CUBIC FORM GEOMETRY FOR HYPERSURFACES OF CENTRO-AFFINE AND GRAPH HYPERSURFACES

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ABSTRACT. We characterize non-degenerate hypersurfaces of centro-affine and graph hypersurfaces of which the cubic form C is divisible by the second fundamental form h.

1. Introduction and main results

One of the most attractive results in classical affine differential geometry is the theorem of Pick and Berwald, stating that a Blaschke hypersurface has vanishing cubic form if and only if it is a non-degenerate hyperquadric. In this way, non-degenerate quadrics are characterized in a differential geometric way.
This theorem has been generalized in many directions, e.g. Simon [6] and Nomizu and Pinkall [3] showed that a non-degenerate hypersurface (with an arbitrary affine structure) of $\mathbb{R}^{n+1}$ with $h \mid C$ is a hyperquadric.

We have analyzed the condition $h \mid C$ for various other immersions. The first results deal with (pseudo-)Riemannian manifolds and extend the main theorem of [2] :

Theorem 1. *Let M^n be a Riemannian submanifold of $S^{n+1}(1)$ with non-degenerate second fundamental form. Let $\widetilde{C}$ be the symmetric traceless part of its cubic form, then $\widetilde{C}$ vanishes identically if and only if M^n is the intersection of the sphere with a non-degenerate quadratic cone which is centered at the origin.*

Theorem 2. *Let M^n be a pseudo-Riemannian submanifold of $S^{n+1}_m(\pm 1) \subset \mathbb{R}^{n+2}_k$ with non-degenerate second fundamental form. Let $\widetilde{C}$ be the symmetric traceless part of its cubic form, then $\widetilde{C}$ vanishes identically if and only if M^n is the intersection of $S^{n+1}_m(\pm 1)$ with a non-degenerate quadratic cone which is centered at the origin.*

In affine differential geometry, non-degenerate hypersurfaces M^n of centro-affine and graph hypersurfaces $\widetilde{M}^{n+1}$ of $\mathbb{R}^{n+2}$ with $h \mid C$ can be characterized as follows

Theorem 3. *Let f be a non-degenerate affine immersion of (M^n, ∇) into $(\widetilde{M}^{n+1}, \widetilde{\nabla})$, where $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ is a centro-affine hypersurface of $\mathbb{R}^{n+2}$ w.r.t. a point o. Let h be the second fundamental form of M^n in $\widetilde{M}^{n+1}$ and C its cubic form. Then $h \mid C$ if and only if $f(M^n)$ is the intersection of $\widetilde{M}^{n+1}$ with a non-degenerate quadratic cone which is centered at the point o.*

Theorem 4. *Let f be a non-degenerate affine immersion of (M^n, ∇) into a graph hypersurface $\widetilde{M}^{n+1}$ of $\mathbb{R}^{n+2}$. Let h be the second fundamental form of M^n in $\widetilde{M}^{n+1}$ and C its*

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cubic form. Then $h \mid C$ if and only if $f(M^n)$ is the intersection of $\widetilde{M}$ with a cylinder on a non-degenerate quadric of which the rulings are parallel to the normal of $\widetilde{M}^{n+1}$.

Corollary 1. *Let f be a non-degenerate affine immersion of (M^n, ∇) into an affine sphere $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ in $\mathbb{R}^{n+2}$. Let h be the second fundamental form of M^n in $\widetilde{M}^{n+1}$ and C its cubic form. If $h \mid C$, then $f(M^n)$ is the intersection of $\widetilde{M}^{n+1}$ with a hyperquadric.*

Corollary 2. *Let $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ be a non-degenerate hyperquadric in $\mathbb{R}^{n+2}$, equipped with its Blaschke normal. Let f be a non-degenerate affine immersion of (M^n, ∇) into $\widetilde{M}^{n+1}$. Then $h \mid C$ if and only if*

(a) *if $\widetilde{M}^{n+1}$ is a central quadric, then $f(M^n)$ is the intersection of $\widetilde{M}^{n+1}$ with a non-degenerate cone which vertex coincides with the center of $\widetilde{M}^{n+1}$, or*

(b) *if $\widetilde{M}^{n+1}$ is a paraboloid, then $f(M^n)$ is the intersection of $\widetilde{M}^{n+1}$ with a cylinder on a non-degenerate quadric of which the rulings are parallel to the normal of $\widetilde{M}^{n+1}$.*

The last result solves a conjecture of Lusala.

