Inductive Types and Exact Completion

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Abstract

Using the theory of exact completions, we show that a specific class of pretopoi, consisting of what we might call "realizability pretopoi", can act as categorical models of certain predicative type theories, including Martin-Löf type theory. Our main theoretical instrument for doing so is a categorical notion, the notion of weak W-types, an "intensional" analogue of the "extensional" notion of W-types introduced in an article by Moerdijk and Palmgren ([6]).

1 Introduction

In this article, we will show how categorical models for certain predicative type theories, especially (extensional) Martin-Löf type theory, can be constructed using the technique of taking exact completions. The categorical models that will interest us, are pretopoi with dependent products and W-types, and hence with a natural number object. As is known from an article by Moerdijk and Palmgren ([6]), these categories can indeed act as such models, and in their internal logic, the constructions familiar from Martin-Löf type theory can be performed.

More concretely, we will specify a set of constraints that a category $\mathbf{C}$ has to satisfy in order for its exact completion to be a pretopos with dependent products, a natural number object and W-types. Before we enumerate this set, let us recall that the exact completion, denoted by $\mathbf{C}_{ex}$ can be constructed by taking formal quotients of pseudo-equivalence relations in $\mathbf{C}$. As will be explained in the next section, it will then have a universal property.

As for the constraints on $\mathbf{C}$, one of them must be that $\mathbf{C}$ is *lextensive*: this notion is introduced in [3] and means that $\mathbf{C}$ has finite limits and finite stable and disjoint coproducts (this result is obtained in [2]). It is also known from the literature that it is necessary for $\mathbf{C}$ to have *weak dependent products* (see [4]). We will also demand that $\mathbf{C}$ has a *weak natural number object* (for a definition, see [1]). In [1], the authors show that if $\mathbf{C}$ satisfies satisfies the demands mentioned so far, $\mathbf{C}_{ex}$ must be a pretopos with dependent products and a natural number object. What is missing is a constraint on $\mathbf{C}$ that will insure that $\mathbf{C}_{ex}$ has W-types.

Finding this additional constraint will help us in connecting this topic with the theory of realizability topoi. If one has a pretopos $\mathbf{E}$, with dependent products and W-types, then one can construct what one might call, for want of a better name, the "effective pretopos" relative to $\mathbf{E}$, denoted here by $\mathbf{Eff}(\mathbf{E})$. This construction takes place in two steps: first, we construct the category $\mathbf{Pass}(\mathbf{E})$ of partitioned assemblies relative to $\mathbf{E}$, and then we take the exact completion of this category. Another goal of this article will be to show that $\mathbf{Eff}(\mathbf{E})$ is again a pretopos with dependent products and W-types. To do this, it will suffice to show that $\mathbf{Pass}(\mathbf{E})$ is a category that satisfies the demands that will insure that its exact completion is such a category. And as it turns out, the following result is known from the literature: $\mathbf{Pass}(\mathbf{E})$ is lextensive, has weak dependent products and a natural number object.

This means that, as for the additional demand, two features are required. Firstly, it has to be strong enough to insure that if $\mathbf{C}$ also satisfies this additional demand, $\mathbf{C}_{ex}$ will have W-types. Secondly, it will have to be weak enough so that $\mathbf{Pass}(\mathbf{E})$ satisfies it for every pretopos $\mathbf{E}$ with dependent products and W-types.

In this article we will introduce the notion of "weak W-type" and we will show that the requirement of having weak W-types meets these two desiderata. This means that for the notion of weak W-type that is to be defined shortly, we can prove the following two theorems:

Theorem 1.1 *If $\mathbf{C}$ is a lextensive category with weak dependent products, a weak natural number object and weak W-types, then $\mathbf{C}_{ex}$ is a pretopos with dependent products and W-types.*

Theorem 1.2 *For every pretopos with dependent products and W-types $\mathbf{E}$, the category $\mathbf{Pass}(\mathbf{E})$ has weak W-types, and so is a lextensive category with weak dependent products and natural number object and weak W-types.*

And these two theorems have the following corrolary:

Corollary 1.3 *For every pretopos with dependent products and W-types $\mathbf{E}$, the category $\mathbf{Eff}(\mathbf{E})$ is pretopos with dependent products and W-types.*

2 Weak W-types

Let us start this section by giving some definitions that are pivotal to this article.

Definition 2.1 For any pretopos $\mathbf{E}$ with dependent products and W-types, the category of *partitioned assemblies* relative to $\mathbf{E}$, denoted $\mathbf{Pass}(\mathbf{E})$, is constructed as follows. If N is the natural number object in $\mathbf{E}$, the objects in $\mathbf{Pass}(\mathbf{E})$ are morphisms in $\mathbf{E}$ having N as their codomain. A map between two such objects $\epsilon_X : X \longrightarrow N$ and $\epsilon_Y : Y \longrightarrow N$ is a map $f : X \longrightarrow Y$ in $\mathbf{E}$ such that the following statement holds in the internal logic of $\mathbf{E}$:

> "There is a partial recursive function with code r such that for all $x \in X$: $r \cdot \epsilon_X(x) = \epsilon_Y(f(x))$."

If a natural number r has this property, we say that *r tracks f*. (Remember that the internal logic of $\mathbf{E}$ is rich enough to do recursion theory in, code partial recursive functions as natural numbers and define Kleene application $\cdot$ on pairs on natural numbers.) ∎

Definition 2.2 The *exact completion* of a category $\mathbf{C}$ with finite limits, denoted by $\mathbf{C}_{ex}$, is characterized, up to natural isomorphism, by the following properties: $\mathbf{C}_{ex}$ is exact and there is a finite limit preserving embedding $\mathbf{y} : \mathbf{C} \longrightarrow \mathbf{C}_{ex}$ such that any finite limit preserving functor from $\mathbf{C}$ to an exact category factors through $\mathbf{y}$. ∎

Definition 2.3 For any pretopos $\mathbf{E}$ with dependent products and W-types, the *effective pretopos* relative to $\mathbf{E}$, denoted by $\mathbf{Eff}(\mathbf{E})$, is the exact completion of $\mathbf{Pass}(\mathbf{E})$. ∎

Definition 2.4 In a pretopos $\mathbf{E}$ with dependent products, for any morphism $f : B \longrightarrow A$, the following functor can be defined:

$$P_f(X) = \sum_{a \in A} X^{f^{-1}(a)}$$

The *W-type* for f in $\mathbf{E}$ is exactly the free P_f-algebra, and therefore exists of an object in $\mathbf{E}$, usually denoted $\mathrm{W}(f)$, together with a P_f-algebra, usually denoted sup. ❑

The following two remarks need to be made concerning this definition. Firstly, if (X, σ_X) is another P_f-algebra, then there is a unique map $\phi : \mathrm{W}(f) \longrightarrow X$ such that

$$\phi(\sup_a(t)) = (\sigma_X)_a(\phi \circ t)$$

for any $a \in A$ and $t : f^{-1}(a) \longrightarrow \mathrm{W}(f)$. We might think of ϕ as defined "by recursion", for this equation specifies the value of ϕ on $\sup_a(t)$, assuming that ϕ has been specified on all values of t. Secondly, any subalgebra $R \subseteq \mathrm{W} = (\mathrm{W}(f), \sup)$ must be equal to W. This means that for any subobject $S \subseteq \mathrm{W}(f)$ for which the following statement in the internal logic holds:

$$\forall a \in A \, \forall t : f^{-1}(a) \longrightarrow \mathrm{W}(f) \, [\, (\forall b \in f^{-1}(a) : t(b) \in S\,) \rightarrow \sup_a(t) \in S\,]$$

the following statement holds as well:

$$\forall w \in \mathrm{W}(f) : w \in S$$

So an "induction principle" complements the possibility of defining maps by recursion.

We will end this section by introducing the notion of weak W-type. But before we can do so, we have to "weaken" the notion of a P_f-algebra. For that purpose, consider the following two functors for a lextensive category $\mathbf{C}$ and an object A in $\mathbf{C}$. Firstly, we have

$$U_A : \mathbf{C}/A \longrightarrow \mathbf{C}$$

defined on an object $p : X \longrightarrow A$ in $\mathbf{C}/A$ as the object X. And secondly, we have

$$A^* : \mathbf{C} \longrightarrow \mathbf{C}/A$$

defined on an object X in $\mathbf{C}$ by sending it to $\mathrm{proj}_A : X \times A \longrightarrow A$, considered as an object in $\mathbf{C}/A$.

We only introduce the latter functor to be able to explain the convention of dropping it in the notation. This means that when we regard an object X in $\mathbf{C}$ as an object in $\mathbf{C}/A$, we are actually talking about $A^*(X)$; and the same convention applies to morphisms.

Definition 2.5 Let X and Y be objects in $\mathbf{C}$. An object Z in $\mathbf{C}$ together with a weak evaluation map $\eta_Z : Z \times Y \longrightarrow X$ is called *a weak version of the exponential* X^Y, if for every map $g : U \times Y \longrightarrow X$ there exists a (not necessarily unique) map $h : U \longrightarrow Z$ such that $g = \eta_Z \circ (h \times Y)$. ❑

Definition 2.6 Let $f : B \longrightarrow A$ be a map in $\mathbf{C}$. A *weak P_f-algebra* is a quadruple $\mathbf{x} = (X, X^*, \sigma_X, \eta_X)$, where X is an object in $\mathbf{C}$ and X^* in $\mathbf{C}/A$, σ_X is a map $U_A X^* \longrightarrow X$ in $\mathbf{C}$ and η_X a map $X^* \times f \longrightarrow X$ in $\mathbf{C}/A$, in such a way that X^* is a weak version of X^f in $\mathbf{C}/A$ with η_X as weak evaluation.

A homomorphism $\mathbf{t}$ of weak P_f-algebra's from $\mathbf{x} = (X, X^*, \sigma_X, \eta_X)$ to $\mathbf{y} = (Y, Y^*, \sigma_Y, \eta_Y)$ consists of a pair of maps (t, t^*), $t : X \longrightarrow Y$ and $t^* : X^* \longrightarrow Y^*$, such that the following diagrams commute:

$$\begin{array}{ccc} UX^* & \xrightarrow{Ut^*} & UY^* \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ X & \xrightarrow[t]{} & Y \end{array} \qquad \begin{array}{ccc} X^* \times f & \xrightarrow{t^* \times f} & Y^* \times f \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ X & \xrightarrow[t]{} & Y \end{array}$$

This defines a category, which we shall denote by $WP_f(\mathbf{C})$. ∎

Remark that a weak P_f-algebra no longer is an algebra. To obtain the notion of a weak W-type, we have to weaken the recursive and inductive properties characteristic of a W-type. For the latter purpose, we need the following auxiliary notions.

