases}$$

Now each game $\Gamma^{(n)}$ has continuous and quasi-concave payoff functions, so it has a Nash equilibrium in pure actions. Standard computation of the best reply correspondence for player 1 gives $B_1^{(n)}(s_2) = \{3\}$ if $s_2 > 3$ and $B_1^{(n)}(s_2) = \{3 - \frac{1}{n}\}$ if $s_2 = 3$. For player 2 the best reply correspondence is given by $B_2^{(n)}(s_1) = \{3\}$. So the (unique) Nash equilibrium for $\Gamma^{(n)}$ is $s^{(n)} := (\bar{s}_1^{(n)}, \bar{s}_2^{(n)}) = (3-\frac{1}{n}, 3)$, which is of course also immediately obvious. Formally, this means also that the *mixed* action profile $\alpha^{(n)} := (\epsilon_{3-\frac{1}{n}}, \epsilon_3)$ is a mixed Nash equilibrium for game $\Gamma^{(n)}$. Observe that $(\bar{s}_1^{(n)}, \bar{s}_2^{(n)}) \to (3,3)$ and $(q_1^{(n)}(\bar{s}^{(n)}), q_2^{(n)}(\bar{s}^{(n)})) = (3 - \frac{1}{2n}, 1 + \frac{1}{2n}) \to (3,1)$ for $n \to \infty$. Trivially, this implies that $\alpha^{(n)} := (\epsilon_{3-\frac{1}{n}}, \epsilon_3)$ converges weakly to $(\alpha_1^*, \alpha_2^*) := (\epsilon_3, \epsilon_3)$ in the sense of probability measures on $[0,3] \times [3,4]$. As this paper will show in far greater generality, these two facts mean that the mixed action profile (α_1^*, α_2^*), together with the value $(3,1)$ for the sharing rule q^* at the point of discontinuity $(s_1,s_2) = (3,3)$, form an equilibrium in the sense of Simon and Zame (1990). Of course, in the present example this can also be deduced *ad hoc* from the fact that $(\bar{s}_1^{(n)}, \bar{s}_2^{(n)})$ is a Nash equilibrium for $\Gamma^{(n)}$ for every $n \in \mathbb{N}$. More precisely, it follows easily from taking limits that $3 \geq q_1^*(s_1,3)$ for every $s_1 \in S_1$ and $1 \geq q_2^*(3,s_2)$ for every $s_2 \in S_2$. So one should set $q_1^*(3,3) := 3$ and $q_2^*(3,3) := 1$. Outside the discontinuity we set $q^* = q$ at any rate, since $Q_q(s_1,s_2)$ is equal to $\{q(s_1,s_2)\}$ there. So the mixed action profile $\alpha^* := (\alpha_1^*, \alpha_2^*) := (\epsilon_3, \epsilon_3)$, together with q^*, meets the equilibrium conditions of Simon and Zame (1990), i.e.,

$$q_1^*(3,3) = \int_{S_1 \times S_2} q_1^* d(\alpha_1^* \times \alpha_2^*) = \max_{s_1 \in S_1} \int_{S_2} q_1^*(s_1,s_2) \alpha_2^*(ds_2)$$

and

$$q_2^*(3,3) = \int_{S_1\times S_2} q_2^* d(\alpha_1^* \times \alpha_2^*) = \max_{s_2\in S_2} \int_{S_1} q_2^*(s_1,s_2)\alpha_1^*(ds_1).$$

The intention of this paper is to provide a general, systematic approach to this kind of generalized equilibrium. Its highlights are as follows:

(i) For N-player games $\Gamma = (S_1,\ldots,S_N;Q_q)$ of the kind considered by Simon and Zame (1990) there always exists a sequence $\{\Gamma^{(n)}\}_n$ of suitably approximating games $\Gamma^{(n)} = (S_1,\ldots,S_N;q^{(n)})$ with continuous payoff profiles $q^{(n)} := (q_1^{(n)},\ldots,q_N^{(n)})$.

(ii) At least a subsequence of the sequence $\{\alpha^{(n)}\}$ of mixed Nash equilibrium profiles of the games $\Gamma^{(n)}$ (such equilibria exist by Glicksberg's theorem) converges weakly to some mixed action profile $\alpha^* := (\alpha_1^*,\ldots,\alpha_N^*)$.

(iii) To this limit profile α^* there corresponds a sharing rule $q^* := (q_1^*,\ldots,q_N^*)$ such that for $\alpha_1^*\times\cdots\times\alpha_N^*$- almost every $s\in S$ the point $q^*(s)$ belongs to the convex hull of the set of all limits $\lim_j q^{(n_j)}(s^j)$ with s^j in the support of the product probability measure $\alpha_1^{(n_j)}\times\cdots\alpha_N^{(n_j)}$.

(iv) The pair (α^*,q^*) is an equilibrium solution in the sense of Simon and Zame (1990) for the game Γ, i.e., $q^*(s)\in Q_q(s)$ for all s and

$$\int_S q_i^*(s)(\alpha_1^*\times\cdots\times\alpha_N^*)(ds) = \max_{s_i\in S_i}\int_{S_{-i}} q_i^*(s_i,s_{-i})(\alpha_1^*\times\cdots\alpha_{i-1}^*\times\alpha_{i+1}^*\times\cdots\times\alpha_N^*)(ds_{-i})$$

for $i=1,\ldots,N$.

