On Fictitious Domain Formulations for Maxwell's Equations

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Abstract

We consider fictitious domain-Lagrange multiplier formulations for variational problems in the space $\boldsymbol{H}(\mathbf{curl};\Omega)$ derived from Maxwell's equations. Boundary conditions and the divergence constraint are imposed weakly by using Lagrange multipliers. Both the time dependent and time harmonic formulations of the Maxwell's equations are considered, and we derive well-posed formulations for both cases. The arising variational problem can be discretized by functions that do not satisfy an a-priori divergence constraint.

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1 Introduction

Partial differential equations arising from computational electromagnetics like Maxwell's equations, still pose very challenging problems in numerical analysis and simulation, partly due to the subtle nature of the relevant function spaces arising in variational formulations. In spite of the significant progress in understanding these equations, that has been achieved during the past years, not much appears to be known about *Ficticious Domain Formulations* (FDF) for Maxwell's equations. In a FDF, the domain of interest $\Omega \subset \mathbb{R}^n$ is embedded into a larger but simpler domain $\square \subset \mathbb{R}^n$. A typical example is $\square = [0,1]^n$. Of course, in general, simple domains support the design of fast numerical methods. Specifically, in the context of Maxwell's equations, simple geometries facilitate a more convenient realization of appropriate discretizations which, due to the above mentioned nature of the relevant function spaces, tend to be quite complex in nature. For instance, isoparametric techniques for adapting trial spaces with incorporated boundary conditions to complex boundaries would typically interfere with the desired structural properties of the trial spaces. In addition, the FDF would help dealing with moving boundaries or treating control problems where boundary values act as a control variable. In such a case, only the discretization of the boundary has to be changed unless complex boundaries require local refinements on $\square$ which, however, would still benefit from the simple geometry of $\square$.

One way of realizing an FDF is to append boundary conditions by *Lagrange multipliers*. Once the saddle point character of the resulting variational problem has been accepted, other

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constraints could, of course, be treated in the same fashion. In most applications, the electric field $\boldsymbol{E}$ is divergence-free, i.e., an extra condition is automatically imposed on the discretization. This can either be enforced directly by using (at least discretely) divergence-free trial functions (as long as they are available) or by incorporating also such constraints with the aid of Lagrange multipliers. In this article, we consider FDFs for variational problems in the spaces $\boldsymbol{H}(\mathbf{curl};\Omega)$ and $\boldsymbol{H}(\mathrm{div};\Omega)$ arising from Maxwell's equations, where not only the boundary conditions are imposed weakly by using Lagrange Multipliers, but also the divergence constraints which can be inhomogeneous, as in the case of the electric field. This allows one to use trial functions that do not have to satisfy a divergence constraint and remain independent of (possibly varying) domain geometries. We consider both the time dependent and the time harmonic formulations of Maxwell's equations.

Of course, there is a well-known price to be paid. As mentioned above, FDFs, or more generally, appending any extra condition weakly in terms of Lagrange multipliers, give rise to *saddle point problems*, which are symmetric but no longer positive definite. This has at least two drawbacks, namely efficient solvers for positive definite systems can no longer be used and, secondly, standard discretizations cannot be chosen arbitrarily but have to fulfill the Ladyshenskaja-Babuška-Brezzi (LBB) condition. Meeting such compatibility conditions, might be a delicate task, in particular, when the extra conditions are non-trivial.

However, recent progress in the analysis of adaptive methods for saddle point problems make the latter issue appear in a somewhat different light. In fact, first in the context of wavelets, [8, 13, 14] and later in [2] for Finite Elements, convergent adaptive algorithms have been constructed where the involved discretization spaces need **not** meet the LBB condition. Moreover, the methods in [8, 13] are proven to converge at an asymptotically optimal rate, i.e., the error is comparable with the error of the best approximation that is obtained by *any* linear combination of N wavelets, when N is the number of adaptively generated degrees of freedom (best N-term approximation). These developments motivate us to address here the FDF in the context of Maxwell's equations. In particular, well posedness (in the sense to be explained in Section 2.3) will be a center of focus, as it is an essential prerequisite for the techniques developed in [8, 13].

Aside from the above aspects, our interest in this subject is fueled by an inherent difficulty that arises when applying a FDF to Maxwell's equations. The electric field belongs to a space which is, in general, *not* a subspace of $\boldsymbol{H}^1(\Omega)$. However, this difference is very small in the sense that under certain mild conditions on the domain this space is embedded in $\boldsymbol{H}^1(\Omega)$. Since a fictitious domain typically gives rise to such an embedding one has to be careful in formulating a fictitious domain approach so as to capture also possible $\boldsymbol{H}^1$-singularities.

The outline of the paper is as follows. Section 2 collects some prerequisites and gives a brief overview of the relevant function spaces, Maxwell's equations and the basic theory of saddle point problems as well as of FDF's. Section 3 is concerned with FDFs for the time dependent Maxwell's equations. We also show an example where a straightforward formulation of a FDF fails since it only captures the smooth $\boldsymbol{H}^1(\Omega)$-part. We derive then a well-posed FDF that indeed preserves the possible $\boldsymbol{H}^1(\Omega)$-singularities. Section 4 is concerned with FDFs for the time harmonic formulation of Maxwell's equation which usually is significantly harder to treat, because the bilinear form in question is in general non-coercive. We show that the formulation derived for the time dependent case does not immediately carry over to a well-posed formulation for the time harmonic non-conducting case. We present a strategy, though, to obtain a well-posed problem.

2 Basic Notation and Facts

We always consider an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with boundary $\Gamma := \partial\Omega$, which consists of m smooth components.

2.1 The Spaces $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$

We use the short hand notation $\partial_i := \frac{\partial}{\partial x_i}$ for $1 \leq i \leq n$ and we will always assume in the sequel that all the involved weak partial derivatives exist in $L_2(\Omega)$. For three-dimensional vector fields $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)^T$, we set

$$\mathbf{curl}\,\boldsymbol{\zeta} := (\partial_2\zeta_3 - \partial_3\zeta_2, \partial_3\zeta_1 - \partial_1\zeta_3, \partial_1\zeta_2 - \partial_2\zeta_1)^T = \nabla \times \boldsymbol{\zeta}. \tag{2.1}$$

As usual, the divergence operator is defined for any vector field $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)^T$ by $\mathrm{div}\,\boldsymbol{\zeta} := \sum_{i=1}^n \partial_i\,\zeta_i = \nabla \cdot \boldsymbol{\zeta}$. For notational convenience, we will always write vector fields and vector valued quantities in boldface characters. Then, for $D \in \{\mathbf{curl}, \mathrm{div}\}$, we define

$$\boldsymbol{H}(D;\Omega) \;:=\; \{\boldsymbol{\zeta} \in \boldsymbol{L}_2(\Omega) :\; D\,\boldsymbol{\zeta} \in L_2(\Omega)\}, \tag{2.2}$$

$$\boldsymbol{V}(D;\Omega) \;:=\; \{\boldsymbol{\zeta} \in \boldsymbol{H}(D;\Omega) :\; D\,\boldsymbol{\zeta} = 0\} = \mathrm{Ker}\,(D). \tag{2.3}$$

We be mainly concerned with the 3D-case here. When nothing else is said, $n = 3$ is assumed. All these spaces are Hilbert spaces with the corresponding graph norm

$$\|\boldsymbol{\zeta}\|^2_{\boldsymbol{H}(D;\Omega)} := \|\boldsymbol{\zeta}\|^2_{0,\Omega} + \|D\,\boldsymbol{\zeta}\|^2_{0,\Omega},$$

where $\|\boldsymbol{\zeta}\|^2_{0,\Omega} = (\boldsymbol{\zeta}, \boldsymbol{\zeta})_{0,\Omega} = \int_\Omega |\boldsymbol{\zeta}|^2$. Finally, we define

$$\boldsymbol{H}_0(D;\Omega) := \mathrm{clos}_{\boldsymbol{H}(D;\Omega)} C_0^\infty(\Omega).$$

Trace Spaces

With $\boldsymbol{n}$ being the outward unit normal on Γ, one can prove that for $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl};\Omega)$, $\boldsymbol{u} \times \boldsymbol{n} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$, and that

$$\boldsymbol{H}_0(\mathbf{curl};\Omega) = \{\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl};\Omega);\; \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \;\text{ on } \Gamma\}.$$

One has the following Green's formula

$$\int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \boldsymbol{\phi} - \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{\phi} = \langle \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\phi} \rangle_\Gamma, \tag{2.4}$$

for all $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega)$ and $\boldsymbol{\phi} \in \boldsymbol{H}^1(\Omega)$, where $\langle \cdot, \cdot \rangle_\Gamma$ is the usual duality form induced by the $\boldsymbol{L}_2(\Gamma)$-inner product. Note that the standard trace mapping

$$\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl};\Omega),\; \boldsymbol{u} \mapsto \boldsymbol{u} \times \boldsymbol{n}, \tag{2.5}$$

is *not* surjective on the space $\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$. The range space of this trace mapping is usually denoted by $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$, see [4, 5, 6, 22] for the analysis of these spaces for general Lipschitz domains.

