Some calculations for Israeli options

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Abstract. Recently Kifer (2000) introduced the concept of an Israeli (or Game) option. That is a general American-type option with the added possibility that the writer may terminate the contract early inducing a payment exceeding the holder's claim had they exercised at that moment. Kifer shows that pricing and hedging of these options reduces to evaluating an optimal stopping problem assocaited with Dynkin games. In this short text we give two examples of perpetual Israeli options where the solutions are explicit.

Key words: Stochastic games, option pricing, fluctuation theory.

JEL Classification: G13, C73

Mathematics Subject Classification (1991): 90A09, 60J40, 90D15.

1 Israeli options

Consider the Black-Scholes market. That is, a market with a risky asset S and a riskless bond, B. The bond evolves according to the dynamic

$$dB_t = rB_t dt \text{ where } r, t \geq 0.$$

The value of the risky asset is written as the process $S = \{S_t : t \geq 0\}$ where

$$S_t = s \exp\{\sigma W_t + \mu t\} \text{ where } s > 0$$

is the initial value of S and $W = \{W_t : t \geq 0\}$ is a Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, P)$ satisfying the usual conditions.

Let $0 < T \leq \infty$. Suppose that $X = \{X_t : t \in [0, T]\}$ and $Y = \{Y_t : t \in [0, T]\}$ be two continuous stochastic processes defined on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ such that with probability one $Y_t \geq X_t$ for all $t \in [0, T]$. The Israeli option, introduced by Kifer (2000), is a contract between a writer and holder at time $t = 0$ such that both have the right to exercise at any $\mathbb{F}$-stopping time before the expiry date T. If the holder exercises, then (s)he may claim the value of X at the exercise date and if the writer exercise, (s)he is obliged to pay to the writer the value of Y at the time of exercise. If neither have exercised at time T and $T < \infty$ then the writer pays the holder the value X_T. If both decide to claim at the same time then the lesser of the two claims is paid. (Note that the assumption that X and Y are continuous processes is not the most generic case but will suffice for the discussion in the sequel). In short, if the holder will exercise with strategy σ and the writer with strategy τ we can conclude that at any moment during the life of the contract, the holder can expect to receive $Z_{\sigma,\tau}$ where

$$Z_{s,t} = X_s \mathbf{1}_{(s \leq t)} + Y_t \mathbf{1}_{(t < s)}.$$

Suppose now that $\mathbb{P}_s$ is the risk-neutral measure for S under the assumption that $S_0 = s$. [Note that standard Black-Scholes theory dictates that this measure exists and is uniquely defined via a Girsanov change of measure]. We shall denote $\mathbb{E}_s$ to be expectation under $\mathbb{P}_s$. The following Theorem is Kifer's pricing result.

Theorem 1 (Kifer) *Suppose that for all $s > 0$*

$$\mathbb{E}_s \left(\sup_{0 \leq t \leq T} e^{-rt} Y_t \right) < \infty$$

and if $T = \infty$ that $\mathbb{P}_s \left(\lim_{t \uparrow \infty} e^{-rt} Y_t = 0 \right) = 1$. Let $\mathcal{T}_{t,T}$ be the class of $\mathbb{F}$-stopping times valued in $[t, T]$. The value of the Israeli option under the Black-Scholes framework is given by $V = \{V_t : t \in [0, T]\}$ where

$$\begin{aligned} V_t &= \text{ess-inf}_{\tau \in \mathcal{T}_{t,T}} \text{ess-sup}_{\sigma \in \mathcal{T}_{t,T}} \mathbb{E}_s \left(e^{-r(\sigma \wedge \tau - t)} Z_{\sigma,\tau} \middle| \mathcal{F}_t \right) \\ &= \text{ess-sup}_{\sigma \in \mathcal{T}_{t,T}} \text{ess-inf}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_s \left(e^{-r(\sigma \wedge \tau - t)} Z_{\sigma,\tau} \middle| \mathcal{F}_t \right) \end{aligned}$$

Further the optimal stopping strategies for the holder and writer respectively are

$$\sigma^* = \inf \{t \in [0, T] : V_t \leq X_t\} \wedge T \text{ and } \tau^* = \inf \{t \geq 0 : V_t \geq Y_t\} \wedge T.$$

The formulae given in this theorem reflect the fact that the essence of this option contract is based on the older theory of Dynkin games or stochastic games; see Friedman (1976) or Dynkin (1969) for example. In this paper we shall perform calculations showing that for certain familiar choices of X and Y exact expressions can be obtained for V when working with perpetual options. The two cases we shall consider are as follows.

Israeli δ-penalty put options. In this case, the holder may claim as a normal American put,

$$X_t = (K - S_t)^+.$$

The writer on the other hand will be assumed to payout the holders claim plus a constant,

$$Y_t = (K - S_t)^+ + \delta \text{ for } \delta > 0.$$

Israeli δ-penalty Russian options. The holder may exercise to take a normal Russian claim,

$$X_t = e^{-\alpha t} \max\left\{ m, \sup_{u\in[0,t]} S_u \right\} \text{ for } \alpha > 0, m > s$$

and the writer is punished by an amount $e^{-\alpha t}\delta S_t$ for annulling the contract early,

$$Y_t = e^{-\alpha t} \left(\max\left\{ m, \sup_{u\in[0,t]} S_u \right\} + \delta S_t \right) \text{ for } \delta > 0.$$

Our strategy is simple. Relying on the results for American put and Russian options (cf. McKean (1965), Shepp and Shiryayev (1995), Graversen and Peškir (1998), Kyprianou and Pistorius (2000) and Avram *et al.* (2002a)) we guess the form of the optimal stopping strategies using heuristic arguments based on fluctuation theory and then show using martingale techniques that the suggested solutions indeed solve the given optimal stopping problems. In the following two sections we deal with the Isreali δ-penalty put and Russian options respectively. We conclude the paper with some remarks about Canadization and the finite expiry case.