As a corollary, this reproduces the existence result of Simon and Zame (1990). However, the present continuous approximation scheme can also be used in actual computations, as shown above, and this would seem not to be directly the case with the discrete approximation scheme followed by Simon and Zame (observe, for instance, that a discrete scheme allows little or no room for purification). Below we also show how the approximation scheme of the present paper provides a systematic approach to the Bertrand duopoly game. Recently, Simon and Zame (1999) and Jackson and Swinkels (2000) have used endogenous sharing rules to obtain new existence results for games with incompete information. Although these papers consider models that are considerably more complicated, it would seem that at least the global features of the approach taken here extend to such games as well.

2. Main results

Discontinuous games, with their associated problems regarding the formulation and existence of appropriate equilibrium notions, form a subject of long-standing interest in economics. The oldest of these, Bertrand's duopoly game, dates back to 1883. Another example is provided by the psychologists' game discussed in the introductory section. Such games are characterized by the fact that they have discontinuities in their payoffs as a function of player's actions. For instance, in Bertrand's duopoly game the firm that sets the lowest price captures the entire market. Several other examples of discontinuous games can be found in Dasgupta and Maskin (1986a,b). In the face of such discontinuity, general theorems about the existence of Nash equilibria for such games would seem to be out of the question. Nevertheless, quite some progress has been made in this direction. Let us mention in this connection the papers by Dasgupta and Maskin (1986a,b), Simon (1987) and Reny (1999). The former paper supposes a particular structure of the payoff discontinuities; this was refined by Simon (1987). In both these papers an approximation by finite games and their associated mixed Nash equilibria is used. In a far-reaching generalization of this work, Reny (1999) showed that the entire question of finding Nash equilibria can be transformed by means of his better-reply security condition. In this way, it is possible to find pure equilibrium existence results as well (this contrasts notably with the two other papers). In a direction that departs considerably from this development, Simon and Zame (1990) introduced a new approach to the existence of equilibria by allowing incomplete determinacy of the payoff function. Using a *payoff correspondence*, which acts as an upper semicontinuous hull of the payoff profile, they manage to restore the essential continuity properties of the payoff structure. The present paper intends to contribute to this particular development. First, we shall make the notions of Simon and Zame (1990) more precise; this is followed immediately by the main new notions and results of this paper.

Let $I := \{1, \ldots, N\}$ be a set of N players (let us merely remark that the approach followed here extends easily to the case of countably many players). Each player $i \in I$ has a nonempty action space S_i, for which we suppose

$$S_i \text{ is metric and compact.} \tag{1}$$

We denote the product space $\Pi_{j=1}S_j$ by S, which is itself also compact and metric; below d_S denotes a metric on S (e.g., the sum metric). As will be tacitly done for all other metric spaces in the sequel as well, S is equipped with the Borel σ-algebra. Player i's *mixed* actions are formed by the set $\mathcal{P}(S_i)$ of all probability measures on S_i. The set of all *mixed action profiles* is $\Pi_{j=1}^N \mathcal{P}(S_j)$. Let $q := (q_1, \ldots, q_N) : S \to \mathbb{R}^N$ be a profile of payoff functions $q_i : S \to \mathbb{R}$ such that

$$q \text{ is Borel measurable and bounded.} \tag{2}$$

The following definition is classical:

Definition 1 A *mixed Nash equilibrium for* q is a mixed action profile $(\alpha_1, \ldots, \alpha_N)$ in $\Pi_{j=1}^N \mathcal{P}(S_j)$ such that for every $i \in I$

$$\int_S q_i d\tilde{\alpha} = \max_{s_i \in S_i} \int_{S_{-i}} q_i(s_i, s_{-i}) \tilde{\alpha}_{-i}(ds_{-i}).$$

Here $\tilde{\alpha}$ denotes the product measure $\alpha_1 \times \cdots \alpha_N$ on S and $\tilde{\alpha}_{-i}$ stands for $\alpha_1 \times \cdots \times \alpha_{i-1} \times \alpha_{i+1} \times \cdots \times \alpha_N$ on $S_{-i} := \Pi_{j \in I, j \neq i} S_j$. Moreover, writing $s = (s_i, s_{-i})$, etc. for $s := (s_1, \ldots, s_N)$, follows the usual abuse of notation in game theory.

Following Simon and Zame (1990), we define the *payoff correspondence* $Q_q : S \to 2^{\mathbb{R}^N}$, corresponding to q above, as follows:

$$Q_q(s) := \text{co (cl gph } q)_s = \text{co } \{z \in \mathbb{R}^N : (s, z) \in \text{cl gph } q\}, \tag{3}$$

where "co" means convex hull, "cl" closure, and where gph q denotes the *graph* $\{(s, q(s)) : s \in S\}$ of q. Note that it is the *section* of cl gph q at s that is convexified; this corresponds to what we previously called "vertical interpolation" for the component functions q_i. It is not hard to verify that Q_q is the smallest upper semicontinuous and convex-valued multifunction of which q is a measurable selection. Observe that by (2) all values of Q_q are contained in some bounded subset of $\mathbb{R}^N$. Therefore, the upper semicontinuity of Q_q is equivalent with the evident property that its graph gph $Q_q := \{(s, z) \in S \times \mathbb{R}^N : z \in Q_q(s)\}$ is closed. The idea behind the introduction of the payoff correspondence Q_q is that, for the purpose of equilibrium existence, discontinuity of the payoff profile q is compensated by the freedom to choose the actual payoff function from this upper semicontinuous hull:

Definition 2 A *mixed Nash equilibrium for the correspondence* Q_q is a mixed action profile $\alpha := (\alpha_1, \ldots, \alpha_N)$ in $\Pi_{j=1}^N \mathcal{P}(S_j)$ for which there exists a measurable selection q^* of Q_q, called the corresponding *endogenous sharing rule*, such that α is a mixed Nash equilibrium for q^*.