Definition 2.7 A weak simple product of a map $c : C \longrightarrow I \times K$ with respect to K consists of a map $w : W \longrightarrow I$ and a map ϵ such that

$$\begin{array}{ccc} W \times K & \xrightarrow{\ \epsilon\ } & C \\ & {\scriptstyle w \times K}\searrow & \downarrow {\scriptstyle c} \\ & & I \times K \end{array} \qquad (1)$$

commutes. Moreover, the pair of w and ϵ is weakly universal with this property: in any situation

$$\begin{array}{ccc} X \times K & \xrightarrow{\ f\ } & C \\ & {\scriptstyle j \times K}\searrow & \downarrow {\scriptstyle c} \\ & & I \times K \end{array}$$

there is a (not necessarily unique) map $f' : X \longrightarrow W$ over I such that $f = \epsilon \circ (f' \times K)$. In this case, we call figure (1) a weak simple product diagram. ∎

Definition 2.8 A map $\mathbf{t} = (t, t^*) : \mathbf{x} \longrightarrow \mathbf{y}$ of weak P_f-algebra's is said to be a weak P_f-subalgebra of $\mathbf{y}$, if for the pullback L in the following diagram in $\mathbf{C}/A$:

$$\begin{array}{ccc} L & \xrightarrow{\ p_1\ } & Y^* \times f \\ {\scriptstyle p_2}\downarrow & & \downarrow {\scriptstyle \eta_Y} \\ X & \xrightarrow[\ t\ ]{} & Y \end{array}$$

the following diagram is a weak simple product diagram:

$$\begin{array}{ccc} X^* \times f & \xrightarrow{\ \alpha_X\ } & L \\ & {\scriptstyle t^* \times f}\searrow & \downarrow {\scriptstyle p_1} \\ & & Y^* \times f \end{array}$$

(with $\alpha_X = \langle t^* \times f, \eta_X \rangle$). ∎

Definition 2.9 A weak W-type $\mathbf{w}$ for a map f is weak P_f-algebra that is (i) weakly initial in $WP_f(\mathbf{C})$ and (ii) is such that every weak P_f-subalgebra $\mathbf{t} : \mathbf{x} \longrightarrow \mathbf{w}$ has a section in $WP_f(\mathbf{C})$. ∎

3 Paths

In this section, we will work towards proving theorem (1.1):

Theorem 3.1 (= Theorem 1.1) *If* $\mathbf{C}$ *is a lextensive category with weak dependent products, a weak natural number object and weak W-types, then* $\mathbf{C}_{ex}$ *is a pretopos with dependent products and W-types.*

We will do so by proving that it suffices to show that $\mathbf{C}_{ex}$ has W-types for all maps lying in the image of $\mathbf{y}$. This is what the proposition below will do for us.

Definition 3.2 A square

$$\begin{array}{ccc} D & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \end{array}$$

in some good category $\mathbf{C}$ is called a *quasi-pullback*, if the map $D \longrightarrow B \times_A C$ is epi. □

Proposition 3.3 *Suppose in a pretopos* $\mathbf{E}$ *with dependent products and a natural number object, we have a diagram of the following form:*

$$\begin{array}{ccc} B' & \xrightarrow{[-]_B} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ A' & \xrightarrow[{[-]_A}]{} & A \end{array} \qquad (2)$$

Suppose furthermore that this diagram is a quasi-pullback and that f' *is a choice map for which there exists a W-type* $W(f')$*. Then there also exists a W-type for* f*.*

Proof : Let $(W', \sup_{W'})$ be the W-type for f'. As is explained in [6], elements w' of W' can be thought of as *trees*. A path in such a tree w' could, from that perspective, be defined as a finite sequence of odd length σ such that: (i) $\sigma(0) = w'$; (ii) for all even $n < \text{length}(\sigma)$ we have that $\sigma(n) \in W'$ and for all odd $n < \text{length}(\sigma)$ we have that $\sigma(n) \in B'$; and (iii) if $\sigma(n) = (\sup_{W'})_{a'}(t')$ for a certain even $n < \text{length}(\sigma)$ and $b' = \sigma(n+1)$, then $f'(b') = a'$ and $t'(b') = \sigma(n+2)$. Notice that we do not demand that paths are complete, i.e., a path may stop even it is possible to climb further upwards in the tree.

As it turns out, this definition makes sense in the internal logic of $\mathbf{E}$, for we can define the subobjects Paths of $(W' + B' + 1)^N$ as consisting of those $\sigma \in (W' + B' + 1)^N$ such that:

- **(1)** $\sigma(0)$ is an element of W'.
- **(2)** If $\sigma(n) \in W'$ for a certain natural number n, then either $\sigma(n+1) \in B'$ or $\sigma(n+1) = *$.
- **(3)** If $\sigma(n) \in W'$ and $\sigma(n+1) \in B'$ for a certain natural number n, and, more specifically, $\sigma(n) = (\sup_{W'})_{a'}(t')$ for certain $a' \in A'$, $t' : (f')^{-1}(a') \longrightarrow W'$, then $f'(\sigma(n+1)) = a'$ and $\sigma(n+2) = t'(\sigma(n+1))$.
- **(4)** If $\sigma(n) = *$ for a certain natural number n, then for all natural numbers $k > n$ also $\sigma(k) = *$.

(Here $*$ denotes the single element of 1.). It is unnecessary to add that for all $\sigma \in \text{Paths}$ there is a natural number $k \in N$ such that $\sigma(k) = *$, for that will be a consequence of the inductive property of W-types.

In the sequel, we will use abbreviating symbolism, like the standard notation for finite sequences and ways of manipulating them (in particular, $*$ for concatenation), in writing down statements in the internal logic. We trust that the reader is ingenious enough to translate these statements in their unabbreviated form, if necessary.

Let furthermore $\text{Paths}_{w'}$ for a certain element $w' \in W'$ be defined as the fiber above w' of the map $l : \text{Paths} \longrightarrow W' : \sigma \mapsto \sigma(0)$, and let ρ be the canonical map from W' to A' via $\sum_{a' \in A'} W'^{(f')^{-1}(a)}$.

We define the following binary relation $\sim$ on W':

$w \sim w' \quad \Leftrightarrow \quad$ For all $\sigma \in \text{Paths}_w$, $\sigma' \in \text{Paths}_{w'}$ and natural numbers n: if $\text{length}(\sigma) = \text{length}(\sigma') = 2n+1$ and $[\sigma(2k+1)]_B = [\sigma'(2k+1)]_B$ for all $k < n$, then $[\rho(\sigma(2n))]_A = [\rho(\sigma(2n))]_A$.

Observe that $\sim$ has the following properties:

(i) $w \sim w' \Rightarrow w' \sim w$

(ii) $w \sim w' \Rightarrow [\rho(w)]_A = [\rho(w')]_A$

(iii) $w \sim w' \Rightarrow$ For every $\sigma \in \text{Paths}_w$ we can find a $\sigma' \in \text{Paths}_{w'}$ having the same length as σ, say $2n+1$, such that for all $k < n$ we have that $[\sigma(2k+1)]_B = [\sigma'(2k+1)]_B$, and for all $k \leq n$ we have that $[\rho(\sigma(2k))]_A = [\rho(\sigma(2k))]_A$.

(iv) $w \sim w', w' \sim w'' \Rightarrow w \sim w''$

The proofs of claims (i) and (ii) are trivial.

(iii) is less easy: suppose $w \sim w'$ and fix a $\sigma \in \text{Paths}_w$. Let $\text{length}(\sigma) = 2n+1$. We now prove with induction that:

$\forall k \leq n$: There is a $\sigma' \in \text{Paths}_{w'}$, having length $2k+1$, such that both $[\sigma(2j+1)]_B = [\sigma'(2j+1)]_B$ for all $j < k$, and $[\rho(\sigma(2j))]_A = [\rho(\sigma'(2j))]_A$ for all $j \leq k$.

For k = 0, there is $\sigma' = \langle w' \rangle$.

Suppose we have constructed a σ'' with length $2k+1$ having the desired property ($k < n$). Pick a b' such that

$$b' \in (f')^{-1}(\rho(\sigma''(2k))) \text{ and } [b']_B = [\sigma(2k+1)]_B$$

(it exists, since 3.2 is a quasi-pullback). If we set $w'' := (\text{sup}_{W'})^{-1}(\sigma''(2k))(b')$, then

$$\sigma' = \sigma'' * \langle b', w'' \rangle \in \text{Paths}_{w'}$$

has the desired property for $k+1$. (We have $[\rho(\sigma(2(k+1)))]_A = [\rho(\sigma'(2(k+1)))]_A$, since $w \sim w'$.)

(iv) follows easily from (iii) and the definition.

We do not have, in general, that $w \sim w$. Set

$$S \quad = \quad \{ w \in W' \mid w \sim w \}$$

Or, equivalently:

$w \in S \quad \Leftrightarrow \quad$ For all $\sigma, \sigma' \in \text{Paths}_w$ and natural numbers n: if $\text{length}(\sigma) = \text{length}(\sigma') = 2n+1$ and $[\sigma(2k+1)]_B = [\sigma'(2k+1)]_B$ for all $k < n$, then $[\rho(\sigma(2n))]_A = [\rho(\sigma(2n))]_A$.

$\sim$ is now an equivalence relation on S, so we can form the quotient $q : S \longrightarrow W$. Observe that we have that

$$w \in S, \sigma \in \mathrm{Paths}_w, 2n+1 = \mathrm{length}(\sigma) \Rightarrow \sigma(2n) \in S$$

Let us also set ($a' \in A'$):

$$S^*_{a'} \quad = \quad \{\, \tau \in (W')^{(f')^{-1}(a')} \mid (\sup_{W'})_{a'}(\tau) \in S \,\}$$

Or, equivalently:

$$\tau \in S^*_{a'} \quad \Leftrightarrow \quad \text{We have for all } b'_0, b'_1 \in (f')^{-1}(a') \text{ that } [b'_0]_B = [b'_1]_B \text{ implies that } \tau(b'_0) \sim \tau(b'_1).$$

Note that it follows from $\tau \in S^*_{a'}$ and $b' \in (f')^{-1}(a')$, that $\tau(b') \in S$.

We clearly have a map $\sigma_S : U_{A'}S^* \longrightarrow S$ in $\mathbf{E}$ making

$$\begin{array}{ccc} U_{A'}S^* & \rightarrowtail & U_{A'}(W')^f \\ {\scriptstyle \sigma_S}\downarrow & & \downarrow{\scriptstyle \sup_{W'}} \\ S & \rightarrowtail & W' \end{array}$$

commute. We will now construct a commuting diagram of the following form:

$$\begin{array}{ccc} U_{A'}S^* & \overset{q^*}{\twoheadrightarrow} & U_A W^f \\ {\scriptstyle \sigma_S}\downarrow & & \downarrow{\scriptstyle \sigma_W} \\ S & \underset{q}{\twoheadrightarrow} & W \end{array} \qquad (3)$$

In pretopoi, there exists a general technique for constructing morphisms. If we want to construct a morphism $g : D \longrightarrow C$ in a Heyting pretopos, we can do this by constructing a subobject $L \subseteq D \times C$ for which the following two statements hold in the internal logic:

(i) $\forall d \in D\, \forall c, c' \in C\, [\, (d,c) \in L \wedge (d,c') \in L \rightarrow c = c' \,]$
(ii) $\forall d \in D\, \exists c \in C\, [\, (d,c) \in L \,]$

In this case we call the subobject L *functional.* It then follows using the axioms of a Heyting pretopos that there exists a map $g : D \longrightarrow C$ having the property that L is the graph of g.