For $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div};\Omega)$, one can prove that the normal trace on the boundary, $\boldsymbol{u} \cdot \boldsymbol{n}$ belongs to $\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ and that this trace mapping is onto, [3]. Furthermore, we have the Greens formula

$$\int_\Omega \boldsymbol{v} \cdot \mathbf{grad}\,\phi + \int_\Omega (\mathrm{div}\,\boldsymbol{v})\phi = \langle \boldsymbol{v} \cdot \boldsymbol{n}, \phi \rangle_\Gamma, \tag{2.6}$$

for all $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega)$ and $\phi \in H^1(\Omega)$.

An Embedding Theorem

The following facts will play a crucial role in the subsequent developments [17, p. 52].

Theorem 2.1 *If the bounded domain* Ω *is either a convex polyhedron or has a* $\mathcal{C}^{1,1}$ *boundary, then the space* $\boldsymbol{H}_0(\mathbf{curl}, \Omega) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ *is continuously embedded in* $\boldsymbol{H}^1(\Omega)$ *and thus equals* $\boldsymbol{H}^1(\Omega)$.

Remark 2.2 *It should be noted that in general* $\boldsymbol{H}^1(\Omega) \subsetneq \boldsymbol{H}(\mathbf{curl}; \Omega) \cap \boldsymbol{H}(\mathrm{div}; \Omega)$. *Indeed, e.g. if the domain has reentrant corners,* $\boldsymbol{H}(\mathbf{curl}; \Omega) \cap \boldsymbol{H}(\mathrm{div}; \Omega)$ *contains* $\boldsymbol{H}^1(\Omega)$*-singularities in the sense that there exists* $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}; \Omega) \cap \boldsymbol{H}(\mathrm{div}; \Omega)$ *but* $\boldsymbol{u} \notin \boldsymbol{H}^1(\Omega)$, *[1].*

Since the gap between these spaces is small, one has to be careful in the construction of any variational formulation for Maxwell's equations, i.e., one has to make sure that the formulation captures the entire field and not only the smooth $\boldsymbol{H}^1(\Omega)$-part.

Another difficulty is the fact that the space $\boldsymbol{H}^1_{\mathrm{Tan}}(\Omega) := \{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\}$ is *closed* in the $\boldsymbol{H}(\mathbf{curl}; \Omega) \cap \boldsymbol{H}(\mathrm{div}; \Omega)$ topology [10, 12]. This means that, if the approximation spaces are subsets of $\boldsymbol{H}^1_{\mathrm{Tan}}(\Omega)$ (i.e., the trial functions are too smooth), whereas the variational formulation is posed in $\boldsymbol{H}(\mathbf{curl}; \Omega) \cap \boldsymbol{H}(\mathrm{div}; \Omega)$, any Galerkin method would converge to the $\boldsymbol{H}^1$-part of the solution. Thus, again $\boldsymbol{H}^1$-singularities would be missed.

These facts complicate the design of appropriate FDFs for Maxwell's equations as we shall see later.

Hodge Decompositions

Hodge decompositions play an important role in the analysis of the spaces $\boldsymbol{H}(\mathbf{curl}; \Omega)$ and $\boldsymbol{H}(\mathrm{div}; \Omega)$. Defining

$$\boldsymbol{H}_1 = \{\mathbf{curl}\ \boldsymbol{u} : \ \boldsymbol{u} \in \boldsymbol{H}^1(\Omega),\ \mathrm{div}\ \boldsymbol{u} = 0,\ (\mathbf{curl}\ \boldsymbol{u}) \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}, \tag{2.7}$$

$$\boldsymbol{H}_2 = \{\mathbf{grad} q : q \in H^1(\Omega)\}, \tag{2.8}$$

the orthogonal decomposition

$$\boldsymbol{L}_2(\Omega) = \boldsymbol{H}_1 \overset{\perp}{\oplus} \boldsymbol{H}_2 \tag{2.9}$$

holds [17].

2.2 Maxwell's Equations

The Time-Dependent Formulation

Let the domain $\Omega \subset \mathbb{R}^3$ be filled with a homogeneous, linear, isotropic dielectric material, characterized by the electric permittivity ε and magnetic permeability μ. These are assumed to be constant. The boundary Γ is assumed to be a perfect electric conductor. The current density is denoted by $\boldsymbol{J}$, and the charge density by ρ. These quantities are related by the fundamental principle of continuity, or charge conservation:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\, \boldsymbol{J} = 0. \tag{2.10}$$

The electric field is denoted by $\boldsymbol{E}$ and the magnetic flux density by $\boldsymbol{B}$. Currents can generate fields and vice versa. Their connections are given by *Maxwell's equations*

$$-\varepsilon \frac{\partial}{\partial t} \boldsymbol{E} + \frac{1}{\mu} \mathbf{curl}\, \boldsymbol{B} = \boldsymbol{J}, \tag{2.11}$$

$$\frac{\partial}{\partial t} \boldsymbol{B} + \mathbf{curl}\, \boldsymbol{E} = \mathbf{0}. \tag{2.12}$$

For a perfectly conducting boundary Γ, the following boundary conditions are imposed

$$\boldsymbol{E} \times \boldsymbol{n} = \mathbf{0} \text{ and } \boldsymbol{B} \cdot \boldsymbol{n} = 0. \tag{2.13}$$

Taking the divergence of these equations and using (2.10) yields

$$\operatorname{div} \boldsymbol{E} = \frac{\rho}{\varepsilon}, \tag{2.14}$$

$$\operatorname{div} \boldsymbol{B} = 0. \tag{2.15}$$

These equations are often included in the set of Maxwell's equations, although they are consequences of (2.11) and (2.12).

Another material parameter is the conductivity σ. In the presence of an electric field $\boldsymbol{E}$, the material will conduct a current with density given by Ohm's law:

$$\boldsymbol{J} = \sigma \boldsymbol{E}. \tag{2.16}$$

Source and sought terms are problem dependent and must be specified explicitly for each particular electromagnetic problem. We shall exclusively deal with problems where a current density $\boldsymbol{J}_{\text{imp}}$ is imposed on the system. For the case of positive conductivity, the current density $\boldsymbol{J}$ that enters (2.11), is composed of two parts:

$$\boldsymbol{J} = \boldsymbol{J}_{\text{imp}} + \sigma \boldsymbol{E}. \tag{2.17}$$

For the non-conducting case $\sigma = 0$ we have $\boldsymbol{J}_{\text{imp}} = \boldsymbol{J}$. As we shall see, the mathematical properties of electromagnetic problems are very different for the two cases $\sigma = 0$ and $\sigma > 0$.

Depending on the smoothness of the boundary, and the smoothness w.r.t. the time variable of the current and charge densities, existence and uniqueness theorems for Maxwell's equations [15, 16] state that for any time instant t,

$$\boldsymbol{E}(\cdot, t), \boldsymbol{B}(\cdot, t) \in \boldsymbol{H}(\mathbf{curl}; \Omega) \cap \boldsymbol{H}(\operatorname{div}; \Omega).$$

A common procedure for the resolution of Maxwell's equations is the elimination of $\boldsymbol{B}$, [1, 7]. This gives rise to a second order differential equation for determining $\boldsymbol{E}$

$$\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \sigma \frac{\partial \boldsymbol{E}}{\partial t} + \frac{1}{\mu} \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{E} = \frac{\partial \boldsymbol{J}_{\text{imp}}}{\partial t}, \tag{2.18}$$

with initial conditions

$$\boldsymbol{E}(\cdot, 0) = \boldsymbol{E}_0, \qquad \operatorname{div} \boldsymbol{E}_0 = \frac{\rho_0}{\varepsilon},$$

$$\frac{\partial \boldsymbol{E}}{\partial t}(\boldsymbol{x}, 0) = \frac{1}{\varepsilon} \left(\boldsymbol{J}(\boldsymbol{x}, 0) + \frac{1}{\mu} \mathbf{curl}\, \boldsymbol{B}_0(\boldsymbol{x}) - \sigma \boldsymbol{E}_0(\boldsymbol{x}) \right), \quad \boldsymbol{x} \in \Omega.$$

Concerning test and trial spaces for a variational formulation, known existence and uniqueness results suggest the space $\boldsymbol{H}_0(\mathbf{curl};\Omega)\cap\boldsymbol{H}(\mathrm{div};\Omega)$ as a natural choice, see e.g. [11]. An alternative approach is based on the observation that the divergence constraints are automatically preserved by the time evolution and do in this sense not arise from first principles. This suggests focusing on the part in $\boldsymbol{H}(\mathbf{curl};\Omega)$ as a natural test and trial space.