As the next result shows, in the classical situation with a continuous payoff profile there is essentially no difference between q and the payoff correspondence Q_q. More generally, this proposition shows that outside any discontinuity point that q may have, the payoff correspondence Q_q is single-valued and coincides with the original payoff profile q:

Proposition 1 *At any point $s \in S$ the following equivalence holds:*

$$q \text{ is continuous at } s \text{ if and only if } Q_q(s) = \{q(s)\}.$$

In particular, if the function $q : S \to \mathbb{R}^N$ is continuous, then $Q_q = \{q\}$; consequently, in that situation any mixed Nash equilibrium for Q_q is a mixed Nash equilibrium for q and vice versa.

We come now to the main new concept of this paper. A *Nash approximation scheme* for the game $\Gamma := (S_1, \ldots, S_N; Q_q)$ is defined to be a sequence $\{(q^{(n)}, \alpha^{(n)})\}_n$ of *approximate payoff profiles* $q^{(n)} : S \to \mathbb{R}^N$ and mixed action profiles $\alpha^{(n)}$ in $\Pi_{j=1}^N \mathcal{P}(S_j)$, $n \in \mathbb{N}$, such that the following hold:

- $q^{(n)} : S \to \mathbb{R}^N$ is *continuous* for every $n \in \mathbb{N}$.
- $\alpha^{(n)}$ is a mixed Nash equilibrium for the game $\Gamma^{(n)} := (S_1, \ldots, S_N; q^{(n)})$ for every $n \in \mathbb{N}$.
- $\lim_{n\to\infty} \sup_{s\in S} \mathrm{dist}((s, q^{(n)}(s)), \mathrm{gph}\ Q_q) = 0$.

The latter property tells us that the graphs gph $q^{(n)}$ of the approximate payoff functions $q^{(n)}$ come arbitrarily close to the graph of the payoff correspondence Q_q. Here $\mathrm{dist}((s, z), Q) := \inf_{(s',z')\in Q} d_S(s, s') + |z - z'|$ defines the distance of a point (s, z) to a set Q in $S \times \mathbb{R}^N$ in the usual way.

Example 1 From plots of $q_i^{(n)}$ and $(Q_q)_i$ in the example from the introductory section (location game) it is already evident that gph $q^{(n)}$ converges to gph Q_q in the above sense. More formally, the following can be said. We have $Q_q(s_1, s_2) = \{q^{(n)}(s_1, s_2)\}$ if $s_2 \neq 3$ or $s_1 < 3 - \frac{1}{n}$. So to determine the distance expression in question it is enough to concentrate on the points $(s_1, 3)$ with $3-\frac{1}{n} \leq s_1 \leq 3$. For any such s_1 let $((s_1, 3), q^{(n)}(s_1, 3))$ be a point in gph $q^{(n)}$; this point has distance at most $\frac{1}{n}$ to the point $((3, 3), (q^{(n)}(s_1, 3))$, which lies in gph Q_q (because of $2 \leq q_1^{(n)}(s_1, 3) \leq 3 - \frac{1}{2n}$ and $1 + \frac{1}{2n} \leq q_2^{(n)}(s_1, 3) \leq 2$). So $\sup_{s\in S} \mathrm{dist}((s, q^{(n)}(s)), \mathrm{gph}\ Q_q) \leq \frac{1}{n} \to 0$. Together with what was shown in the introductory section about $\alpha^{(n)} := (\epsilon_{3-\frac{1}{n}}, \epsilon_3)$, it follows that $\{(q^{(n)}, \alpha^{(n)}\}_n$ is a Nash approximation scheme.

Proposition 2 *Under (1)–(2) there exists a Nash approximation scheme.*

Recall that the *limes superior* (in the sense of Kuratowski) of the sequence of graphs $\{\mathrm{gph}\ q^{(n)}\}_n$ is defined as the set of all limit points of $\{\mathrm{gph}\ q^{(n)}\}_n$, i.e., the set of all $(s, z) \in S\times\mathbb{R}^N$ for which there exist a subsequence $\{\mathrm{gph}\ q^{n_j}\}_j$ and corresponding $(s^j, z^j) \in \mathrm{gph}\ q^{(n_j)}$ such that (s^j, z^j) converges to (s, z). This set will be denoted by $\mathrm{Ls}_n\mathrm{gph}\ q^{(n)}$. Since $S \times \mathbb{R}^N$ is a metric space, one has the following representation for this limes superior:

$$\mathrm{Ls}_n\mathrm{gph}\ q^{(n)} = \cap_{p=1}^{\infty}\mathrm{cl}\ \cup_{n\geq p} \mathrm{gph}\ q^{(n)}. \tag{4}$$

Recall also from Billingsley (1968) or Dellacherie and Meyer (1975) that the *weak* topology on the set $\mathcal{P}(S_i)$ of all probability measures on S_i is the weakest topology on $\mathcal{P}(S_i)$ for which the mapping $\nu \mapsto \int_{S_i} c\, d\nu$ is continuous for every bounded and continuous function $c : S_i \to \mathbb{R}$. By Assumption 1 the set $\mathcal{P}(S_i)$ is metric and compact for the weak topology. Thus, we can also describe the weak topology in terms of sequential convergence (and this is what is done in a major part of Billingsley (1968)): a sequence $\{\nu_n\}_n$ in $\mathcal{P}(S_i)$ converges weakly to ν_0 if the integrals $\int_{S_i} c\, d\nu_n$ converge to $\int_{S_i} c\, d\nu_0$ for every bounded and continuous function $c : S_i \to \mathbb{R}$. The set of all mixed action profiles in our model is the Cartesian product $\Pi_{i\in I}\mathcal{P}(S_i)$, and from now on we equip it with the weak product topology.