2. Preliminaries

Let M^n and $\widetilde{M}^{n+1}$ be manifolds with a torsion-free affine connection ∇ resp. $\widetilde{\nabla}$. Let $f : M^n \to \widetilde{M}^{n+1}$ be an immersion for which there is a transversal vector field ξ s.t.

$$\widetilde{\nabla}_X f_*(Y) \;=\; f_*(\nabla_X Y) + h(X,Y)\xi \tag{2.1}$$

for all $X, Y \in \mathfrak{X}(M^n)$. Then f is said to be an affine immersion of (M, ∇) into $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ and h is called the affine second fundamental form. For an affine immersion f of (M^n, ∇) into $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ with transversal vector field ξ, the (affine) shape operator S and the transversal connection form τ are defined by

$$\widetilde{\nabla}_X \xi \;=\; -f_*(S(X)) + \tau(X)\xi. \tag{2.2}$$

Given an immersion $f : M \to (\widetilde{M}, \widetilde{\nabla})$, one can always choose an transversal section ξ and with (2.1) induce a connection ∇ on M; then with this choice of ξ, $f : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ is an affine immersion.

M^n is called non-degenerate if h is non-degenerate (and this condition is independent of the choice of ξ).

Remark. A cone in $\mathbb{R}^n$ is a set consisting of half-lines emanating from some point v, the vertex of the cone. A quadratic cone Q with vertex v is called non-degenerate if it does not contain a straight line, i.e. there exists an affine coordinate system $\{x^1, \dots, x^n\}$ on $\mathbb{R}^n$ in which Q is given by $\sum_{i=1}^n a_i(x^i - v^i)^2 = 0$ and $x^n > v^n$ with $a_i \in \mathbb{R}_0$.
Note that this terminology does not correspond to the above definition. Indeed, for the inclusion $\iota : Q \hookrightarrow \mathbb{R}^{n+1}$ the second fundamental form of a cone Q is always degenerate.

The cubic form of (M^n, ∇) in $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ is defined by

$$C(X,Y,Z) := (\nabla_X h)(Y,Z) + \tau(X)h(Y,Z)\ ,$$

for $X, Y, Z \in \mathfrak{X}(M^n)$. By the Codazzi equation for h, the cubic form is totally symmetric.

The cubic form is called divisible by h (see [3]) if there exists a one-form ρ such that for all $X, Y, Z \in \mathfrak{X}(M^n)$,

$$C(X,Y,Z) = \rho(X)h(Y,Z) + \rho(Y)h(Z,X) + \rho(Z)h(X,Y),$$

and this property is denoted by $h \mid C$. One can show that this property does not depend on the choice of the transversal vector field.

Now take $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ to be $(\mathbb{R}^{n+1}, D)$, the affine space with its usual flat connection and fix an affine coordinate system $\{x^1, \ldots, x^{n+1}\}$ on $\mathbb{R}^{n+1}$. Let M^n be a hypersurface of $\mathbb{R}^{n+1}$. If the position vector is at each point x of M^n transversal to the tangent space of M^n at x, one can take $\xi = -x$ and consider the induced connection on M^n given by (2.1); with this choice of ξ, (M^n, ∇) is called a centro-affine hypersurface.

Let $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ be $(\mathbb{R}^{n+1}, D)$. A hypersurface M is called a graph hypersurface if the connection ∇ on M is induced by a constant transversal vector field ξ, i.e. $D_X\xi = 0$. Taking $\xi = (0, \ldots, 0, 1)$ there is an affine coordinate system $(x^1, \ldots, x^n, x^{n+1})$ on $\mathbb{R}^{n+1}$ such that M is locally given by

$$\{(x^1, \ldots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} \mid (x^1, \ldots, x^n) \in U \text{ and } x^{n+1} = F(x^1, \ldots, x^n)\},$$

where U is a connected open part of $\mathbb{R}^n$ and F is a smooth function on U.

If $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ is equipped with a parallel volume element ω (e.g. $(\mathbb{R}^{n+1}, D)$ with its volume form given by the determinant), and M is non-degenerate, there exists (up to sign) a unique choice of ξ such that

(i) $\tau = 0$, or equivalently $\nabla\theta = 0$, and
(ii) θ coincides with the volume element ω_h of the non-degenerate metric h,

where θ denotes the induced volume form given by

$$\theta(X_1, \ldots, X_n) = \omega(f_*(X_1), \ldots, f_*(X_n), \xi)$$

for $X_1, \ldots, X_n \in \mathfrak{X}(M)$. This choice of ξ is called the Blaschke normal and (M^n, ∇) is called a Blaschke hypersurface.

An improper affine hypersphere is a Blaschke hypersurface for which the shape operator is identically zero. If the shape operator of a Blaschke hypersurface M^n is a constant nonzero multiple of the identity, M^n is called a proper affine hypersphere.