Having chosen some test space $\boldsymbol{X}(\Omega)$ and approximating the derivatives with respect to time by a standard backward difference, leads, by (2.4), to a variational problem at the time step t_n: Find $\boldsymbol{E}^n = \boldsymbol{E}(\cdot, t_n) \in \boldsymbol{X}(\Omega)$ such that

$$(\mathbf{curl}\,\boldsymbol{E}^n, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega} + \kappa(\boldsymbol{E}^n, \boldsymbol{v})_{0,\Omega} = (\boldsymbol{F}^n, \boldsymbol{v})_{0,\Omega}, \tag{2.19}$$

for all $\boldsymbol{v} \in \boldsymbol{X}(\Omega)$, where the constant κ is positive, and depends on ε, μ, σ as well as on the time step Δt_n. $\boldsymbol{F}^n$ is a vector field depending on $\boldsymbol{J}_{\mathrm{imp}}$, $\boldsymbol{E}^{n-1}$ and on $\boldsymbol{E}^{n-2}$.

The Time-Harmonic Formulation

Maxwell's equations in the time-harmonic formulation are obtained by assuming that all involved quantities have a sinusoidal time variation with angular frequency ω. If $\boldsymbol{g}(\boldsymbol{x}, t)$ is a field quantity, we describe it as

$$\boldsymbol{g}(\boldsymbol{x}, t) = \mathrm{Re}\left(\tilde{\boldsymbol{g}}(\boldsymbol{x})\mathrm{e}^{-i\omega t}\right),$$

where $\tilde{\boldsymbol{g}}(\boldsymbol{x})$ is a complex valued amplitude, only depending on spatial variables. Upon inserting $\tilde{\boldsymbol{E}}(\boldsymbol{x})\mathrm{e}^{-i\omega t}$ and $\tilde{\boldsymbol{B}}(\boldsymbol{x})\mathrm{e}^{-i\omega t}$ into Maxwell's equations in the time domain, one obtains

$$\frac{1}{\mu}\mathbf{curl}\tilde{\boldsymbol{B}} + i\varepsilon\omega\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{J}} + \sigma\tilde{\boldsymbol{E}}, \tag{2.20}$$

$$\mathbf{curl}\tilde{\boldsymbol{E}} - i\omega\tilde{\boldsymbol{B}} = \mathbf{0}, \tag{2.21}$$

$$\mathrm{div}\tilde{\boldsymbol{E}} = \frac{1}{\varepsilon}\rho, \tag{2.22}$$

$$\mathrm{div}\tilde{\boldsymbol{B}} = 0. \tag{2.23}$$

Once $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{B}}$ are known, the physical fields are determined by taking the real parts of $\tilde{\boldsymbol{E}}(\boldsymbol{x})\mathrm{e}^{-i\omega t}$ resp. $\tilde{\boldsymbol{B}}(\boldsymbol{x})\mathrm{e}^{-i\omega t}$. A common procedure for the time-harmonic formulation is to take the **curl** of (2.21) and insert it into (2.20), which yields

$$\mathbf{curl}\,\mathbf{curl}\,\tilde{\boldsymbol{E}} - \omega^2\mu\left(i\frac{\sigma}{\omega} + \varepsilon\right)\tilde{\boldsymbol{E}} = -i\omega\mu\boldsymbol{J}, \qquad \mathrm{div}\,\tilde{\boldsymbol{E}} = \frac{\rho}{\varepsilon},$$

whose variational formulation in the chosen test and trial space $\boldsymbol{X}(\Omega)$ reads, on account of (2.4), as

$$(\mathbf{curl}\,\tilde{\boldsymbol{E}}, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega} - \omega^2\mu\left(i\frac{\sigma}{\omega} + \varepsilon\right)(\tilde{\boldsymbol{E}}, \boldsymbol{v})_{0,\Omega} = -i\omega\mu(\boldsymbol{J}, \boldsymbol{v})_{0,\Omega}, \quad \boldsymbol{v} \in \boldsymbol{X}(\Omega). \tag{2.24}$$

The treatment of the divergence condition depends on the choice of $\boldsymbol{X}(\Omega)$, as we shall see below.

As we have seen, both formulations lead to a problem involving the bilinear form

$$a(\boldsymbol{u}, \boldsymbol{v}) := (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega} + \kappa\,(\boldsymbol{u}, \boldsymbol{v})_{0,\Omega} \tag{2.25}$$

for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{X}(\Omega)$. The parameter κ may be either a positive real or a complex (and possibly negative) constant. As we shall see below, the treatment of (2.25) significantly differs for these two cases. In the remainder of this paper we shall investigate the well-posedness of FDFs involving the bilinear form (2.25).

2.3 Saddle Point Problems

As indicated before, we wish to enforce boundary conditions and even further constraints weakly with the aid of Lagrange multipliers. This leads to saddle point problems. For the convenience of the reader we will collect a few facts about saddle point problems that will be frequently used below.

Let X and M be Hilbert spaces with duals X' resp. M' such that $X \hookrightarrow H_X \hookrightarrow X'$, $M \hookrightarrow H_M \hookrightarrow M'$ for two Hilbert spaces H_X, H_M inducing the dualities. Let $a(\cdot,\cdot) : X \times X \to \mathbb{C}$ and $b(\cdot,\cdot) : X \times M \to \mathbb{C}$ be continuous bilinear forms. The general format of a saddle point problem reads as follows: Given $f \in X'$, $g \in B(X) \subseteq M'$, find $(u,p) \in X \times M$ as the solution of

$$\begin{aligned} a(u,v) \quad +b(v,p) &= \langle f, v\rangle,\ v \in X, &(2.26)\\ b(u,q) \qquad\qquad &= \langle g, q\rangle,\ q \in M. &(2.27)\end{aligned}$$

Define the operators $A \in \mathcal{L}(X,X')$, $B \in \mathcal{L}(X,M')$ and $B' \in \mathcal{L}(M,X')$ as follows

$$\langle Au, v\rangle := a(u,v), \quad \langle Bu, p\rangle := b(u,p) =: \langle u, B'p\rangle,$$

for $u,v \in X$ and $p \in M$. In operator form, the system (2.26) and (2.27) can be equivalently written as

$$\mathcal{L}U := \begin{pmatrix} A & B' \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} =: F. \tag{2.28}$$

The problem (2.28) is said to be *well-posed* if the operator $\mathcal{L}$ induces a norm isomorphism of $V := X \times M$ onto its dual $V' = X' \times M'$, i.e., there exist constants $0 < c \le C < \infty$ such that for any $U \in V$,

$$c\|U\|_V \le \|\mathcal{L}U\|_{V'} \le C\|U\|_V. \tag{2.29}$$

The characterization of well-posedness for real-valued bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ in terms of *inf–sup conditions* is due to F. Brezzi (see [3, Thm. II.1.1]). It's counterpart for complex-valued bilinear forms reads as follows (see [3] and [17, 23, 25]).

Theorem 2.3 *Define* $\ker(B) := \{v \in X : Bv = 0 \text{ in } M'\}$*. The saddle point problem (2.28) is well-posed if and only if the following conditions hold for some positive constants* $\alpha, \beta > 0$*:*

$$\inf_{v\in\ker(B)} \sup_{u\in\ker(B)} \frac{|a(u,v)|}{\|u\|_X\,\|v\|_X} \ge \alpha > 0, \tag{2.30}$$

$$\inf_{u\in\ker(B)} \sup_{v\in\ker(B)} \frac{|a(u,v)|}{\|u\|_X,\|v\|_X} \ge \alpha > 0, \tag{2.31}$$

$$\inf_{p\in M} \sup_{u\in X} \frac{|b(u,p)|}{\|u\|_X\,\|p\|_M} \ge \beta > 0. \tag{2.32}$$

Moreover, A is onto if and only if (2.30) *is valid, A is one-to-one if and only if* (2.31) *holds and* (2.32) *is equivalent to* $\operatorname{Ran}(B)$ *being closed in M'.*

3 FDFs for the Time-Dependent Formulation

Recall from Section 2 that boundary constraints involve the tangential components $\gamma_\tau(\boldsymbol{u}) := \boldsymbol{u}|_\Gamma \times \boldsymbol{n}$ which belong to the space $\boldsymbol{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma;\Gamma)$.

