CUBIC FORM GEOMETRY FOR HYPERSURFACES OF CENTRO-AFFINE AND GRAPH HYPERSURFACES

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Abstract. We characterize non-degenerate hypersurfaces of centro-affine and graph hypersurfaces of which the cubic form $C$ is divisible by the second fundamental form $h$.

1. Introduction and Main Results

One of the most attractive results in classical affine differential geometry is the theorem of Pick and Berwald, stating that a Blaschke hypersurface has vanishing cubic form if and only if it is a non-degenerate hyperquadric. In this way, non-degenerate quadrics are characterized in a differential geometric way.

This theorem has been generalized in many directions, e.g. Simon [6] and Nomizu and Pinkall [3] showed that a non-degenerate hypersurface (with an arbitrary affine structure) of $\mathbb{R}^{n+1}$ with $h \mid C$ is a hyperquadric.

We have analyzed the condition $h \mid C$ for various other immersions. The first results deal with (pseudo-)Riemannian manifolds and extend the main theorem of [2]:

Theorem 1. Let $M^n$ be a Riemannian submanifold of $S^{n+1}(1)$ with non-degenerate second fundamental form. Let $\bar{C}$ be the symmetric traceless part of its cubic form, then $\bar{C}$ vanishes identically if and only if $M^n$ is the intersection of the sphere with a non-degenerate quadratic cone which is centered at the origin.

Theorem 2. Let $M^n$ be a pseudo-Riemannian submanifold of $S^{n+1}_m(\pm 1) \subset \mathbb{R}^{n+2}_m$ with non-degenerate second fundamental form. Let $\bar{C}$ be the symmetric traceless part of its cubic form, then $\bar{C}$ vanishes identically if and only if $M^n$ is the intersection of $S^{n+1}_m(\pm 1)$ with a non-degenerate quadratic cone which is centered at the origin.

In affine differential geometry, non-degenerate hypersurfaces $M^n$ of centro-affine and graph hypersurfaces $\bar{M}^{n+1}$ of $\mathbb{R}^{n+2}$ with $h \mid C$ can be characterized as follows:

Theorem 3. Let $f$ be a non-degenerate affine immersion of $(M^n, \nabla)$ into $(\bar{M}^{n+1}, \bar{\nabla})$, where $(\bar{M}^{n+1}, \bar{\nabla})$ is a centro-affine hypersurface of $\mathbb{R}^{n+2}$ w.r.t. a point $o$. Let $h$ be the second fundamental form of $M^n$ in $\bar{M}^{n+1}$ and $C$ its cubic form. Then $h \mid C$ if and only if $f(M^n)$ is the intersection of $\bar{M}^{n+1}$ with a non-degenerate quadratic cone which is centered at the point $o$.

Theorem 4. Let $f$ be a non-degenerate affine immersion of $(M^n, \nabla)$ into a graph hypersurface $\bar{M}^{n+1}$ of $\mathbb{R}^{n+2}$. Let $h$ be the second fundamental form of $M^n$ in $\bar{M}^{n+1}$ and $C$ its...
cubic form. Then \( h \mid C \) if and only if \( f(M^n) \) is the intersection of \( \widetilde{M} \) with a cylinder on a non-degenerate quadric of which the rulings are parallel to the normal of \( \widetilde{M}^{n+1} \).

**Corollary 1.** Let \( f \) be a non-degenerate affine immersion of \((M^n, \nabla)\) into an affine sphere \((\widetilde{M}^{n+1}, \widetilde{\nabla})\) in \( \mathbb{R}^{n+2} \). Let \( h \) be the second fundamental form of \( M^n \) in \( \widetilde{M}^{n+1} \) and \( C \) its cubic form. If \( h \mid C \), then \( f(M^n) \) is the intersection of \( \widetilde{M}^{n+1} \) with a hyperquadric.

