New Examples of Willmore Surfaces in S^n

Haizhong Li * and Luc Vrancken †

Abstract

A surface $x : M \to S^n$ is called a Willmore surface if it is a critical surface of the Willmore functional $\int_M (S - 2H^2)dv$, where H is the mean curvature and S is the square of the length of the second fundamental form. It is well-known that any minimal surface is a Willmore surface. The first non-minimal example of a flat Willmore surface in higher codimension was obtained by Ejiri. This example which can be viewed as a tensor product immersion of $S^1(1)$ and a particular small circle in $S^2(1)$, and therefore is contained in $S^5(1)$ gives a negative answer to a question by Weiner. In this paper we generalize the above mentioned example by investigating Willmore surfaces in $S^n(1)$ which can be obtained as a tensor product immersion of two curves. We in particular show that in this case too, one of the curves has to be $S^1(1)$, whereas the other one is contained either in $S^2(1)$ or in $S^3(1)$. In the first case, we explicitly determine the immersion in terms of elliptic functions, thus constructing infinetely many new non-minimal flat Willmore surfaces in S^5. Also in the latter case we explicitly include examples.

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1 Introduction

Let $x : M \to S^n$ be a surface in the n-dimensional unit sphere S^n. If h^α_{ij} denotes the components of the second fundamental form of M, S denotes the square of the length of the second fundamental form, $\mathbf{H}$ denotes the mean curvature vector and H denotes the mean curvature of M, then we have

$$S = \sum_\alpha \sum_{i,j} (h^\alpha_{ij})^2, \quad \mathbf{H} = \sum_\alpha H^\alpha e_\alpha, \quad H^\alpha = \frac{1}{2} \sum_k h^\alpha_{kk}, \quad H = \|\mathbf{H}\|,$$

where e_α $(3 \le \alpha \le n)$ are orthonormal normal vector fields of M in S^n.

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We define the following non-negative function on M

$$\rho^2 = S - 2H^2, \tag{1.1}$$

which vanishes exactly at the umbilic points of M.

The Willmore functional is then the following non-negative functional (see [1], [4] or [23])

$$W(x) = \int_M \rho^2 dv = \int_M (S - 2H^2) dv. \tag{1.2}$$

It was shown in [4] (see also [22] and [24]) that this functional is invariant under conformal transformations of S^n. The Willmore conjecture states that $W(x) \geq 4\pi^2$ for all immersed tori $x : M \to S^3$. The conjecture has been proved in some conformal classes by Li and Yau [17], Montiel and Ros [19]. The conjecture is also known to be true for flat tori (see Chen [5]) and tori whose images under stereographic projection are surfaces of revolution in R^3 (see Langer and Singer [13], Hertrich-Jeromin and Pinkall [12]). It is a natural idea to approach the Willmore conjecture by studying the critical surfaces of the Willmore functional $W(x)$. A surface in S^n is called a Willmore surface if it is a critical surface of the above Willmore functional.

Let M be a surface in S^n, it was proved by R. Bryant in case $n = 3$ (see [1]) and by J. Weiner in [23] in the general case $n \geq 3$ that M is a Willmore surface if and only if

$$\Delta^\perp \mathbf{H} + \sum_{\alpha,\beta,i,j} h_{ij}^\alpha h_{ij}^\beta H^\beta e_\alpha - 2H^2 \mathbf{H} = 0, \quad 3 \leq \alpha \leq n, \tag{1.3}$$

where $\Delta^\perp$ is the Laplacian in the normal bundle NM.

Note that every minimal surface in S^n is trivially a solution of (1.3). Therefore, Willmore surfaces are a generalisation of minimal surfaces in a sphere. We note that Pinkall [20] constructed some compact non-minimal flat Willmore surfaces in S^3. and that in [11] the authors studied the geometry of m-dimensional Willmore submanifolds in S^n. One of the first examples of a flat non-minimal Willmore surface in higher codimension was obtained by Ejiri in [10]. He showed that the following surface:

$$\begin{aligned} x(t,s) = (&\sqrt{\tfrac{2}{3}}\cos t, \sqrt{\tfrac{2}{3}}\sin t, \sqrt{\tfrac{1}{3}}\cos(\sqrt{3}s)\cos t \\ &\sqrt{\tfrac{1}{3}}\cos(\sqrt{3}s)\sin t, \sqrt{\tfrac{1}{3}}\sin(\sqrt{3}s)\cos t, \sqrt{\tfrac{1}{3}}\sin(\sqrt{3}s)\sin t). \end{aligned}$$

is a Willmore surface. Note that the above immersion can be seen as a tensor product $\alpha \otimes \beta$ of the curve

$$\alpha(t) = (\cos t, \sin t)$$

with the curve

$$\beta(s) = (\sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{1}{3}}\cos(\sqrt{3}s), \sqrt{\tfrac{1}{3}}\sin(\sqrt{3}s)).$$

The main purpose of this paper is to investigate in a systematical way when a tensor product of two curves is a Willmore surface. A systematic study of the geometric properties of a tensor product was started in [6] and [7]. Further properties were obtained in amongst others [8] and [9]. The paper is organized as follows. In Section 2, we recall some basic facts about tensor product immersions of curves and the condition for the tensor product imemrsion to be a Willmore surface in terms of the curves α and β. A further investigation of the obtained condition is done in Section 3. Amongst others we show that:

Theorem 1. *Let $\alpha : \mathbb{R} \to S^1 \subset \mathbb{R}^2 : t \mapsto (\cos t, \sin t)$ and let $\beta : I \subset \mathbb{R} \to S^3(1) \subset R^4$ be an arclength parametrized regular curve whose Frenet curvatures (considered as a curve in $S^3(1)$), satisfy*

- $k_2 k_1^2 = c$,
- $k_1'' - k_1 + \frac{1}{2}k_1^3 - k_1 k_2^2 = 0$,

where c is an arbitrary constant. Then $\alpha \otimes \beta$ is a flat Willmore surface in S^7. Moreover, the mean curvature H and ρ^2 satisfy

$$H^2 = \frac{1}{4}(k_1(s))^2, \qquad \rho^2 = 2 + \frac{1}{2}(k_1(s))^2. \tag{1.4}$$

Conversely, let $M \to S^n$ be a Willmore surface which can be written as a tensor product of 2 curves. Then, either

(i) $n = 3$ and M is congruent with the surface obtained by taking $\alpha : \mathbb{R} \to S^1 \subset \mathbb{R}^2 : t \mapsto (\cos t, \sin t)$ and $\beta : I \subset \mathbb{R} \to S^1(1) \subset R^2 : s \mapsto (\cos s, \sin s)$, i.e. by assuming $k_1 = k_2 = 0$ in the above differential equations;

(ii) $n = 5$ and M is congruent with a surface obtained by taking $\alpha : \mathbb{R} \to S^1 \subset \mathbb{R}^2 : t \mapsto (\cos t, \sin t)$ and $\beta : I \subset \mathbb{R} \to S^2(1) \subset R^3$ an arclength parametrized regular curve whose curvature k_1 is a positive function satisfying

$$k_1'' - k_1 + \tfrac{1}{2}k_1^3 = 0,$$

i.e. by assuming $k_2 = 0$ in the above differential equations;

(iii) $n = 7$ and $\alpha : \mathbb{R} \to S^1 \subset \mathbb{R}^2 : t \mapsto (\cos t, \sin t)$ and let $\beta : I \subset \mathbb{R} \to S^3(1) \subset R^4$ be an arclength parametrized regular curve whose curvatures are nowhere vanishing solutions of the above differential equations;

(iv) $n > 7$ and M is obtained as the composition of a totally geodesic immersion of S^7 into S^n together with one of the previous examples.