3.1 An Essential Obstruction

To our knowledge, not much is known so far on FDFs for Maxwell's equations. The issue is somewhat more delicate than in other situations due to a complication caused by the above mentioned subtle difference between the spaces $\{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) : \gamma_\tau(\boldsymbol{u}) = \boldsymbol{0}\}$ and $\boldsymbol{H}_0(\mathbf{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div};\Omega)$. We will clarify the consequences of this fact first for the problem (2.25) with $\kappa \in \mathbb{R}^+$ and $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \cap \boldsymbol{V}(\mathrm{div};\Omega)$. Without loss of generality we can assume that

$$\Omega \subset \square = (0,a)^n, \qquad a \in \mathbb{R}^+,$$

where the 'simple domain' $\square$ will serve as the fictitious domain. Moreover, for simplicity, we will use *periodic* functions on $\square$, see e.g. [18]. Let us denote by

$$\boldsymbol{\mathcal{C}}^\infty_{\mathrm{per}}(\square) := \{\boldsymbol{v} \in \boldsymbol{\mathcal{C}}^\infty : \boldsymbol{v}(\boldsymbol{x} + a\boldsymbol{k}) = \boldsymbol{v}(\boldsymbol{x}),\ \boldsymbol{k} \in \mathbb{Z}^3\}$$

the a-periodic infinitely differentiable vector fields. Then for any function space $\boldsymbol{X}$ we denote by $\boldsymbol{X}_{\mathrm{per}}(\square)$ the closure of $\boldsymbol{\mathcal{C}}^\infty_{\mathrm{per}}(\square)$ in the $\boldsymbol{X}$-norm: $\boldsymbol{X}_{\mathrm{per}}(\square) := \mathrm{clos}_{\boldsymbol{X}}\, \boldsymbol{\mathcal{C}}^\infty_{\mathrm{per}}(\square)$. Note that also other boundary conditions on $\square$ can be considered. In fact, the subsequent analysis also applies to homogeneous Dirichlet boundary conditions on $\square$.

One would then be tempted to employ the following FDF:

Problem 3.1 *Given $\tilde{\boldsymbol{f}} \in \boldsymbol{X}_{\mathrm{per}}(\square)'$, $\boldsymbol{X}(\square) := \boldsymbol{H}(\mathbf{curl};\square) \cap \boldsymbol{H}(\mathrm{div};\square)$, determine $(\boldsymbol{u}, \phi, \boldsymbol{\lambda}) \in \boldsymbol{X}_{\mathrm{per}}(\square) \times L_2(\square) \times (\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'$ such that*

$$\begin{aligned}
a(\boldsymbol{u},\boldsymbol{v})_\square + (\phi, \mathrm{div}\,\boldsymbol{v})_{0,\square} \quad &+ \langle \boldsymbol{\lambda}, \gamma_\tau(\boldsymbol{v})\rangle_\Gamma = (\tilde{\boldsymbol{f}}, \boldsymbol{v})_{0,\square}, && \boldsymbol{v} \in \boldsymbol{X}_{\mathrm{per}}(\square),\\
(\mathrm{div}\,\boldsymbol{u}, \psi)_{0,\square} \quad &= 0, && \psi \in L_2(\square),\\
\langle \boldsymbol{\mu}, \gamma_\tau(\boldsymbol{u})\rangle_\Gamma \quad &= 0, && \boldsymbol{\mu} \in (\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'.
\end{aligned}$$

The reason why this approach would in general fail can be explained as follows. On one hand, if Problem 3.1 has a solution, its first component $\boldsymbol{u}$ belongs to $\boldsymbol{X}_{\mathrm{per}}(\square)$. But there is a counterpart to Theorem 2.1 for $\boldsymbol{X}_{\mathrm{per}}(\square)$.

Lemma 3.2 *The space $\boldsymbol{X}_{\mathrm{per}}(\square)$ is continuously embedded in $\boldsymbol{H}^1(\square)$.*

Proof: Using integration by parts, we obtain for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{\mathcal{C}}^\infty(\square)_{\mathrm{per}}$

$$\int_\square \mathbf{grad}\,\boldsymbol{u} : \mathbf{grad}\,\boldsymbol{v} = \int_\square \mathrm{div}\,\boldsymbol{u}\ \mathrm{div}\,\boldsymbol{v} + \int_\square \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v}. \tag{3.1}$$

The assertion follows then by standard density arguments. ∎

Hence also the restriction $\boldsymbol{u}|_\Omega$ of a solution belongs to $\boldsymbol{H}^1(\Omega)$. On the other hand, as already noted above, the electric field $\boldsymbol{E}$ solving Maxwell's equations may be an element of $\boldsymbol{H}(\mathbf{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div};\Omega)$, but, depending on Ω may have $\boldsymbol{H}^1$-singularities, see Remark 2.2.

The above reasoning, combined with Theorem 2.1 actually shows the following somewhat more general fact.

Remark 3.3 *Assume that the physical domain Ω is embedded in a convex polyhedron $\square$. Then FDFs for (2.25) that fall in either one of the following categories:*

- *test and trial functions belong to* $\boldsymbol{X}_{\text{per}}(\square)$ *defined above;*
- *test and trial functions have global* $\boldsymbol{H}(\mathbf{curl};\square) \cap \boldsymbol{H}(div;\square)$ *regularity and have vanishing tangential components* $\boldsymbol{u} \times \boldsymbol{n}_\square$ *on* $\partial\square$*;*

fail in the sense that they may not capture possible $\boldsymbol{H}^1$*-singularities, i.e. parts of the solution that do not belong to* $\boldsymbol{H}^1(\Omega)$*.*

Remark 3.4 *Since trivially* $(\boldsymbol{H}(\mathbf{curl};\square) \cap \boldsymbol{V}(\text{div};\square))_{\text{per}} \hookrightarrow (\boldsymbol{H}(\mathbf{curl};\square) \cap \boldsymbol{H}(\text{div};\square))_{\text{per}}$*, the availability of divergence-free trial functions on* $\square$ *(as e.g. divergence-free wavelet bases, see [21, 24]) would be no remedy.*

In this context, we mention the following FDF for a scattering problem, i.e., for an exterior boundary value problem proposed in [9]. Defining the bilinear form

$$a(\boldsymbol{u},\boldsymbol{v})_\square = (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_\square + \kappa(\boldsymbol{u},\boldsymbol{v})_\square, \quad \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\square), \tag{3.2}$$

the following variational problem is analyzed in [9]:
Given $(\boldsymbol{J},\boldsymbol{\eta}) \in (\boldsymbol{H}(\mathbf{curl};\square))' \times (\boldsymbol{H}^{-\frac{1}{2}}(\text{div}_\Gamma;\Gamma))'$, find $(\boldsymbol{u},\boldsymbol{\lambda}) \in \boldsymbol{H}(\mathbf{curl};\square) \times (\boldsymbol{H}^{-\frac{1}{2}}(\text{div}_\Gamma;\Gamma))'$ such that

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v})_\square \quad + \langle \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\lambda}\rangle_\Gamma &= (\boldsymbol{J}_\square, \boldsymbol{v}), \quad && \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\square), && (3.3)\\ \langle \boldsymbol{u} \times \boldsymbol{n}, \boldsymbol{\xi}\rangle \quad &= \langle \boldsymbol{\eta}, \boldsymbol{\xi}\rangle_\Gamma, && \boldsymbol{\eta} \in \boldsymbol{H}^{-\frac{1}{2}}(\text{div}_\Gamma;\Gamma), && (3.4) \end{aligned}$$

where $\boldsymbol{J}_\square$ is some extension of $\boldsymbol{J}$ from Ω to $\square$, and $\langle\cdot,\cdot\rangle_\Gamma$ is the duality form on the boundary. For the definition of div_Γ, we again refer to [4, 5, 6]. In [9], this problem is discretized using Nedelec elements on a fixed grid for $\square$, imposing zero boundary conditions on the tangential components $\boldsymbol{u} \times \boldsymbol{n}|_\square$ on $\partial\square$. Due to the use of the Nedelec elements on $\square$ the solution would have in the limit zero divergence on $\square$ so that Remark 3.3 applies. Of course, this problem may disappear when suitable absorbing boundary conditions are employed instead, which would correspond anyway to the nature of the scattering problem.