**Corollary 2.** Let \((\widetilde{M}^{n+1}, \widetilde{\nabla})\) be a non-degenerate hyperquadric in \( \mathbb{R}^{n+2} \), equipped with its Blaschke normal. Let \( f \) be a non-degenerate affine immersion of \((M^n, \nabla)\) into \( \widetilde{M}^{n+1} \). Then \( h \mid C \) if and only if

(a) if \( \widetilde{M}^{n+1} \) is a central quadric, then \( f(M^n) \) is the intersection of \( \widetilde{M}^{n+1} \) with a non-degenerate cone which vertex coincides with the center of \( \widetilde{M}^{n+1} \), or

(b) if \( \widetilde{M}^{n+1} \) is a paraboloid, then \( f(M^n) \) is the intersection of \( \widetilde{M}^{n+1} \) with a cylinder on a non-degenerate quadric of which the rulings are parallel to the normal of \( \widetilde{M}^{n+1} \).

The last result solves a conjecture of Lusala.

2. Preliminaries

Let \( M^n \) and \( \widetilde{M}^{n+1} \) be manifolds with a torsion-free affine connection \( \nabla \) resp. \( \widetilde{\nabla} \). Let \( f : M^n \to \widetilde{M}^{n+1} \) be an immersion for which there is a transversal vector field \( \xi \) s.t.

\[
\widetilde{\nabla}_X f_*(Y) = f_*(\nabla_X Y) + h(X, Y)\xi
\]

for all \( X, Y \in \mathcal{X}(M^n) \). Then \( f \) is said to be an affine immersion of \((M, \nabla)\) into \((\widetilde{M}^{n+1}, \widetilde{\nabla})\) and \( h \) is called the affine second fundamental form. For an affine immersion \( f \) of \((M^n, \nabla)\) into \((\widetilde{M}^{n+1}, \widetilde{\nabla})\) with transversal vector field \( \xi \), the (affine) shape operator \( S \) and the transversal connection form \( \tau \) are defined by

\[
\widetilde{\nabla}_X \xi = -f_*(S(X)) + \tau(X)\xi.
\]

Given an immersion \( f : M \to (\widetilde{M}, \widetilde{\nabla}) \), one can always choose an transversal section \( \xi \) and with \( (2.1) \) induce a connection \( \nabla \) on \( M \); then with this choice of \( \xi \), \( f : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla}) \) is an affine immersion.

\( M^n \) is called non-degenerate if \( h \) is non-degenerate (and this condition is independent of the choice of \( \xi \)).

**Remark.** A cone in \( \mathbb{R}^n \) is a set consisting of half-lines emanating from some point \( v \), the vertex of the cone. A quadratic cone \( Q \) with vertex \( v \) is called non-degenerate if it does not contain a straight line, i.e. there exists an affine coordinate system \( \{x^1, \ldots, x^n\} \) on \( \mathbb{R}^n \) in which \( Q \) is given by \( \sum_{i=1}^n a_i(x^i - v^i)^2 = 0 \) and \( x^n > v^n \) with \( a_i \in \mathbb{R}_0 \).

Note that this terminology does not correspond to the above definition. Indeed, for the inclusion \( \iota : Q \hookrightarrow \mathbb{R}^{n+1} \) the second fundamental form of a cone \( Q \) is always degenerate.

The cubic form of \((M^n, \nabla)\) in \((\widetilde{M}^{n+1}, \widetilde{\nabla})\) is defined by

\[
C(X, Y, Z) := (\nabla_X h)(Y, Z) + \tau(X)h(Y, Z),
\]
for $X, Y, Z \in \mathfrak{X}(M^n)$. By the Codazzi equation for $h$, the cubic form is totally symmetric.

The cubic form is called divisible by $h$ (see [3]) if there exists a one-form $\rho$ such that for all $X, Y, Z \in \mathfrak{X}(M^n)$,

$$C(X, Y, Z) = \rho(X)h(Y, Z) + \rho(Y)h(Z, X) + \rho(Z)h(X, Y),$$

and this property is denoted by $h \mid C$. One can show that this property does not depend on the choice of the transversal vector field.