2 Perpetual Israeli δ-penalty puts

For reflection, let us consider the case of the perpetual American put option with the same parameter K. In this case it is know that the option is given by the process $\{v^A(S_t) : t \geq 0\}$ where

$$v^A(s) = \sup_{\sigma\in\mathcal{T}_{0,\infty}} \mathbb{E}_s\left(e^{-r\tau}(K - S_\tau)^+\right)$$

which may otherwise be expressed as

$$v^A(s) = \begin{cases} (K-s) & s \in (0, s^*] \\ (K-s^*)(s^*/s)^{2r/\sigma^2} & s \in (s^*, \infty) \end{cases}$$

with

$$s^* = \frac{K}{(1+\sigma^2/2r)}.$$

Further the optimal stopping strategy is $\sigma_{s^*} = \inf\{t \geq 0 : S_t \leq s^*\}$.

The logic behind this solution is as follows. The holder is interested in stopping to claim as small a value of S as possible. On the other hand, if (s)he waits too long for this to happen, then (s)he will be punished through the exponential discounting. The compromise is to stop at some boundary close to zero. Suppose now that the holder has not yet exercised, then the remaining time to expiry in this option is still infinite suggesting the solution is time invariant, that is to say the boundary is a fixed level.

Now let us turn our attention to the Israeli δ-penalty put. In this case, the holder is still interested in stopping as close to zero as possible but without waiting too long. From the writer's perspective, there is a chance to exercise when the value of the asset S is small enough to make $(K - S_\tau)^+ = 0$ in which case, they are only left with the burden of a payment of the form $\delta e^{-r\tau}$. The later this can happen the better. If the initial value of the risky asset is to the left of K then it would seem rational to cancel the contract as soon as S hits K. On the other hand, if the initial value of the risky asset is to the right of K then it would seem rational to wait until the first moment that $S_t = K$. Again, the perpetual nature of the option suggests a time invariant approach to the writers strategy. The conclusion would seem to be a hitting problem of the set $(0, k^*) \cup \{K\}$ for some choice of k^*. This will turn out to be the case providing the value of δ is not too large. Beyond a certain value of δ it would not seem efficient for the writer to exercise at all. We shall show in this case that the solution is, as one would expect, the same as the American put.

Theorem 2 *Let* $\gamma = \left(r/\sigma^2 + 1/2\right)$ *and define*

$$\delta^* = v^A(K) = \frac{K}{2\gamma}\left(\frac{2\gamma-1}{2\gamma}\right)^{(2\gamma-1)}.$$

(i) *If* $\delta \geq \delta^*$ *then the perpetual Israeli* δ*-penalty put option is nothing more than an American put option, that is, the writer will never exercise.*

(ii) *If* $\delta < \delta^*$ *then the perpetual Israeli* δ*-Put option has value process such that* $V_t = I^P(S_t)$ *where* $I^P(s)$ *is given by*

$$\begin{array}{cc} K - s & s \in (0, k^*] \\ \begin{array}{c}(K-k^*)\left(\frac{s}{k^*}\right)^{-(\gamma-1)} \frac{(s/K)^\gamma - (s/K)^{-\gamma}}{(k^*/K)^\gamma - (k^*/K)^{-\gamma}} \\ +\delta\left(\frac{s}{K}\right)^{-(\gamma-1)} \frac{(s/k^*)^{-\gamma} - (s/k^*)^\gamma}{(k^*/K)^\gamma - (k^*/K)^{-\gamma}}\end{array} & s \in (k^*, K) \\ \delta\left(\frac{s}{K}\right)^{-(2\gamma-1)} & s \in [K, \infty) \end{array}$$

and the optimal stopping strategies for the holder and writer respectively are

$$\sigma^* = \inf\{t \geq 0 : S_t \leq k^*\} \text{ and } \tau^* = \inf\{t \geq 0 : S_t = K\}$$

where k^*/K *is the solution in* $(0,1)$ *to the equation*

$$y^{2\gamma} + 2\gamma - 1 = 2\gamma\left(1 + \frac{\delta}{K}\right)y$$

Proof. The value process V can be simplified on account of the Markovian character of the claim structure. Indeed $V_t = I^P(S_t)$ almost surely where $I^P(s)$ is given by

$$\inf_{\tau \in \mathcal{T}_{0,\infty}} \sup_{\sigma \in \mathcal{T}_{0,\infty}} \mathbb{E}_s\left(e^{-r(\sigma\wedge\tau)}\left[(K - S_\sigma)\mathbf{1}_{(\sigma\leq\tau)} + \{(K - S_\tau) + \delta\}\mathbf{1}_{(\sigma>\tau)}\right]\right)$$

or indeed

$$\sup_{\sigma\in\mathcal{T}_{0,\infty}}\inf_{\tau\in\mathcal{T}_{0,\infty}}\mathbb{E}_s\left(e^{-r(\sigma\wedge\tau)}\left[(K-S_\sigma)\mathbf{1}_{(\sigma\leq\tau)}+\{(K-S_\tau)+\delta\}\mathbf{1}_{(\sigma>\tau)}\right]\right).$$