Note that the Clifford torus, which is the only minimal example corresponds to the case that $n = 3$, whereas Ejiri's example which lies in S^5 corresponds to the case that $k_2 = 0$ and $k_1 = \sqrt{2}$. The above differential equation for a curve has already been studied (see

[14], with parameter $\lambda = 4$). Using their results, we explicitly solve the above system of differential equations for the curvatures k_1 and k_2 in terms of elliptic functions. We then show, in the case that $n = 5$, how to obtain the curve explicitly in terms of elliptic functions and their integrals. As a consequence we obtain infinitely many new examples of Willmore tori in $S^5(1)$ which generalise Ejiri's example. In particular we show the following:

Theorem 2. *Let* $\alpha : \mathbb{R} \to S^1 \subset R^2 : t \mapsto (\cos t, \sin t)$ *and*

$$\beta : \mathbb{R} \to S^2 \subset \mathbb{R}^3 : s \mapsto \begin{pmatrix} \sqrt{\frac{(2-k^2)}{(3-3k^2+k^4)}}\,\mathrm{dn}(\frac{1}{\sqrt{2-k^2}}s, k) \\ \sqrt{\frac{(3-3k^2+k^4)-4(2-k^2)\,\mathrm{dn}^2(\frac{1}{\sqrt{2-k^2}}s,k)}{(3-3k^2+k^4)}}\cos(\phi(s)) \\ \sqrt{\frac{(3-3k^2+k^4)-4(2-k^2)\,\mathrm{dn}^2(\frac{1}{\sqrt{2-k^2}}s,k)}{(3-3k^2+k^4)}}\sin(\phi(s)), \end{pmatrix}$$

where

$$\phi(s) = -\int_0^s \frac{\sqrt{3-3k^2+k^4}}{1-k^2} \frac{1 + \frac{k^2}{1-k^2}\,\mathrm{sn}^2(\frac{u}{\sqrt{2-k^2}}, k)}{1 - \frac{k^2(2-k^2)}{(1-k^2)^2}\,\mathrm{sn}^2(\frac{u}{\sqrt{2-k^2}}, k)} du,$$

where sn *and* dn *are Jacobi elliptic function, see [2], and the parameter* k *with* $0 < k < 1$ *is chosen such that*

$$T_1(k) = 4\sqrt{(3 - 3k^2 + k^4)(2 - k^2)}K(k) + 2\pi(1 - \Lambda_0(\arcsin(1 - k^2), k)),$$

is a rational multiple of 2π*, where* Λ_0 *denotes the Heumann Lambda function and* $K(k)$ *is the complete elliptic integral of the first kind. Then* $\alpha \otimes \beta$ *is a Willmore torus which lies linearly full in* $S^5(1)$.

Theorem 3. *Let* $\alpha : \mathbb{R} \to S^1 \subset R^2 : t \mapsto (\cos t, \sin t)$ *and*

$$\beta : \mathbb{R} \to S^2 \subset \mathbb{R}^3 : s \mapsto \begin{pmatrix} k\sqrt{\frac{(2k^2-1)}{(3k^4-3k^2+1)}}\,\mathrm{cn}(\frac{1}{\sqrt{2k^2-1}}s, k) \\ \sqrt{\frac{(3k^4-3k^2+1)-k^2(2k^2-1)\,\mathrm{cn}^2(\frac{1}{\sqrt{2k^2-1}}s,k)}{(3k^4-3k^2+1)}}\cos(\phi(s)) \\ \sqrt{\frac{(3k^4-3k^2+1)-k^2(2k^2-1)\,\mathrm{cn}^2(\frac{1}{\sqrt{2k^2-1}}s,k)}{(3k^4-3k^2+1)}}\sin(\phi(s)), \end{pmatrix}$$

where

$$\phi(s) = -\int_0^s \sqrt{1 - 3k^2 + k^4} \frac{(1 - k^2) - k^2\,\mathrm{sn}(\frac{u}{\sqrt{2k^2-1}})}{(1 - k^2)^2 + k^2(2k^2 - 1)\,\mathrm{sn}(\frac{u}{\sqrt{2k^2-1}})} du,$$

where sn *and* cn *are Jacobi elliptic function, and the parameter* k *with* $\frac{1}{\sqrt{2}} < k < 1$ *is chosen such that*

$$T_2(k) = 4\frac{(2k^2 - 1)^{3/2}}{\sqrt{1 - 3k^2 + 3k^4}}K(k) - 2\pi(1 - \Lambda_0(\arcsin(\tfrac{1-k^2}{1-3k^2+3k^4}), k))$$

is a rational multiple of 2π*. Then* $\alpha \otimes \beta$ *is a Willmore torus which lies linearly full in* $S^5(1)$.

We also show that all flat tori in $S^5(1)$ which can be obtained as a tensor product immersion of curves can be obtained in this way. Finally, in Section 5, we include some explicit examples of flat Willmore tori in $S^7(1)$.

We would like to remark that in [15] and [16], the first author proved the following pinching result

Theorem 4. *([15],[16]) Let $x : M \to S^n$ be a compact Willmore surface in S^n. Then we have*

$$\int_M \rho^2(c(n) - \rho^2)dv \leq 0,$$

where

$$c(n) = \begin{cases} 2 & \text{if } n = 3 \\ \frac{4}{3} & \text{if } n \geq 4. \end{cases}$$

If

$$0 \leq \rho^2 \leq c(n),$$

then either $\rho^2 \equiv 0$ and $x(M)$ is totally umbilical, or $\rho^2 \equiv c(n)$. In the latter case, either $n = 3$ and $x(M)$ is the Clifford minimal torus; or $n = 4$ and $x(M)$ is the Veronese surface.

2 Preliminaries

First, we recall some elementary properties of the tensor product. Let $v = {}^t(v_1, \ldots, v_p) \in \mathbb{R}^p$ and $w = {}^t(w_1, \ldots, w_q) \in \mathbb{R}^q$, then $v \otimes w$ is the element of $\mathbb{R}^{pq} \cong \mathbb{R}^{p \times q}$ defined by

$$v \otimes w = v.\,{}^t w = (v_1 w_1, \ldots, v_1 w_q, v_2 w_1, \ldots, v_p w_q),$$

where we identify $\mathbb{R}^{p\times q}$ and $\mathbb{R}^{pq}$ in a natural way. If we denote by $\{e_1, \ldots, e_p\}$ and $\{f_1, \ldots, f_q\}$ the standard basis for respectively $\mathbb{R}^p$ and $\mathbb{R}^q$, it immediately follows that the vectors $e_{(ij)} = e_i \otimes f_j$, where $i = 1, \ldots, p$ and $j = 1, \ldots, q$ form a basis for $\mathbb{R}^{pq}$. We denote by $A_{(ij)}$ the component of a vector $A \in \mathbb{R}^{pq}$ in the direction of $e_i \otimes f_j$.