We can now state the theorem of Pick and Berwald :

Theorem 5. *Let $f : (M^n, \nabla) \to (\mathbb{R}^{n+1}, D)$ be a Blaschke hypersurface. If its cubic form C vanishes identically, then $f(M^n)$ is a hyperquadric in $\mathbb{R}^{n+1}$.*

This theorem has been generalized in many directions; e.g.

Theorem 6 ([6]; [3])**.** *Let $f : (M^n, \nabla) \to (\mathbb{R}^{n+1}, D)$ be a non-degenerate immersion. If C is divisible by h, then $f(M^n)$ lies in a hyperquadric.*

In [2], Lusala considered the following question within the context of Riemannian geometry. Given a hypersurface (M^n, ∇) of a space-form $(\widetilde{M}^{n+1}(c), \widetilde{\nabla})$ with non-degenerate second fundamental form h, put $C(X,Y,Z) = (\nabla h)(X,Y,Z)$ for X,Y,Z in $\mathfrak{X}(M)$ (here the ξ from above is taken to be the unit normal, so τ vanishes identically). Since h is non-degenerate, we can define the Tchebychev vector field by

$$\begin{aligned} h(T,u) &= \frac{1}{n} tr_h C(u,\cdot,\cdot) \\ &= \frac{1}{n}\sum_{i=1}^{n} \epsilon_i C(u,e_i,e_i) \text{ for all } u \ , \end{aligned}$$

where the e_i form an orthonormal frame w.r.t. h; i.e. $h(e_i,e_j) = \epsilon_i \delta_{ij}$ with $\epsilon_i = \pm 1$. The symmetric traceless part $\widetilde{C}$ of C is given by

$$\widetilde{C}(X,Y,Z) = C(X,Y,Z) - \frac{n}{n+2}\left(h(X,Y)h(Z,T) + h(Z,X)h(Y,T) + h(Y,Z)h(X,T)\right)$$

for X,Y,Z in $\mathfrak{X}(M^n)$.
One can now try to classify hypersurfaces for which $\widetilde{C}$ vanishes identically; this generalizes the notion of parallel submanifolds, defined by $\nabla h \equiv 0$. In [2], this has been done for M^2 in $S^3(1)$.
For more information about parallel submanifolds, we refer to the survey [1].

Remark. It turns out that hypersurfaces of $(\mathbb{R}^n, D)$ with non-degenerate second fundamental form for which $\widetilde{C}$ vanishes identically, are, by the theorem of Pick and Berwald, non-degenerate hyperquadrics, cf. the proof of Theorem 1.

3. Proofs

3.1. **Proof of Theorem 3.** Let $\widetilde{M}^{n+1}$ be a centro-affine hypersurface of $\mathbb{R}^{n+2}$ w.r.t. the point o and let f be an immersion of M^n into $\widetilde{M}^{n+1}$. Since our considerations will be local, we may assume $M^n \subset \widetilde{M}^{n+1}$ and we will use the notation M and $\widetilde{M}$ throughout for the local situation.

Define $F : \widetilde{M} \subset \mathbb{R}^{n+2} \to \mathbb{R}^{n+2} : x \mapsto \lambda(x)x$ with $\lambda > 0$ such that the image of $\widetilde{M}$ under F is contained in a hyperplane of $\mathbb{R}^{n+2}$ not passing through o. We will denote this hyperplane with $\mathbb{R}^{n+1}$.
Denote the image of M under F with $\overline{M}$.

We have the following immersions

- $(M,\nabla) \hookrightarrow (\widetilde{M},\widetilde{\nabla})$ with transversal vector field ξ, fundamental form h, shape operator S and transversal connection form τ,
- $(\widetilde{M},\widetilde{\nabla}) \hookrightarrow (\mathbb{R}^{n+2}, D)$ with transversal vector field $-x$ and fundamental form $\widetilde{h}$ and
- $F : (M,\overline{\nabla}) \hookrightarrow (\mathbb{R}^{n+1}, D)$ with transversal vector field $\overline{\xi} = F_*(\xi)$, fundamental form $\overline{h}$, shape operator $\overline{S}$ and transversal connection form $\overline{\tau}$.

