COMPLEXITY OF TORUS BUNDLES OVER THE CIRCLE WITH MONODROMY $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^n$

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Abstract. We find the exact values of complexity for an infinite series of 3-manifolds. Namely, by calculating hyperbolic volumes, we show that $c(N_n) = 2n$, where c is the complexity of a 3-manifold and N_n is the total space of the punctured torus bundle over S^1 with monodromy $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^n$. We also apply a recent result of Matveev and Pervova to show that $c(M_n) \geq 2Cn$ with $C \approx 0.598$, where a compact manifold M_n is the total space of the torus bundle over S^1 with the same monodromy as N_n, and discuss an approach to the conjecture $c(M_n) = 2n + 5$ based on the equality $c(N_n) = 2n$.

The notion of complexity of 3-dimensional manifolds (see Definition 4 below) was introduced by S. Matveev, see [5]. Upper bounds for complexity can easily be obtained. On the other hand, no lower bounds were known until recently, except for only several hundreds of manifolds of small complexity, where a full case-by-case analysis can be performed by a computer [6, 7]; thus, neither exact values nor even reasonable lower bounds were known for any infinite class of 3-manifolds. First meaningful lower bounds of the complexity were obtained in 2001, see [8].

In the present paper, the exact value of complexity is found for an infinite series of 3-manifolds. This is done in Section 3 for the manifolds N_n that are n-fold covers of the figure eight knot complement N_1; alternatively, N_n can be described as the total space of the punctured torus bundle over the circle with monodromy $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^n$. In Section 2 we apply the aforementioned lower bound [8] to the total space M_n of (compact) torus bundle over S^1 with the same monodromy. Section 1 contains necessary definitions.

1. Definitions.

In this section, we recall some definitions following [5, 6]. By K denote the 1-dimensional skeleton of the tetrahedron, which is just the clique (that is, the complete graph) with 4 vertices. Note that K is homeomorphic to a circle with three radii.

Definition 1. A compact 2-dimensional polyhedron is called *almost simple* if the link of its every point can be embedded in K. An almost simple polyhedron P is said to be *simple* if the link of each point of P is homeomorphic to either a circle or a circle with a diameter or the whole graph K. A point of an almost simple polyhedron is *non-singular* if its link is homeomorphic to a circle, it is said to be a *triple point* if its link is homeomorphic to a circle with a diameter, and it is called a *vertex* if its link is homeomorphic to K. The set of singular points of a simple polyhedron P (i.e., the union of the vertices and the triple lines) is called its *singular graph* and is denoted by SP.

It is easy to see that any compact subpolyhedron of an almost simple polyhedron is almost simple as well. Neighborhoods of non-singular and triple points of a simple polyhedron are shown in Fig. 1 a, b; Fig. 1 c–f represents four equivalent ways of looking at vertices; in particular, Fig. 1 e shows the cone over the 1-dimensional skeleton of the tetrahedron.

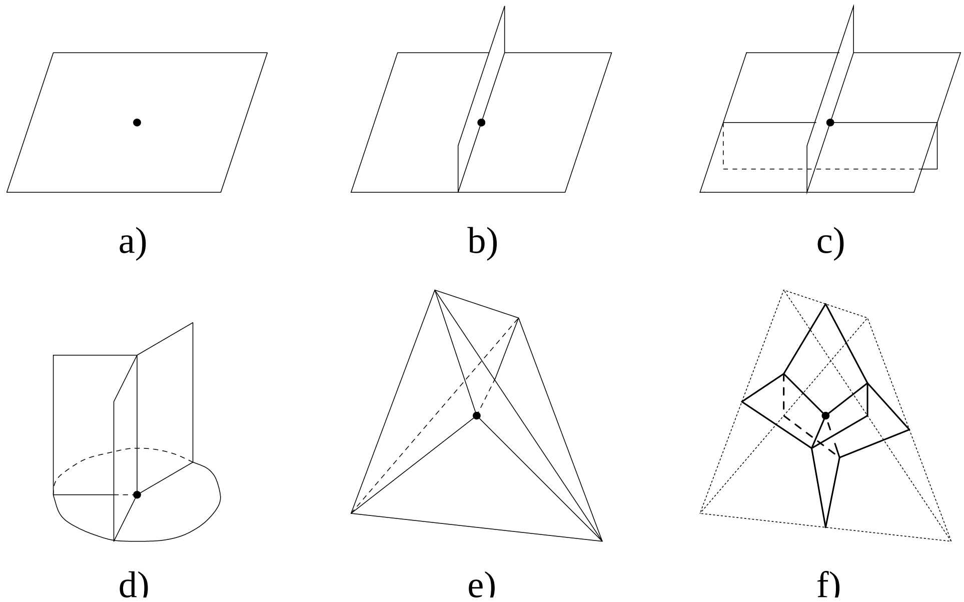

FIGURE 1. Nonsingular (a) and triple (b) points; ways of looking at vertices (c–f)

Definition 2. A simple polyhedron P with at least one vertex is said to be *special* if it contains no closed triple lines (without vertices) and every connected component of $P \setminus SP$ is a 2-dimensional cell.