It then follows that

$$\begin{aligned} < v \otimes w, v' \otimes w' > &= \sum_{i=1}^{p} \sum_{j=1}^{q} (v.\,{}^t w)_{(ij)} (v'.\,{}^t w')_{(ij)} \\ &= \sum_{i=1}^{p} \sum_{j=1}^{q} v_i w_j v'_i w'_j \\ &= (\sum_{i=1}^{p} v_i v'_i)(\sum_{j=1}^{q} w_j w'_j) \\ &= < v, v' >< w, w' > . \end{aligned}$$

Note that if A and B are linear transformations of $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, we have

$$\begin{aligned} Av \otimes Aw &= (Av).{}^t(Bw) \\ &= Av.{}^tw\,{}^tB \\ &= Cv \otimes w, \end{aligned}$$

where C is the linear tranformation of $\mathbb{R}^{pq}$ determined by

$$Ce_{(ij)} = \sum_{k=1}^{p}\sum_{\ell=1}^{q} A_{ik}B_{j\ell}e_{(k\ell)}.$$

In particular, we see that if A and B are orthogonal matrices, then so is the matrix C. Indeed, denoting by $C_{(ij)(k\ell)}$ the component of $Ce_{(ij)}$ in the direction of $e_{(k\ell)}$, we have that

$$\begin{aligned} \sum_{k=1}^{p}\sum_{\ell=1}^{q} C_{(ij)(k\ell)}C_{(i'j')(k\ell)} &= \sum_{k=1}^{p}\sum_{\ell=1}^{q} A_{ik}B_{j\ell}A_{i'k}B_{j'\ell} \\ &= \sum_{k=1}^{p} A_{ik}A_{i'k}\sum_{\ell=1}^{q} B_{j\ell}B_{j'\ell} \\ &= \delta_{ii'}\delta_{jj'} \\ &= \delta_{(ii'),(jj')}. \end{aligned}$$

As a consequence the tensor product behaves nicely with respect to orthogonal tranformation in the base spaces.

Now, let $r : M^2 \to S^n \subset \mathbb{R}^{n+1}$ be an immersion of a surface into the n-dimensional sphere with radius 1. We say that r is a tensor product immersion if there exists curves $\alpha : I \to \mathbb{R}^p$ and $\beta : J \to R^q$, with $pq = n+1$, such that, if necessary after applying an orthogonal transformation, $r(M) = \alpha(I) \otimes \beta(J)$. Throughout this paper we will denote by t the variable on I and by s the variable on J. As there is no confusion possible, we denote by α', α'', α_i' derivatives of α (or its components) with respect to t. Similarly, we denote by β', β'', β_j' derivatives of β (or its components) with respect to s. Note that as

$$\alpha(I) \otimes \beta(J) \subset S^n,$$

we have for all t and s that $\|\alpha(t) \otimes \beta(s)\|^2 = 1$, which implies that

$$< \alpha(t), \alpha(t) >< \beta(s), \beta(s) >= 1.$$

Hence there exist positive constants c_1 and c_2 with $c_1^2c_2^2 = 1$ such that

$$\begin{aligned} &< \alpha(t), \alpha(t) >= c_1^2, \\ &< \beta(s), \beta(s) >= c_2^2. \end{aligned}$$

Therefore replacing α and β by

$$\tilde{\alpha} = \frac{1}{c_1}\alpha,$$
$$\tilde{\beta} = \frac{1}{c_2}\beta,$$

it still follows that $r(M) = \tilde{\alpha}(I) \otimes \tilde{\beta}(J)$. Moreover $\tilde{\alpha}$ and $\tilde{\beta}$ satisfy $< \tilde{\alpha}, \tilde{\alpha} >= 1 =< \tilde{\beta}, \tilde{\beta} >$. As M is a surface, it also follows that the vectors $(\tilde{\alpha} \otimes \tilde{\beta})_t = \tilde{\alpha}' \otimes \tilde{\beta}$ and $(\tilde{\alpha} \otimes \tilde{\beta})_s = \tilde{\alpha} \otimes \tilde{\beta}'$ are linearly independent. This implies in particular that

$$< \tilde{\alpha}', \tilde{\alpha}' > \neq 0 \neq < \tilde{\beta}', \tilde{\beta}' > .$$

Hence $\tilde{\alpha}$ and $\tilde{\beta}$ are regular curves and can therefore be arc length parametrized. Thus, the following lemma follows immediately:

Lemma 2.1. *Let $r : M^2 \to S^n \subset \mathbb{R}^{n+1}$ be a tensor product immersion of 2 curves. Then there exist arclength parametrized curves $\alpha : I \to S^{p-1}(1)$ and $\beta : J \to S^{q-1}(1)$, with $pq = n-1$ such that after applying an orthogonal transformation of S^n,*

$$r(M) = \alpha(I) \otimes \beta(J).$$

As a first property of tensor product immersions in a sphere of radius 1, we obtain the following:

Lemma 2.2. *Let $r : M \to S^n \subset \mathbb{R}^{n+1}$ be a tensor product immersion of 2 curves. Then M is flat. More precisely if $\alpha : I \to S^{p-1}(1)$ and $\beta : J \to S^{q-1}(1)$ are the curves given in Lemma 2.1, then t and s are flat coordinates on the surface, i.e. we have that*

$$< r_t, r_t >= 1 =< r_s, r_s >$$

and

$$< r_t, r_s >= 0.$$

Proof. Let α and β be arclength parametrized as in Lemma 2.1. Clearly, we have that

$$r_t = \alpha' \otimes \beta,$$
$$r_s = \alpha \otimes \beta',$$

As α and β are arclength parametrized curves in a unit sphere it immediately follows that

$$< r_t, r_t >= 1 =< r_s, r_s > \qquad\qquad < r_t, r_s >= 0,$$

which completes the proof. $\square$

Now, in the remainder of this section we want to express the condition that M is a Willmore surface in terms of the curves α and β. First note that

$$r_{tt} = \alpha'' \otimes \beta,$$
$$r_{ss} = \alpha \otimes \beta'',$$
$$r_{st} = \alpha' \otimes \beta',$$

and that by the properties of the tensor product the above vectors are orthogonal to $r_t = \alpha' \otimes \beta$ and $r_s = \alpha \otimes \beta'$. Also, we have that the component in the direction of the normal of the sphere in $\mathbb{R}^{n+1}$ can be computed as follows:

$$\begin{aligned} < r_{tt}, r > &=< \alpha'' \otimes \beta, \alpha \otimes \beta > \\ &=< \alpha'', \alpha >< \beta, \beta > \\ &= - < \alpha', \alpha' >= -1. \end{aligned}$$

Similar computations will be made frequently throughout this section. It now follows immediately that the second fundamental form h (of M^2 in $S^n(1)$) is given by:

$$h(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = r_{tt} + r, \quad h(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}) = r_{ts}, \quad h(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}) = r_{ss} + r. \tag{2.1}$$