(i) Suppose that $\delta>\delta^*$. Taking the value function $v^A(s)$ recall the well established facts that

$$\left\{e^{-rt}v^A(S_t):t\geq 0\right\}\text{ and }\left\{e^{-r(t\wedge\sigma_{s^*})}v^A(S_{t\wedge\sigma_{s^*}}):t\geq 0\right\}$$

are a supermartingale and martingale respectively where $\sigma_{s^*}=\inf\{t\geq 0:S_t=s^*\}$. Since $\delta>\delta^*$ it follows that

$$(K-s)^+\leq v^A(s)<(K-s)^++\delta \tag{1}$$

(a sketch may help) and hence leaning on Doob's Optional Stopping Theorem

$$\begin{aligned}
v^A(s) &= \inf_{\tau\in\mathcal{T}_{0,\infty}}E_s\left(e^{-r(\tau\wedge\sigma_{s^*})}v^A(S_{\tau\wedge\sigma_{s^*}})\right)\\
&\leq \inf_{\tau\in\mathcal{T}_{0,\infty}}E_s\left(e^{-r(\tau\wedge\sigma_{s^*})}\left[(K-S_{\sigma_{s^*}})\mathbf{1}_{(\sigma_{s^*}\leq\tau)}+\{(K-S_\tau)+\delta\}\mathbf{1}_{(\sigma_{s^*}>\tau)}\right]\right)\\
&\leq \sup_{\sigma\in\mathcal{T}_{0,\infty}}\inf_{\tau\in\mathcal{T}_{0,\infty}}E_s\left(e^{-r(\tau\wedge\sigma)}\left[(K-S_\sigma)\mathbf{1}_{(\sigma\leq\tau)}+\{(K-S_\tau)+\delta\}\mathbf{1}_{(\sigma>\tau)}\right]\right)\\
&\leq \sup_{\sigma\in\mathcal{T}_{0,\infty}}E_s\left(e^{-r\sigma}(K-S_\sigma)\right)\\
&\leq \sup_{\sigma\in\mathcal{T}_{0,\infty}}E_s\left(e^{-r\sigma}v^A(S_\sigma)\right)\\
&\leq v^A(s).
\end{aligned}$$

The first equality follows from the martingale property. The first inequality follows from (1), the third inequality uses the fact that the infimum can be no greater than the expectation evaluated at $\tau=\infty$, the fourth uses (1) again and the final inequality uses the supermartingale property. Note also that the order of the supremum and infimum in the second inequality above can also be reversed by starting from the right hand side and reasoning in a similar manner towards the left hand side. It follows that $I^P(s)=v^A(s)$.

(ii) Let us now suppose then that $\delta\leq\delta^*$. We thus need to conclude that $I^P(s)$ is indeed of the given form and further is achieved by stopping at $\sigma^*\wedge\tau^*$. To this end, define for general $k\leq K$

$$v(s)=\mathbb{E}_s\left(e^{-r(\sigma_k\wedge\tau_K)}Z_{\sigma_k,\tau_K}\right)$$

where

$$\sigma_k=\inf\{t\geq 0:S_t\leq k\}\text{ and }\tau_K=\inf\{t\geq 0:S_t=K\}.$$

We can write

$$v(s)=\left\{\begin{array}{cc}K-s & s\in(0,k]\\ (K-k)\mathbb{E}_s\left(e^{-r\sigma_k}\mathbf{1}_{(\sigma_k\leq\tau_K)}\right)+\mathbb{E}_s\left(\delta e^{-r\tau_K}\mathbf{1}_{(\sigma_k>\tau_K)}\right) & s\in(k,K)\\ \delta\mathbb{E}_s\left(e^{-r\tau_K}\right) & s\in[K,\infty)\end{array}\right..$$

The expectations in the previous expression are the classic objects of study from the two sided exit problem of Brownian motion; see for example Borodin and Salaminen (1996) or Karatzas and Shreve (1988). Filling in we have $v(s)$ is equal to

$$\begin{array}{cc} K-s & s\in(0,k] \\ (K-k)\left(\frac{s}{k}\right)^{-(\gamma-1)}\frac{(s/K)^{\gamma}-(s/K)^{-\gamma}}{(k/K)^{\gamma}-(k/K)^{-\gamma}}+\delta\left(\frac{s}{K}\right)^{-(\gamma-1)}\frac{(s/k)^{-\gamma}-(s/k)^{\gamma}}{(k/K)^{\gamma}-(k/K)^{-\gamma}} & s\in(k,K) \\ \delta\left(\frac{s}{K}\right)^{-(2\gamma-1)} & s\in[K,\infty) \end{array}$$

where $\gamma=(r/\sigma^2+1/2)$. Note that there is continuity at $s=k$ and $s=K$. We shall also note for later use that it is immediate from the two sided exit problem that

$$\left\{e^{-r(t\wedge\tau_K\wedge\sigma_k)}v\left(S_{t\wedge\tau_K\wedge\sigma_k}\right):t\geq 0\right\}$$

is a $\mathbb{P}_s$-martingale, alternatively that $(\mathcal{L}-r)\,v\,(s)=0$ on (k,K) where $\mathcal{L}$ is the infinitesimal generator of the process $(S,\mathbb{P})$. Similarly, from the one sided exit problem, it follows that when $s\geq K$

$$\left\{e^{-r(t\wedge\tau_K)}v\left(S_{t\wedge\tau_K}\right):t\geq 0\right\}$$

is a $\mathbb{P}_s$-martingale from which it follows that $(\mathcal{L}-r)\,v\,(s)=0$ on (K,∞). Finally we can add to these variational equalities that by a trivial computation $(\mathcal{L}-r)\,v\,(s)\leq 0$ on $(0,k)$.