Theorem 1 *Under (1)–(2) every Nash approximation scheme $\{(q^{(n)}, \alpha^{(n)})\}_n$ yields in the limit a mixed Nash equilibrium $\alpha^* := (\alpha_1^*, \ldots, \alpha_N^*)$ for Q_q as follows:*

(i) There exists a subsequence $\{\alpha^{(n')}\}_{n'}$ of $\{\alpha^{(n)}\}_n$ that weakly converges in $\Pi_{j=1}^N \mathcal{P}(S_j)$ to some limit mixed profile α^*.

(ii) This profile α^* is a mixed Nash equilibrium for Q_q with a corresponding endogenous sharing rule q^* such that

$$q^*(s) \in \mathrm{co}\ (\mathrm{Ls}_n\mathrm{gph}\ q^{(n)})_s \text{ for every } s \in S,$$

and

$$q^*(s) \in \mathrm{co}\ L(s) \text{ for } \tilde{\alpha}^*\text{-almost every } s \text{ in } S.$$

Here $\tilde{\alpha}^* := \alpha_1^* \times \cdots \times \alpha_N^*$ and $L(s)$ is the set of all $z \in \mathbb{R}^N$ for which there exists a subsequence $\{(q^{(n_k)}(s^{(n_k)})\}_k$ of $\{(q^{(n)}(s^{(n)})\}_n$ with $s^{(n_k)} \to s$, $q^{(n_k)}(s^{(n_k)}) \to z$ and with $s^{(n_k)} \in \mathrm{supp}\ \tilde{\alpha}^{(n_k)}$ for every k.

Recall here that the *support* of a probability measure ν in $\mathcal{P}(S)$ is defined by supp $\nu := \cap\{F : F \subset S, F$ closed and $\nu(F) = 1\}$. Concatenation of Proposition 2 and Theorem 1 gives the existence theorem on p. 865 of Simon and Zame (1990):

Corollary 1 *Under (1)–(2) there exists a Nash equilibrium for Q_q.*

We conclude this section by demonstrating the effectiveness of Nash approximation schemes for the computation of the equilibrium in a Bertrand duopoly game. Although such equilibria can, of course, be determined in an *ad hoc* fashion, the present approach offers the attraction of being systematic.

To make our calculations easy, we shall assume linear demand $D(s_i) := a - s_i$ and per-unit production cost c with $c < a$. More elaborate versions can be produced easily. The profit of firm $i \in I := \{1, 2\}$ is $q_i(s_1, s_2) := (s_i - c)(a - s_i)$ if $s_i < s_j$ and $q_i(s_1, s_2) := 0$ if $s_i > s_j$ (here $j = 2$ if $i = 1$ and $j = 1$ if $i = 2$ and s_i, s_j are the price variables). In addition, for obvious reasons the price variables must satisfy $s_i, s_j \geq 0$ and $s_i, s_j \leq a$. A discontinuity occurs when $s_i = s_j$, that is, when both firms charge the same price. It is standard in the literature to allocate total demand evenly in this case: $q_i(s_1, s_1) := \frac{1}{2}(s_1 - c)(a - s_1)$, but note that there is no compelling economic reason for such a division of the demand. In any case, this is irrelevant for the payoff correspondence, since (3) gives here

$$Q_q(s_1, s_1) := \{(\lambda(s_1 - c)(a - s_1), (1 - \lambda)(s_1 - c)(a - s_1) : 0 \leq \lambda \leq 1\}.$$

Here λ, the fraction of the total demand that goes to firm 1, forms an additional parameter of the model. As follows by Proposition 1, in all other points of $S := [0, a]^2$ the correspondence Q_q coincides with $\{q\}$.

We start a Nash approximation scheme by defining the continuous function $q^{(n)} : S \to \mathbb{R}^2$ by

$$q_i^{(n)}(s_i, s_j) := \begin{cases} q_i(s_i, s_j) & \text{if } s_i \leq s_j - \frac{1}{n} \text{ or } s_i \geq s_j + \frac{1}{n} \\ vs_i + w & \text{if } s_j - \frac{1}{n} < s_i < s_j + \frac{1}{n} \end{cases}$$

Here v, w are determined by $v(s_j - \frac{1}{n}) + w = q_i(s_j - \frac{1}{n}, s_j)$ and $v(s_j + \frac{1}{n}) + w = p_i(s_j + \frac{1}{n}, s_j) = 0$ (i.e., $v = -\frac{n}{2}(s_j - \frac{1}{n} - c)(a - s_j + \frac{1}{n})$ and $w = \frac{n}{2}(s_j - \frac{1}{n} - c)(a - s_j + \frac{1}{n})(s_j + \frac{1}{n})$). It is easy to check that this scheme is as specified in Lemma 1: if $s_j - \frac{1}{n} < s_i < s_j + \frac{1}{n}$ then the distance of $(s_i, vs_i + w)$ to gph Q_q is at most $\frac{2}{n}$.