3.2 Preserving $\boldsymbol{H}^1$-Singularities

We present now an alternative FDF which preserves the possible $\boldsymbol{H}^1$-singularities of the solutions to Maxwell's equations. The above observations suggest as a remedy to require only $\boldsymbol{H}(\mathbf{curl};\square)$ regularity and impose the divergence constraint only on Ω which is reflected by the following

Problem 3.5 *Given* $(\boldsymbol{J},f) \in (\boldsymbol{H}_0(\mathbf{curl};\Omega))' \times H^{-1}(\Omega)$*, find a pair* $(\boldsymbol{u},p) \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \times H_0^1(\Omega)$ *such that*

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) + (\boldsymbol{v}, \mathbf{grad}\,p)_{0,\Omega} &= (\boldsymbol{J},\boldsymbol{v})_{0,\Omega}, \quad && \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega),\\ (\boldsymbol{u}, \mathbf{grad}\,q)_{0,\Omega} &= (f,q)_{0,\Omega}, && q \in H_0^1(\Omega), \end{aligned}$$

where we assume again first that $\kappa > 0$ in (2.25). The next step is to append boundary conditions. Recalling (3.2), we consider

Problem 3.6 *Given the triple* $(\boldsymbol{J}, f, \boldsymbol{\eta}) \in (\boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\Box))' \times H^{-1}(\Omega) \times \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$*, find a triple* $(\boldsymbol{u}, p, \boldsymbol{\lambda}) \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\Box) \times H_0^1(\Omega) \times (\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'$ *so that*

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v})_\Box \quad + (\boldsymbol{v}, \mathbf{grad}\, p)_{0,\Omega} + \langle \gamma_\tau(\boldsymbol{v}), \boldsymbol{\lambda}\rangle_\Gamma &= (\boldsymbol{J}, \boldsymbol{v})_{0,\Box}, & \boldsymbol{v} &\in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\Box),\\ (\boldsymbol{u}, \mathbf{grad}\, q)_{0,\Omega} &= (f, q)_\Omega, & q &\in H_0^1(\Omega),\\ \langle \gamma_\tau(\boldsymbol{u}), \boldsymbol{\mu}\rangle_\Gamma &= \langle \boldsymbol{\eta}, \boldsymbol{\mu}\rangle_\Gamma, & \boldsymbol{\mu} &\in (\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))', \end{aligned}$$

where as above $\gamma_\tau(\boldsymbol{u}) = \boldsymbol{u}|_\Gamma \times \boldsymbol{n}$.

The main result of this section can be stated as follows.

Theorem 3.7 *The Problems 3.5 and 3.6 are well-posed and have unique solutions. The first two components of the solutions to both problems coincide on* Ω *when the boundary conditions in 3.6 are homogeneous.*

The remainder of this section is devoted to the proof of Theorem 3.7. We begin with some prerequisites that will eventually allow us to apply Theorem 2.3. As before, we assume always that Ω is some bounded, simply connected polyhedral Lipschitz domain with boundary $\Gamma = \partial\Omega$. Recall the orthogonal decomposition of $\boldsymbol{L}_2(\Omega)$ in (2.9). A similar decomposition also holds for $\boldsymbol{H}(\mathbf{curl};\Omega)$. In fact, defining

$$\begin{aligned} \boldsymbol{X}_1(\Omega) &:= \left\{\mathbf{curl}\,\boldsymbol{u} :\ \boldsymbol{u} \in \boldsymbol{H}^1(\Omega),\ \mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl};\Omega),\ \mathrm{div}\,\boldsymbol{u} = 0,\ (\mathbf{curl}\ \boldsymbol{u})\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\right\},\\ \boldsymbol{X}_2(\Omega) &:= \left\{\mathbf{grad}\,\phi :\ \phi \in H^1(\Omega)\right\}, \end{aligned}$$

we record the following observation.

Lemma 3.8 *The spaces* $\boldsymbol{X}_1(\Omega)$ *and* $\boldsymbol{X}_2(\Omega)$ *form an orthogonal decomposition of* $\boldsymbol{H}(\mathbf{curl};\Omega)$

$$\boldsymbol{H}(\mathbf{curl};\Omega) = \boldsymbol{X}_1(\Omega) \oplus \boldsymbol{X}_2(\Omega). \tag{3.5}$$

Proof: Given $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl};\Omega)$, write $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$ according to (2.9). It is clear that $\boldsymbol{u}_i \in \boldsymbol{X}_i(\Omega)$, $i = 1, 2$. Since $\mathbf{curl}\,\boldsymbol{u}_2 = \mathbf{0}$, $\boldsymbol{u}_1 \perp \boldsymbol{u}_2$ in $\boldsymbol{H}(\mathbf{curl};\Omega)$. ☐

Now we embed Ω into a larger, simpler domain $\Box$. Problems 3.5 and 3.6 suggest considering the following operators.

Lemma 3.9 *The operators* $\mathrm{Div} :\ \boldsymbol{X}_2(\Omega) \to H^{-1}(\Omega)$ *and* $\mathrm{Div}_\Omega : \boldsymbol{X}_2(\Box) \to H^{-1}(\Omega)$ *defined by*

$$\langle \mathrm{Div}\,\boldsymbol{u}, p\rangle := (\boldsymbol{u}, \mathbf{grad}\, p)_{0,\Omega}, \quad \langle \mathrm{Div}_\Omega\,\boldsymbol{u}, p\rangle := (\boldsymbol{u}, \mathbf{grad}\, p)_{0,\Omega}, \quad p \in H_0^1(\Omega), \tag{3.6}$$

are bounded and onto.

Proof: The boundedness is clear. For $f \in H^{-1}(\Omega)$ the boundary value problem

$$-\Delta\phi = f \quad \text{on } \Omega, \quad \phi = 0 \quad \text{on } \Gamma = \partial\Omega,$$

has a unique solution $\phi \in H_0^1(\Omega)$. This means $\mathbf{grad}\,\phi \in \boldsymbol{X}_2(\Omega)$ and $\mathrm{Div}\,\mathbf{grad}\,\phi = f$. To confirm also the surjectivity of Div_Ω, it suffices to extend ϕ by zero from Ω to $\Box$. ☐

Lemma 3.10 *The spaces* $\boldsymbol{H}_0(\mathbf{curl};\Omega)\cap\boldsymbol{V}(\mathrm{div};\Omega)$ *and*

$$\boldsymbol{H}_{\mathrm{per}}(\Box,\Omega) := \{\boldsymbol{u}\in\boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\Box) : \mathrm{Div}_\Omega\,\boldsymbol{u}=0, \boldsymbol{u}\times\boldsymbol{n}=0 \text{ on } \Gamma\}$$

are closed in $\boldsymbol{H}(\mathbf{curl};\Omega)$ *and* $\boldsymbol{H}(\mathbf{curl};\Box)$, *respectively.*

Proof: The trace mapping $\gamma_\tau(\boldsymbol{u})=\boldsymbol{u}|_\Gamma\times\boldsymbol{n}$ is bounded and, by Lemma 3.9, so are the operators Div and Div_Ω. The spaces defined above are the kernels of these operators. □

From now on, we will frequently use the notation $A\lesssim B$ which means that there exists a constant $c>$ such that $A\le c\,B$, uniformly in all parameters on which A and B may depend. The following observation is based on the results in [6].

Lemma 3.11 *There exists a continuous extension operator* $E:\boldsymbol{H}(\mathbf{curl};\Omega)\to\boldsymbol{H}(\mathbf{curl};\mathbb{R}^3)$.