From this we deduce that the mean curvature vector $\mathbf{H}$ satisfies

$$\mathbf{H} = \frac{1}{2}(r_{tt} + r_{ss}) + r, \qquad \|\mathbf{H}\|^2 = \frac{1}{4}|\alpha''(t)|^2 + \frac{1}{4}|\beta''(s)|^2 - \frac{1}{2}. \tag{2.2}$$

Now we first prove

Lemma 2.3. *Denote by $\Delta^\perp$ the Laplacian with respect to the normal connection of M^2 in S^n. Then, we have*

$$\begin{aligned} \Delta^\perp \mathbf{H} \;=\; & \tfrac{1}{2}(r_{tttt} + r_{ssss}) + r_{sstt} + \tfrac{1}{2}(1 + \|\alpha''(t)\|^2) r_{tt} \\ & + \tfrac{1}{2}(1 + \|\beta''(s)\|^2) r_{ss} + \tfrac{3}{4}[(\|\alpha''(t)\|^2)_t r_t + (\|\beta''(s)\|^2)_s r_s]. \end{aligned} \tag{2.3}$$

Proof. Note that $\nabla^\perp_{\frac{\partial}{\partial t}} H$ denotes the normal component of H_t in the sphere, i.e. that part of H_t which is normal to the tangent space spanned by r_s, r_t and the position vector r. As α and β are arclength parametrized curves in a unit sphere, it follows straightforwardly from the tensor product that:

$$\begin{aligned} < r_{ttt}, r > &=< \alpha''' \otimes \beta, \alpha \otimes \beta > \\ &=< \alpha''', \alpha > \\ &= - < \alpha'', \alpha' >= 0, \end{aligned}$$

where we also used that $< \alpha'', \alpha >= - < \alpha', \alpha' >= -1$. Similarly, it follows that

$$\begin{array}{lll} < r_{ttt}, r >= 0, & < r_{ttt}, r_t >= -\|\alpha''\|^2, & < r_{ttt}, r_s >= 0, \\ < r_{tts}, r >= 0, & < r_{tts}, r_t >= 0, & < r_{tts}, r_s >= -1, \\ < r_{sst}, r >= 0, & < r_{sst}, r_t >= -1, & < r_{sst}, r_s >= 0, \\ < r_{sss}, r >= 0, & < r_{sss}, r_t >= 0, & < r_{sss}, r_s >= -\|\beta''\|^2. \end{array}$$

Using the above formulas together with the fact that r, r_s and r_t are three mutually orthonormal vectors, it now follows from the definition of $\nabla^\perp\mathbf{H}$ and $\mathbf{H}=\frac{1}{2}(r_{tt}+r_{ss})+r$, that

$$\begin{aligned}\nabla^\perp_{\frac{\partial}{\partial t}}\mathbf{H} &= \mathbf{H}_t-\tfrac{1}{2}[1-\|\alpha''(t)\|^2]r_t\\ &= \tfrac{1}{2}(r_{ttt}+r_{sst})+r_t-\tfrac{1}{2}[1-\|\alpha''(t)\|^2]r_t,\end{aligned}\tag{2.4}$$

$$\begin{aligned}\nabla^\perp_{\frac{\partial}{\partial s}}\mathbf{H} &= \mathbf{H}_s-\tfrac{1}{2}[1-\|\beta''(s)\|^2]r_s\\ &= \tfrac{1}{2}(r_{tts}+r_{sss})+r_s-\tfrac{1}{2}[1-\|\beta''(s)\|^2]r_s,\end{aligned}\tag{2.5}$$

Similarly, it also follows that

$$\begin{aligned}\nabla^\perp_{\frac{\partial}{\partial t}}(\nabla^\perp_{\frac{\partial}{\partial t}}\mathbf{H}) &= (\nabla^\perp_{\frac{\partial}{\partial t}}\mathbf{H})_t+\tfrac{1}{4}(\|\alpha''(t)\|^2)_t r_t\\ &= \tfrac{1}{2}r_{tttt}+\tfrac{1}{2}r_{ttss}+\tfrac{1}{2}(1+\|\alpha''(t)\|^2)r_{tt}+\tfrac{3}{4}(\|\alpha''(t)\|^2)_t r_t,\end{aligned}\tag{2.6}$$

$$\begin{aligned}\nabla^\perp_{\frac{\partial}{\partial s}}(\nabla^\perp_{\frac{\partial}{\partial s}}\mathbf{H}) &= (\nabla^\perp_{\frac{\partial}{\partial s}}\mathbf{H})_s+\tfrac{1}{4}(\|\beta''(s)\|^2)_s r_s\\ &= \tfrac{1}{2}r_{ssss}+\tfrac{1}{2}r_{sstt}+\tfrac{1}{2}(1+\|\beta''(s)\|^2)r_{ss}+\tfrac{3}{4}(\|\beta''(s)\|^2)_s r_s.\end{aligned}\tag{2.7}$$

Lemma 2.3 now follows directly from (2.6) and (2.7). □

Now we can show that:

Proposition 2.1. *A tensor product immersion is a Willmore surface if and only if curves $\alpha(t)$ and $\beta(s)$ satisfy*

$$\begin{aligned}&\tfrac{1}{2}(r_{tttt}+r_{ssss})+r_{sstt}+(\tfrac{1}{2}+\tfrac{3}{4}\|\alpha''(t)\|^2-\tfrac{1}{4}\|\beta''(s)\|^2)r_{tt}\\ &+(\tfrac{1}{2}+\tfrac{3}{4}\|\beta''(s)\|^2-\tfrac{1}{4}\|\alpha''(t)\|^2)r_{ss}+\tfrac{3}{4}(\|\alpha''(t)\|^2)_t r_t+\tfrac{3}{4}(\|\beta''(s)\|^2)_s r_s=\mathbf{0}.\end{aligned}\tag{2.8}$$

Proof. Using the properties of the tensor product, as well as (2.1) and (2.2), we have

$$\begin{aligned}\sum_{\alpha,\beta}h^\alpha_{11}h^\beta_{11}H^\beta e_\alpha &= <\mathbf{H},h(\tfrac{\partial}{\partial t},\tfrac{\partial}{\partial t})>h(\tfrac{\partial}{\partial t},\tfrac{\partial}{\partial t})\\ &= <\tfrac{1}{2}(r_{tt}+r_{ss})+r,r_{tt}+r>(r_{tt}+r)\\ &= \tfrac{1}{2}[\|\alpha''(t)\|^2-1](r_{tt}+r),\end{aligned}\tag{2.9}$$

$$\begin{aligned}\sum_{\alpha,\beta}h^\alpha_{22}h^\beta_{22}H^\beta e_\alpha &= <\mathbf{H},h(\tfrac{\partial}{\partial s},\tfrac{\partial}{\partial s})>h(\tfrac{\partial}{\partial s},\tfrac{\partial}{\partial s})\\ &= <\tfrac{1}{2}(r_{tt}+r_{ss})+r,r_{ss}+r>(r_{ss}+r)\\ &= \tfrac{1}{2}[\|\beta''(s)\|^2-1](r_{ss}+r),\end{aligned}\tag{2.10}$$

$$\begin{aligned}\sum_{\alpha,\beta}h^\alpha_{12}h^\beta_{12}H^\beta e_\alpha &= <\mathbf{H},h(\tfrac{\partial}{\partial t},\tfrac{\partial}{\partial s})>h(\tfrac{\partial}{\partial t},\tfrac{\partial}{\partial s})\\ &= <\tfrac{1}{2}(r_{tt}+r_{ss})+r,r_{ts}>r_{ts}\\ &= 0.\end{aligned}\tag{2.11}$$