In this way, we construct a morphism $q^* : S^* \longrightarrow W^f$ in $\mathbf{E}$ by noting that the subobject

$$Q \quad = \quad \{\, (g,h) \in US^* \times UW^f \mid Q(g,h) \,\}$$

where $Q(g,h)$ is the statement:

> If $g \in S^*_{a'}$ and $h \in W^{f^{-1}(a)}$ for certain $a' \in A'$ and $a \in A$, then $[a']_A = a$ and for all $b' \in (f')^{-1}(a') : q(g(b')) = h([b']_B)$.

is functional. (This is not hard to see.) The map q^* so constructed is epi: for let h be an arbitrary element of $f^{-1}(a) \longrightarrow W$ for a certain $a \in A$. Pick an $a' \in A'$ such that $[a']_A = a$. We have

$$\forall b' \in (f')^{-1}(a')\, \exists s \in S : \; q(s) = h([b']_B)$$

since q is epi. Since f' is a choice map, we have a map $g : (f')^{-1}(a') \longrightarrow S$ such that $q(g(b')) = h([b']_B)$ for all $b' \in (f')^{-1}(a')$. If $b'_0, b'_1 \in (f')^{-1}(a')$ are such that $[b'_0]_B = [b'_1]_B$, then

$$q(g(b'_0)) = h([b'_0]_B) = h([b'_1]_B) = q(g(b'_1))$$

so $g(b'_0) \sim g(b'_1)$. This means that $g \in S^*_{a'}$ and hence that $(g, h) \in Q$. Since h was arbitrary, this means that q^* is epi.

We now construct $\sigma_W : U_A W^f \longrightarrow W$ in **E** by showing that

$$\Sigma_W \quad = \quad \{\, (h, w) \in UW^f \times W \mid \Sigma_W(h, w) \,\}$$

is functional. Here $\Sigma_W(h, w)$ is the statement "There is a $g \in US^*$ such that $q^*(g) = h$ and $q\sigma_S(g) = w$." That

$$\forall h \in UW^f \, \exists w \in W : (h, w) \in \Sigma_W$$

follows easily from the fact that q^* is epi. Let us now show that

$$\forall g_0, g_1 \in S^* : q^*(g_0) = q^*(g_1) \Rightarrow q\sigma_S(g_0) = q\sigma_S(g_1)$$

From this it follows that

$$\forall h \in W^f \, \forall w, w' \in W : (h, w) \in \Sigma_W, (h, w') \in \Sigma_W \Rightarrow w = w'$$

and that (3.3) commutes.

Let for certain $a'_0, a'_1 \in A'$ elements $g_0 : (f')^{-1}(a'_0) \longrightarrow W' \in S^*$ and $g_1 : (f')^{-1}(a'_1) \longrightarrow W' \in S^*$ be given such that $q^*(g_0) = q^*(g_1)$. This implies that $[a'_0]_A = [a'_1]_A$ and that

$$\forall b'_0 \in f^{-1}(a'_0), b'_1 \in f^{-1}(a'_1) : [b'_0]_B = [b'_1]_B \Rightarrow g_0(b'_0) \sim g_1(b'_1)$$

From this it follows that $\sigma_S(g_0) \sim \sigma_S(g_1)$, as the reader can check for himself. So Σ_W is functional, and (3.3) commutes for the map σ_W just constructed.

I now claim that the P_f-algebra $\mathbf{w} = (W, \sigma_W : \sum_{a \in A} W^{f^{-1}(a)} \longrightarrow W)$ is actually the W-type for f. For let

$$\mathbf{x} = (X, \sigma_X : \sum_{a \in A} X^{f^{-1}(a)} \longrightarrow X)$$

be a P_f-algebra. We introduce the following notation: we write $\sigma \sim \tau$ for $\sigma, \tau \in$ Paths if and only if for some $n \in N$: length$(\sigma) =$ length$(\tau) = 2n + 1$ and $\sigma(2n) \sim \tau(2n)$. And we define the following subset L of $S \times X$:

$$L \quad = \quad \{\, (s, x) \in S \times X \mid \text{There exists } g : \text{Paths}_s \longrightarrow X \text{ that is a witness for } (s, x). \,\}$$

And a map $g : \text{Paths}_s \longrightarrow X$ is a witness for (s, x) if it satisfies the following three demands:

- **(i)** If $\sigma, \tau \in \text{Paths}_s$ are such that $\sigma \sim \tau$, then $g(\sigma) = g(\tau)$.
- **(ii)** For all $\sigma \in \text{Paths}_s$ and natural numbers n with the properties that $\sigma(n) = (\sigma_S)_{a'}(t')$ for certain $a' \in A, t' \in S^*_{a'}$, that the length of σ equals $n + 1$, and that

  $$m([b']_B) = g(\sigma * \langle b', t'b' \rangle) \qquad (b' \in (f')^{-1}(a'))$$

  defines a map $f^{-1}(a) \longrightarrow X$, we have that $g(\sigma) = (\sigma_X)(m)$.
- **(iii)** $g(\langle \sigma \rangle) = x$.

(It should be noted that the function m mentioned in (ii) indeed defines a function $f^{-1}(a) \longrightarrow X$, if (i) is satisfied. For if $b'_0, b'_1 \in (f')^{-1}(a')$ are such that $[b'_0]_B = [b'_1]_B$, then $t'b'_0 \sim t'b'_1$ since $t' \in S^*_{a'}$. So $g(\sigma * \langle b'_0, t'b'_0\rangle) = g(\sigma * \langle b'_1, t'b'_1\rangle)$ according to (i).)

Lemma 3.4 *Suppose for elements $s, s' \in S$ we have functions $g : Paths_s \longrightarrow X$ and $h : Paths_{s'} \longrightarrow X$ satisfying conditions (i) and (ii) from the definition of a witness. If $\sigma \sim \tau$ for elements $\sigma \in Paths_s$ and $\tau \in Paths_{s'}$, then $g(\sigma) = h(\tau)$.*

Proof : The proof of this lemma uses the fact that:

Lemma 3.5 *If R is a subobject of S satisfying*

$$\forall a' \in A' \, \forall t' \in S^*_{a'} \, [\forall b' \in (f')^{-1}(a') : t'b' \in R \Rightarrow (\sigma_S)t' \in R]$$

then $R = S$ as subobjects of S.

Proof : Let $K = \{ w \in W' \,|\, w \in S \rightarrow w \in R\}$. It is easy to prove, using the inductive property of W-types, that $K = W'$ and hence $R = S$ as subobjects of S. ∎

Let

$$M \quad = \quad \{ s_0 \in S \,|\, M(s_0) \}$$

where $M(s_0)$ is the following condition:

> For all $s_a \in S$ and $\sigma \in \text{Paths}_{s_a}$: if length$(\sigma) = n+1$ and $\sigma(n) = s_0$, then for all $s_b \in S$ and $\tau \in \text{Paths}_{s_b}$ such that $\sigma \sim \tau$ and for all $g : \text{Paths}_{s_a} \longrightarrow X$ and $h : \text{Paths}_{s_b} \longrightarrow X$ having properties (i) and (ii): $g(\sigma) = h(\tau)$.

We prove that $M = S$ as subobjects of S, using the previous lemma. Then the desired result follows immediately.

Let $a'_0 \in A$ and $t'_0 : (f')^{-1}(a'_0) \longrightarrow W' \in S^*$ be such that for all $b'_0 \in (f')^{-1}(a'_0)$ we have that $t'b'_0 \in M$. I want to show that $s_0 = (\sigma_S)t'_0 \in M$, so take arbitrary $s_a, s_b \in S$ and $\sigma \in \text{Paths}_{s_a}, n \in N$ with length$(\sigma) = n+1$ and $\sigma(n) = s_0$ and $\tau \in \text{Paths}_{s_b}$ such that $\sigma \sim \tau$ and $g : \text{Paths}_{s_a} \longrightarrow X$ and $h : \text{Paths}_{s_b} \longrightarrow X$ satisfying conditions (i) and (ii).

Let $s_1 = \tau(n)$. Since $\sigma_{W'}$ is iso, s_1 is of the form $(\sigma_{W'})_{a'_1}(t'_1)$. The fact that $a'_0 \sim a'_1$ implies that $[a'_0]_A = [a'_1]_A$ and that

$$\forall b'_0 \in (f')^{-1}(a'_0), \, b'_1 \in (f')^{-1}(a'_1) : [b'_0]_B = [b'_1]_B \Rightarrow t'_0 b'_0 \sim t'_1 b'_1$$

So for $b'_0 \in (f')^{-1}(a'_0)$, $b'_1 \in (f')^{-1}(a'_1)$ such that $[b'_0]_B = [b'_1]_B$, we have that

$$\sigma * \langle b'_0, t'_0 b'_0\rangle \sim \tau * \langle b'_1, t'_1 b'_1\rangle$$

and hence $g(\sigma * \langle b'_0, t'_0 b'_0) = h(\tau * \langle b'_1, t'_1 b'_1\rangle)$. Using the fact that both g and h satisfy (ii), this implies that $g(\sigma) = h(\tau)$. Since all the choices made were arbitrary, this shows that $s_0 \in M$.

We conclude that $M = S$ and that the lemma holds. ∎

This lemma has the following easy consequence:

Lemma 3.6 *If $(s, x) \in L$ and $(s', x') \in L$ and $s \sim s'$, then $x = x'$. If $(s, x) \in L$, the function $g : Paths_s \longrightarrow X$ "witnessing" this fact is unique.*

Lemma 3.7 *For every $s \in S$ there exists a (unique) $x \in X$ such that $(s, x) \in L$.*

Proof : Let $M = \{ s \in S \mid \exists x \in X : (s, x) \in L \}$. We prove, using lemma (3.5), that $M = S$ as subobjects of S. The desired results follows immediately from this.

Let $a' \in A'$ and $t' \in S^*_{a'}$ be such that for all $b' \in (f')^{-1}(a')$ we have that $t'b' \in M$. This means that there are unique $x_{b'}$ such that $(t'b', x_{b'}) \in L$ and unique $g_{b'} : \text{Paths}_{t'b'} \longrightarrow X$ witnessing this fact. Let

$$m([b']_B) = x_{b'} \qquad (b' \in (f')^{-1}(a'))$$

This is a valid definition of a map $f^{-1}(a) \longrightarrow X$ ($a = [a']_A$): for if $b'_0, b'_1 \in (f')^{-1}(a')$ are such that $[b'_0]_B = [b'_1]_B$, then $t'b'_0 \sim t'b'_1$, since $t' \in S^*_{a'}$, and hence $x_{b'_0} = x_{b'_1}$ (see lemma (3.6)).

Set $s = (\sigma_S)_{a'}(t')$ and $x = (\sigma_X)_a(m)$. Define $g : \text{Paths}_s \longrightarrow X$ as:

$$\begin{array}{rcll} g(\langle s \rangle) & = & x & \\ g(\langle s, b' \rangle * \sigma) & = & g_{b'}(\sigma) & \qquad (b' \in (f')^{-1}(a')) \end{array}$$

This g witnesses the fact that $(s, x) \in L$. So $s = (\sigma_S)(t') \in M$. ∎

This last lemma implies that we have a map $l : S \longrightarrow X$ having L as its graph. If we define $l^* : U_{A'}S^* \longrightarrow U_A X^f$ in **E** by

$$l^*(t')([b']_B) = l(t'(b'))$$

for all $a' \in A', t' \in S^*_{a'}, b' \in (f')^{-1}(a')$, then the diagram in **E**

$$\begin{array}{ccc} U_{A'}S^* & \xrightarrow{l^*} & U_A X^f \\ \sigma_S \downarrow & & \downarrow \sigma_X \\ S & \xrightarrow[l]{} & X \end{array}$$

commutes. In fact, it factors as:

$$\begin{array}{ccccc} US^* & \xrightarrow{q^*} & UW^f & \xrightarrow{U\xi^*} & UX^f \\ \sigma_S \downarrow & & \downarrow \sigma_W & & \downarrow \sigma_X \\ S & \xrightarrow[q]{} & W & \xrightarrow[\xi]{} & X \end{array}$$

(Both these assertions are not hard to verify.)