Proof: Let $\boldsymbol{f}\in\boldsymbol{H}(\mathbf{curl};\Omega)$ be given. Embed Ω in an open ball B and denote by $\Gamma_\cap$ the common interface between Ω and $B\setminus\overline{\Omega}$. Let $\tilde{\boldsymbol{f}}\in\boldsymbol{H}(\mathbf{curl};B/\overline{\Omega})$ be the the continuous lifting given in [6] that satisfies $\tilde{\boldsymbol{f}}\times\boldsymbol{n}_B=0$ on ∂B as well as

$$\tilde{\boldsymbol{f}}\times\boldsymbol{n}_{\Gamma_\cap}=-\boldsymbol{f}\times\boldsymbol{n}_\Omega,\quad \|\tilde{\boldsymbol{f}}\|_{\boldsymbol{H}(\mathbf{curl};B\setminus\overline{\Omega})}\lesssim\|\tilde{\boldsymbol{f}}\times\boldsymbol{n}_B\|_{\boldsymbol{H}^{-\frac12}(\mathrm{div}_{\partial(B\setminus\overline{\Omega})};\partial(B\setminus\overline{\Omega}))}. \tag{3.7}$$

Here the vectors $\boldsymbol{n}_{\Gamma_\cap}=-\boldsymbol{n}_\Omega$, $\boldsymbol{n}_B$ are the outward unit normals on the interface $\Gamma_\cap$ as part of the boundary of $B\setminus\overline{\Omega}$, respectively ∂B. Now define

$$E(\boldsymbol{f}) := \begin{cases} \boldsymbol{f}\,, & \text{in } \Omega,\\ \tilde{\boldsymbol{f}}\,, & \text{in } B/\overline{\Omega},\\ 0\,, & \text{in } \mathbb{R}^3/B.\end{cases}$$

Due to the continuity of the tangential components $E(\boldsymbol{f})$ belongs to $\boldsymbol{H}(\mathbf{curl};\mathbb{R}^3)$. By the boundedness of the lifting (3.7), we obtain

$$\begin{aligned}\|E(\boldsymbol{f})\|_{\boldsymbol{H}(\mathbf{curl};\mathbb{R}^3)} &\lesssim \|\boldsymbol{f}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}+\|\tilde{\boldsymbol{f}}\times\boldsymbol{n}_B\|_{\boldsymbol{H}^{-\frac12}(\mathrm{div}_{\partial(B/\overline{\Omega})};\partial(B/\overline{\Omega}))}\\ &= \|\boldsymbol{f}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}+\|\boldsymbol{f}\times\boldsymbol{n}_\Omega\|_{\boldsymbol{H}^{-\frac12}(\mathrm{div}_\Gamma;\Gamma)}\\ &\lesssim \|\boldsymbol{f}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)},\end{aligned}$$

where we have used the boundedness of the trace in the last step. This proves the claim. □

The last ingredient concerns the inf-sup condition (2.32) in Theorem 2.3.

Lemma 3.12 *There exists a positive constant* $\beta>0$ *such that the inf-sup condition*

$$\inf_{\boldsymbol{\lambda}\in(\boldsymbol{H}^{-\frac12}(\mathrm{div}_\Gamma;\Gamma))'}\ \sup_{\boldsymbol{u}\in\boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\Box)}\frac{\langle\boldsymbol{\lambda},\boldsymbol{u}\times\boldsymbol{n}\rangle_\Gamma}{\|\boldsymbol{\lambda}\|_{(\boldsymbol{H}^{-\frac12}(\mathrm{div}_\Gamma;\Gamma))'}\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Box)}}\ge\beta \tag{3.8}$$

holds.

Proof: The proof is similar in spirit to [19]. Since the extension in Lemma 3.11 is bounded, the above extension from $\boldsymbol{H}(\mathbf{curl};\Omega)$ to $\boldsymbol{H}(\mathbf{curl};\mathbb{R}^3)$ also yields a bounded extension to the space $\boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\Box)$ provided that $B \subset \Box$. Fix any $\boldsymbol{\lambda} \in (\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'$. Using the surjectivity and boundedness of the trace mapping

$$\boldsymbol{H}(\mathbf{curl};\Omega) \ni \boldsymbol{u} \mapsto \gamma_\tau(\boldsymbol{u}) = \boldsymbol{u}|_\Gamma \times \boldsymbol{n} \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma), \tag{3.9}$$

see [6], and finally the boundedness of the lifting map (see Lemma 3.11), we obtain

$$\begin{aligned}
\|\boldsymbol{\lambda}\|_{(\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'} &= \sup_{\boldsymbol{\mu}\in\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)} \frac{\langle\boldsymbol{\lambda},\boldsymbol{\mu}\rangle_\Gamma}{\|\mu\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)}} = \sup_{\boldsymbol{u}\in\boldsymbol{H}(\mathbf{curl};\Omega)} \frac{\langle\boldsymbol{\lambda},\gamma_\tau(\boldsymbol{u})\rangle_\Gamma}{\|\gamma_\tau(\boldsymbol{u})\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)}} \\
&\lesssim \sup_{\boldsymbol{u}\in\boldsymbol{H}(\mathbf{curl};\Omega)} \frac{\langle\boldsymbol{\lambda},\gamma_\tau(\boldsymbol{u})\rangle_\Gamma}{\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}} \lesssim \sup_{\boldsymbol{u}\in\boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\Box)} \frac{\langle\boldsymbol{\lambda},\gamma_\tau(\boldsymbol{u})\rangle_\Gamma}{\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Box)}},
\end{aligned}$$

which is the desired inf-sup condition. □

Proof of Theorem 3.7: Let us consider first Problem 3.5. Since $\kappa > 0$, the bilinear form $a(\cdot,\cdot)$ in (2.25) is coercive on $\boldsymbol{H}(\mathbf{curl};\Omega)$, hence also on the subspace $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. This implies the validity of (2.30) and (2.31). The inf-sup condition (2.32) corresponding to the divergence constraint in Problem 3.5 is equivalent to the closedness of the range of the divergence mapping Div_Ω, which is indeed ensured by Lemma 3.9. Moreover, the corresponding mapping is onto. Thus, Problem 3.5 is, by Theorem 2.3, well-posed and thus has a unique solution.

Now, we investigate Problem 3.6. The bilinear form $a(\cdot,\cdot)_\Box$ in (3.2) is coercive on $\boldsymbol{H}(\mathbf{curl};\Box)$ and hence also on the closed subspace $\boldsymbol{H}_{\mathrm{per}}(\Box,\Omega) = \{\boldsymbol{u} \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\Box) : \mathrm{Div}_\Omega\, \boldsymbol{u} = 0,\ \boldsymbol{u}\times\boldsymbol{n} = 0 \text{ on } \Gamma\}$ which is the kernel of the operator B defined by the Lagrange multipliers in Problem 3.6. The inf-sup condition corresponding to the trace mapping in Problem 3.6 is fulfilled by Lemma 3.12. Moreover, the operator $\mathcal{L}$ induced by the variational problem: Given $(\boldsymbol{J}, f) \in (\boldsymbol{H}_{\mathrm{per}}(\Box,\Omega))' \times H^{-1}(\Omega)$, find a pair $(\boldsymbol{u}, p) \in \boldsymbol{H}_{\mathrm{per}}(\Box,\Omega) \times H^1_0(\Omega)$ such that

$$\begin{aligned}
a(\boldsymbol{u},\boldsymbol{v})_\Box + (\boldsymbol{v}, \mathbf{grad}\, p)_{0,\Omega} &= (\boldsymbol{J},\boldsymbol{v})_{0,\Omega}, \quad \boldsymbol{v} \in \boldsymbol{H}_{\mathrm{per}}(\Box,\Omega), \\
(\boldsymbol{u}, \mathbf{grad}\, q)_{0,\Omega} &= (f,q)_{0,\Omega}, \quad q \in H^1_0(\Omega),
\end{aligned}$$

is by the preceeding remarks a norm isomorphism from the space $\boldsymbol{H}_{\mathrm{per}}(\Box,\Omega) \times H^1_0(\Omega)$ onto its dual. This implies the validity of (2.30) and (2.31) for the bilinear form corresponding to $\mathcal{L}$. Thus, again Theorem 2.3 yields that Problem 3.6 is well-posed and hence has a unique solution. □

4 FDFs for the Time-Harmonic Formulation

Finally, we discuss properly defined FDFs also for the time-harmonic case. Again, we need some preparations.

Definition 4.1 *We say that $\lambda > 0$ is a* Maxwell eigenvalue, *if there exists a non-zero function $\boldsymbol{u} \in \boldsymbol{V}_0(\Omega) := \boldsymbol{H}_0(\mathbf{curl};\Omega) \cap \boldsymbol{V}(\mathrm{div};\Omega)$ such that*

$$(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega} = \lambda(\boldsymbol{u},\boldsymbol{v})_{0,\Omega}, \quad \boldsymbol{v} \in \boldsymbol{V}_0(\Omega).$$

One can prove that the Maxwell eigenvalues form a discrete subset of $\mathbb{R}$, [11]. As in (2.24), we define the bilinear form $a(\cdot,\cdot) : \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathbf{curl};\Omega) \mapsto \mathbb{C}$ by

$$a(\boldsymbol{u},\boldsymbol{v}) := (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega} - \omega^2\mu\left(\varepsilon + i\frac{\sigma}{\omega}\right)(\boldsymbol{u},\boldsymbol{v})_{0,\Omega}. \tag{4.1}$$

Remark 4.2 *For $\sigma > 0$, there exists a positive constant $\alpha > 0$ such that we have the inf-sup condition*

$$\inf_{\boldsymbol{u}\in\boldsymbol{V}_0(\Omega)} \sup_{\boldsymbol{v}\in\boldsymbol{V}_0(\Omega)} \frac{|a(\boldsymbol{u},\boldsymbol{v})|}{||\boldsymbol{u}||_{\boldsymbol{H}(\mathbf{curl};\Omega)}||\boldsymbol{v}||_{\boldsymbol{H}(\mathbf{curl};\Omega)}} \geq \alpha.$$

For $\sigma = 0$ it holds as well, provided that $\lambda := \omega^2\mu\varepsilon$ is not a Maxwell eigenvalue.