Substituting (2.2), (2.3) and (2.9)-(2.11) into the Willmore equation (1.3), we get Proposition 2.1. □

Next, we want to simplify (2.8), which depends on both curves α and β simultaneously as much as possible in order to determine explicitly which curves α and β solve such equation, i.e. we want to obtain independent expressions for the curves α and β. For that purpose, we introduce auxiliary functions A and B respectively defined by

$$A = \frac{1}{2} + \frac{3}{4}\|\alpha''(t)\|^2 - \frac{1}{4}\|\beta''(s)\|^2, \tag{2.12}$$

$$B = \frac{1}{2} + \frac{3}{4}\|\beta''(s)\|^2 - \frac{1}{4}\|\alpha''(t)\|^2, \tag{2.13}$$

Then (2.8) is equivalent to

$$\begin{aligned} &\tfrac{1}{2}\alpha_i^{(4)}(t)\beta_j(s) + \tfrac{1}{2}\alpha_i(t)\beta_j^{(4)}(s) + \alpha_i''(t)\beta_j''(s) + \tfrac{3}{4}(\|\alpha''(t)\|^2)_t\alpha_i'(t)\beta_j(s) \\ &\quad + \tfrac{3}{4}(\|\beta''(s)\|^2)_s\alpha_i(t)\beta_j'(s) + A\cdot\alpha_i''(t)\beta_j(s) + B\cdot\alpha_i(t)\beta_j''(s) = 0, \end{aligned} \tag{2.14}$$

for all $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Multiplying now the above expression with β_j and taking the summation over j, we obtain

$$\begin{aligned} &\tfrac{1}{2}\alpha_i^{(4)}(t) + \tfrac{1}{2}\|\beta''(s)\|^2\alpha_i(t) - \alpha_i''(t) \\ &\quad + \tfrac{3}{4}(\|\alpha''(t)\|^2)_t\alpha_i'(t) + A\cdot\alpha_i''(t) - B\cdot\alpha_i(t) = 0, \end{aligned} \tag{2.15}$$

which is valid for all $i = 1, \ldots, p$. Substituting now (2.12) and (2.13) into (2.15), we have for all $i = 1, \ldots, p$ that

$$\begin{aligned} &\tfrac{1}{2}\alpha_i^{(4)}(t) + \tfrac{1}{2}\|\beta''(s)\|^2\alpha_i(t) - \alpha_i''(t) + \tfrac{3}{4}(\|\alpha''(t)\|^2)_t\alpha_i'(t) \\ &\quad -[\tfrac{1}{2} + \tfrac{3}{4}\|\beta''(s)\|^2 - \tfrac{1}{4}\|\alpha''(t)\|^2]\alpha_i(t) + [\tfrac{1}{2} + \tfrac{3}{4}\|\alpha''(t)\|^2 - \tfrac{1}{4}\|\beta''(s)\|^2]\alpha_i''(t) = 0, \end{aligned}$$

which can still be rewritten as

$$\begin{aligned} &\|\beta''(s)\|^2(-\tfrac{1}{4}\alpha_i(t) - \tfrac{1}{4}\alpha_i''(t)) + \tfrac{1}{2}\alpha_i^{(4)}(t) - \tfrac{1}{2}\alpha_i''(t) \\ &\quad + \tfrac{3}{4}(\|\alpha''(t)\|^2)_t\alpha_i'(t) - \tfrac{1}{2}\alpha_i(t) + \tfrac{1}{4}\|\alpha''(t)\|^2\alpha_i(t) + \tfrac{3}{4}\|\alpha''(t)\|^2\alpha_i''(t) = 0. \end{aligned} \tag{2.16}$$

Of course, by interchanging the roles of the two curves α and β, we also obtain for all $j = 1, \ldots, q$ that

$$\begin{aligned} &\|\alpha''(t)\|^2(-\tfrac{1}{4}\beta_j(s) - \tfrac{1}{4}\beta_j''(s)) + \tfrac{1}{2}\beta_j^{(4)}(s) - \tfrac{1}{2}\beta_j''(s) \\ &\quad + \tfrac{3}{4}(\|\beta''(s)\|^2)_s\beta_j'(s) - \tfrac{1}{2}\beta_j(s) + \tfrac{1}{4}\|\beta''(s)\|^2\beta_j(s) + \tfrac{3}{4}\|\beta''(s)\|^2\beta_j''(s) = 0. \end{aligned} \tag{2.17}$$

From this we can prove:

Lemma 2.4. *Let M be a tensor product immersion generated by curves α and β as in Lemma 2.1. Then, if necessary after exchanging the role of α and β, we may assume that α is a great circle in S^{p-1}, and thus after applying an orthogonal transformation of S^{p-1}, we may assume that*

$$\alpha(t) = (\cos t, \sin t, 0, \ldots, 0).$$

Proof. Assume that neither α nor β is a great circle, i.e., $\alpha_i(t)+\alpha_i''(t)\neq 0$, $\beta_j(s)+\beta_j''(s)\neq 0$. Then it follows from (2.17) that $\|\alpha''\|$ is a constant. Similarly, it follows from (2.16) that $\|\beta''\|$ is constant too. Substituting now the expression for $\alpha_i^{(4)}$ and $\beta_j^{(4)}$ obtained from respectively (2.16) and (2.17) in (2.14) we find for all i and j that

$$\begin{aligned}
0 &= \tfrac{1}{2}\alpha_i^{(4)}\beta_j + \tfrac{1}{2}\alpha_i\beta_j^{(4)} + \alpha_i''\beta_j'' + A\alpha_i''\beta_j + B\alpha_i\beta_j'' \\
&= -\tfrac{1}{2}\|\beta''\|\alpha_i\beta_j + \alpha_i''\beta_j + B\alpha_i\beta_j + \tfrac{1}{2}\alpha_i\beta_j^{(4)} + \alpha_i''\beta_j'' + B\alpha_i\beta_j'' \\
&= -\tfrac{1}{2}\|\beta''\|\alpha_i\beta_j + \alpha_i''\beta_j + B\alpha_i\beta_j - \tfrac{1}{2}\|\alpha''\|\alpha_i\beta_j + \alpha_i\beta_j'' + A\alpha_i\beta_j + \alpha_i''\beta_j'' \\
&= \alpha_i\beta_j + \alpha_i''\beta_j + \alpha_i\beta_j'' + \alpha_i''\beta_j'' \\
&= (\alpha_i+\alpha_i'')(\beta_j+\beta_j''),
\end{aligned}$$

which contradicts the fact that neither α nor β is a great circle. $\square$

The consequences that the curve α is a great circle are further investigated in the next section. Of course as by properties of the tensor product, an orthogonal transformation of the curves induces an orthogonal transformation of the surface itself, we may assume that

$$\alpha(t) = (\cos t, \sin t, 0, \ldots, 0).$$

3 Proof of Theorem 1

As we have shown in the previous section, we may assume that

$$\alpha = (\mathrm{cos}t, \mathrm{sin}t, 0, \ldots, 0), \tag{3.1}$$

Putting (3.1) into (2.14), it is straightforward to check that the Willmore equation (2.14) reduces to

$$\beta_j^{(4)}(s) + \frac{3}{2}(\|\beta''(s)\|^2 - 1)\beta_j''(s) + \frac{3}{2}(\|\beta''(s)\|^2)_s\beta_j'(s) - (\frac{3}{2} - \frac{1}{2}\|\beta''(s)\|^2)\beta_j(s) = 0, \tag{3.2}$$

where $j = 1, \ldots, q$.