It remains to show the uniqueness of ξ. It will follow from the following claim: if we have another commuting diagram in **E** of the form

$$\begin{array}{ccc} US^* & \xrightarrow{m^*} & UX^f \\ \sigma_S \downarrow & & \downarrow \sigma_X \\ S & \xrightarrow[m]{} & X \end{array}$$

and m^* is such that for all $a' \in A', t' \in S^*_{a'}$ and $b' \in (f')^{-1}(a')$ we have that $m^*(t')([b']_B) = m(t'b)$, then $s_0 \sim s_1$ ($s_0, s_1 \in S$) implies that $l(s_0) = m(s_1)$.

So let $R = \{ s_0 \in S \mid \forall s_1 \in S : s_0 \sim s_1 \Rightarrow l(s_0) = m(s_1) \}$. We show, using lemma (3.5), that $R = S$ as subobjects of S. This will immediately yield the desired result.

Let $s_0 = (\sigma_S)_{a'_0}(t'_0)$ for certain $a'_0 \in A$ and $t'_0 : (f')^{-1}(a'_0) \longrightarrow W' \in S^*$, and suppose that for all $b'_0 \in (f')^{-1}(a'_0)$ we have that $t'_0 b'_0 \in R$. Let s_1 be an arbitrary element of S such that $s_0 \sim s_1$. We know that s_1 is of the form $(\sigma_S)_{a'_1}(t'_1)$ for certain $a'_1 \in A', t'_1 \in S^*_{a'}$. From $s_0 \sim s_1$ we deduce that $[a'_0]_A = [a'_1]_A$ and that

$$\forall b'_0 \in (f')^{-1}(a'_0),\, b'_1 \in (f')^{-1}(a'_1) : [b'_0]_B = [b'_1]_B \Rightarrow t'_0 b'_0 \sim t'_1 b'_1$$

So for all $b'_0 \in (f')^{-1}(a'_0)$, $b'_1 \in (f')^{-1}(a'_1)$ such that $[b'_0]_B = [b'_1]_B$, we have that $l(t'_0 b'_0) = m(t'_1 b'_0)$. This implies that the functions $w_0, w_1 : f^{-1}(a) \longrightarrow X$ $(a = [a'_0]_A = [a'_1]_A)$ defined by:

$$\begin{array}{rcl} w_0([b_0]_B) & = & l(t'_0 b'_0) \\ w_1([b_1]_B) & = & m(t'_1 b'_0) \end{array}$$

are equal. So we conclude that:

$$\begin{array}{rcl} l(s_0) & = & l((\sigma_S)t'_0) \\ & = & \sigma_X(l^* t'_0) \\ & = & \sigma_X(w_0) \\ & = & \sigma_X(w_1) \\ & = & \sigma_X(m^* t'_1) \\ & = & m((\sigma_S)t'_1) \\ & = & m(s_1) \end{array}$$

Wrapping up, we see that ξ is indeed unique. This completes the proof of the fact that **w** is the W-type for f in **C**. □

4 Constructing W-types in the exact completion

In this section we will complete our proof of the first main result of this paper (theorem (1.1)). In view of the reduction effected in the previous section, it will suffice to show:

Theorem 4.1 *Suppose* **C** *is a lextensive category with weak dependent products and a weak natural number object. If* **C** *has a weak W-type for a map* f *in* **C**, *then* $\mathbf{C}_{ex}$ *has a strong W-type for the map* $\mathbf{y}f$.

Having the following auxiliary notions at hand, will facilitate the exposition of its proof.

Definition 4.2 For an arbitrary map $f : B \longrightarrow A$, we define the notion of a P_f*-structure* in **C**. A P_f-structure is a quadruple $\mathbf{x} = (X, X^*, \sigma_X, \eta_X)$ with X an object in **C**, X^* an object in $\mathbf{C}/A$, σ_X a map $U_A(X^*) \longrightarrow X$ in **C** and η_X a map $X^* \times f \longrightarrow X$ in $\mathbf{C}/A$. A homomorphism of P_f-structures from $\mathbf{x} = (X, X^*, \sigma_X, \eta_X)$ to $\mathbf{y} = (Y, Y^*, \sigma_Y, \eta_Y)$ is a pair $\mathbf{t} = (t, t^*)$, where t is a map in **C** from X to Y, and t^* is a map from X^* to Y^* in $\mathbf{C}/A$. Furthermore, the following diagrams should commute:

$$\begin{array}{ccc} UX^* & \xrightarrow{Ut^*} & UY^* \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ X & \xrightarrow[t]{} & Y \end{array} \qquad \begin{array}{ccc} X^* \times f & \xrightarrow{t^* \times f} & Y^* \times f \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ X & \xrightarrow[t]{} & Y \end{array}$$

It is easy to see that this defines a category, one we shall call $P_f(\mathbf{C})$. □

The category of weak P_f-algebra's is a full subcategory of the category of P_f-structures. Therefore morphisms in $WP_f(\mathbf{C})$ can be considered as morphisms in $P_f(\mathbf{C})$, and if this is convenient, we will do so. And since (strong, i.e. ordinary) P_f-algebra's can be regarded as weak P_f-algebra's, they can also be considered as P_f-structures.

Definition 4.3 A map $\mathbf{t} : \mathbf{x} \longrightarrow \mathbf{y}$ in $P_f(\mathbf{C})$ is said to be a *weak P_f-substructure map*, if for the pullback L in this diagram in $\mathbf{C}/A$:

$$\begin{array}{ccc} L & \xrightarrow{p_1} & Y^* \times f \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ X & \xrightarrow[t]{} & Y \end{array}$$

the following diagram is a weak simple product diagram:

$$\begin{array}{ccc} X^* \times f & \xrightarrow{\alpha_X} & L \\ & {\scriptstyle t^* \times f}\searrow & \downarrow{\scriptstyle p_1} \\ & & Y^* \times f \end{array}$$

Here $\alpha_X = \langle t^* \times f, \eta_X \rangle$. □

Concerning these notions, we have the following two lemma. Their proofs are easy and omitted.

Lemma 4.4 *If $\mathbf{t} : \mathbf{x} \longrightarrow \mathbf{y}$ is a weak P_f-substructure map and $\mathbf{y}$ is a weak P_f-subalgebra, then so is $\mathbf{x}$.*

Lemma 4.5 *If $\mathbf{w} = (W, W^*, \sigma_W, \eta_W)$ is a weak W-type for f in a good category $\mathbf{C}$ with weak dependent products, then σ_W has a section.*

The proof of theorem (4.1) will proceed in two steps. First we will show that if $\mathbf{w} = (W, W^*, \sigma_W, \eta_W)$ is a weak W-type in $\mathbf{C}$ for a map $f : B \longrightarrow A$, then $\mathbf{yw} = (\mathbf{y}W, \mathbf{y}W^*, \mathbf{y}\sigma_W, \mathbf{y}\eta_W)$ has the following properties (in the remainder of this section, we will drop the occurences of $\mathbf{y}$; we trust that the reader will not get confused):

(i) The canonical map $q_1^* : W^* \longrightarrow W^f$ fitting into the diagram

$$\begin{array}{ccc} W^* \times f & \xrightarrow{q_1^* \times f} & W^f \times f \\ {\scriptstyle \eta_W}\searrow & & \swarrow{\scriptstyle \mathrm{ev}} \\ & W & \end{array}$$

is epi.

(ii) For any subobject $R \subseteq W$: if it holds for every $a \in A$ and $\tau \in W_a^*$ that

$$(\forall b \in f^{-1}(a) : \mathrm{proj}_W(\eta_W(\tau, b)) \in R) \rightarrow \sigma_W(\tau) \in R$$

then $R = W$ as subobjects of W.

(iii) If $\mathbf{k} = (K, \sigma_K : \sum_{a \in A} K^{f^{-1}(a)} \longrightarrow K)$ is a P_f-type, then there exists a (not necessarily unique) P_f-structure map $\mathbf{t} : \mathbf{w} \longrightarrow \mathbf{k}$.

(iv) σ_W has a section s.

As our second step, we will show that if we have a P_f-structure in a pretopos $\mathbf{E}$ with dependent products and a natural number object for a choice map $f : B \longrightarrow A$

$$\mathbf{w} = (W, W^*, \sigma_W, \eta_W)$$

having the above four properties, then there is a (strong, i.e. ordinary) W-type for f. Taken together, the two steps will establish the proposition.

The following sequence of lemma's will suffice for taking our first step. So suppose $\mathbf{C}$ is a lextensive category with weak dependent products and a weak natural number object, and suppose that $\mathbf{w} = (W, W^*, \sigma_W, \eta_W)$ is a weak W-type for a map $f : B \longrightarrow A$ in $\mathbf{C}$.

In these lemma's, the following facts, familiar from the theory of exact completions, will repeatedly be used. In the exact completion, the objects in the image of $\mathbf{y}$ are, up to iso, the *projectives* of this category. And every object in the exact completion can be covered with a regular epi by such a projective. (See, for instance, [5].)

Lemma 4.6 *The canonical map $q_1^* : W^* \longrightarrow W^f$ in $\mathbf{C}_{ex}/A$ is epi. (The fact that q_1^* is canonical means that the diagram*

$$\begin{array}{ccc} W^* \times f & \xrightarrow{q_1^* \times f} & W^f \times f \\ & {\scriptstyle \eta_W} \searrow \quad \swarrow {\scriptstyle ev} & \\ & W & \end{array}$$

commutes.)

Proof : Since $\mathbf{w}$ was a weak P_f-algebra, we know that W^* was a weak version of W^f in $\mathbf{C}/A$. We can now define the equivalence relation

$$R_a = \{ (g, h) \in W_a^* \times W_a^* \mid \forall b \in f^{-1}(a) : \eta_W(g, b) = \eta_W(h, b) \}$$

on W_a^* ($a \in A$) in $\mathbf{C}_{ex}/A$. It is not difficult to see that the quotient W^*/R in $\mathbf{C}_{ex}/A$ is a strong version of W^f. So $W^*/R \cong W^f$ and it follows that $q_1^* : W^* \longrightarrow (A^*W)^f$ is epi. ∎

Lemma 4.7 *For any weak P_f-algebra $\mathbf{k} = (K, K^*, \sigma_K, \eta_K)$ in $\mathbf{C}_{ex}$ there is a P_f-structure map $\mathbf{t} : \mathbf{w} \longrightarrow \mathbf{k}$.*

Proof : Let $r : R \twoheadrightarrow K$ be a regular cover by an object R in the image of $\mathbf{y}$. Form the exponential R^f in $\mathbf{C}_{ex}/A$. Since K^* is a weak version of K^f in $\mathbf{C}_{ex}/A$, we have a map $r_2^* : R^f \longrightarrow K^*$ making

$$\begin{array}{ccc} R^f \times f & \xrightarrow{r_2^* \times f} & K^* \times f \\ {\scriptstyle \mathrm{ev}_R} \downarrow & & \downarrow {\scriptstyle \eta_K} \\ R & \underset{r}{\twoheadrightarrow} & K \end{array}$$

commute. Let $r_1^* : R^* \twoheadrightarrow R^f$ be a regular cover by an object in the image of $\mathbf{y}$. Now R^* can be seen as a weak version of R^f in $\mathbf{C}/A$, with $\eta_R = \mathrm{ev}_R \circ r_1^*$.