For $V \in T_p\widetilde{M}$, one has

$$F_*(V) = \lambda V + V(\lambda)x.$$

For $X, Y \in T_pM$ we have

$$\begin{aligned} D_X F_*(Y) &= D_X(\lambda Y + Y(\lambda)x) \\ &= \lambda D_X Y + X(\lambda)Y + Y(\lambda)X + X(Y(\lambda))x \end{aligned}$$

and, because of

$$D_X Y = \nabla_X Y + h(X,Y)\xi + \widetilde{h}(X,Y)(-x),$$

we find

$$D_X F_*(Y) = \lambda\left(\nabla_X Y + h(X,Y)\xi - \widetilde{h}(X,Y)x\right) + X(\lambda)Y + Y(\lambda)X + X(Y(\lambda))x.$$

Keeping in mind that $F_*(\xi) = \lambda\xi + \xi(\lambda)x$, we also obtain

$$\begin{aligned} D_X F_*(Y) &= F_*(\overline{\nabla}_X Y) + \overline{h}(X,Y)(\lambda\xi + \xi(\lambda)x) \\ &= \lambda\overline{\nabla}_X Y + X(Y(\lambda))x + \overline{h}(X,Y)(\lambda\xi + \xi(\lambda)x) \\ &= \lambda\overline{\nabla}_X Y + \overline{h}(X,Y)\lambda\xi + (X(Y(\lambda)) + \overline{h}(X,Y)\xi(\lambda))x. \end{aligned}$$

By comparing the coefficients of ξ and x and taking the tangential part of both expressions for $D_X F_*(Y)$, we find

$$\begin{aligned} h(X,Y) &= \overline{h}(X,Y) \\ -\lambda\widetilde{h}(X,Y) &= \overline{h}(X,Y)\xi(\lambda) \\ \lambda\overline{\nabla}_X Y &= \lambda\nabla_X Y + X(\lambda)Y + Y(\lambda)X\ , \end{aligned}$$

in particular

$$\overline{\nabla}_X Y = \nabla_X Y + \rho(X)Y + \rho(Y)X,$$

where $\rho = d\log\lambda$.
Calculating $D_X F_*(\xi)$ in two ways gives

$$\begin{aligned} D_X F_*(\xi) &= D_X(\lambda\xi + \xi(\lambda)x) \\ &= X(\lambda)\xi + \xi(\lambda)X + X(\xi(\lambda))x + \lambda(-S(X) + \tau(X)\xi) \\ D_X F_*(\xi) &= -F_*(\overline{S}(X)) + \overline{\tau}(X)F_*(\xi) \\ &= -\lambda\overline{S}(X) - (\overline{S}(X))(\lambda)x + \overline{\tau}(X)(\lambda\xi + \xi(\lambda)x))\ . \end{aligned}$$

Equating the component of ξ in these formulae gives

$$\lambda\overline{\tau}(X) = X(\lambda) + \lambda\tau(X),$$

hence

$$\overline{\tau}(X) = \rho(X) + \tau(X).$$

The cubic form of $\overline{M}$ in $\mathbb{R}^{n+1}$ is given by

$$\begin{aligned}
\overline{C}(X,Y,Z) &= (\overline{\nabla}_X\overline{h})(Y,Z)+\overline{\tau}(X)\overline{h}(Y,Z)\\
&= (\overline{\nabla}_X\overline{h})(Y,Z)+(\rho(X)+\tau(X))h(Y,Z)\\
&= X(h(Y,Z))+(\rho(X)+\tau(X))h(Y,Z)\\
&\quad -\overline{h}(\nabla_X Y+\rho(X)Y+\rho(Y)X,Z)\\
&\quad -\overline{h}(Y,\nabla_X Z+\rho(X)Z+\rho(Z)X)\\
&= X(h(Y,Z))+\rho(X)h(Y,Z)+\tau(X)h(Y,Z)\\
&\quad -h(\nabla_X Y,Z)-\rho(X)h(Y,Z)-\rho(Y)h(X,Z)\\
&\quad -h(Y,\nabla_X Z)-\rho(X)h(Y,Z)-\rho(Z)h(Y,X)\\
&= X(h(Y,Z))-h(\nabla_X Y,Z)-h(Y,\nabla_X Z)+\tau(X)h(Y,Z)\\
&\quad -\rho(X)h(Y,Z)-\rho(Y)h(X,Z)-\rho(Z)h(X,Y)\\
&= (\nabla_X h)(Y,Z)+\tau(X)h(Y,Z)\\
&\quad -\rho(X)h(Y,Z)-\rho(Y)h(Z,X)-\rho(Z)h(X,Y)\\
&= C(X,Y,Z)-\rho(X)h(Y,Z)-\rho(Y)h(Z,X)-\rho(Z)h(X,Y).
\end{aligned}$$

Now assume that the cubic form C of M^n in $\widetilde{M}^{n+1}$ is divisible by h. Since $\overline{h}$ and h coincide, we obtain that $\overline{h} \mid \overline{C}$, hence, by Theorem 6, $\overline{M}$ is an open part of a non-degenerate hyperquadric of $\mathbb{R}^{n+1}$.