Definition 3. A polyhedron $P \subset \operatorname{Int} M$ is called a *spine* of a compact 3-dimensional manifold M if $M \setminus P$ is homeomorphic to $\partial M \times (0,1]$ (if $\partial M \neq 0$) or to an open 3-cell (if $\partial M = 0$). In other words, P is a spine of M if a manifold M with boundary (or a closed manifold M punctured at one point) can be collapsed onto P. A spine P of a 3-manifold M is said to be *almost simple*, *simple*, or *special* if it is an almost simple, simple, or special polyhedron, respectively.

Given a special spine P of a compact manifold M^3, one can construct a dual singular triangulation of M^3 with one vertex (lying in the middle of the 3-cell $M \setminus P$), see Fig. 1 f; if M is a manifold with connected boundary, the same construction gives a triangulation of the one-point compactification of $M \setminus \partial M$. In both cases, there is a one-to-one correspondence between vertices of P and tetrahedra of the triangulation.

Definition 4. The *complexity* $c(M)$ of a compact 3-manifold M is the minimal possible number of vertices of an almost simple spine of M. An almost simple spine with the smallest possible number of vertices is said to be a *minimal* spine of M.

Theorem 1 [5]. *Let M be an orientable irreducible 3-manifold with incompressible (or empty) boundary and without essential annuli. If $c(M) > 0$ (that is, if M is different from (possibly punctured) S^3, $\mathbb{R}P^3$, and $L_{3,1}$), then any minimal almost simple spine of M is special.*

Thus, if M is as in Theorem 1, then $c(M)$ is equal to the minimal number of tetrahedra in a singular triangulation of M. By the way, this implies that $c(M) \geq \|M\|$, where $\|M\|$ stands for the Gromov norm of M (see [4]), whenever M is a compact 3-manifold that satisfies the assumptions of Theorem 1. We do not know any manifold M such that $c(M) = \|M\|$.

2. Torus bundles over S^1: a lower bound.

Theorem 2 [8]. *Let M be a compact irreducible orientable 3-manifold non-homeomorphic to S^3, $\mathbb{R}P^3$, and $L_{3,1}$. Then $c(M) \geq 2\log_5 |\mathrm{Tor}(H_1(M,\mathbb{Z}))| + \beta_1(M,\mathbb{Z}) - 1$.*

Corollary 1 [8]. *For lens spaces, we have $c(L_{p,q}) \geq 2\log_5 p - 1$.*

In particular, $c(L_n) \geq 2\log_5 \varphi_n - 1$, where L_n stands for $L_{\varphi_n,\varphi_{n-1}}$ and φ_n denotes the nth Fibonacci number. Since $\varphi_n = (((\sqrt{5}+1)/2)^n + ((-\sqrt{5}+1)/2)^n)/\sqrt{5}$, we have $c(L_n) \geq C_n n - 2$ with $C_n = \frac{2}{n}\log_5(\sqrt{5}\varphi_n)$ tending to $C = 2\log_5((\sqrt{5}+1)/2) \approx 0.598$ as $n \to \infty$, which is a fairly good estimate, since $c(L_n) \leq n-4$ whenever $n \geq 4$, see [6].

Theorem 2 can be successfully applied to some other 3-manifolds. Let us denote by M_n the total space of the T^2-bundle over S^1 with monodromy A^n, where $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. A short calculation shows that $|\mathrm{Tor}(H_1(M_n,\mathbb{Z}))| = \pm\det(A^n - I) = \pm(\det A^n - \mathrm{Tr}\, A^n + 1)$; since $A^n = \begin{pmatrix} \varphi_{2n+1} & \varphi_{2n} \\ \varphi_{2n} & \varphi_{2n-1} \end{pmatrix}$ and $\det A^n = 1$, we have $|\mathrm{Tor}(H_1(M_n,\mathbb{Z}))| = \varphi_{2n+1} + \varphi_{2n-1} - 2$. Taking into account that $\beta_1(M_n,\mathbb{Z}) = 1$, we get the following estimate in a similar way.