Since $U_A(R^*)$ is projective, we have a morphism σ_R in $\mathbf{C}_{ex}$ such that

$$\begin{array}{ccc} UR^*) & \xrightarrow{Ur^*} & UK^* \\ {\scriptstyle \sigma_R} \downarrow & & \downarrow {\scriptstyle \sigma_K} \\ R & \underset{r}{\twoheadrightarrow} & K \end{array}$$

commutes ($r^* = r_2^* \circ r_1^*$). Now $\mathbf{r} = (R, R^*, \sigma_R, \eta_R)$ can be seen as a weak P_f-algebra in $\mathbf{C}$, and $\mathbf{r} = (r, r^*)$ is a P_f-structure map in $\mathbf{C}_{ex}$. But the former implies that there is a P_f-structure map $\xi : \mathbf{w} \longrightarrow \mathbf{r}$. So $\mathbf{t} := \mathbf{r} \circ \xi$ is a P_f-structure map from $\mathbf{w}$ to $\mathbf{k}$. ▯

Lemma 4.8 *If* $\mathbf{r} = (R, R^*, \sigma_R, \eta_R)$ *is a* P_f*-structure in* $\mathbf{C}_{ex}$ *and* $\mathbf{t} : \mathbf{r} \longrightarrow \mathbf{w}$ *is a weak* P_f*-substructure map, then* $\mathbf{t}$ *has a section in* $P_f(\mathbf{C}_{ex})$.

Proof : Since $\mathbf{t}$ is a weak P_f-substructure map, we know that if we form in $\mathbf{C}_{ex}/A$ the pullback

$$\begin{array}{ccc} L & \xrightarrow{p_1} & W^* \times f \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \eta_W} \\ K & \xrightarrow[t]{} & W \end{array}$$

then the following is a weak simple product diagram:

$$\begin{array}{ccc} R^* \times f & \xrightarrow{\alpha_R} & L \\ & {\scriptstyle t^* \times f}\searrow & \downarrow{\scriptstyle p_1} \\ & & W^* \times f \end{array}$$

(with $\alpha_R = \langle (t^* \times f), \eta_R \rangle\rangle$). Let $\xi : K \longrightarrow\!\!\!\!\rightarrow R$ be a regular cover by an object in the image of $\mathbf{y}$. Now consider the following diagram:

$$\begin{array}{ccccc} L' & \xrightarrow{l_1} & L & \xrightarrow{p_1} & W^* \times f \\ {\scriptstyle l_2}\downarrow & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \eta_W} \\ K & \xrightarrow[\xi]{} & R & \xrightarrow[t]{} & W \end{array}$$

Since the objects K, W and $W^* \times f$ lie in the image of $\mathbf{y}$, and since this functor preserves pullbacks, we may assume that L' also lies in the image of $\mathbf{y}$.

Construct the following pullback:

$$\begin{array}{ccc} L'' & \xrightarrow{j_2} & L' \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle l_1} \\ R^* \times f & \xrightarrow[\alpha_R]{} & L \end{array}$$

And construct the strong version of $\Pi_{\pi_1}(j_1)$ in $\mathbf{C}_{ex}/A$ (where π_1 is the projection $R^* \times f \longrightarrow R^*$). This means that we have an object K_0^* with maps $\xi_2^* : K_0^* \longrightarrow R^*$ and $\alpha_{K_0} : K_0^* \times f \longrightarrow L''$ such that

$$\begin{array}{ccc} K_0^* \times f & \xrightarrow{\alpha_{K_0}} & L'' \\ & {\scriptstyle \xi_2^* \times f}\searrow & \downarrow{\scriptstyle j_1} \\ & & R^* \times f \end{array}$$

is a strong simple product diagram (the notion of a strong simple product is defined in the obvious way).

It is not hard to verify that

$$\begin{array}{ccc} K_0^* \times f & \xrightarrow{j_2 \circ \alpha_{K_0}} & L' \\ & \searrow^{(t^* \circ \xi_2^*) \times f} & \downarrow p_1 \circ l_1 \\ & & W^* \times f \end{array}$$

is a weak simple product diagram. Now let $\xi_1^* : K^* \twoheadrightarrow K_0^*$ be a regular cover by an element in the image of $\mathbf{y}$. This implies that

$$\begin{array}{ccc} K^* \times f & \xrightarrow{\alpha_K} & L' \\ & \searrow^{(t^* \circ \xi^*) \times f} & \downarrow p_1 \circ l_1 \\ & & W^* \times f \end{array} \qquad (4)$$

with $\alpha_K = j_2 \circ \alpha_{K_0} \circ (\xi_1^* \times f)$ and $\xi^* = \xi_2^* \circ \xi_1^*$, can be seen as a weak simple product diagram in $\mathbf{C}/A$.

Using the fact that K^* is projective, we construct a map σ_K making

$$\begin{array}{ccc} UK^* & \xrightarrow{U\xi^*} & UR^* \\ \sigma_K \downarrow & & \downarrow \sigma_R \\ K & \xrightarrow[\xi]{} & R \end{array}$$

commutative. This means that we have a P_f-structure $\mathbf{k} = (K, K^*, \sigma_K, \eta_K = l_2 \circ \alpha_K)$ in $\mathbf{C}_{ex}$, that can also be seen as a P_f-structure in $\mathbf{C}$, and a P_f-structure map $\xi = (\xi, \xi^*)$ in $\mathbf{C}_{ex}$. Now $\mathbf{t} \circ \xi$ can be seen as a P_f-structure map in $\mathbf{C}$, and it is actually a weak P_f-substructure map in $\mathbf{C}$ (since (4) is a weak simple product diagram). Therefore $\mathbf{k}$ can be seen as a weak P_f-algebra in $\mathbf{C}$, and since $\mathbf{w}$ is a weak W-type in $\mathbf{C}$, we have a P_f-structure map $\mathbf{s}'$ such that $(\mathbf{t} \circ \xi) \circ \mathbf{s}' = \mathbf{id}_W$ in $P_f(\mathbf{C})$ and $P_f(\mathbf{C}_{ex})$. So $\mathbf{s} = \xi \circ \mathbf{s}'$ is a P_f-structure map in $\mathbf{C}_{ex}$ that is a section of $\mathbf{t}$. ∎

Corollary 4.9 *Let $R \subseteq W$ be a subobject in $\mathbf{C}_{ex}$ and assume that the following statement holds in the internal logic of $\mathbf{C}_{ex}$:*

$$\forall a \in A\, \forall \tau \in W_a^* \, [\, (\forall b \in f^{-1}(a) : \mathit{proj}_W(\eta_W(\tau, b)) \in R\,) \rightarrow \sigma_W(\tau) \in R\,] \qquad (5)$$

Then $R = W$ as subobjects of W.

Proof : Define the following object in $\mathbf{C}_{ex}/A$: for any $a \in A$

$$R_a^* = \{\, \tau \in W_a^* \mid \forall b \in f^{-1}(a) : \mathrm{proj}_W(\eta_W(\tau, b)) \in R\,\}$$

The validity of statement (5) implies that for the canonical map $j^* : R^* \rightarrowtail W^*$, $\sigma_W \circ U_A j^*$ factors through R. So we have a map σ_R making

$$\begin{array}{ccc} UR^* & \xrightarrow{Uj^*} & UW^* \\ \sigma_R \downarrow & & \downarrow \sigma_W \\ R & \xrightarrow[j]{} & W \end{array}$$

commutative. By the very definition of R^*, the map $\eta_W \circ (j^* \times f)$ factors through A^*R, so we have a map η_R making

$$\begin{array}{ccc} R^* \times f & \xrightarrow{j^* \times f} & W^* \times f \\ {\scriptstyle \eta_R}\downarrow & & \downarrow{\scriptstyle \eta_W} \\ R & \xrightarrow[j]{\qquad} & W \end{array}$$

commute. So $\mathbf{r} = (R, R^*, \sigma_R, \eta_R)$ is a P_f-structure in $\mathbf{C}_{ex}$ and $\mathbf{j} = (j, j^*)$ is a P_f-structure map. It is actually a weak P_f-substructure map, so $\mathbf{j}$ has a section $\mathbf{s} : \mathbf{w} \longrightarrow \mathbf{r}$. This implies that j is iso, and $R = W$ as subobjects. ❑

Now we will take our second step. So suppose we have a P_f-structure $\mathbf{w} = (W, W^*, \sigma_W, \eta_W)$ having the four properties mentioned on page 12 in a pretopos $\mathbf{E}$ with dependent products and a natural number object for a choice map $f : B \longrightarrow A$. Now, even though the elements cannot, generally speaking, be thought of as trees, we can still define the subobject Paths of $(W + B + 1)^N$ as consisting of those $\sigma \in (W + B + 1)^N$ such that:

(1) $\sigma(0)$ is an element of W.

(2) If $\sigma(n) \in W$ for a certain natural number n, then either $\sigma(n+1) \in B$ or $\sigma(n+1) = *$.

(3) If $\sigma(n) \in W$ and $\sigma(n+1) \in B$ for a certain natural number n, and, more specifically, $\sigma(n) = (\sigma_W)_a(\tau)$ for certain $a \in A$, $\tau \in W^*_a$, then $f(\sigma(n+1)) = a$ and $\sigma(n+2) = (\text{proj}_W \circ \eta_W)(\tau, \sigma(n+1))$.

(4) If $\sigma(n) = *$ for a certain natural number n, then for all natural numbers $k > n$ also $\sigma(k) = *$.

It is again unnecessary to require that for all $\sigma \in$ Paths there is a natural number $k \in N$ such that $\sigma(k) = *$, for we can show that this is true using property (ii). And we let Paths_w for a certain element $w \in W$ denote the fiber above w of the map $l : \text{Paths} \longrightarrow W : \sigma \mapsto \sigma(0)$, and let ρ be the composition of the canonical map $U_A W^* \longrightarrow A$ and s (s is the map mentioned in property (iv)).

If the elements of W really were trees, then the following equivalence relation:

$w \sim w' \quad \Leftrightarrow \quad$ There is a map $h : \text{Paths}_w \longrightarrow \text{Paths}_{w'}$ satisfying the condition ♣.

where ♣ is the condition

> $h : \text{Paths}_w \longrightarrow \text{Paths}_{w'}$ is a bijection and for all $\sigma \in \text{Paths}_w$ we have that $\text{length}(\sigma) = \text{length}(h(\sigma))$ and, if n is the natural number such that $2n+1 = \text{length}(\sigma) = \text{length}(h(\sigma))$, for all $k < n$ that $\sigma(2k+1) = (h(\sigma))(2k+1)$, and for all $k \leq n$ that $\rho(\sigma(2k)) = \rho(h(\sigma)(2k))$.

would be the identity. The way to proceed is to force $\sim$ to be the identity, by dividing out by this equivalence relation. So let $W' = W/\sim$, and let $q : W \longrightarrow W'$ be the quotient map. We will show that W' can be turned into a full-blooded W-type for the map f.