3.2. **Proof of Theorem 4.** Let $\widetilde{M}^{n+1}$ be a graph hypersurface of $\mathbb{R}^{n+2}$ and let f be a non-degenerate affine immersion of M^n into $\widetilde{M}^{n+1}$.

One can proceed locally as in the previous proof by considering a projection π of the graph hypersurface $\widetilde{M}$ on a hyperplane $\mathbb{R}^{n+1}$ that does not contain the direction of the normal of $\widetilde{M}$, where the projection takes place along the normals of $\widetilde{M}$. Calculating the data $(\overline{\nabla},\overline{h},\overline{\tau})$ for the projection $\overline{M}=\pi(M)$ in terms of those of M reveals $\overline{\nabla}=\nabla, \overline{h}=h$ and $\overline{\tau}=\tau$, hence $\overline{C}=C$. Since $\overline{h}=h$, $\overline{M}$ is a non-degenerate hyperquadric of $\mathbb{R}^{n+1}$ if $h \mid C$.

For the first corollary, note that a proper affine sphere is a centro-affine hypersurface and that an improper affine sphere is a graph hypersurface (see e.g. [4]). For the second corollary, note that a central hyperquadric with its Blaschke structure is a centro-affine hypersurface w.r.t. the center of the hyperquadric and that a paraboloid with its Blaschke structure is an improper affine sphere.

3.3. **Proof of Theorem 1 and 2.** To prove Theorem 1, we first note

Lemma 1. *For the immersion $S^{n+1}(1) \hookrightarrow (\mathbb{R}^{n+2}, D)$, the Blaschke normal and the Euclidean unit normal coincide.*

To complete the proof of Theorem 1, it now suffices to observe that the conditions $\widetilde{C} \equiv 0$ and $h \mid C$ are equivalent and then use Theorem 3. If $\widetilde{C} \equiv 0$, then with $\rho = \frac{n}{n+2}h(T,\cdot)$,

$$C(X,Y,Z) = \rho(X)h(Y,Z)+\rho(Y)h(Z,X)+\rho(Z)h(X,Y). \tag{3.1}$$

On the other hand, assume that there exists a one-form ρ such that (3.1) holds. Choose an orthonormal frame $(e_1,\ldots,e_n)$ w.r.t. h, i.e. $h(e_i,e_j)=\epsilon_i\delta_{ij}$ with $\epsilon_i=\pm1$. With

$X = \sum_{i=1}^n X^i e_i$, we have

$$\begin{aligned} h(T,X) &= \frac{1}{n} tr_h C(X,\cdot,\cdot) \;=\; \frac{1}{n}\sum_{i=1}^n \epsilon_i C(X,e_i,e_i) \\ &= \frac{1}{n}\sum_{i=1}^n \epsilon_i \left(\rho(X)\epsilon_i + 2\rho(e_i)h(X,e_i)\right) \\ &= \frac{1}{n}\sum_{i=1}^n \epsilon_i \left(\rho(X)\epsilon_i + 2\rho(e_i)X^i\epsilon_i\right) \\ &= \frac{1}{n}\sum_{i=1}^n \left(\rho(X) + 2\rho(X^i e_i)\right) \;=\; \frac{n+2}{n}\rho(X)\ . \end{aligned}$$

For $|\{k,l,m\}| = 3$, $\widetilde{C}(e_k,e_l,e_m) = 0$, since the e_i form an orthonormal frame and by (3.1), $C(e_i,e_j,e_k) = 0$. Since $\widetilde{C}$ is totally symmetric, it remains to show $\widetilde{C}(e_i,e_j,e_j) = 0$ (we do not exclude the case $i = j$). This is now straightforward :

$$\begin{aligned} \widetilde{C}(e_i,e_j,e_j) &= C(e_i,e_j,e_j) - \frac{n}{n+2}\left(h(e_i,T)h(e_j,e_j) + 2h(e_j,T)h(e_i,e_j)\right) \\ &= C(e_i,e_j,e_j) - \frac{n}{n+2}\left(\frac{n+2}{n}\rho(e_i)h(e_j,e_j) + 2\frac{n+2}{n}\rho(e_j)h(e_i,e_j)\right) \\ &= \rho(e_i)h(e_j,e_j) + 2\rho(e_j)h(e_i,e_j) - \left(\rho(e_i)h(e_j,e_j) + 2\rho(e_j)h(e_i,e_j)\right) \;=\; 0. \end{aligned}$$

Since Theorem 6 holds true when the signature of the standard metric on $\mathbb{R}^{n+2}$ is changed, the proof of Theorem 2 is similar.

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