To determine the Nash equilibrium profile $(\alpha_1^{(n)}, \alpha_2^{(n)})$ for each approximating game $\Gamma^{(n)}$, we may use purification, since each $q_i^{(n)}(s_i, s_j)$ is quasi-concave in the variable s_i (actually, less is required for such purification; cf. Balder (2001b)). Afterwards, to regain the terms of Theorem 1, we return to mixed actions by means of point measures. In pure actions the best reply correspondence $B_i^{(n)} : [0, a] \to 2^{[0,a]}$ for player i in game $\Gamma^{(n)}$ is given by

$$B_i(s_j) := \begin{cases} \{s_i \in [0, a] : s_i \geq s_j + \frac{1}{n}\} & \text{if } s_j \leq c + \frac{1}{n} \\ \{s_j - \frac{1}{n}\} & \text{if } c + \frac{1}{n} < s_j \leq \frac{a+c}{2} + \frac{1}{n} \\ \{\frac{a+c}{2}\} & \text{if } s_j > \frac{a+c}{2} + \frac{1}{n} \end{cases}$$

The fixed points $\bar{s}^{(n)}$ of this correspondence are easily seen to form the following set S_0: the union of all pairs $(s_2 + \frac{1}{n}, s_2)$ with $s_2 \in (c, c + \frac{1}{n}]$ and all pairs $(s_1, s_1 + \frac{1}{n})$ with $s_1 \in (c, c + \frac{1}{n}]$. Evidently, no matter which pairs of pure actions we choose at stage n, they converge to (c, c) for $n \to \infty$. In parallel, at stage n every pair $(\alpha_1^{(n)}, \alpha_2^{(n)})$ of point measures $\alpha_i^{(n)} := \epsilon_{\bar{s}_i^{(n)}}$ is a mixed equilibrium profile for the approximate game $\Gamma^{(n)}$ and, regardless of the choice of $\bar{s}^{(n)}$ in S_0, these profiles converge weakly to $\alpha^* := (\epsilon_c, \epsilon_c)$ as $n \to \infty$. By Theorem 1 α^* is a Nash equilibrium for a measurable profile $q^* : S \to \mathbb{R}^2$. For this q^* we have $q^*(s_1, s_2) = q(s_1, s_2)$ if $s_1 \neq s_2$ by Proposition 1. Moreover, Theorem 1 also implies $q^*(c, c) = (0, 0)$. Together, this tells us the following: $0 = q_1^*(c, c) \geq q_1^*(s_1, c)$ for every $s_1 \in [0, a]$ and $0 = q_2^*(c, c) \geq q_2^*(c, s_2)$ for every $s_2 \in [0, a]$. This outcome agrees with the classical one, even though the precise form of profit sharing when equal prices are imposed was left unspecified.

Prompted by this example and the one given in the introduction, we observe that it would be interesting to find general conditions under which quasi-concavity of the q_i can be transferred to the approximate payoffs $q_i^{(n)}$ (rather than having to find this out *ad hoc* in each example).

3. Proofs

Proof of Proposition 1. Fix $s \in S$ and suppose q is continuous in the point s. Then $(\text{cl gph } q)_s = \{q(s)\}$, for if $(s,z) \in \text{cl gph } q$ then $z = q(s)$ by continuity of q. Conversely, suppose $(\text{cl gph } q)_s = \{q(s)\}$ and let $s^k \to s$. By the given boundedness of q (see (2)), the sequence $\{q(s^k)\}_k$ is bounded. It is enough to show that every limit point $\bar{z}$ of $\{q(s^k)\}_k$ equals $q(s)$. But for any such $\bar{z}$ the point $(s, \bar{z})$ belongs to cl gph q, so $\bar{z} \in \{q(s)\}$. QED

Of course, boundedness of q is indispensible for Proposition 1 (e.g., for $S := [0,1]$ consider $q(s) := 1/s$ if $s > 0$ and $q(0) := 0$).

Lemma 1 *There exists a sequence $\{q^{(n)}\}_n$ of continuous functions $q^{(n)} : S \to \mathbb{R}^N$ such that*

$$\lim_{n\to\infty} \sup_{s\in S} \text{dist}((s, q^{(n)}(s)), \text{gph } Q_q) = 0.$$

This lemma is a direct application of Cellina's approximate continuous selection theorem (see Theorem 1 on p. 84 of Aubin and Cellina). This result applies, since Q_q is an upper semicontinuous multifunction with convex values, as we already observed following (3).

Lemma 2 *For every $n \in \mathbb{N}$ there exists a mixed Nash equilibrium profile $\alpha^{(n)}$ for $q^{(n)}$.*

Since $q_i^{(n)}$ is continuous for every $i \in I$, Lemma 2 is the classical existence result of Glicksberg (1952).

Proof of Proposition 2. Combine Lemmas 1 and 2.

Lemma 3 *The sequence $\{\alpha^{(n)}\}_n$ contains a subsequence $\{\alpha^{(n')}\}_{n'}$ such that for every $i \in I$ the following holds: $\alpha_i^{(n')}$ converges weakly to some α_i^* in $\mathcal{P}(S_i)$ for $n' \to \infty$.*

Since each S_i is compact metric, we have by Theorem III.60 in Dellacherie and Meyer (1975) that each space $\mathcal{P}(S_i)$ is compact and metrizable for the weak topology (for instance, the Prohorov metric can be used for this purpose; cf. Billingsley (1968)). Hence, each $\mathcal{P}(S_i)$ is also sequentially weakly compact, so the result follows with ease. To save on notation, we shall pretend without loss of generality that the sequences $\{\alpha_i^{(n)}\}_n$ converge as a whole to α_i^*, $i \in I$.