Corollary 2. *$c(M_n) \geq 2C_n n$, where $C_n = \frac{1}{n}\log_5(\varphi_{2n+1} + \varphi_{2n-1} - 2)$.*

Note that $C_n \to C = \log_5((\sqrt{5}+1)/2)^2 \approx 0.598$ as $n \to \infty$, which is as good as in the previous example since $c(M_n) \leq 2n+5$, see [1]. Combining this with the inequalities $c(M_n) \geq 7$ and $C_n > 0.597$ whenever $n \geq 6$ (none of the M_n is contained in the list of 3-manifolds up to complexity 6, see [6]; in fact, all compact 3-manifolds up to complexity 6 are elliptic except for the flat manifolds, which all have complexity 6, while all the manifolds M_n are Sol-manifolds), we get $c(M_n) > 1.19n$ for all $n \geq 1$. We believe that $c(M_n) = 2n+5$ (see [1]) and $c(L_n) = n-4$ (see [5, 6]). Note that $\|M_n\| = 0$, because there is an obvious action of S^1 on M_n.

3. Punctured torus bundles: exact values.

This is the main section of the paper. Here we find the exact values of $c(N_n)$ for an infinite series of 3-manifolds; to the best of our knowledge, this is the first result of this kind. The manifolds N_n, $n \in \mathbb{N}$, are the total spaces of the punctured torus bundles over S^1 with monodromy A^n, where $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

The manifold N_1 has been studied extensively. It is well known to be hyperbolic and homeomorphic to the figure eight knot complement, see [9, Chap. 4]. A special spine P of N_1 with two vertices, four edges, and two hexagonal 2-cells is represented by Fig. 2 a; the picture shows the boundary of the neighborhood in P of the singular graph SP (which consists of two vertices and four triple lines; note that the triple lines themselves are not drawn). Since the spine P is special (and thus both components of $P \setminus SP$ are disks), the picture contains enough information to reconstruct P. This spine coincides with the spine P_0 constructed in [1, §2.3]. Figures 2 a and 2 c are taken from [2, §4]; many other pictures related to the manifold N_1 can be found in [3, Chap. 8].

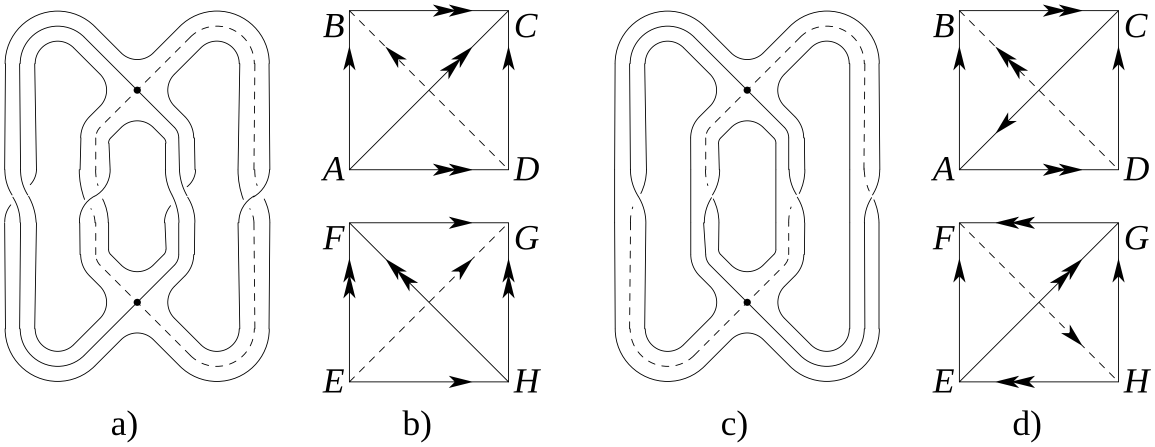

FIGURE 2. Noncompact hyperbolic 3-manifolds of complexity 2

Theorem 3. *The equality $c(N_n) = 2n$ holds.*

Proof. Any special spine defines its dual decomposition of the manifold into tetrahedra. For N_1, this decomposition is shown on Fig. 2 b. It consists of two tetrahedra (each contains one vertex of the spine), glued together so that there are two edges (dual to the hexagons of the spine); each of them is incident to six dihedral angles of the tetrahedra; it is described in detail in [9, Chap. 1]. The gluing pattern can be reconstructed from Fig. 2 a; it is $ABC \longleftrightarrow EHF$, $BAD \longleftrightarrow GEF$, $CDA \longleftrightarrow GFH$, $DCB \longleftrightarrow EGH$. Thus, the edges marked by single arrows are glued together, those marked by double arrows are glued together, too, and the direction of the arrows is respected.