Now take $(\tilde{M}^{n+1}, \tilde{\nabla})$ to be $(\mathbb{R}^{n+1}, D)$, the affine space with its usual flat connection and fix an affine coordinate system $\{x^1, \ldots, x^{n+1}\}$ on $\mathbb{R}^{n+1}$. Let $M^n$ be a hypersurface of $\mathbb{R}^{n+1}$. If the position vector is at each point $x$ of $M^n$ transversal to the tangent space of $M^n$ at $x$, one can take $\xi = -x$ and consider the induced connection on $M^n$ given by (2.1); with this choice of $\xi$, $(M^n, \nabla)$ is called a centro-affine hypersurface.

Let $(\tilde{M}^{n+1}, \tilde{\nabla}) = (\mathbb{R}^{n+1}, D)$. A hypersurface $M$ is called a graph hypersurface if the connection $\nabla$ on $M$ is induced by a constant transversal vector field $\xi$, i.e. $D_X \xi = 0$. Taking $\xi = (0, \ldots, 0, 1)$ there is an affine coordinate system $(x^1, \ldots, x^n, x^{n+1})$ on $\mathbb{R}^{n+1}$ such that $M$ is locally given by

$$\{(x^1, \ldots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} | (x^1, \ldots, x^n) \in U \text{ and } x^{n+1} = F(x^1, \ldots, x^n)\},$$

where $U$ is a connected open part of $\mathbb{R}^n$ and $F$ is a smooth function on $U$.

If $(\tilde{M}^{n+1}, \tilde{\nabla})$ is equipped with a parallel volume element $\omega$ (e.g. $(\mathbb{R}^{n+1}, D)$ with its volume form given by the determinant), and $M$ is non-degenerate, there exists (up to sign) a unique choice of $\xi$ such that

(i) $\tau = 0$, or equivalently $\nabla \theta = 0$, and

(ii) $\theta$ coincides with the volume element $\omega_h$ of the non-degenerate metric $h$,

where $\theta$ denotes the induced volume form given by

$$\theta(X_1, \ldots, X_n) = \omega(f_*(X_1), \ldots, f_*(X_n), \xi)$$

for $X_1, \ldots, X_n \in \mathfrak{X}(M)$. This choice of $\xi$ is called the Blaschke normal and $(M^n, \nabla)$ is called a Blaschke hypersurface.

An improper affine hypersphere is a Blaschke hypersurface for which the shape operator is identically zero. If the shape operator of a Blaschke hypersurface $M^n$ is a constant nonzero multiple of the identity, $M^n$ is called a proper affine hypersphere.

We can now state the theorem of Pick and Berwald:

**Theorem 5.** Let $f : (M^n, \nabla) \to (\mathbb{R}^{n+1}, D)$ be a Blaschke hypersurface. If its cubic form $C$ vanishes identically, then $f(M^n)$ is a hyperquadric in $\mathbb{R}^{n+1}$.

This theorem has been generalized in many directions; e.g.

**Theorem 6 ([6]; [3]).** Let $f : (M^n, \nabla) \to (\mathbb{R}^{n+1}, D)$ be a non-degenerate immersion. If $C$ is divisible by $h$, then $f(M^n)$ lies in a hyperquadric.