For $s<K$ we want to deduce that $\left\{e^{-r(t\wedge\tau_K)}v\left(S_{t\wedge\tau_K}\right):t\geq 0\right\}$ $\mathbb{P}_s$-is a supermartingale by applying the Itô formula. The minimum requirement of smoothness on v we can allow then is that k is chosen to be a special value k^* such that $v'(k^*)=-1$. That is, there is continuity in v' at k^*. A rather tedious calculation reveals that this condition on k^* amounts to finding a solution in $(0,K)$ to the equation

$$\left(\frac{k^*}{K}\right)^{2\gamma}+2\gamma-1=2\gamma\left(1+\frac{\delta}{K}\right)\left(\frac{k^*}{K}\right). \tag{2}$$

Note that if $\delta=\delta^*$ then the solution is easily seen on inspection to be $k^*=s^*=K\,(2\gamma-1)\,/2\gamma$. Further, as δ decreases the solution k^* increases until $\delta=0$ where the solution becomes $k^*=K$. It can be further checked that with $k=k^*$, it is also true that v is a convex function on $(0,\infty)$ such that

$$(K-s)^+\leq v\,(s)\leq(K-s)^++\delta. \tag{3}$$

Since now $v(s)\in C^1(0,K)\cup C^2(0,K)\backslash\{k^*\}$ and $(\mathcal{L}-r)\,v\,(s)\leq 0$ on $(0,K)\backslash\{k^*\}$ we can apply the Itô formula to the process

$$\left\{e^{-r(t\wedge\tau_K)}v\left(S_{t\wedge\tau_K}\right):t\geq 0\right\}$$

and deduce that it is a $\mathbb{P}_s$-supermartingale. It also follows from Itô's rule for convex functions that on $t \le \sigma_{k^*}$

$$\begin{aligned} d\left[e^{-rt} v\left(S_t\right)\right] &= e^{-rt}\left(\mathcal{L}-r\right) v\left(S_t\right) dt \\ &\quad + e^{-rt}\left(v'\left(K^+\right) - v'\left(K^-\right)\right) L_t^K + dM_t \end{aligned}$$

where L^K is the local time at K of S, $v'(K^+)$ and $v'(K^-)$ are the right and left first derivatives of v at K and M_t is a pure martingale term (cf. Karatzas and Shreve Problem 3.6.24). Since $v'\left(K^+\right) - v'\left(K^-\right) \ge 0$ (becasue of convexity) and $\left(\mathcal{L}-r\right) v\left(s\right) = 0$ on $(k^*, \infty)\backslash\{K\}$ it follows that

$$\left\{e^{-r(t\wedge\sigma_{k^*})} v\left(S_{t\wedge\sigma_{k^*}}\right) : t \ge 0\right\}$$

is a $\mathbb{P}_s$-submartingale. With the previous observations concerning martingales we now have

$$\begin{aligned} v\left(s\right) &\le \inf_{\tau\in\mathcal{T}_{0,\infty}} E_s\left(e^{-r(\tau\wedge\sigma_{k^*})} v\left(S_{\tau\wedge\sigma_{k^*}}\right)\right) \\ &\le \inf_{\tau\in\mathcal{T}_{0,\infty}} E_s\left(e^{-r(\tau\wedge\sigma_{s^*})}\left[\left(K - S_{\sigma_{k^*}}\right)\mathbf{1}_{(\sigma_{k^*}<\tau)} + \left\{\left(K - S_\tau\right) + \delta\right\}\mathbf{1}_{(\tau\le\sigma_{k^*})}\right]\right) \\ &\le \sup_{\sigma\in\mathcal{T}_{0,\infty}} \inf_{\tau\in\mathcal{T}_{0,\infty}} E_s\left(e^{-r(\tau\wedge\sigma)}\left[\left(K - S_\sigma\right)\mathbf{1}_{(\sigma<\tau)} + \left\{\left(K - S_\tau\right) + \delta\right\}\mathbf{1}_{(\tau\le\sigma)}\right]\right) \\ &\le \sup_{\sigma\in\mathcal{T}_{0,\infty}} E_s\left(e^{-r(\tau_K\wedge\sigma)}\left[\left(K - S_\sigma\right)\mathbf{1}_{(\sigma<\tau_K)} + \left\{\left(K - S_{\tau_K}\right) + \delta\right\}\mathbf{1}_{(\tau_K\le\sigma)}\right]\right) \\ &\le \sup_{\sigma\in\mathcal{T}_{0,\infty}} E_s\left(e^{-r(\tau_K\wedge\sigma)}\left[v\left(S_{\tau_K\wedge\sigma}\right)\right]\right) \\ &\le v\left(s\right). \end{aligned}$$

We have used the submartingale property in the first inequality, (3) in the second. For the third inequality we have used the fact that the infimum of the expectation over τ is no greater that the expectation evaluated at τ_K. The fourth inequality uses (3) again and the fifth uses the supermartingale property. Again the order of the supremum and infimum can be exchanged by starting from the right hand side and working the inequalities in reverse. It follows that $I^P(s) = v\left(s\right)$. ■

Remark 3 There is an intuitive way to see the results that have appeared in Theorem 2. Consider in the same diagram the graph of $(K-s)^+, v^A\left(s\right)$ and $(K-s)^+ + \delta$. Given that the writer now has the possibility of removing the rights of the holder, one should expect to see that $I^P\left(s\right)$ is bounded above by the smaller of $v^A\left(s\right)$ and $(K-s)^+ + \delta$. On the other hand, it is also clear that if the writer is to exercise at all, then they should do it when $s \ge K$. With this in mind, when $\delta \ge \delta^*$ a possibility that would make sense is that the writer never exercises and hence the option is nothing more than an American put. When $\delta < \delta^*$ and the graphs of $v^A\left(s\right)$ and $(K-s)^+ + \delta$ cross over one another, things change. The shape of $I^P(s)$ could be imagined to be the result of the following deformation of $v^A(s)$. Slowly decrease δ from a large value, so that the curve

$(K-s)^{+}+\delta$ pushes down on $v^{A}(s)$ at the contact point $s=K$ reshaping it. A non-smooth 'angle' will form at $s=K$ and the smooth join to the line $(K-s)$ will also be dragged forwards.