Note that by the above differetial equation $\beta(s)$, $\beta'(s)$, $\beta''(s), \beta^{(3)}(s), \beta^{(4)}(s)$ are linear dependent. This implies that by applying an orthogonal transformation, we may assume that curve β is contained in a totally geodesic $S^3(1)$ in $S^{q-1}(1)$. Let us assume that β does not contain a piece of a great circle. For such a curve, we have, on an open dense subset of the parameter domain, the following Frenet formulas:

$$\left\{\begin{aligned}
&\beta(s) = t_0 \\
&\beta'(s) = t_1 \\
&\beta''(s) = t_1' = -t_0 + k_1 t_2 \\
&t_2' = -k_1 t_1 + k_2 t_3 \\
&t_3' = -k_2 t_2,
\end{aligned}\right. \tag{3.3}$$

where $\{t_0, t_1, t_2, t_3\}$ is an orthonormal moving frame along the curve and $k_1 > 0$ on the previously mentioned open and dense subset.

Thus we have

$$\begin{cases} \beta(s) = t_0 \\ \beta'(s) = t_1 \\ \beta''(s) = -t_0 + k_1 t_2 \\ \beta'''(s) = -t_1 + k_1' t_2 - (k_1)^2 t_1 + k_1 k_2 t_3 \\ \beta^{(4)}(s) = (1 + k_1^2) t_0 - 3k_1 k_1' t_1 + (k_1'' - k_1 - k_1^3 - k_1 k_2^2) t_2 + (2k_1' k_2 + k_1 k_2') t_3. \end{cases} \tag{3.4}$$

Noting that

$$\|\beta''(s)\|^2 = 1 + (k_1)^2,$$

the Willmore equation (3.2) becomes

$$\beta^{(4)}(s) = -\frac{3}{2}(k_1)^3 t_2 - 3k_1 k_1' t_1 + (1 + (k_1)^2) t_0. \tag{3.5}$$

Combining (3.4) with (3.5), using that t_0, t_1, t_2 and t_3 are linearly independent, we know that the surface M is a Willmore surface if and only if the curvature $k_1(s)$ and $k_2(s)$ of curve $\beta(s)$ satisfy

$$k_2(k_1)^2 = c, \qquad k_1'' - k_1 + \frac{1}{2}(k_1)^3 - k_1(k_2)^2 = 0, \tag{3.6}$$

where c is an arbitrary constant. Note that the above system of differential equations corresponds to the one in (1.3) of [14] with $G = 1$ and $\lambda = 4$.

Remark that the above differential equations imply that if k_2 is nonzero somewhere, it is nonzero everywhere, as in that case the constant c is nonzero. This corresponds to the case that the curve β is linearly full in $S^3(1)$, i.e. is not contained in a totally geodesic $S^2(1)$ in $S^3(1)$. However, if the curve is contained in a totally geodesic $S^2(1)$, k_2 vanishes identically and therefore $c = 0$ and k_1 is determined by the following differential equation:

$$k_1'' - k_1 + \tfrac{1}{2} k_1^3 = 0, \tag{3.7}$$

which can be solved explicitly using elliptic functions. Using elementary properties of elliptic functions of [2], see also Table 2.7(c) of [14], we obtain after a translation of the s-coordinate that either:

(i) $k_1(s) = 0$, i.e. the curve β is a great circle;

(ii) $k_1(s) = \sqrt{2}$, in which case it follows from (3.4) that β is congruent with a circle of radius $\sqrt{\tfrac{1}{3}}$ in $S^2(1)$. It is clear that integrating this leads to

$$\beta(s) = (\sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{1}{3}}\cos(\sqrt{3}s), \sqrt{\tfrac{1}{3}}\sin(\sqrt{3}s),$$

this gives Ejiri's example;

(iii) $k_1(s) = \frac{2}{\sqrt{2-k^2}} \operatorname{dn}(\frac{1}{\sqrt{2-k^2}} s, k)$, where $0 < k < 1$;

(iv) $k_1(s) = \frac{2k}{\sqrt{2k^2-1}} \operatorname{cn}(\sqrt{\frac{1}{2k^2-1}} s, k)$, where $\sqrt{\frac{1}{2}} < k < 1$;

(v) $k_1(s) = 2 \operatorname{sech}(s)$,

where dn, cn and sn are the Jacobi elliptic functions ([2]). Note that the first solution corresponds to a great circle and that the last solution can be seen as a limit case of the third or fourth one for k approaching 1. Also, the second solution can be seen as a limit of the third one with k approaching 0. As it is clear that the above solutions cannot be joint together differentiably, Theorem 1 now follows immediately from the fact that the tensor product immersion behaves nicely with respect to orthogonal transformations in the base spaces.

To conclude this section, we remark that (2.2), (3.1) and (3.4) imply that $H^2 = (k_1(s))^2/4$. From the Gauss equation, we get $\rho^2 = 2 + \frac{1}{2}(k_1(s))^2$. Thus $H \equiv 0$ if and only if $k_1(s) \equiv 0$, which implies that $r : M \to S^n$ is congruent with the Clifford minimal torus in $S^3(1)$.

4 New examples of Willmore tori in $S^5(1)$

Throughout this section we will assume that the curvature of the curve β is determined by (3.7). First note that as $\beta'' + \beta \neq 0$, it follows that $\alpha \otimes \beta$, $\alpha' \otimes \beta$, $\alpha \otimes \beta'$, $\alpha' \otimes \beta'$, $\alpha \otimes (\beta'' + \beta)$, $\alpha' \otimes (\beta'' + \beta)$ are 6 orthogonal vectors, implying that the tensor product of $\alpha \otimes \beta$ is linearly full in S^5. From the discussion in the proof of Theorem 1, we only need to study the following cases:

$$k_1(s) = \frac{2}{\sqrt{2-k^2}} \operatorname{dn}(\frac{1}{\sqrt{2-k^2}} s, k), \qquad 0 < k < 1, \tag{4.1}$$

$$k_1(s) = \frac{2k}{\sqrt{2k^2-1}} \operatorname{cn}(\sqrt{\frac{1}{2k^2-1}} s, k), \qquad \sqrt{\frac{1}{2}} < k < 1. \tag{4.2}$$

Note that the other cases, which are linearly full in $S^2(1)$, can be obtained by taking the appropriate limits. Curves corresponding to (4.1) are called curves of Type 1, whereas curves corresponding to (4.2) are called curves of Type 2. In both cases, the curve β itself is then determined from (3.4) by:

$$\beta'''(s) - \frac{k_1'}{k_1}(\beta''(s) + \beta(s)) + (1 + k_1^2)\beta'(s) = 0. \tag{4.3}$$

As the curvature k_1 is determined as solution of

$$k_1''(s) = k_1 - \frac{1}{2}k_1^3,$$

it follows that

$$k_1'''(s) = k_1' - \frac{3}{2}k_1^2 k_1'.$$

It now follows the quite remarkable fact that the curvature of the curve itself is a solution of the equation (4.3). This means that if necessary after applying an orthogonal transformation of $S^2(1)$, we may assume that

$$\beta(s) = (dk_1(s), \sqrt{1-d^2k_1^2}\cos\phi(s), \sqrt{1-d^2k_1^2}\sin\phi(s)), \tag{4.4}$$

where d is a positive constant and $\phi(0)=0$.