We now come to the heart of the proof of Theorem 1, which involves the construction of the sharing rule from a limit product probability measure π^*. Similar constructions are well-known in Young measure theory and its generalizations; they go back to work of L.C. Young and E.J. McShane around 1940 – e.g., see Balder (2000, 2001a). Without loss of generality we shall assume that all graphs gph $q^{(n)}$ have a distance to gph Q_q of at most 1; in view of the above Lemma 1 we can do so. Then $q^{(n)} : S \to Z$ for every $n \in \mathbb{N}$, where $Z := [-\|q\|_\infty - 1, \|q\|_\infty + 1]^N$ with $\|q\|_\infty := \sup_{s\in S} |q(s)| < +\infty$. We form the following probability measures on $S \times Z$, which are a classical instrument in Young measure theory. For every Borel subset A of S and every Borel subset B of Z we define

$$\pi^{(n)}(A \times B) := \tilde{\alpha}^{(n)}(A \cap \{s \in S : q^{(n)}(s) \in B\}), \tag{5}$$

and we remember that this completely determines the product measure $\pi^{(n)} \in \mathcal{P}(S \times Z)$, because the measurable rectangles $A \times B$ generate the Borel σ-algebra on $S \times Z$.

Lemma 4 *The sequence $\{\pi^{(n)}\}_n$ contains a subsequence $\{\pi^{(n')}\}_{n'}$ which converges weakly to some $\pi^* \in \mathcal{P}(S \times Z)$.*

Since $S \times Z$ is compact and metric, the proof runs just as that of Lemma 3. Again we shall pretend in the remainder, without any loss of generality, that $\{\pi^{(n)}\}$ as a whole converges to π^*. Our next result can also be obtained as a direct application of Theorem 2.7 in Balder (2000).

Lemma 5 *The support* supp π^* *of the probability measure* π^* *is contained in the set* $\mathrm{Ls}_n \mathrm{supp}\ \pi^{(n)}$ *of all limit points of the supports of the measures* $\pi^{(n)}$*,* $n \in \mathbb{N}$*.*

PROOF. Let $F_p := \mathrm{cl}\ \cup_{n \geq p} \mathrm{supp}\ q^{(n)}$, for $p \in \mathbb{N}$. This is a closed set, so by the portmanteau theorem for weak convergence (see Theorem 2.1 of Billingsley (1968)) and Lemma 4 we have $\limsup_n \pi^{(n)}(F_p) \leq \pi^*(F_p)$. The left hand side here equals 1, so we conclude $\pi^*(F_p) = 1$. Since $\mathrm{Ls}_n \mathrm{supp}\ \pi^{(n)} = \cap_p F_p$ by (4, it follows that $\pi^*(\mathrm{Ls}_n \mathrm{supp}\ \pi^{(n)}) = 1$. It only remains to note that $\mathrm{Ls}_n \mathrm{supp}\ \pi^{(n)}$ is closed. QED

Lemma 6 *The support of every probability measure* $\pi^{(n)}$*,* $n \in \mathbb{N}$*, is given by*

$$\mathrm{supp}\ \pi^{(n)} = \{(s, q^{(n)}(s)) \in S \times Z : s \in \mathrm{supp}\ \tilde{\alpha}^{(n)}\} \subset \mathrm{gph}\ q^{(n)}.$$

PROOF. Denote the set in the middle by $\tilde{F}$. This set is clearly closed and by Proposition III.2.1 in Neveu (1965) it follows that $\pi^{(n)}(\tilde{F}) = \tilde{\alpha}^{(n)}(\mathrm{supp}\ \tilde{\alpha}^{(n)}) = 1$. So $\tilde{F}$ contains supp $\pi^{(n)}$. Conversely, the set F, defined to consist of all $s \in S$ for which $(s, q^{(n)}(s)) \in \mathrm{supp}\ \pi^{(n)}$, is a closed set in S. Since $\pi^{(n)}(\mathrm{supp}\ \pi^{(n)}) = 1$, it follows by Proposition III.2.1 of Neveu (1965) that $q^{(n)}(s) \in (\mathrm{supp}\ \pi^{(n)})_s$ for $\tilde{\alpha}^{(n)}$-almost every s in S. This is to say that $s \in F$ for $\tilde{\alpha}^{(n)}$-almost every s in S, so $\alpha^{(n)}(F) = 1$. It follows that supp $\alpha^{(n)} \subset F$, which in turn implies $\tilde{F} \subset \mathrm{supp}\ \pi^{(n)}$. QED

Combining Lemmas 5 and 6, we get:

Lemma 7 *The following inclusion holds:* $\mathrm{supp}\ \pi^* \subset L := \mathrm{Ls}_n\{(s, q^{(n)}(s)) \in S \times Z : s \in \mathrm{supp}\ \tilde{\alpha}^{(n)}\}$.

Lemma 8 *The probability measure* π^* *can be decomposed as follows: there exists a transition probability* η^* *with respect to* S *and* Z *such that for every pair of Borel sets* $A \subset S$ *and* $B \subset Z$

$$\pi^*(A \times B) = \int_A \eta^*(s; B) \tilde{\alpha}^*(ds).$$

PROOF. We invoke a classical disintegration theorem (see Valadier (1973)). The only observation still to be made is the following. By weak convergence of $\pi^{(n)}$ to π^* it follows that marginal probabilities of $\pi^{(n)}$ converge weakly to the corresponding marginal of π^*. Now observe that each $\pi^{(n)}$ has $\tilde{\alpha}^{(n)}$ as its marginal probability on S, because $\pi^{(n)}(A \times Z) = \tilde{\alpha}^{(n)}(A)$ by (5). So it follows that $\tilde{\alpha}^*$, which we know to be the weak limit of the $\tilde{\alpha}^{(n)}$, is the marginal probability of π^* on S. The classical disintegration theorem then guarantees the existence of the transition probability η^* for which the stated identity is true. QED