3 Perpetual Israeli δ-penalty Russian

We begin again by considering the older relative of this option, the perpetual Russian option. Recall the Russian option has value given by

$$\text{ess-sup}_{\sigma\in\mathcal{T}_{t,\infty}}\mathbb{E}_s\left(e^{-r(\sigma-t)}e^{-\alpha\sigma}\max\left\{e^{m},\overline{S}_{\sigma}\right\}\right).$$

By using a second change of measure (over and above moving to the risk neutral measure)

$$\left.\frac{d\widetilde{\mathbb{P}}_s}{d\mathbb{P}_s}\right|_{\mathcal{F}_t}=\frac{e^{-rt}S_t}{s}$$

and defining $\widetilde{\mathbb{P}}_{m/s}(\cdot)=\widetilde{\mathbb{P}}_s\left(\cdot|\overline{S}_0=m\right)$, Shepp and Shiryayev (1995) shown that the value of the option can be more neatly written as

$$\left\{e^{-\alpha t}S_t v^{R}(\Psi_t):t\geq 0\right\}$$

where $\Psi=\{\Psi_t=\overline{S}_t/S_t:t\geq 0\}$ and

$$v^{R}(\psi)=\text{ess-sup}_{\sigma\in\mathcal{T}_{0,\infty}}\widetilde{\mathbb{E}}_{\psi}\left(e^{-\alpha\sigma}\Psi_{\sigma}\right).$$

Further, with $\gamma=\left(r/\sigma^{2}+1/2\right)$ as before and $\eta:=\sqrt{2\alpha/\sigma^{2}+\gamma^{2}}$ $v^{R}(\psi)$ can be written as

$$\begin{cases}(\psi_*/2\eta)\left[(\gamma+\eta-1)\left(\psi/\psi_*\right)^{\gamma-\eta}+(1-\gamma+\eta)\left(\psi/\psi_*\right)^{\gamma+\eta}\right] & \psi\in[1,\psi_*]\\ \psi & \psi\in(\psi_*,\infty)\end{cases},$$

where

$$\psi_*=\left(\frac{\gamma+\eta}{\eta-\gamma}\cdot\frac{\eta-\gamma+1}{\gamma+\eta-1}\right)^{1/2\eta}.$$

Further, the optimal stopping strategy is given by $\sigma_{\psi_*}=\inf\{t\geq 0:\Psi_t\geq\psi_*\}$.

The logic behind this result is as follows. The holder is interested in the supremum of the value of the risky asset reaching a high level. However waiting too long for this to happen will again will count against the holder because of the exponential weighting in the payout. If S experiences an excursion from $\overline{S}$ which is large, then the holder will wait a long time for the supremum to increase before the excursion is completed and thus will be penalized.

Let us assume temporarily that $\alpha=0$. When moving to the perpetual Israeli δ-penalty Russian option, it would seem that the holder's intentions should not change if they are to act reasonably. On the other hand, the writer would like to protect themselves against large values of $\overline{S}$. To do this it would seem logical to exercise once the value of S gets too high. Indeed with in an initial value of $\overline{S}$ being m, prudence would suggest it is better to call the contract off once the value of the risky asset hits m. In both cases, the perpetual nature of the option preserves the time invariance of their stopping strategies. With the obvious restriction that δ is not too large we shall show that this is indeed the case. When δ takes large values, it would not seem rational for the writer to exercise at all, in which case we have returned to the case of the Russian option.

Theorem 4 *Define*

$$\delta_* = v^R(1) - 1 = \frac{(\eta+\gamma-1)\,\psi_*^{\eta-\gamma+1} + (1+\eta-\gamma)\psi_*^{-\gamma-\eta+1} - 2\eta}{2\eta}.$$

(i) *Let $\delta \geq \delta_*$ and $\alpha > 0$ then the perpetual Israeli δ-penalty Russian option is nothing more than the perpetual Russian option. That is, the writer's strategy will be to never exercise.*

(ii) *Let $\delta < \delta_*$ and $\alpha \geq 0$. Define k_* as the solution in $[1,\infty)$ to*

$$(\gamma+\eta-1)\,y^{\eta-\gamma+1} + (\eta-\gamma+1)y^{-(\eta+\gamma-1)} = 2\eta(1+\delta).$$

If

$$2\eta k_*^{-\gamma+1} - (1+\delta)\left[(\eta-\gamma)k_*^{\eta} + (\eta+\gamma)k_*^{-\eta}\right] \geq 0 \tag{4}$$

then $V_t = e^{-\alpha t} S_t I^R(\Psi_t)$ where

$$I^R(\psi) = \begin{cases} k_*\left(\frac{\psi}{k_*}\right)^{\gamma} \frac{\psi^{\eta}-\psi^{-\eta}}{k_*^{\eta}-k_*^{-\eta}} + (1+\delta)\,\psi^{\gamma}\frac{(\psi/k_*)^{-\eta}-(\psi/k_*)^{\eta}}{k_*^{\eta}-k_*^{-\eta}} & 1 \leq \psi < k_* \\ \psi & \psi \geq k_*. \end{cases}$$

Further the optimal stopping strategies for the holder and writer respectively are

$$\sigma^* = \inf\{t \geq 0 : \Psi_t \geq k_*\} \text{ and } \tau^* = \inf\{t \geq 0 : \Psi_t = 1\}.$$

Remark 5 Like the proof of the Israeli δ-penalty put option, the method of proof in the second part of the above theorem is to show that $\{e^{-\alpha t} I^R(\Psi_t) : t \geq 0\}$ is a martingale, supermartingale and submartingale when stopped at $\sigma^* \wedge \tau^*$, τ^* and σ^* respectively. The strange technical condition (4) guarentees that the submartingale status can be affirmed. It it is little work to verify that this condition holds when for example $\alpha = 0$. This is quite a natural situation as to some extent the parameter α is a superfluous distraction here. In principle, its presence is merely for the purpose of guarenteeing that the optimal stopping problem associated with the Russian option has a solution (cf. Shepp and Shiryayev (1995)).