Now, assume that β is a curve of Type 1. Then, expressing that β' is an arclength parametrized curve yields:

$$\phi'(s) = \sqrt{\frac{-(2-k^2)^2-4d^2(1-k^2)+8d^2(2-k^2)\,\mathrm{dn}(\frac{s}{\sqrt{2-k^2}},k)^2-4d^2\,\mathrm{dn}(\frac{s}{\sqrt{2-k^2}},k)^4}{2-k^2-4d^2\,\mathrm{dn}(\frac{s}{\sqrt{2-k^2}},k)^2}}.$$

Substituting now β in the differential equation (4.3) (and evaluating the result at $s=0$), we find after a computer computation using the program Mathematica that the constant d is related to the modulus of the elliptic function by

$$d = \frac{2-k^2}{\sqrt{3-3k^2+k^4}}.$$

Substituting this expression now in the equations for ϕ' it follows that

$$\phi'(s) = \pm 1\frac{1}{\sqrt{3-3k^2+k^4}}\frac{2-k^2-\mathrm{dn}(\frac{s}{\sqrt{2-k^2}},k)^2}{1-\frac{2-k^2}{3-3k^2+k^4}\,\mathrm{dn}(\frac{s}{\sqrt{2-k^2}},k)^2}. \tag{4.5}$$

Clearly, by an orthogonal transformation we may take the negative sign in the above expression. Moreover, replacing dn by sn, we get that the function ϕ' is determined by

$$\phi'(s) = \frac{\sqrt{3-3k^2+k^4}}{1-k^2}\frac{1+\frac{k^2}{1-k^2}\,\mathrm{sn}^2(\frac{s}{\sqrt{2-k^2}},k)}{1-\frac{k^2(2-k^2)}{(1-k^2)^2}\,\mathrm{sn}^2(\frac{s}{\sqrt{2-k^2}},k)}. \tag{4.6}$$

As sn, *cn* and *dn* are periodic function with period $4K(k)$, where $K(k)$ is the complete elliptic integral of the first kind ([2]), the resulting curve is closed provided that

$$I_1(k) = \int_0^{4K(k)\sqrt{2-k^2}} \phi'(s)ds,$$

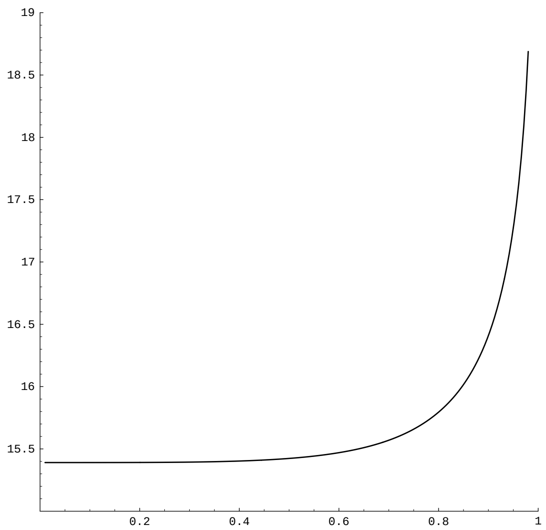
is a rational multiple of 2π. The above integral can be expressed in terms of the Heumann-Lambda function Λ_0. Indeed from [2], formula 411.6, together with the fact that the integral is periodic with period $K(k)\sqrt{2-k^2}$, the latter following immediately from

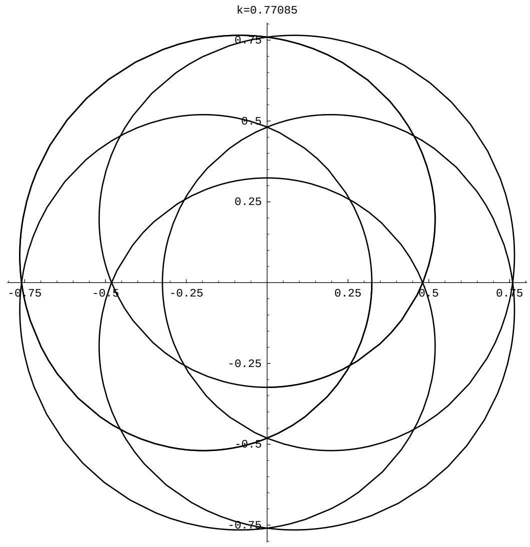
$$\begin{aligned}\mathrm{sn}(u+2K,k) &= -\,\mathrm{sn}(u,k)\\ \mathrm{sn}(2K-u,k) &= -\,\mathrm{sn}(-u,k) = \mathrm{sn}(u,k),\end{aligned}$$

it follows that

$$T_1(k) = 4\sqrt{(3-3k^2+k^4)(2-k^2)}K + 2\pi(1-\Lambda_0(\arcsin(1-k^2),k)), \tag{4.7}$$

where Λ_0 denotes the Heumann Lambda function. As $T_1(k)$ is a continuous function, we obtain infinitely many closed curves β in S^2 and thus infinitely many new examples of Willmore tori in $S^5(1)$. This also proves Theorem 2. Below, we include some pictures of the function $T_1(k)$, as well as one of the curves β (their projection on the yz-plane) for a suitable value of k.





If β is a curve of Type 2, we proceed in the same way. First, we deduce that

$$d = \frac{2k^2-1}{2\sqrt{1-3k^2+3k^4}}$$

Next, we find that the function ϕ, eventually after an orthogonal transformation is determined by the differential equation:

$$\phi'(s) = -\sqrt{1-3k^2+k^4}\frac{(1-k^2)-k^2\,\mathrm{sn}(\frac{t}{\sqrt{2k^2-1}})}{(1-k^2)^2+k^2(2k^2-1)\,\mathrm{sn}(\frac{t}{\sqrt{2k^2-1}})} \tag{4.8}$$

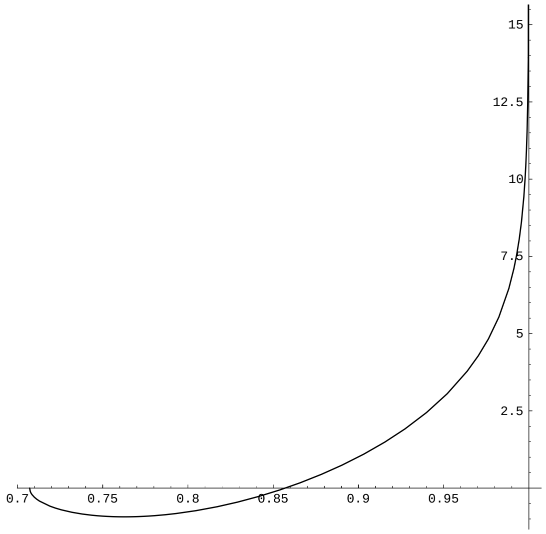
from which it follows that the resulting curve is periodic provided