In fact, for $\sigma > 0$, the claim is a trivial consequence of the coercivity of $a(\cdot,\cdot)$ as a bilinear form on $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, [20]. For $\sigma = 0$, the operator A induced by $a(\cdot,\cdot)$ in (4.1) *is injective on $\boldsymbol{V}_0(\Omega)$ since λ is assumed not to be a Maxwell eigenvalue. Moreover, A is selfadjoint, so that A is also surjective, hence one-to-one, which is the desired inf-sup-condition by* (2.31).

We arrive at the following saddle-point formulation realizing the divergence constraint on Ω for the time-harmonic problem.

Problem 4.3 *Given $(\boldsymbol{J}, f) \in (\boldsymbol{H}_0(\mathbf{curl};\Omega))' \times H^{-1}(\Omega)$, find a pair $(\boldsymbol{u}, p) \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \times H_0^1(\Omega)$ such that*

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) + (\boldsymbol{v}, \mathbf{grad}\,p)_{0,\Omega} &= i\omega\mu\,(\boldsymbol{J},\boldsymbol{v})_{0,\Omega}, \quad && \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega),\\ (\boldsymbol{u}, \mathbf{grad}\,q)_{0,\Omega} &= \quad (f,q)_{0,\Omega}, && q \in H_0^1(\Omega), \end{aligned}$$

where $a(\cdot,\cdot)$ is defined by (4.1).

Theorem 4.4 *Assume that $\sigma > 0$ or, when $\sigma = 0$, that $\omega^2\mu\varepsilon$ is not a Maxwell eigenvalue. Then, Problem 4.3 is well-posed.*

Proof: By Remark 4.2 the bilinear form $a(\cdot,\cdot)$ satisfies the required inf-sup condition, and the claim follows from Lemma 3.9 and Theorem 2.3. □

Boundary Conditions

Above we have imposed only the divergence constraint weakly in terms of Lagrange multipliers, which still requires incorporating the boundary conditions into a discretization in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. In order to treat them with the aid of Lagrange multipliers, we extend the bilinear form in (4.1) from Ω to $\Box$ as follows: For $\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Box)$, let as in (3.2)

$$a(\boldsymbol{u},\boldsymbol{v})_{\Box} := (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{0,\Box} - \omega^2\mu\left(\varepsilon + i\frac{\sigma}{\omega}\right)(\boldsymbol{u},\boldsymbol{v})_{0,\Box}. \tag{4.2}$$

The resulting variational problem then reads:

Problem 4.5 *Given the triple $(\boldsymbol{J}, f, \boldsymbol{\eta}) \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\Box)' \times H^{-1}(\Omega) \times \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$, find a triple $(\boldsymbol{u}, p, \boldsymbol{\lambda}) \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\Box) \times H_0^1(\Omega) \times (\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'$ so that*

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v})_{\Box} \quad + (\boldsymbol{v}, \mathbf{grad}\,p)_{0,\Omega} + \langle \boldsymbol{v}\times\boldsymbol{n}, \boldsymbol{\lambda}\rangle_\Gamma &= (\boldsymbol{J},\boldsymbol{v})_{0,\Box}, \quad && \boldsymbol{v} \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\Box),\\ (\boldsymbol{u}, \mathbf{grad}\,q)_{0,\Omega} &= (f,q)_\Omega, && q \in H_0^1(\Omega),\\ \langle \boldsymbol{u}\times\boldsymbol{n}, \boldsymbol{\mu}\rangle_\Gamma &= \langle \boldsymbol{\eta}, \boldsymbol{\mu}\rangle_\Gamma, && \boldsymbol{\mu} \in (\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'. \end{aligned}$$

Let us start by the case $\sigma > 0$.

Proposition 4.6 *When $\sigma > 0$ the bilinear form $a(\cdot,\cdot)$ in (4.1) is coercive on $\boldsymbol{H}(\mathbf{curl}; G)$ for any domain G.*

The proof of this fact is probably known, see, e.g. [20] for a closely related situation. Since we are not aware of a precise reference and since the argument is short we include it for convenience. To this end, we need the following technical lemma.

Lemma 4.7 *Given η, $\xi > 0$, there exists a $\alpha > 0$ so that*

$$f(x, y) := x^2 + (\eta^2 + \xi^2)y^2 - 2\eta xy - \alpha^2(x^2 + y^2 + 2xy) > 0$$

for all $x, y \in \mathbb{R}$ such that $xy \geq 0$, $x^2 + y^2 > 0$.

Proof: Setting $x = 0$, $y > 0$, respectively $x > 0$, $y = 0$, shows that necessary conditions on α are $\alpha < 1$ and $\alpha^2 < \eta^2 + \xi^2$. Moreover, these conditions ensure that f is positive in a neighborhood of $\{(x, y) : \; x = 0 \text{ or } y = 0\}$. Then, the claim follows if one can find $\alpha > 0$ so that for any fixed $x > 0$, the function $g_x(y) := f(x, y)$ has no real zeros. Straightforward manipulations reveal that this is indeed the case if

$$r(\alpha) := -\alpha^2(2\eta + \eta^2 + \xi^2 + 1) + \xi^2 > 0.$$

Since r has only one positive zero α^*, we can choose $0 < \alpha < \min(1, \sqrt{\eta^2 + \xi^2}, \alpha^*)$. □

We can use this observation to prove Proposition 4.6 as follows.

Proof: For $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}; G)$, define $x(\boldsymbol{u}) := \|\mathbf{curl}\ \boldsymbol{u}\|^2_{0,G}$, $y(\boldsymbol{u}) := \|\boldsymbol{u}\|^2_{0,G}$ and $\eta := \omega^2\mu\varepsilon$, $\xi := \mu\omega\sigma$. Note that $a(\boldsymbol{u}, \boldsymbol{u}) = x(\boldsymbol{u}) - (\eta + i\xi)y(\boldsymbol{u}) = x(\boldsymbol{u}) - \eta y(\boldsymbol{u}) - i\xi y(\boldsymbol{u})$ as well as $\|\boldsymbol{u}\|^2_{\boldsymbol{H}(\mathbf{curl};G)} = x(\boldsymbol{u}) + y(\boldsymbol{u})$. It is then readily seen that the bilinear form $a(\cdot,\cdot)$ is coercive in the sense that $|a(\cdot,\cdot)| \geq \alpha \|\cdot\|^2_{\boldsymbol{H}(\mathbf{curl};G)}$ if and only if there exists a constant $\alpha > 0$ such that

$$(x(\boldsymbol{u}) - \eta y(\boldsymbol{u}))^2 + \xi^2 y(\boldsymbol{u})^2 > \alpha^2(x(\boldsymbol{u})^2 + y(\boldsymbol{u})^2 + 2x(\boldsymbol{u})y(\boldsymbol{u})),$$

which follows from Lemma 4.7. □

Now, we are prepared to show the desired result.

Corollary 4.8 *Problem 4.5 is well-posed for $\sigma > 0$.*

Proof: Again we wish to apply Theorem 2.3. On account of Lemma 3.12, it remains to verify the validity of the conditions (2.30) and (2.31) for the form $a(\cdot,\cdot)_\Box$. Since by Lemma 4.6 the bilinear form in (4.2) is, in particular, coercive on $\boldsymbol{H}_{\text{per}}(\mathbf{curl}; \Box)$, (2.30) and (2.31) obviously hold. □

The question of well-posedness of Problem 4.5 for the non-conducting case $\sigma = 0$ turns out to be less straightforward since $a(\cdot,\cdot)_\Box$ is no longer coercive. We assume throughout this section that that $\lambda := \omega^2\mu\varepsilon$ is not a Maxwell eigenvalue with respect to Ω in the sense of Definition 4.1. This is a reasonable assumption, since otherwise the original Maxwell problem would not be well-posed. However, since $\sigma = 0$, it may very well happen that the equation $(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_\Box = \lambda(\boldsymbol{u}, \boldsymbol{v})_\Box$, $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_{\text{per}}(\mathbf{curl}; \Box)$, has a non-trivial solution $\boldsymbol{u} \neq \boldsymbol{0}$ on $\Box$. We shall show that this can be avoided by judiceously choosing $\Box$.