Remark 6 Part (ii) of the above theorem shows that the Israeli δ-penalty Russian option is infact a perpetual double barrier option in disguise.

Proof of Theorem 4. First note that the value process V can be simplified in a similar way to the Russian option. Indeed we can use the measure $\widetilde{\mathbb{P}}_{m/s}$ in a similar way to deduce that $V_t = e^{-\alpha t} S_t I^R(\Psi_t)$ where $I^R(\psi)$ is given by

$$\sup_{\sigma \in \mathcal{T}_{0,\infty}} \inf_{\tau \in \mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha\sigma} \Psi_\sigma \mathbf{1}_{(\tau \geq \sigma)} + e^{-\alpha\tau} (\Psi_\tau + \delta) \mathbf{1}_{(\tau < \sigma)} \right)$$

or indeed

$$\inf_{\tau \in \mathcal{T}_{0,\infty}} \sup_{\sigma \in \mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha\sigma} \Psi_\sigma \mathbf{1}_{(\tau \geq \sigma)} + e^{-\alpha\tau} (\Psi_\tau + \delta) \mathbf{1}_{(\tau < \sigma)} \right).$$

(i) Suppose now that $\delta \geq \delta_*$ and $\alpha > 0$ Note that when this happens, we have that

$$\psi \leq v^R(\psi) < \delta + \psi \tag{5}$$

(a quick sketch may help). Recall from well established facts concerning the Russian option

$$\left\{ e^{-\alpha(t \wedge \sigma_{\psi_*})} v^R \left(\Psi_{t \wedge \sigma_{\psi_*}} \right) : t \geq 0 \right\} \text{ and } \left\{ e^{-\alpha t} v^R(\Psi_t) : t \geq 0 \right\}$$

are a $\widetilde{\mathbb{P}}_{m/s}$-martingale and a $\widetilde{\mathbb{P}}_{m/s}$-supermartingale respectively. With these two pieces of information we can deduce that $v^R(\psi) = I^P(\psi)$ as follows:

$$\begin{aligned}
v^R(\psi) &\geq \sup_{\sigma \in \mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha\sigma} v^R(\Psi_\sigma) \right) \\
&\geq \sup_{\sigma \in \mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha\sigma} \Psi_\sigma \right) \\
&\geq \inf_{\tau \in \mathcal{T}_{0,\infty}} \sup_{\sigma \in \mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha\sigma} \Psi_\sigma \mathbf{1}_{(\sigma \leq \tau)} + e^{-\alpha\tau} (\Psi_\tau + \delta) \mathbf{1}_{(\sigma > \tau)} \right) \\
&\geq \inf_{\tau \in \mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha\sigma_{\psi_*}} \Psi_{\sigma_{\psi_*}} \mathbf{1}_{(\sigma_{\psi_*} \leq \tau)} + e^{-\alpha\tau} (\Psi_\tau + \delta) \mathbf{1}_{(\sigma_{\psi_*} > \tau)} \right) \\
&\geq \inf_{\tau \in \mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha(\sigma_{\psi_*} \wedge \tau)} v^R \left(\Psi_{\sigma_{\psi_*} \wedge \tau} \right) \right) \\
&= v^R(\psi).
\end{aligned}$$

The first inequality follows by the supermartingale property associated with v^R, the second by (5). The third inequality is a lower bound on the second which one can consider to have the same form of expectation as in the third inequality except with $\tau = \infty$. Note that under $\widetilde{\mathbb{P}}_\psi$ the state 1 is positive recurrent for the process Ψ and hence $\limsup_{t \uparrow \infty} e^{-\alpha t} \Psi_t = 0$ $\widetilde{\mathbb{P}}_\psi$-almost surely (cf. Shepp and Shiryayev (1995)). The fifth inequality uses again (5) together with the definition of σ_{ψ_*} and the final inequality is a consequence of the martingale property associated with v^R.

(ii) Let us assume that $\delta < \delta_*$ and $\alpha \geq 0$ then we want to show that $I^R(\psi)$ is of the specified form. To this end let us define for $k > 1$

$$\tau_1 = \inf\{t \geq 0 : \Psi_t = 1\} \text{ and } \sigma_k = \inf\{t \geq 0 : \Psi_t \geq k\}$$

and for $1 \leq \psi < \infty$ the function

$$
\begin{aligned}
v(\psi) &= \widetilde{\mathbb{E}}_{\psi}\left(e^{-\alpha\sigma_k}\Psi_{\sigma_k}\mathbf{1}_{(\tau_1>\sigma_k)} + e^{-\alpha\tau_1}\left(\Psi_{\tau_1}+\delta\right)\mathbf{1}_{(\tau_1<\sigma_k)}\right) \\
&= \begin{cases} \widetilde{\mathbb{E}}_{\psi}\left(ke^{-\alpha\sigma_k}\mathbf{1}_{(\tau_x>\sigma_k)} + e^{-\alpha\tau_1}\left(1+\delta\right)\mathbf{1}_{(\tau_1<\sigma_k)}\right) & 1 \leq \psi < k \\ \psi & \psi \geq k \end{cases} .
\end{aligned}
$$