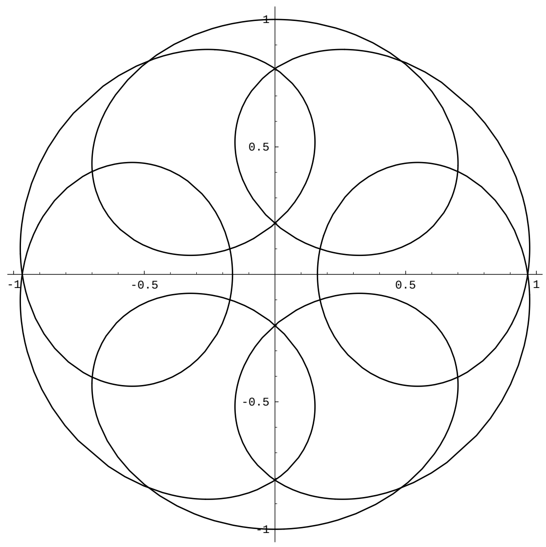
$$T_2(k) = -\int_0^{4K(k)\sqrt{2k^2-1}} \sqrt{1-3k^2+k^4}\frac{(1-k^2)-k^2\,\mathrm{sn}(\frac{t}{\sqrt{2k^2-1}})}{(1-k^2)^2+k^2(2k^2-1)\,\mathrm{sn}(\frac{t}{\sqrt{2k^2-1}})},$$

is a rational multiple of 2π. Again, the above integral can be computed in terms of the Heumann-Lambda function resulting in:

$$T_2(k) = 4\frac{(2k^2-1)^{3/2}}{\sqrt{1-3k^2+3k^4}}K(k) - 2\pi(1-\Lambda_0(\arcsin(\tfrac{1-k^2}{1-3k^2+3k^4}),k)). \tag{4.9}$$

As $T_2(k)$ is a non constant continuous function we obtain infinitely many examples of closed curves β and thus of Willmore tori in $S^5(1)$. This completes the proof of Theorem 3. Again, we include some pictures of the function $T_2(k)$ and of a curve β corresponding to suitable values of $T_2(k)$.





5 Examples of flat Willmore torii in $S^7(1)$

In case that the curve β is contained in $S^2(1)$, we have already obtained an expression for the curvature of the curve β in terms of elliptic functions and used this relation in the previous section in order to construct new explicit examples of Willmore tori in $S^5(1)$. Also in the case that β is linear full in $S^3(1)$, the equation (3.6) can be solved explicitly in terms of elliptic functions by following the ideas developed in Section 2 of [13]. Note that as in this case c is a nonzero constant, k_1 is a stricly positive function. First, we assume that k_1 (and therefore also k_2) are nonzero constants. It then follows that k_1 is determined by

$$\tfrac{1}{2}k_1^2 - c^2k_1^{-4} = 1,$$

From this, we get that

$$c^2 = \tfrac{k_1^2-2}{2}k_1^4,$$

which implies $k_1^2 > 2$. Then the solution of (3.4) is

$$\beta(s) = (\lambda\cos(\theta_1 s), \lambda\sin(\theta_1 s), \mu\cos(\theta_2 s), \mu\sin(\theta_2 s)),$$

where

$$\lambda = \{\frac{1}{2}(1 - \frac{3k_1^2 - 4}{\sqrt{9(k_1)^4 - 8(k_1)^2 + 16}})\}^{\frac{1}{2}},$$

$$\mu = \{\frac{1}{2}(1 + \frac{3k_1^2 - 4}{\sqrt{9(k_1)^4 - 8(k_1)^2 + 16}})\}^{\frac{1}{2}},$$

$$\theta_1 = \frac{1}{2}\sqrt{3(k_1)^2 + \sqrt{9(k_1)^4 - 8(k_1)^2 + 16}},$$

$$\theta_2 = \frac{1}{2}\sqrt{3(k_1)^2 - \sqrt{9(k_1)^4 - 8(k_1)^2 + 16}}.$$

We get the following non-minimal flat Willmore tori $\mathbf{r} : M \to S^7$

$$\begin{aligned}\mathbf{r} \;=\; & (\lambda\cos(\theta_1 s)\cos t, \lambda\cos(\theta_1 s)\sin t, \lambda\sin(\theta_1 s)\cos t, \lambda\sin(\theta_1 s)\sin t,\\ & \mu\cos(\theta_2 s)\cos t, \mu\cos(\theta_2 s)\sin t, \mu\sin(\theta_2 s)\cos t, \mu\sin(\theta_2 s)\sin t).\end{aligned}$$

Next we assume that k_1 is not a constant. We then introduce a positive function u by $u(s) = k_1(s)^2$. As $u'(s) = 2k_1k_1'$, it follows that

$$\begin{aligned}u''(s) &= 2(k_1')^2 + 2k_1k_1''\\ &= \tfrac{1}{2}\tfrac{(u')^2}{u} + 2k_1(k_1 - \tfrac{1}{2}k_1^3 + c^2(k_1)^{-3})\\ &= \tfrac{1}{2}\tfrac{(u')^2}{u} + 2u - u^2 + 2c^2u^{-1},\end{aligned}$$

which we can still rewrite as:

$$((u')^2u^{-1})' = (4 - 2u + 4c^2u^{-2})u'.$$

Solving the above equation, we find that there exists a constant d such that

$$(u')^2 = 4u^2 - u^3 - 4c^2 + du. \tag{5.1}$$

As the right hand side of (5.1) is positive for $u \to -\infty$ and negative for $u = 0$ or $u \to +\infty$ and because $u(s)$ is a positive function (and thus there are values of s such that $u(s)$ is positive and the righthandside of (5.1) is positive), it follows that the righthandside polynomial in u has three real roots: 1 negative $-\alpha_1$ and 2 positive $\alpha_2 < \alpha_3$. Clearly, we must have that

$$\alpha_1 = \alpha_2 + \alpha_3 - 4. \qquad\qquad 4c^2 = \alpha_1\alpha_2\alpha_3$$

This implies that $\alpha_2 + \alpha_3 - 4 > 4$. It is then well known that the solution u can be expressed as

$$k_1^2(s) = u(s) = \alpha_3(1 - \tfrac{\alpha_3-\alpha_2}{\alpha_3+\alpha_2}\,\mathrm{sn}^2(\tfrac{1}{2}\sqrt{\alpha_3+\alpha_1}s, \sqrt{\tfrac{\alpha_3-\alpha_2}{\alpha_3+\alpha_1}})).$$

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Haizhong Li (present address)
Institut für Mathematik
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin, Germany
Email: hli@sfb288.math.tu-berlin.de

Luc Vrancken
Department of Mathematics
Universiteit Utrecht
Budapeslaan 6, 3584 CD Utrecht
The Netherlands
Email:vrancken@math.uu.nl

Haizhong Li (permanent address)

Department of Mathematical Sciences
Tsinghua University
Beijing, 100084
People's Republic of China
Email: hli@math.tsinghua.edu.cn