Our first task is to define a strong P_f-algebra structure on W'. Basically, we only have to give a map

$$\sigma_{W'} : \sum_{a \in A} W'^{f^{-1}(a)} = U_A(W')^f \longrightarrow W'$$

We claim that there is such a map, making

$$\begin{array}{ccc} UW^* & \xrightarrow{Uq^*} & U(W')^f \\ \sigma_W \downarrow & & \downarrow \sigma_{W'} \\ W & \xrightarrow[q]{} & W' \end{array}$$

commute. (Here q^* is the composition of $q_1^* : W^* \longrightarrow W^f$ and $q_2^* = q^f : W^f \longrightarrow (W')^f$, and hence epi.)

We prove this claim by showing that the subobject

$$J \quad = \quad \{ (\tau, w') \in U(W')^f \times W' \mid J(\tau, w') \}$$

where $J(\tau, w')$ is the statement

> There exists a $\sigma \in UW^*$ such that both $(Uq^*)(\sigma) = \tau$ and $(q \circ \sigma_W)(\sigma) = w'$.

is functional. The validity of the statement

$$\forall \tau \in U(W')^f \, \exists w' \in W' : (\tau, w') \in J$$

follows immediately from the fact that Uq^* is epi.

So suppose that $\tau \in U(W')^f, w', w'' \in W'$ are such that $(\tau, w') \in J$ and $(\tau, w'') \in J$. Now we can find $\sigma', \sigma'' \in UW^*$ such that:

$$\begin{array}{lll} (Uq^*)(\sigma') = \tau & \text{and} & (q \circ \sigma_W)(\sigma') = w', \text{ as well as} \\ (Uq^*)(\sigma'') = \tau & \text{and} & (q \circ \sigma_W)(\sigma'') = w''. \end{array}$$

So we have that $(Uq^*)(\sigma') = (Uq^*)(\sigma'')$, and if $\tau \in (W')^{f^{-1}(a)}$, this implies that for all $b \in f^{-1}(a)$:

$$(\text{proj}_W \circ \eta_W)(\sigma', b) \sim (\text{proj}_W \circ \eta_W)(\sigma'', b)$$

Now, using choice for f, we can select for every b in $f^{-1}(a)$ a map

$$h_b : \text{Paths}_{(\text{proj}_W \circ \eta_W)(\sigma', b)} \longrightarrow \text{Paths}_{(\text{proj}_W \circ \eta_W)(\sigma'', b)}$$

that satisfies the condition ♣.

Define a map $h : \text{Paths}_{(\sigma_W)\sigma'} \longrightarrow \text{Paths}_{(\sigma_W)\sigma''}$ as follows:

$$\begin{array}{rcl} h(\langle (\sigma_W)\sigma' \rangle) & = & \langle (\sigma_W)\sigma'' \rangle \text{ and} \\ h(\langle (\sigma_W)\sigma', b \rangle * \sigma) & = & \langle (\sigma_W)\sigma'', b \rangle * h_b(\sigma) \end{array}$$

It is easy to see that h is well-defined and conforms to the constraint ♣. So $(\sigma_W)\sigma' \sim (\sigma_W)\sigma''$, and hence $w' = (q \circ \sigma_W)\sigma' = (q \circ \sigma_W)\sigma'' = w''$.

This proves that J is functional, and hence that we have a map

$$\sigma_{W'} : \sum_{a \in A} W'^{f^{-1}(a)} \longrightarrow W'$$

in $\mathbf{C}$ having J as graph and making the desired diagram commute. This means that we have defined a strong P_f-algebra structure $\mathbf{w}'$ on W' (and notice that $\mathbf{q} = (q, q^*)$ is a P_f-structure map). Now we have to prove that this strong P_f-algebra is the strong W-type for f in $\mathbf{C}$.

So let

$$\mathbf{x} \;=\; (X, \sigma_X : \sum_{a \in A} X^{f^{-1}(a)} \longrightarrow X)$$

be another strong P_f-algebra in $\mathbf{C}$. We have to construct a strong P_f-algebra map $\mathbf{t} : \mathbf{w}' \longrightarrow \mathbf{x}$. We know that we have a P_f-structure map $\mathbf{j} = (j, j^*) : \mathbf{w} \longrightarrow \mathbf{x}$. I claim that $\mathbf{j}$ factors through $\mathbf{q}$. In order to prove this, we have to show that

$$w \sim w' \Rightarrow j(w) = j(w')$$

So let $R = \{\, w \in W \,|\, \forall w' \in W : w \sim w' \Rightarrow j(w) = j(w') \,\}$. We prove that $R = W$ as subobjects of W using the inductive property (ii) of $\mathbf{w}$.

Suppose $w = (\sigma_W)_a(\tau)$ for certain $a \in A$ and $\tau \in W^*_a$, and assume furthermore that

$$\forall b \in f^{-1}(a) : (\mathrm{proj}_W \circ \eta_W)(\tau', b) \in R$$

Let w' be an arbitrary element of W such that $w \sim w'$. We know that w' is of the form $(\sigma_W)_{a'}(\tau')$ for some $a' \in A, \tau' \in W^*_{a'}$ (remember that σ_W has a section). From the fact that $w \sim w'$ we deduce that $a = a'$ and that

$$\forall b \in f^{-1}(a) : (\mathrm{proj}_W \circ \eta_W)(\tau, b) \sim (\mathrm{proj}_W \circ \eta_W)(\tau, b)$$

Now it follows from the induction hypothesis that $(j \circ \mathrm{proj}_W \circ \eta_W)(\tau, b) = (j \circ \mathrm{proj}_W \circ \eta_W)(\tau', b)$ for all $b \in f^{-1}(a)$. But this implies that the following functions $f^{-1}(a) \longrightarrow X$

$$\begin{array}{rcl} (Uj^*)(\tau) & = & \lambda b \in f^{-1}(a).(j \circ \mathrm{proj}_W \circ \eta_W)(\tau, b) \\ (Uj^*)(\tau') & = & \lambda b \in f^{-1}(a).(j \circ \mathrm{proj}_W \circ \eta_W)(\tau', b) \end{array}$$

are equal, and hence that

$$\begin{array}{rcl} j(w) & = & (j\sigma_W)_a(\tau) \\ & = & \sigma_X(Uj^*(\tau)) \\ & = & \sigma_X(Uj^*(\tau')) \\ & = & (j\sigma_W)_a(\tau') \\ & = & j(w') \end{array}$$

This completes the induction hypothesis. Therefore we can conclude that $R = W$ as subobjects of W.

We conclude that there exists a map $t : W' \longrightarrow X$ such that $t \circ q = j$ in $\mathbf{C}$. Let $t^* = t^f$. We prove that t is actually a strong P_f-algebra map. The following diagram in $\mathbf{E}/A$ commutes:

$$\begin{array}{ccc} (W')^f \times f & \xrightarrow{t^* \times f} & X^f \times f \\ \eta_{W'} \downarrow & & \downarrow \eta_X \\ W & \xrightarrow[t]{} & X \end{array}$$

As does the righthand square of the following diagram in $\mathbf{E}$:

$$\begin{array}{ccccc} UW^* & \xrightarrow{Uq^*} & U(W')^f & \xrightarrow{Ut^*} & UX^f \\ \sigma_W \downarrow & & \downarrow \sigma_{W'} & & \downarrow \sigma_X \\ W & \xrightarrow[q]{} & W' & \xrightarrow[t]{} & X \end{array} \qquad (6)$$

And for the following reason: the map $\xi : W^* \longrightarrow X^f$ making

$$\begin{array}{ccc} W^* \times f & \xrightarrow{\xi \times f} & X^f \times f \\ \eta_W \downarrow & & \downarrow \eta_X \\ W & \xrightarrow[j = t \circ q]{} & X \end{array}$$

commute, is unique. Since both j^* and $t^* \circ q^*$ have this property, we know that $t^* \circ q^* = j^*$ and hence that the outer rectangle in (6) commutes. But since the lefthand square in (6) commutes, and Uq^* is epi, the same holds for the righthand square in (6). The conclusion is that t is really a strong P_f-algebra map.

We now prove that t is the unique strong P_f-algebra map from $\mathbf{w}'$ to $\mathbf{x}$. This follows from the following claim: if $\mathbf{k} = (k, k^*)$ is another P_f-structure map from $\mathbf{w}'$ to $\mathbf{x}$, then:

$$w \sim w' \Rightarrow j(w) = k(w')$$

We again prove this using the "inductive" property (ii) of $\mathbf{w}$.

So let $R = \{w \in W \,|\, \forall w' \in W : w \sim w' \Rightarrow j(w) = k(w')\}$. Assume that $w \in W$ equals $(\sigma_W)_a(\tau)$ for a certain $a \in A$ and $\tau \in W_a^*$ such that

$$\forall b \in f^{-1}(a) : (\mathrm{proj}_W \circ \eta_W)(\tau, b) \in R$$

Let w' be an arbitrary element of W such that $w \sim w'$. w' is of the form $(\sigma_W)_a(\tau')$ for a particular $\tau' \in W_a^*$. Since $w \sim w'$, we have that

$$\forall b \in f^{-1}(a) : (\mathrm{proj}_W \circ \eta_W)(\tau, b) \sim (\mathrm{proj}_W \circ \eta_W)(\tau', b)$$

So we know that for all $b \in f^{-1}(a)$ we have that

$$(j \circ \mathrm{proj}_W)(\tau, b) = (k \circ \mathrm{proj}_W)(\tau', b)$$

But this implies that the functions $(Uj^*)(\tau)$ and $(Uk^*)(\tau')$ are equal and that

$$\begin{aligned} j(w) &= (j\sigma_W)_a(\tau) \\ &= \sigma_X(Uj^*(\tau)) \\ &= \sigma_X(Uk^*(\tau')) \\ &= (k\sigma_W)_a(\tau') \\ &= k(w') \end{aligned}$$

So $w \in R$.

We conclude that $R = W$ as subobjects of W, and hence that $j = k$. This means that the map t really is unique and that therefore $\mathbf{w}'$ is a strong W-type. So we have taken our second step, and we have completed the proof of proposition (4.1). And with it, we have also proven theorem (1.1).

5 Constructing weak W-types in the category of partitioned assemblies

In this final section, we want to prove

Theorem 5.1 (= Corollary 1.3) *For all pretopoi* **E** *with dependent products and W-types, we have that* **Eff(E)** *is also a pretopos with dependent products and W-types.*

In view of theorem (1.1), it is sufficient to prove the following result:

Theorem 5.2 (= Theorem 1.2) *For all pretopoi* **E** *with dependent products and W-types, we have that* **Pass(E)** *has weak W-types.*

For this purpose, let

$$f : (\epsilon_B : B \longrightarrow N) \longrightarrow (\epsilon_A : A \longrightarrow N)$$

be an arbitrary map in **Pass(E)**, and let f also be the name of the corresponding map in **E**. We will first construct a weak P_f-algebra for f in **Pass(E)** and then show that it is actually the weak W-type for f.