Note that by construction (that is to say by virtue of the fact that v is the linear sum of solutions to a two sided exit problem for Ψ) we have that $\left(\widetilde{\mathcal{L}}-\alpha\right)v\left(\psi\right) = 0$ for $\psi \in (1,k)$ and $\left(\widetilde{\mathcal{L}}-\alpha\right)v\left(\psi\right) \leq 0$ for $\psi \in (k,\infty)$ where $\widetilde{\mathcal{L}}$ is the infinitesimal generator of $\left(\Psi,\widetilde{\mathbb{P}}\right)$. [To see this recall that $e^{-t\wedge\tau_1\wedge\sigma_k}v(\Psi_{t\wedge\tau_1\wedge\sigma_k})$ is a $\widetilde{\mathbb{P}}_{\psi}$-martingale and apply the Itô forumla]. We will show that for an appropriate choice of k, $v(\psi) = I^R\left(\psi\right)$. The expectations in the right hand side of the above equation can be evaluated using again fluctuation theory. Note that

$$
\sigma^{-1}\log\left(\overline{S}_t/S_t\right) = \left(\overline{\beta}_t - \beta_t\right) \tag{6}
$$

where under $\widetilde{\mathbb{P}}_{m/s}$, β is a Brownian motion with drift $\sigma\gamma$ where γ was defined in the previous section as $\left(r/\sigma^2+1/2\right)$. Using this information, we can use the usual two sided exit problem for Brownian motion to deduce that in fact

$$
v(\psi) = \begin{cases} k\left(\frac{\psi}{k}\right)^{\gamma}\frac{\psi^{\eta}-\psi^{-\eta}}{k^{\eta}-k^{-\eta}} + (1+\delta)\,\psi^{\gamma}\frac{(\psi/k)^{-\eta}-(\psi/k)^{\eta}}{k^{\eta}-k^{-\eta}} & 1 \leq \psi < k \\ \psi & \psi \geq k \end{cases} .
$$

We would again like to apply Itô's formula to $v(\Psi_{t\wedge\tau_1})$ in which case we will need at least continuity in v' at k. Again a tedious calculation reveals that by requiring that $k = k_*$ where k_* is the solution to

$$
\left(\gamma+\eta-1\right)k_*^{\eta-\gamma+1} + (\eta-\gamma+1)k_*^{-(\eta+\gamma-1)} = 2\eta(1+\delta), \tag{7}
$$

then $v'(k_*) = 1$ and v is a convex function on $[1,\infty)$ satisfying $v'(1) \geq 0$ when condition (4) holds and further

$$
\psi \leq v\left(\psi\right) \leq \psi + \delta. \tag{8}
$$

Note that when $\delta = \delta_*$ the solution to (7) is $k_* = \psi_*$ and as δ decreases then so does the value of k_* until finally at $\delta = 0$, $k_* = 1$.

With all the afore mentioned properties of $v(\psi)$ in mind for the choice $k = k_*$, applications of Itô's formula thus yield that

$$
\left\{e^{-\alpha(t\wedge\tau_1\wedge\sigma_{k_*})}v\left(\Psi_{t\wedge\tau_1\wedge\sigma_{k_*}}\right) : t \geq 0\right\} \text{ and } \left\{e^{-\alpha(t\wedge\tau_1)}v\left(\Psi_{t\wedge\tau_1}\right) : t \geq 0\right\}
$$

are a $\widetilde{\mathbb{P}}_{\psi}$-martingale and a $\widetilde{\mathbb{P}}_{\psi}$-supermartingale respectively. Further, Itô-Stieltjes calculus reveals that on $t \leq \sigma_{k_*}$, the non-martingale part of $d\left[e^{-\alpha t}v\left(\Psi_t\right)\right]$ takes the form

$$
e^{-\alpha t}\left(\widetilde{\mathcal{L}}-\alpha\right)v\left(\Psi_t\right)dt + e^{-\alpha t}S_t^{-1}v'\left(\Psi_t\right)d\overline{S}_t
$$

(cf. Shepp and Shiryayev (1995)). Since $\overline{S}_t$ only increases when $\Psi_t = 1$ it follows that we could replace $v'(\Psi_t)$ by $v'(1) \geq 0$ in the above calculation. The consequence of this is that

$$\left\{ e^{-\alpha(t\wedge\sigma_{k_*})} v\left(\Psi_{t\wedge\sigma_{k_*}}\right) : t \geq 0 \right\}$$

is a $\widetilde{\mathbb{P}}_\psi$-submartingale. The proof of the theorem is now completed in a familiar way. That is

$$\begin{aligned}
v(\psi) &\geq \sup_{\sigma\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha(\tau_1\wedge\sigma)} v\left(\Psi_{\tau_1\wedge\sigma}\right)\right) \\
&\geq \sup_{\sigma\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha\sigma}\Psi_\sigma \mathbf{1}_{(\tau_1>\sigma)} + e^{-\alpha\tau_1}\left(\Psi_{\tau_1}+\delta\right)\mathbf{1}_{(\tau_1<\sigma)}\right) \\
&\geq \inf_{\tau\in\mathcal{T}_{0,\infty}} \sup_{\sigma\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha\sigma}\Psi_\sigma \mathbf{1}_{(\tau>\sigma)} + e^{-\alpha\tau}\left(\Psi_{\tau}+\delta\right)\mathbf{1}_{(\tau<\sigma)}\right) \\
&\geq \inf_{\tau\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha\sigma_{k_*}}\Psi_{\sigma_{k_*}} \mathbf{1}_{(\tau>\sigma)} + e^{-\alpha\tau}\left(\Psi_{\tau}+\delta\right)\mathbf{1}_{(\tau<\sigma_{k_*})}\right) \\
&\geq \inf_{\tau\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha\sigma_{k_*}}\Psi_{\sigma_{k_*}} \mathbf{1}_{(\tau>\sigma)} + e^{-\alpha\tau}\left(\Psi_{\tau}+\delta\right)\mathbf{1}_{(\tau<\sigma_{k_*})}\right) \\
&\geq \inf_{\tau\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha(\tau\wedge\sigma_{k_*})} v\left(\Psi_{\tau\wedge\sigma_{k_*}}\right)\right) \\
&\geq v(\psi)
\end{aligned}$$

and a similar sequence of inequalities going the other way which proves that $v(\psi) = I^R(\psi)$. ■

Remark 7 Once again there is a diagrammatic intuitive explanation for these results. Consider a plot of the functions ψ, $v^R(\psi)$ and $\psi+\delta$. The holder having fewer rights than a holder of a Russian option means we expect the function $I^R(\psi)$ to be bounded above by the lesser of $v^R(\psi)$ and $\psi+\delta$. When $\delta \geq \delta_*$ then a possibility is $v^R(\psi)$ itself. For the case that $\delta < \delta_*$ one could imagine slowly decreasing δ from a large value. As it meets the curve $v^R(\psi)$ at $\psi = 1$ it pushes it down and begins to distort its shape. The zero gradient at $\psi = 1$ is lost and the point of smooth join with the function ψ is decreased in value.