We note that $\eta^*(s; \cdot)$ above can also be interpreted as the distribution of the payoff vector z under π^*, conditional upon the outcome s of the players' actions under that same probability distribution. The previous lemma will now be sharpened:

Lemma 9 *There exists a transition probability* δ^* *with respect to* S *and* Z *such that such that for every pair of Borel sets* $A \subset S$ *and* $B \subset Z$

$$\pi^*(A \times B) = \int_A \delta^*(s; B) \tilde{\alpha}^*(ds)$$

and $\delta^*(s; G_s) = 1$ *for every* $s \in S$*, where* $G := \mathrm{Ls}_n \mathrm{gph}\ q^{(n)}$*. Moreover,* $\delta^*(s; L_s) = 1$ *for* $\tilde{\alpha}^*$*-almost every* s *in* S*.*

PROOF. Of course, Lemmas 5 and 6 imply that L coincides with $\mathrm{Ls}_n \mathrm{supp}\ \pi^{(n)}$. By $\mathrm{supp}\ \pi^{(n)} \subset \mathrm{gph}\ q^{(n)}$ (see Lemma 6), this implies $L \subset G$. Let η^* be as in Lemma 8. From Lemma 5 we know $\pi^*(L) = 1$. Hence Lemma 8 implies

$$1 = \pi^*(L) = \int_S \eta^*(s; L_s) \tilde{\alpha}^*(ds),$$

by simple properties of product measures (see Proposition III.2.1 of Neveu (1965)). Since $\eta^*(s; L_s) \leq 1$ for all $s \in S$, it follows that the Borel set $M := \{s \in S : \eta^*(s; L_s) < 1\}$ has $\tilde{\alpha}^*$-measure zero. By (4) and Lemma 4.1 of Balder (1996), the correspondence $s \mapsto G_s$ is measurable in the sense of Theorem III.2 of Castaing and Valadier (1977). So by the measurable selection Theorem III.6 in Castaing and Valadier (1977), there exists a Borel measurable function $\hat{q} : S \to Z$ with $\hat{q}(s) \in G_s$ for every $s \in S$. Since $L \subset G$, the desired transition probability δ^* is now obtained by defining

$$\delta^*(s; B) := \begin{cases} \eta^*(s; B) & \text{if } s \in S \backslash M \\ 1 & \text{if } s \in M \text{ and } \hat{q}(s) \in B \\ 0 & \text{if } s \in M \text{ and } \hat{q}(s) \notin B \end{cases}$$

In words, δ^* is the modification of η^* that is obtained by making it equal to the point measure at $\hat{q}(s)$ for s in the exceptional set M. QED

Clearly, just as with η^*, we can interpret $\delta^*(s; \cdot)$ as the conditional distribution of the payoff vector, given that s is the outcome of the players' actions under π^*. In the next lemma we take the corresponding conditional expectation of the payoff vector.

Lemma 10 *For every $s \in S$ the integral $\int_Z z\, \delta^*(s; dz)$ defines an element $q^{**}(s)$ in $\text{co } G_s \subset Q_q(s)$. Moreover, $q^{**} : S \to Z$, thus defined, is a measurable function and $q^{**}(s) \in \text{co } L_s$ for $\tilde{\alpha}^*$-almost every s in S.*

PROOF. Lemma 1 gives $G := \text{Ls}_n \text{gph } q^{(n)} \subset \text{gph } Q_q$ by closedness of gph Q_q. Hence, co $G_s \subset Q_q(s)$ by convexity of the set $Q_q(s)$. Existence of the integral defining $q^{**}(s)$ is just a consequence of boundedness of the set Z. Measurability of q^{**} follows by standard properties of transition probabilities (see Proposition III.2.1 in Neveu (1965)). By Lemma 9 and a well-known property of expectations (see Pfanzagl (1974)) the point $q^{**}(s)$, i.e., the expectation of $\delta^*(s)$, belongs to the convex hull of G_s for every $s \in S$ and to the convex hull of L_s for $\tilde{\alpha}^*$-a.e. s in S. QED

Lemma 11 *For every $i \in I$ there exists a Borel set N_i in S_i, $\alpha_i^*(N_i) = 0$, such that for every $s_i \in S_i \backslash N_i$*

$$\int_S q^{**}(s) \tilde{\alpha}^*(ds) \geq \int_{S_{-i}} q_i^{**}(s_i, s_{-i}) \tilde{\alpha}_{-i}^*(ds_{-i}). \tag{6}$$

PROOF. Fix $i \in I$. The definition of $\pi^{(n)}$ gives

$$\int_S q_i^{(n)} d\tilde{\alpha}^{(n)} = \int_{S \times Z} z_i\, \pi^{(n)}(d(s, z))$$

for every $n \in \mathbb{N}$. Since the coordinate projection $(s, z) \mapsto z_i$ is continuous and bounded on $S \times Z$, it follows by weak convergence of $\pi^{(n)}$ to π^* (see Lemma 4) that

$$\lim_{n \to \infty} \int_S q_i^{(n)} d\tilde{\alpha}^{(n)} = \int_{S \times Z} z_i\, \pi^*(d(s, z)) = \int_S [\int_Z z^i\, \delta^*(s; dz)] \tilde{\alpha}^*(ds) = \int_S q_i^{**} d\tilde{\alpha}^*,$$

where the second identity holds by Lemma 8 and the third by Lemma 10. Let B be an arbitrary Borel set in S_i. Then obviously for every $n \in \mathbb{N}$

$$\alpha_i^{(n)}(B) \int_S q^{(n)} d\tilde{\alpha}^{(n)} \geq \int_{B \times S_{-i} \times Z} q_i^{(n)} d\tilde{\alpha}^{(n)}, \tag{7}$$

because $\alpha^{(n)}$ is a mixed Nash equilibrium profile (Lemma 2). We claim that this implies

$$\alpha_i^*(B) \int_S q_i^{**} d\tilde{\alpha}^* \geq \int_{B \times S_{-i}} q_i^{**} d\tilde{\alpha}^*. \tag{8}$$