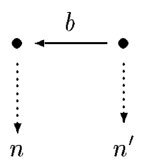
Let $W(f)$ be the W-type for f in **E**, and let Paths be the object in **E** defined as on page 5 (minus the accents): as explained there, we might think of it as the set of paths in the trees that are the elements of $W(f)$. In order to construct the weak P_f-algebra, we need the notion of decorated tree in the internal logic of **E**. For any element $w \in \mathrm{W}(f)$, a member κ of N is called a *decoration* of w if κ is such that for every $\sigma \in \mathrm{Paths}_w$ and $n = \mathrm{length}(\sigma)$ there is a function $c : \{0, 2, \ldots, n-1\} \longrightarrow N$ such that: (i) $c(0) = \kappa$; (ii) if for some even $k < n$, $a \in A$ and $t : f^{-1}(a) \longrightarrow \mathrm{W}(f)$, we have that $\sigma(k) = \mathrm{sup}_a(t)$, then $j_0(c(k)) = \epsilon_A(a)$; and (iii) for all even $k < n-1$, we have $j_1(c(k)) \cdot \epsilon_B(\sigma(k+1)) = c(k+2)$. (Here j is a pairing function for the natural numbers, and j_0 and j_1 are its associated projections.)

The following comment is perhaps helpful in clarifying this definition: every such κ determines a function $\mathrm{Paths}_w \longrightarrow N$, one I shall also denote κ. On an element $\sigma \in \mathrm{Paths}_w$ this function is defined as follows: let $n = \mathrm{length}(\sigma)$ and $c : \{0, 2, \ldots, n-1\} \longrightarrow N$ be a function satisfying the conditions (i)-(iii) given above. Now we set $\kappa(\sigma) := c(n-1)$, and this is well-defined, since the function c satisfying (i)-(iii) is necessarily unique.

Conversely, we can consider a function $\kappa : \mathrm{Paths}_w \longrightarrow N$ and this determines an element of N by taking $\kappa(\langle w \rangle)$. In fact, this natural number is a decoration if and only if κ satisfies the following constraint:

> For all $\sigma \in \mathrm{Paths}_w$, $a \in A$, $t : f^{-1}(a) \longrightarrow W$ and natural numbers m, n_0, n_1: if $\sigma(m) = \mathrm{sup}_a(t)$, $m = \mathrm{length}(\sigma) - 1$ and $\kappa(\sigma) = j(n_0, n_1)$, then (i) $\epsilon_A(a) = n_0$ and (ii) $n_1 \cdot \epsilon_B(b) = \kappa(\sigma * \langle b, tb \rangle)$ for all $b \in f^{-1}(a)$.

In the case **E** = **Sets** this has the following, more intuitive, significance: we think of κ as a function that assigns to every *node* in the tree w a natural number, in such a way that if n is assigned to a node that is labelled by a certain element $a \in A$, then $j_0(n) = \epsilon_A(a)$ and that if



is an edge in the tree w that is labelled by the element $b \in B$, then we require for the natural numbers n assigned to the end point and n' assigned to the begin point that $j_1(n) \cdot \epsilon_B(b) = n'$.

In the sequel we will frequently exploit the presence of the functional perspective by using the notation $\kappa(\sigma)$ in writing down formulas, even if κ is just an element of N.

So we also define the notion of a decorated tree in the internal logic of **E**, being a pair consisting of an element $w \in \mathrm{W}(f)$ together with a decoration for it. Let us call the object of all these pairs W. This object can easily be turned into an object in $\mathbf{Pass}(\mathbf{E})$ by defining a morphism $\epsilon_W : W \longrightarrow N$ as follows:

$$\epsilon_W(w, \kappa) = \kappa = \kappa(\langle w \rangle)$$

Now we let W together with ϵ_W be the first component of the quadruple that is to be the weak W-type.

In order to define the second component of the quadruple, we first define the object W^* in $\mathbf{E}/A$ as follows ($a \in A$):

$$W^*_a \quad = \quad \{ (g, n) \in W^{f^{-1}(a)} \times N \mid W^*(g, n) \}$$

where $W^*(g, n)$ is the statement

> n codes a pair (n_0, n_1), that is, $n = j(n_0, n_1)$, where n_0 equals $\epsilon_A(a)$ and n_1 tracks g, meaning that for all $b \in f^{-1}(a)$ we have: $n_1 \cdot \epsilon_B(b) = \epsilon_W(g(b))$.

By setting $\epsilon_{W^*}(g, n) = n$, we turn W^* into an object in $\mathbf{Pass}(\mathbf{E})$, and also in $\mathbf{Pass}(\mathbf{E})/(\epsilon_A : A \longrightarrow N)$. And if $\eta_W : W^* \times f \longrightarrow W$ in $\mathbf{Pass}(\mathbf{E})/(\epsilon_A : A \longrightarrow N)$ is given by

$$\eta_W((g, n), b) \quad = \quad (g(b), f(b))$$

then W^* is a weak version of W^f with η_W as weak evaluation.

Therefore we let η_W be the fourth component of the quadruple, and the construction will be complete once we have defined a map $\sigma_W : U_{\epsilon_A}(W^*) \longrightarrow W$. So let us suppose that we have an element (g, n) in W^*_a. Then we have for every $b \in f^{-1}(a)$ elements $(w_b, \kappa_b) = g(b)$ in W. We now define $\sigma_W(g, n)$ to be (w, κ), where

$$w \quad = \quad \sup_a(\lambda b \in f^{-1}(a).w_b)$$

and κ equals n. (It is not hard to see that κ is a decoration of w.)

This completes the definition of the weak P_f-algebra

$$\mathbf{w} \quad = \quad (W, W^*, \sigma_W, \epsilon_W)$$

in $\mathbf{Pass}(\mathbf{E})$. We will now show that it is a weak W-type. First, we demonstrate that it is weakly initial in $WP_f(\mathbf{Pass}(\mathbf{E}))$.

Proposition 5.3 *The weak P_f-algebra $\mathbf{w}$ constructed in this section is weakly initial in $WP_f(\mathbf{Pass}(\mathbf{E}))$.*

Proof : Suppose $\mathbf{x} = (X, X^*, \sigma_X, \eta_X)$ is a weak P_f-algebra in $\mathbf{Pass}(\mathbf{E})$. We have to construct a map $\mathbf{t} = (t, t^*) : \mathbf{w} \longrightarrow \mathbf{x}$. Without loss of generality, we may assume that X^*, $\epsilon_{X^*} : X^* \longrightarrow N$ and $\eta_X : X^* \times f \longrightarrow X$ are of the following form: for any $a \in A$

$$X^*_a \quad = \quad \{(h, m) \in X^{f^{-1}(a)} \times N \mid X^*(h, m) \}$$

where $X^*(h, m)$ is the statement:

> m codes a pair (m_0, m_1) in such a way that $m_0 = \epsilon_A(a)$ and m_1 tracks g, i.e., for all $b \in f^{-1}(a)$: $m_1 \cdot \epsilon_B(b) = \epsilon_X(g(b))$.

and $\epsilon_{X^*}(h,m) = m$ and $\eta_X((h,m),b) = (h(b), f(b))$.

We first construct the natural number r that is to track t. Suppose s is a natural number tracking σ_X, and H is a function that yields for the codes of partial recursive functions the code of their composition, i.e., for all codes of partial recursive functions p and q and natural numbers n

$$H(p,q)\cdot n \quad = \quad p\cdot(q\cdot n)$$

We then obtain r as a solution of

$$r\cdot n \quad = \quad s(j(j_0(n), H(r, j_1(n))))$$

using the First Recursion Theorem.

We construct the map t by defining a suitable functional subobject of $W\times X$. We set:

$$L \quad = \quad \{\, ((w,\kappa),x)\in W\times X \mid L((w,\kappa),x)\,\}$$

Where $L((w,\kappa),x)$ is the statement: "There exists a function $g : \mathrm{Paths}_w \longrightarrow X$ that is a witness for $((w,\kappa),x)$." And a witness is defined as follows: we call $g : \mathrm{Paths}_w \longrightarrow X$ a witness for $((w,\kappa),x)$ if the following three conditions hold:

(i) $r\cdot\kappa(\sigma) = \epsilon_X(g(\sigma))$ for all $\sigma\in\mathrm{Paths}_w$.

(ii) For all $\sigma\in\mathrm{Paths}_w$ and natural numbers n, if $\sigma(n) = \mathrm{sup}_a(t)$ and $\mathrm{length}(\sigma) = n+1$, then we have for the element (h,m) in X^*_a defined by

$$\begin{aligned} h &= \lambda b\in f^{-1}(a).\, g(\sigma * \langle b, tb\rangle) \\ m &= j(\,\epsilon_A(a), H(r, j_1(\kappa(\sigma)))\,) \end{aligned}$$

that $\sigma_X(h,m) = g(\sigma)$.

(iii) $x = g(\langle w\rangle)$.

(If constraint (i) is satisfied, the element (h,m) mentioned in (ii) really belongs to X^*_a: for if b is in $f^{-1}(a)$, then

$$\begin{aligned} H(r, j_1\kappa(\sigma))\cdot\epsilon_B(b) &= r\cdot(\,j_1\kappa(\sigma)\cdot\epsilon_B(b)\,) \\ &= r\cdot\kappa(\sigma * \langle b, t(b)\rangle) \qquad (\kappa \text{ is a decoration}) \\ &= \epsilon_X(g(\sigma * \langle b, tb\rangle)) \end{aligned}$$

So $H(r, j_1\kappa(\sigma))$ really tracks h.) The way to understand this definition is by thinking of g as a restriction of t to those elements that are "decorated subtrees" of (w,κ).

We prove that L is functional, by establishing a sequence of lemma's.

Lemma 5.4 *If $(w,\kappa)\in W$ and g,h are functions $\mathit{Paths}_w \longrightarrow X$ satisfying constraints (i) and (ii) in the definition of a witness, then $g = h$.*

Proof : Let

$$R \quad = \quad \{\, w\in \mathrm{W}(f) \mid R(w)\,\}$$

where $R(w)$ is the condition

> For all decorations κ of w and functions $g,h : \mathrm{Paths}_w \longrightarrow X$ satisfying constraints (i) and (ii) for being a witness: $g = h$.

We prove, using transfinite induction for W-types, that $R = \mathrm{W}(f)$.

So let $w = \sup_a(t)$ for some $a \in A$ and $t : f^{-1}(a) \longrightarrow \mathrm{W}(f)$ and assume that for all $b \in f^{-1}(a)$ the element $t(b)$ of $\mathrm{W}(f)$ lies in R. Let κ be a decoration of w and let $g, h : \mathrm{Paths}_w \longrightarrow X$ be functions satisfying (i) and (ii).

Notice that for every element $b \in f^{-1}(a)$, if we define $\kappa_b = \kappa(\langle w, b, t(b)\rangle)$ and $g_b, h_b : \mathrm{Paths}_{t(b)} \longrightarrow X$ by

$$\begin{aligned} g_b(\sigma) &= g(\langle w, b\rangle * \sigma) \\ h_b(\sigma) &= h(\langle w, b\rangle * \sigma) \end{aligned}$$

for every $\sigma \in \mathrm{Paths}_{t(b)}$, then κ_b is a decoration of $t(b)$ and g_b and h_b both satisfy (i) and (ii) for $(t(b), \kappa_b)$. Since we assumed that $t(b) \in R$ for every $b \in f^{-1}(a)$, we also have that $g_b = h_b$ for every $b \in f^{-1}(a)$.

Now let finally $m := H(r, j_1\kappa(\langle w\rangle))$ and remark that it follows from the assumption that g and h satisfy constraint (ii), that:

$$\begin{aligned} g(\langle w\rangle) &= \sigma_X(\lambda b \in f^{-1}(a).\, g(\langle w, b, t(b)\rangle), m) \\ &= \sigma_X(\lambda b \in f^{-1}(a).\, h(\langle w, b, t(b)\rangle), m) \\ &= h(\langle w\rangle) \end{aligned}$$

This shows that $g = h$.