In [2], Lusala considered the following question within the context of Riemannian geometry. Given a hypersurface $(M^n, \nabla)$ of a space-form $(\tilde{M}^{n+1}(e), \tilde{\nabla})$ with non-degenerate second
fundamental form $h$, put $C(X, Y, Z) = (\nabla h)(X, Y, Z)$ for $X, Y, Z$ in $\mathfrak{X}(M)$ (here the $\xi$ from above is taken to be the unit normal, so $\tau$ vanishes identically). Since $h$ is non-degenerate, we can define the Tchebychev vector field by

$$h(T, u) = \frac{1}{n} tr_h C(u, \cdot, \cdot)$$

$$= \frac{1}{n} \sum_{i=1}^{n} e_i C(u, e_i, e_i)$$

for all $u$, where the $e_i$ form an orthonormal frame w.r.t. $h$; i.e. $h(e_i, e_j) = \epsilon_i \delta_{ij}$ with $\epsilon_i = \pm 1$. The symmetric traceless part $\tilde{C}$ of $C$ is given by

$$\tilde{C}(X, Y, Z) = C(X, Y, Z) - \frac{n}{n + 2} (h(X, Y)h(Z, T) + h(Z, X)h(Y, T) + h(Y, Z)h(X, T))$$

for $X, Y, Z$ in $\mathfrak{X}(M^n)$.

One can now try to classify hypersurfaces for which $\tilde{C}$ vanishes identically; this generalizes the notion of parallel submanifolds, defined by $\nabla h \equiv 0$. In [2], this has been done for $M^2$ in $S^3(1)$.

For more information about parallel submanifolds, we refer to the survey [1].

**Remark.** It turns out that hypersurfaces of $(\mathbb{R}^n, D)$ with non-degenerate second fundamental form for which $\tilde{C}$ vanishes identically, are, by the theorem of Pick and Berwald, non-degenerate hyperquadrics, cf. the proof of Theorem 1.

3. **Proofs**

3.1. **Proof of Theorem 3.** Let $\tilde{M}^{n+1}$ be a centro-affine hypersurface of $\mathbb{R}^{n+2}$ w.r.t. the point $o$ and let $f$ be an immersion of $M^n$ into $\tilde{M}^{n+1}$. Since our considerations will be local, we may assume $M^n \subset \tilde{M}^{n+1}$ and we will use the notation $M$ and $\tilde{M}$ throughout for the local situation.

Define $F : \tilde{M} \subset \mathbb{R}^{n+2} \to \mathbb{R}^{n+2} : x \mapsto \lambda(x)x$ with $\lambda > 0$ such that the image of $\tilde{M}$ under $F$ is contained in a hyperplane of $\mathbb{R}^{n+2}$ not passing through $o$. We will denote this hyperplane with $\mathbb{R}^{n+1}$.

Denote the image of $M$ under $F$ with $\overline{M}$.

We have the following immersions

- $(M, \nabla) \hookrightarrow (\tilde{M}, \tilde{\nabla})$ with transversal vector field $\xi$, fundamental form $h$, shape operator $S$ and transversal connection form $\tau$,
- $(\tilde{M}, \tilde{\nabla}) \hookrightarrow (\mathbb{R}^{n+2}, D)$ with transversal vector field $-x$ and fundamental form $\tilde{h}$ and
- $F : (M, \nabla) \hookrightarrow (\mathbb{R}^{n+1}, D)$ with transversal vector field $\tilde{\xi} = F_*(\xi)$, fundamental form $\tilde{h}$, shape operator $\overline{S}$ and transversal connection form $\overline{\tau}$.

For $V \in T_p M$, one has

$$F_*(V) = \lambda V + V(\lambda)x.$$
For $X, Y \in T_pM$ we have

\[
D_X F_*(Y) = D_X (\lambda Y + Y(\lambda)x) \\
= \lambda D_X Y + X(\lambda)Y + Y(\lambda)X + X(Y(\lambda))x
\]

and, because of

\[
D_X Y = \nabla_X Y + h(X,Y)\xi + \tilde{h}(X,Y)(-x),
\]

we find

\[
D_X F_*(Y) = \lambda \left( \nabla_X Y + h(X,Y)\xi - \tilde{h}(X,Y)x \right) + X(\lambda)Y + Y(\lambda)X + X(Y(\lambda))x.
\]