It is enough to prove this claim, for (6) follows from it with ease, in view of the arbitrariness of the Borel subset B of S_i. First, suppose in addition that the boundary ∂B of B satisfies $\alpha_i^*(\partial B) = 0$.

In that case (8) follows immediately from (7) by standard properties of weak convergence, in view of Lemma 3 (observe that the boundary of $B \times S_{-i}$, that is $\partial B \times S_{-i}$, has $\tilde{\alpha}^*$-measure zero). From this, it is not hard to extend the validity of (8) to any Borel set B in S_i. First, if B is closed we let B^ϵ be the set of all $s \in S_i$ whose distance to B is (strictly) less than $\epsilon > 0$. The boundaries of the sets B^ϵ, which are all disjoint, can have positive α_i^*-measure for at most countably many ϵ. So for all other ϵ the set B^ϵ is of the type for which (8) was shown to hold. By taking a countable collection (ϵ_n) of these, with $\epsilon_n \to 0$, the claimed inequality (8) now follows from the fact that the intersection of all B^{ϵ_n}, $n \in \mathbb{N}$ is the set B (the latter holds by closedness of B). Second, the probability measure α_i^* on S_i is regular by Billingsley (1968), Theorem 1.1. So for general Borel B in S_i there exists for every $\epsilon > 0$ a closed subset F^ϵ of B with $\alpha_i^*(B \backslash F^\epsilon) < \epsilon$. Then for each F^ϵ the inequality (8) is valid by the above; hence its validity for B follows by a simple approximation argument. This concludes the proof of the claim and of the lemma as a whole. QED

Lemma 12 *Let $q^* : S \to Z$ be defined as follows: for $i \in I$ and $s = (s_i, s_{-i})$ set $q^*(s) := p_i(s)$ if $s_i \in N_i$ and if there is no $j \neq i$ for which $s \in N_j \times S_{-j}$; otherwise, set $q^*(s) := q^{**}(s)$. Here p^i is a fixed measurable selection of $s \mapsto \text{argmin}_{z \in G_s} z_i$ Then α^* is a Nash equilibrium for q^*. Also, $q^*(s) \in \text{co } G_s$ for every $s \in S$ and $q^*(s) \in L(s)$ for $\tilde{\alpha}^*$-almost every s in S.*

PROOF. A similar modification of q^{**} was given by Simon and Zame (1990), who used Q_q instead of the present more precise $s \mapsto G_s$. Observe first that p^i is well-defined: by Lemmas 4.1 and 4.2 of Balder (1996) the correspondence $s \mapsto \text{argmin}_{z \in G_s} z_i$ is measurable in the sense of Theorem III.2 of Castaing and Valadier (1977), so we can again apply the measurable selection Theorem III.6 of that same reference. Fix $i \in I$ and $s_i \in S_i$. We shall argue that (6) remains valid if q^{**} is replaced by q^*. If $s_i \in S_i \backslash N_i$, then $q^{**}(s_i, s_{-i})$ and $q^*(s_i, s_{-i})$ can only differ if s_{-i} belongs to the union of the sections $(N_j \times S_{-j})_{s_i}$, $j \neq i$. But this union clearly has measure zero under the product measure $\tilde{\alpha}_{-i}^*$ (which contains a factor α_j^* of course), so the right hand side of (6) is not affected if we change q^{**} into q^* and for the left hand side this is entirely obvious. If $s_i \in N_i$, then $q^{**}(s_i, s_{-i})$ is only different from $q^*(s_i, s_{-i})$ if there is no $j \neq i$ with $(s_i, s_{-i}) \in N_j \times S_{-j}$. It remains to show that for those points the change from q^{**} to q^* means a change downward, i.e., $q_i^*(s_i, s_{-i}) = p_i^i(s_i, s_{-i}) \leq q_i^{**}(s_i, s_{-i})$. Since $q^{**}(s) \in \text{co } G_s$ (by Lemma 10) this follows from the elementary identity

$$p_i^i(s) = \inf_{z \in G_s} z_i = \inf_{z \in \text{co } G_s} z_i.$$

We conclude that α^* is a mixed Nash equilibrium for q^*. Finally, note that $p^i(s) \in G_s$ by definition and that by Lemma 10 both $q^{**}(s) \in \text{co } G_s$ for all s and $q^{**}(s) \in L_s$ for $\tilde{\alpha}^*$-a.e. s. It remains to observe that L_s, the section at s of $L := \text{Ls}_n\{(s, q^{(n)}(s)) \in S \times Z : s \in \text{supp } \tilde{\alpha}^{(n)}\}$, coincides with the set $L(s)$ as defined in Theorem 1. So the modification q^* is as stated. QED

PROOF OF THEOREM 1. Part (i) follows from Lemma 3 and part (ii) follows by Lemma 12.

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