4 Conclusion

Israeli options generalized the concept of the American option in that they give the writer the opportunity to cancel the contract. Seeing this as lesser rights from the point of view of the holder, a given Israeli option should be more cheaply priced than an assoicated American option. Based on this fact, Kifer (2000) has argued that they serve as an interesting derivative in the financial markets and offers a generic pricing formula. We have shown here that for two familiar claim structures, the put and Russian, within a perpetual context, Kifer's pricing formula reduces to explicit expressions and the optimal stopping times of the holder and writer reduce to intuitively appealing strategies.

One can also consider the solution to these problems in the context of a free boundary problem. For example, the Israeli δ-penalty put is the solution to

$$\begin{aligned} (\mathcal{L}-r)\, I^P(s) &= 0 \text{ on } (k^*, K)\cup(K,\infty) \\ I^P(s) &= (K-s) \text{ on } (0,k^*) \\ dI^P(k^*)/ds &= -1 \\ I^P(K) &= \delta \wedge v^A(K) \\ \lim_{s\uparrow\infty} I^P(s) &= 0 \end{aligned}$$

where k^* is to be determined. Following the terminology of Carr (1988), Canadizing an Israeli option would mean replacing a finite expiry date T by an independent exponential random variable with some rate $\lambda > 0$. If one were to proceed with the Canadized version of the Israeli δ-penalty put, then taking a free boundary perspective, one could solve the following problem for the value function $I^{CP}(s)$

$$\begin{aligned} (\mathcal{L}-r-\lambda)\, v(s) &= -\lambda(K-s)^+ \text{ on } (c^*, K)\cup(K,\infty) \\ v(s) &= (K-s) \text{ on } (0,c^*) \\ dv(c^*)/ds &= -1 \\ v(K) &= \delta \wedge v^{CA}(K) \\ \lim_{s\uparrow\infty} v\,(s) &= 0 \end{aligned}$$

where $v^{CA}(K)$ is the value of the Canadized American put and c^* is to be determined. Alternitively one could adress the problem using fluctuation theory and martingales by assuming the solution takes the form

$$\begin{aligned} &\mathbb{E}_s\left(e^{-(r+\lambda)(\tau_K\wedge\sigma_{c^*})}\left[(K - S_{\sigma_{c^*}\wedge\tau_K}) + \delta\mathbf{1}_{(\sigma_{c^*}>\tau_K)}\right]\right) \\ &+\lambda\mathbb{E}_s\left(\int_0^{\tau_K\wedge\sigma_{c^*}} e^{-(r+\lambda)u}\,(K-S_u)^+\,du\right) \end{aligned}$$

Similar remarks can be made for Candized Israeli δ-penalty Russian options.

If one were to consider the two examples we have dealt with in this paper but for finite expiry, the optimal stopping times for writer and holder would likely be time dependent and yet more difficult to characterize than for American put and Russian options. However in forthcoming work by the author and W.Schoutens, a result in Rogers (2001) concerning dual representation for American options has been generalized to the case of Israeli options. This enables simulation of option prices without having to deal with optimal crossing boundaries such as is necessary when solving free boundary problems.

On a final note, it is worth remarking that given the exact analytical expressions obtained in Avram *et. al.* (2002a,b), it is likely that one can re-employ the methods presented here to deal with the same options under spectrally negative and phase-type models.

5 Acknowledgments

I would like to express thanks to Chris Rogers for a valuable conversation.

References

[1] Avram, F. Kyprianou, A.E. and Pistorius, M.R. (2002a) Exit problems for spectrally negative Lévy processes and applications to Russian, American and Canadized options. *Submitted and Preprint University of Utrecht*

[2] Avram, F. Asmussen, S. and Pistorius, M.R. (2002b) Russian options under phase type exponential Lévy models. *Preprint University of Utrecht.*

[3] Borodin, A.N. and Salaminen, P. (1996) *Handbook of Brownian motion - facts and formulae.* Probability and its Applications. Birkhäuser.

[4] Carr, P. (1998) Randomization and the American put. *Rev. Fin. Stud.* **11**, 597–626.

[5] Dynkin, E.B. Game variant problem on optimal stopping. *Soviet Math, Dokl.* **10**, 270–274.

[6] Friedman, A. (1976) *Stochastic differential equations and applications. Vol II.* Academic Press.

[7] Graversen, S.E. and Peškir, G. (1998) On the Russian option: the expected waiting time. Theory Probab. Appl. **42**, 416–425.

[8] Karatzas, I. and Shreve, S. (1988) *Brownian motion and stochastic calculus. 2nd Edition.* Springer New York.

[9] Kifer, Yu. (2000) Game options. *Fin. Stoch. 4, 443-463.*

[10] Kyprianou, A.E. and Pistorius, M.R. (2000) Perpeutal options and canadization through fluctuation theory. *Utrecht University Preprint.*

[11] McKean, H. (1965) Appendix: A free boundary problem for the heat equation arising from a problem of mathematical economics. *Ind. Man. Rev.* **6**, 32–39.

[12] Rogers (2001) Monte Carlo evaluation of American options. *Preprint.*

[13] Shepp, L. and Shiryayev, A.N. (1995) A new look at pricing of the Russian option. *Th. Probab. Appl.* **39**, 103–119.