Keeping in mind that $F_*(\xi) = \lambda \xi + \xi(\lambda)x$, we also obtain

\[
D_X F_*(Y) = F_*(\nabla_X Y) + \tilde{h}(X,Y)(\lambda \xi + \xi(\lambda)x) \\
= \lambda \nabla_X Y + X(Y(\lambda))x + \tilde{h}(X,Y)(\lambda \xi + \xi(\lambda)x) \\
= \lambda \nabla_X Y + \tilde{h}(X,Y)\lambda \xi + (X(Y(\lambda)) + \tilde{h}(X,Y)\xi(\lambda))x.
\]

By comparing the coefficients of $\xi$ and $x$ and taking the tangential part of both expressions for $D_X F_*(Y)$, we find

\[
h(X,Y) = \tilde{h}(X,Y) \\
- \lambda \tilde{h}(X,Y) = \tilde{h}(X,Y)\xi(\lambda) \\
\lambda \nabla_X Y = \lambda \nabla_X Y + X(\lambda)Y + Y(\lambda)X,
\]

in particular

\[
\nabla_X Y = \nabla_X Y + \rho(X)Y + \rho(Y)X,
\]

where $\rho = d \log \lambda$.

Calculating $D_X F_*(\xi)$ in two ways gives

\[
D_X F_*(\xi) = D_X (\lambda \xi + \xi(\lambda)x) \\
= X(\lambda)\xi + \xi(\lambda)X + X(\xi(\lambda))x + \lambda(-S(X) + \tau(X)\xi) \\
D_X F_*(\xi) = -F_*(\overline{S}(X)) + \overline{\tau}(X)F_*(\xi) \\
= -\lambda\overline{S}(X) - (\overline{S}(X))(\lambda)x + \overline{\tau}(X)(\lambda \xi + \xi(\lambda)x).
\]

Equating the component of $\xi$ in these formulae gives

\[
\lambda \overline{\tau}(X) = X(\lambda) + \lambda \tau(X),
\]

hence

\[
\overline{\tau}(X) = \rho(X) + \tau(X).
\]
The cubic form of $\overline{M}$ in $\mathbb{R}^{n+1}$ is given by
\[
\overline{C}(X, Y, Z) = (\nabla_X \overline{h})(Y, Z) + \tau(X) \overline{h}(Y, Z) \\
= (\nabla_X \overline{h})(Y, Z) + (\rho(X) + \tau(X))h(Y, Z) \\
= X(h(Y, Z)) + (\rho(X) + \tau(X))h(Y, Z) \\
= X(h(Y, Z)) + \rho(X)h(Y, Z) + \tau(X)h(Y, Z) \\
= X(h(Y, Z)) + \tau(X)h(Y, Z) \\
= C(X, Y, Z) - \rho(X)h(Y, Z) - \rho(Y)h(Z, X) - \rho(Z)h(X, Y).
\]

Now assume that the cubic form $C$ of $M^n$ in $\widetilde{M}^{n+1}$ is divisible by $h$. Since $\overline{h}$ and $h$ coincide, we obtain that $\overline{h} \mid \overline{C}$, hence, by Theorem 6, $\overline{M}$ is an open part of a non-degenerate hyperquadric of $\mathbb{R}^{n+1}$.

3.2. Proof of Theorem 4. Let $\widetilde{M}^{n+1}$ be a graph hypersurface of $\mathbb{R}^{n+2}$ and let $f$ be a non-degenerate affine immersion of $M^n$ into $\widetilde{M}^{n+1}$.

One can proceed locally as in the previous proof by considering a projection $\pi$ of the graph hypersurface $M$ on a hyperplane $\mathbb{R}^{n+1}$ that does not contain the direction of the normal of $\tilde{M}$, where the projection takes place along the normals of $\tilde{M}$. Calculating the data $(\nabla, \overline{h}, \tau)$ for the projection $\overline{M} = \pi(M)$ in terms of those of $M$ reveals $\nabla = \nabla, \overline{h} = h$ and $\tau = \tau$, hence $\overline{C} = C$. Since $\overline{h} = h$, $\overline{M}$ is a non-degenerate hyperquadric of $\mathbb{R}^{n+1}$ if $h \mid C$.