A complete hyperbolic structure on N_1 comes from that on two tetrahedra considered as regular ideal tetrahedra in H^3. All their dihedral angles are equal to $\pi/3$ [9], so the sum of the dihedral angles incident to an edge equals 2π for both edges, which means that the hyperbolic structure described above is well defined. Among the ideal tetrahedra in H^3, the regular one has the maximal volume $V \approx 1.0149$, see, e.g., [9, Chap. 7]. Therefore, the manifold N_1 admits a hyperbolic structure of volume $2V$.

Since there is an n-fold covering $p\colon N_n \to N_1$, the polyhedron $p^{-1}(P)$ is a special spine of N_n with $2n$ vertices, so we have $c(N_n) \leq 2n$; again, that spine coincides with the one constructed in [1, §2.3]. For the same reason, the manifold N_n admits a complete hyperbolic structure of volume $2nV$. Now we have to prove the inequality $c(N_n) \geq 2n$.

The manifolds N_n satisfy the hypotheses of Theorem 1. Thus, their minimal spines are special. So, if a minimal spine of N_n contains k vertices, then there is a (singular) triangulation of N_n formed by k tetrahedra. Straightening them, we get a triangulation of a fundamental domain for $\pi_1(N_n)$ in H^3, which has volume $2nV$, into k ideal tetrahedra (which may overlap). Since the volume of any ideal tetrahedron in H^3 does not exceed V, we get $k \geq 2n$. $\square$

Remarks. 1. In fact, we have shown that $c(M) \geq \left\lceil \frac{\mathrm{Vol}(M)}{V} \right\rceil$ for any hyperbolic manifold M^3, either compact or noncompact, orientable or not. We know no examples of compact hyperbolic 3-manifolds for which this estimate is sharp. On the other hand, there exist compact orientable hyperbolic 3-manifolds of volume 0.94... and 0.98..., while the complexity of any compact orientable hyperbolic 3-manifold is at least 9, see [2]. Moreover, there exist infinitely many compact hyperbolic 3-manifolds such that their volume is less than $2V$ [2, 9]; their list contains manifolds of arbitrary large complexity, because there are only finitely many irreducible 3-manifolds of complexity bounded by any integer N, see [5].

2. There exists one more noncompact orientable hyperbolic 3-manifold of volume $2V$ and complexity 2, see [2]. Its minimal special spine and corresponding triangulation are shown on Fig. 2 c, d. The gluing pattern is $ABC \longleftrightarrow FHE$, $BAD \longleftrightarrow FEG$, $CDA \longleftrightarrow HFG$, $DCB \longleftrightarrow HGE$. The complexity of any n-fold covering space of this manifold is again equal to $2n$. The proof of this statement repeats that of Theorem 3.

3. Let us return to the manifolds M_n considered in Section 2. Consider a minimal triangulation of M_n (dual to its minimal spine P, which is special by Theorem 1). Since there is only one vertex (dual to the 3-cell $M \setminus P$), all the edges of the triangulation are loops. They generate the group $\pi_1(M)$. Therefore, at least one of them has a nonzero image under the projection $p_*\colon \pi_1(M_n) \to \pi_1(S^1)$. Let us suppose for a moment that there is an edge e that is isotopic to the section of the fibration $p\colon M_n \to S^1$. Let σ be the 2-component of P dual to e. Put $P' = P \setminus \sigma$. Then P' is an almost simple spine of the manifold $M_n \setminus e = N_n$. By Theorem 3, P' contains at least $2n$ vertices. Consequently, $P = P' \cup \sigma$ has at least $2n+2$ vertices, which is close to the conjectured value $c(M_n) = 2n+5$, see [1] (indeed, if adding σ to P' does not increase the number of vertices, then $\partial\sigma$ is a closed triple line and the spine P is not minimal by virtue of Theorem 1; if all vertices of P belonging to $\partial\sigma$ are different but their number is less than 4, then a simplification move [5–7] can be applied, and P is not a minimal spine; finally, one can show that the case where $\partial\sigma$ passes through some vertex of P more than once but does not pass through any other vertex is impossible). However, it remains unclear why such an edge e should exist in a triangulation dual to arbitrary minimal spine of M_n.

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