We conclude that $R = \mathrm{W}(f)$ and that the lemma indeed holds. ❑

This lemma has the following trivial consequence:

Lemma 5.5 *If $((w, \kappa), x) \in L$ and $((w, \kappa), x') \in L$, then $x = x'$.*

We furthermore claim:

Lemma 5.6 *For all $(w, \kappa) \in W$ there is a $x \in X$ such that $((w, \kappa), x) \in L$.*

Proof : Let

$$R \;=\; \{\, w \in \mathrm{W}(f) \mid R(w) \,\}$$

where $R(w)$ is the condition:

For all decorations κ of w there exists a $x \in X$ such that $((w, \kappa), x) \in L$.

We show, again using transfinite induction for W-types, that $R = W$.

So let $w = \sup_a(t)$ for some $a \in A$ and $t : f^{-1}(a) \longrightarrow \mathrm{W}(f)$ and assume that for all b in $f^{-1}(a)$: $t(b) \in R$. Let κ be a decoration of w, and it again follows that for every $b \in f^{-1}(a)$ the element κ_b of N defined by $\kappa_b = \kappa(\langle w, b, tb\rangle)$ is a decoration of $t(b)$. Observe that if we set $v = \lambda b \in f^{-1}(a).(t(b), \kappa_b)$, then $(v, \kappa) \in W_a^*$ and $\sigma_W(v, \kappa) = (w, \kappa)$.

But from the fact that $t(b) \in R$ it now follows that there are (necessarily unique, by lemma (5.5)) $x_b \in X$ such that $((t(b), \kappa_b), x_b) \in L$. This in turn implies that there are for every $b \in f^{-1}(a)$ (also necessarily unique, by lemma (5.4)) $g_b : \mathrm{Paths}_{t(b)} \longrightarrow X$ that are witnesses for $((t(b), \kappa_b), x_b)$.

Now define h and m as follows:

$$\begin{aligned} h &= \lambda b \in f^{-1}(a).x_b \\ m &= j(j_0(\kappa), H(r, j_1(\kappa))) \end{aligned}$$

Now $(h, m) \in X_a^*$, for $H(r, j_1(\kappa))$ tracks h, as the following calculation shows:

$$\begin{array}{rcll}
H(r, j_1(\kappa)) \cdot \epsilon_B(b) & = & r \cdot (j_1(\kappa) \cdot \epsilon_B(b)) & \text{(definition } H\text{)} \\
& = & r \cdot \epsilon_W(v(b)) & \text{(definition } X^*\text{)} \\
& = & r \cdot \epsilon_W(t(b), \kappa_b) & \text{(definition } v\text{)} \\
& = & r \cdot \kappa_b(\langle t(b) \rangle) & \text{(definition } \epsilon_W\text{)} \\
& = & \epsilon_X(g_b(\langle t(b) \rangle)) & (g_b \text{ satisfies constraint (i))} \\
& = & \epsilon_X(x_b) & (g_b \text{ satisfies constraint (ii))}
\end{array}$$

Set $x := \sigma_X(h, m)$, and define $g : \text{Paths}_w \longrightarrow X$ as follows:

$$\begin{array}{rcl}
g(\langle w \rangle) & = & x \\
g(\langle w, b \rangle * \sigma) & = & g_b(\sigma)
\end{array}$$

It is easily verified that g satisfies constraints (ii) and (iii) for being a witness for $((w, \kappa), x)$. But g also satisfies constraint (i), as follows from the following calculation and the remark that all g_b satisfy constraint (i).

$$\begin{array}{rcl}
\epsilon_X(g(\langle w \rangle)) & = & \epsilon_X(x) \\
& = & \epsilon_X(\sigma_X(h, m)) \\
& = & s(\epsilon_{X^*}(h, m)) \\
& = & s(m) \\
& = & s(j(j_0(\kappa), H(r, j_1(\kappa)))) \\
& = & r \cdot \kappa \\
& = & r \cdot \kappa(\langle w \rangle)
\end{array}$$

This shows that $((w, \kappa), x) \in L$, and hence that $w \in R$.

We conclude that $R = W$. ∎

Wrapping up, we see that we have shown that the subobject L is functional. It follows that we have a map $t : W \longrightarrow X$ in **E** such that L is the graph of t. It is not hard to see that t is also a map in **Pass(E)**, since it is tracked by r. For if $(w, \kappa) \in W$, then $((w, \kappa), t(w, \kappa)) \in L$, and so there exists a map $g : \text{Paths}_w \longrightarrow X$ that is a witness for this element. This means that:

$$r \cdot \epsilon_W(w, \kappa) = r \cdot \kappa(\langle w \rangle) = \epsilon_X(g(\langle w \rangle)) = \epsilon_X(t(w, \kappa))$$

To complete the definition of $\mathbf{t}$, we define $t_a^* : W_a^* \longrightarrow X_a^*$ by:

$$t_a^*(v, n) \quad = \quad (t \circ v, j(j_0(n), H(r, j_1(n))))$$

It remains to be shown that the following diagrams commute:

$$\begin{array}{ccc}
W^* \times f & \xrightarrow{t^* \times f} & X^* \times f \\
\eta_W \downarrow & & \downarrow \eta_X \\
W & \xrightarrow[t]{} & X
\end{array}
\qquad
\begin{array}{ccc}
UW^* & \xrightarrow{Ut^*} & UX^* \\
\sigma_W \downarrow & & \downarrow \sigma_X \\
W & \xrightarrow[t]{} & X
\end{array}$$

It is easy to see that the first diagram commutes, so we concentrate on the second. Suppose $(v, n) \in W_a^*$ for some $a \in A$. We know that for all b in $f^{-1}(a)$, we have that $(v(b), t(v(b))) \in L$. Reasoning along the same lines as in lemma (5.4), we can show that for

$$\begin{array}{rcl}
h & = & t \circ v \\
m & = & j(j_0(n), H(r, j_1(n)))
\end{array}$$

we have that $(h,m) \in X^*_a$ and $(\sigma_W(v,n), \sigma_X(h,m)) \in L$. But since $(h,m) = t^*_a(v,n)$, this implies that $\sigma_X t^*(v,n) = t\sigma_W(g,n)$.

So $\mathbf{t} = (t, t^*)$ is a homomorphism of weak P_f-algebra's. We conclude that $\mathbf{w}$ is indeed weakly initial in $WP_f(\mathbf{Pass}(\mathbf{E}))$. ∎

The following proposition will complete the proof of the fact that $\mathbf{w}$ is a weak W-type for f in $\mathbf{Pass}(\mathbf{E})$.

Proposition 5.7 *If* $\mathbf{k} : \mathbf{x} \longrightarrow \mathbf{w}$ *is a weak* P_f*-subalgebra, then* $\mathbf{k}$ *has a section.*

Proof : Let $\mathbf{k} : \mathbf{x} \longrightarrow \mathbf{w}$ be a weak P_f-subalgebra. Form the following pullback in $\mathbf{Pass}(\mathbf{E})/(\epsilon_A : A \longrightarrow N)$:

$$\begin{array}{ccc} L & \xrightarrow{p_1} & W^* \times f \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \eta_W} \\ X & \xrightarrow[k]{} & W \end{array}$$

We may without loss of generality assume that in $\mathbf{x} = (X, X^*, \sigma_X, \eta_X)$ the second component X^* and the fourth component η_X are of the following form, and that k^* is given as follows. For any $a \in A$ and $\tau \in W^*_a$:

$$X^*_\tau \quad = \quad \{\, (h,m) \in L^f \times N \mid X^*(h,m) \,\}$$

where $X^*(h,m)$ is the statement

> For all b in $f^{-1}(a)$ we have that $p_1 g(b) = (\tau, b)$ and m codes a pair (m_0, m_1) such that (i) $\epsilon_A(a) = m_0$; and (ii) m_1 tracks $\mathrm{proj}_X \circ p_2 \circ h$, i.e., for all $b \in f^{-1}(a)$: $m_1 \cdot \epsilon_B(b) = \epsilon_X((\mathrm{proj}_X \circ p_2 \circ g)(b))$.

and $X^*_a = \coprod_{\tau \in W^*_a} X^*_\tau$, and $k^*_a : X^*_a \longrightarrow W^*_a$ is the projection, and $\epsilon_{X^*} : X^* \longrightarrow N$ is defined by $\epsilon_{X^*}(\tau, r) = \epsilon_{W^*}(\tau)$, and finally $\eta_X : X^* \times f \longrightarrow X$ is given by $\eta_X((\tau,(h,m)), b) = (p_2 \circ h)(b)$.

The proof of this proposition is now a variation of that on the previous. We again want to construct a map $s : W \longrightarrow X$, and the natural number r that is to track this map is constructed in the same way as in that proof.

We again define the subobject L, and what a means for a function

$$g : \mathrm{Paths}_w \longrightarrow X$$

to be a witness, almost in the same way, but adding an extra constraint:

$$\textbf{(iv)} \quad k(x) = (w, \kappa)$$

The proofs of the lemma's (5.4-6) now carry over almost verbatim. In the proof of lemma (5.6) we have to make the following slight adaptations. Let h and m now be given by:

$$\begin{array}{rcl} h & = & \lambda b \in f^{-1}(a).(((v,\kappa), b), (a, x_b)) \text{ and} \\ m & = & j(j_0(\kappa), H(r, j_1(\kappa))) \end{array}$$

We now have that $((v,\kappa),(h,m)) \in X^*_a$, since the fact that $((t(b), \kappa_b), x_b) \in L$ now also implies that $k(x_b) = (w_b, \kappa_b) = v(b)$. So if we now set $x := \sigma_X((v,\kappa),(h,m))$ and let $g : \mathrm{Paths}_w \longrightarrow X$ again be given by:

$$\begin{array}{rcl} g(\langle w \rangle) & = & x \\ g(\langle w, b \rangle * \sigma) & = & g_b(\sigma) \end{array}$$

g again satisfies the "old" constraints (i)-(iii). It also satisfies the new constraint (iv), since:

$$\begin{aligned} k(x) &= k\sigma_X((v,\kappa),(h,m)) \\ &= \sigma_W k^*((v,\kappa),(h,m)) \\ &= \sigma_W(v,\kappa) \\ &= (w,\kappa) \end{aligned}$$

This means that the subobject L is again functional. Let $s : W \longrightarrow X$ in $\mathbf{E}$ be such that L is the graph of s. Now s is a section of k by construction.

The map $s^* : W^* \longrightarrow X^*$ is constructed as follows. We set

$$s^*_a(v,n) = ((v,n),(h,m))$$

for every $a \in A$ and $(v,n) \in W^*_a$, with

$$\begin{aligned} h &= \lambda b \in f^{-1}(a).(\,((v,n),b),(a,(s\circ v)(b))\,) \text{ and} \\ m &= j(j_0(n), H(r, j_1(n))) \end{aligned}$$

It is now not hard to see that $\mathbf{s} = (s,s^*)$ is a weak P_f-algebra map in $\mathbf{Pass}(\mathbf{E})$ and that $\mathbf{s}$ is a section of $\mathbf{k}$. This completes the proof. □

This means that we have also established corollary (1.3).

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