Let us again assume that the problem is scaled in such a way that $\Omega \subset \Box := (0, a)^3$ for a sufficiently large $a \in \mathbb{R}$.

Theorem 4.9 *Suppose that $\lambda := \omega^2\mu\varepsilon$ is not a Maxwell eigenvalue with respect to Ω. Choose $\Box := (0,a)^3$ such that*

$$\lambda \neq \frac{4\pi^2}{a^2}(m^2+n^2+p^2) \tag{4.3}$$

for all m, n, $p \in \mathbb{N}$. Then, Problem 4.5 is well-posed.

Note that (4.3) can *always* be satisfied. In fact, if $\pi^2/\lambda \in \mathbb{Q}$ choose $a \in \mathbb{R}\setminus\mathbb{Q}$, otherwise some $a \in \mathbb{Q}$ such that $\Omega \subset \Box$. In order to prove Theorem 4.9, we consider the following auxiliary problem:

Problem 4.10 *Find $\boldsymbol{u} \in \boldsymbol{H}_{\text{per}}(\mathbf{curl};\Box)$ such that*

$$(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_\Box = \lambda(\boldsymbol{u},\boldsymbol{v})_\Box, \qquad \boldsymbol{v} \in \boldsymbol{H}_{\text{per}}(\mathbf{curl};\Box). \tag{4.4}$$

Lemma 4.11 *Suppose that $\lambda > 0$ is not a Maxwell eigenvalue. Then, Problem 4.5 is well-posed if Problem 4.10 has only the trivial solution $\boldsymbol{u} \equiv \mathbf{0}$.*

Proof: If Problem 4.10 has only the trivial solution $\boldsymbol{u} \equiv \mathbf{0}$, the operator induced by $a(\cdot,\cdot)_\Box$ is injective on $\boldsymbol{H}_{\text{per}}(\mathbf{curl};\Box)$ and thus the inf-sup-condition (2.31) is satisfied. Hence Lemma 3.9 and Lemma 3.12 show that Problem 4.5 is well-posed. $\Box$

We shall next identify the non-trivial solutions of (4.4), i.e., eigenfunctions.

Lemma 4.12 *There exists a family of eigenfields for the* **curl curl** *operator on $\boldsymbol{C}^\infty_{\text{per}}(\Box)$ with eigenvalues*

$$\lambda_{m,n,p} := \frac{4\pi^2}{a^2}(m^2+n^2+p^2), \quad m,n,p \in \mathbb{N}. \tag{4.5}$$

Moreover, this family is orthogonal and complete in $\boldsymbol{L}_2(\Box)$, i.e., the set of finite linear combinations of eigenfields is dense in $\boldsymbol{L}_2(\Box)$.

Proof: We rely on the identity $\mathbf{curl}\,\mathbf{curl} = \mathbf{grad}\,\text{div} - \Delta$, which is valid since we consider smooth functions. Define

$$k_m := \frac{2\pi m}{a}, \tag{4.6}$$

where $m \in \mathbb{N}$, and consider

$$u_3^{(1)}(x_1,x_2,x_3)_{m,n,p} := \sin(k_m x_1)\,\sin(k_n x_2)\,\sin(k_p x_3), \tag{4.7}$$

where $m,n,p \in \mathbb{N}$. In order to enforce $\boldsymbol{u}^{(1)}(\boldsymbol{x})_{m,n,p} := (u_1^{(1)}(\boldsymbol{x})_{m,n,p}, u_2^{(1)}(\boldsymbol{x})_{m,n,p}, u_3^{(1)}(\boldsymbol{x})_{m,n,p})$ to be divergence-free we set

$$u_1^{(1)}(\boldsymbol{x})_{m,n,p} := -\tfrac{mp}{m^2+n^2}\cos(k_m x_1)\,\sin(k_n x_2)\,\cos(k_p x_3), \tag{4.8}$$

$$u_2^{(1)}(\boldsymbol{x})_{m,n,p} := -\tfrac{np}{m^2+n^2}\sin(k_m x_1)\,\cos(k_n x_2)\,\cos(k_p x_3). \tag{4.9}$$

It is easy to see that

$$\Delta u_i^{(1)}(\boldsymbol{x})_{m,n,p} = -\frac{4\pi^2}{a^2}(m^2+n^2+p^2)\;u_i^{(1)}(\boldsymbol{x})_{m,n,p}, \quad i = 1,2,3. \tag{4.10}$$

Hence $\boldsymbol{u}^{(1)}(\boldsymbol{x})_{m,n,p}$ is an eigenfield of **curl curl** with the desired eigenvalue. Define now

$$u_3^{(2)}(\boldsymbol{x})_{m,n,p} := \sin(k_m x_1)\,\sin(k_n x_2)\,\cos(k_p x_3). \tag{4.11}$$

In the same manner as just outlined, one can choose the remaining components of $\boldsymbol{u}^{(2)}(\boldsymbol{x})_{m,n,p}$ so that it becomes divergence-free and is an eigenfield of the Laplacian, hence also for **curl curl**. Letting

$$\begin{aligned} u_3^{(3)}(\boldsymbol{x})_{m,n,p} &:= \sin(k_m x_1)\,\cos(k_n x_2)\,\sin(k_p x_3), \\ u_3^{(4)}(\boldsymbol{x})_{m,n,p} &:= \sin(k_m x_1)\,\cos(k_n x_2)\,\cos(k_p x_3), \\ \vdots\; &\;\vdots \\ u_3^{(8)}(\boldsymbol{x})_{m,n,p} &:= \cos(k_m x_1)\,\cos(k_n x_2)\,\cos(k_p x_3), \end{aligned}$$

one obtains for all components all necessary combinations of sines and cosines to form a basis for $\boldsymbol{L}_2(\Box)$. Because of the constants in (4.8,4.9), the basis is orthogonal, but not orthonormal. Since the family

$$(\boldsymbol{u}^{(1)}(\boldsymbol{x})_{m,n,p}, \boldsymbol{u}^{(2)}(\boldsymbol{x})_{m,n,p}, \ldots, \boldsymbol{u}^{(8)}(\boldsymbol{x})_{m,n,p}), \quad m,n,p \in \mathbb{N}, \tag{4.12}$$

is just a scaled version of the canonical Fourier basis of $\boldsymbol{L}_2(\Box)$, it is complete in the sense that any function in $\boldsymbol{L}_2(\Box)$ can be approximated arbitrarily well by a finite linear combination of elements in (4.12). □

Proposition 4.13 *All eigenvalues of* (4.4) *are given by* (4.5)*.*

Proof: Let $\boldsymbol{u}_{m,n,p}$ be any of the eigenfunctions $\boldsymbol{u}^{(i)}(\boldsymbol{x})_{m,n,p}$, $i = 1, \ldots, 8$, arising in the proof of Lemma 4.12. Integrating by parts leads to

$$(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_{m,n,p}, \boldsymbol{v})_\Box = (\mathbf{curl}\,\boldsymbol{u}_{m,n,p}, \mathbf{curl}\,\boldsymbol{v})_\Box = \lambda_{m,n,p}(\boldsymbol{u}_{m,n,p}, \boldsymbol{v})_\Box, \tag{4.13}$$

for all $\boldsymbol{v} \in \boldsymbol{H}_{\text{per}}(\mathbf{curl}; \Box)$ and hence the eigenvalues (4.5) are also eigenvalues of (4.4).

On the other hand, assume now that $(\boldsymbol{u}, \lambda)$ is an eigenpair of (4.4), where $\lambda \neq \lambda_{m,n,p}$ for all $m,n,p \in \mathbb{N}$. Then for any element $\boldsymbol{u}_{m,n,p}$ of the basis (4.12) one obtains

$$\lambda(\boldsymbol{u}, \boldsymbol{u}_{m,n,p})_\Box = (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{u}_{m,n,p})_\Box = (\boldsymbol{u}, \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_{m,n,p})_\Box = \lambda_{m,n,p}(\boldsymbol{u}, \boldsymbol{u}_{m,n,p})_\Box.$$

Since $\lambda \neq \lambda_{m,n,p}$ for every $m,n,p \in \mathbb{N}$, $\boldsymbol{u}$ is perpendicular to the dense set of finite linear combinations of elements from (4.12), hence $\boldsymbol{u} = \mathbf{0}$. □

Proof of Theorem 4.9: If (4.3) holds, Lemma 4.12 and Proposition 4.13 imply that λ cannot be an eigenvalue of (4.4), i.e., Problem 4.10 and hence Problem 4.10 only have the trivial solution $\boldsymbol{u} = \mathbf{0}$. Hence, by Lemma 4.11, Problem 4.5 is well-posed. □

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