For the first corollary, note that a proper affine sphere is a centro-affine hypersurface and that an improper affine sphere is a graph hypersurface (see e.g. [4]). For the second corollary, note that a central hyperquadric with its Blaschke structure is a centro-affine hypersurface w.r.t. the center of the hyperquadric and that a paraboloid with its Blaschke structure is an improper affine sphere.

3.3. Proof of Theorem 1 and 2. To prove Theorem 1, we first note

**Lemma 1.** For the immersion $S^{n+1}(1) \hookrightarrow (\mathbb{R}^{n+2}, D)$, the Blaschke normal and the Euclidean unit normal coincide.

To complete the proof of Theorem 1, it now suffices to observe that the conditions $\tilde{C} \equiv 0$ and $h \mid C$ are equivalent and then use Theorem 3. If $\tilde{C} \equiv 0$, then with $\rho = \frac{n}{n+2}h(T, \cdot)$,
\[
C(X, Y, Z) = \rho(X)h(Y, Z) + \rho(Y)h(Z, X) + \rho(Z)h(X, Y).
\]

On the other hand, assume that there exists a one-form $\rho$ such that (3.1) holds. Choose an orthonormal frame $(e_1, \ldots, e_n)$ w.r.t. $h$, i.e. $h(e_i, e_j) = \epsilon_i \delta_{ij}$ with $\epsilon_i = \pm 1$. With
\[ X = \sum_{i=1}^{n} X^i e_i, \text{ we have} \]
\[
    h(T, X) = \frac{1}{n} \text{tr}_h C(X, \cdot, \cdot) = \frac{1}{n} \sum_{i=1}^{n} \epsilon_i C(X, e_i, e_i) \\
    = \frac{1}{n} \sum_{i=1}^{n} \epsilon_i (\rho(X) e_i + 2 \rho(e_i) h(X, e_i)) \\
    = \frac{1}{n} \sum_{i=1}^{n} \epsilon_i (\rho(X) e_i + 2 \rho(e_i) X^i e_i) \\
    = \frac{1}{n} \sum_{i=1}^{n} (\rho(X) + 2 \rho(X^i e_i)) = \frac{n+2}{n} \rho(X).
\]

For \(|\{k, l, m\}| = 3, \tilde{C}(e_k, e_l, e_m) = 0, \text{ since the } e_i \text{ form an orthonormal frame and by (3.1)},
\]
\[ C(e_i, e_j, e_k) = 0. \text{ Since } \tilde{C} \text{ is totally symmetric, it remains to show } \tilde{C}(e_i, e_j, e_j) = 0 \text{ (we do not exclude the case } i = j). \text{ This is now straightforward :} \]
\[
    \tilde{C}(e_i, e_j, e_j) = C(e_i, e_j, e_j) - \frac{n}{n+2} \left( h(e_i, T) h(e_j, e_j) + 2 h(e_j, T) h(e_i, e_j) \right) \\
    = C(e_i, e_j, e_j) - \frac{n}{n+2} \left( \frac{n+2}{n} \rho(e_i) h(e_j, e_j) + 2 \frac{n+2}{n} \rho(e_j) h(e_i, e_j) \right) \\
    = \rho(e_i) h(e_j, e_j) + 2 \rho(e_j) h(e_i, e_j) - (\rho(e_i) h(e_j, e_j) + 2 \rho(e_j) h(e_i, e_j)) = 0.
\]

Since Theorem 6 holds true when the signature of the standard metric on \(\mathbb{R}^{n+2} \) is changed, the proof of Theorem 2 is similar.

REFERENCES


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