Linear Splines and their Derivatives on Uniform Simplicial Partitions of Polytopes

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Abstract

Superconvergence of the gradient for the linear simplicial finite element method applied to elliptic equations is a well-known feature in one, two, and three space dimensions. In this paper we show that, in fact, there exists an elegant proof of this feature independent of the space dimension. As a result, superconvergence for dimensions four and up is proved simultaneously. The key ingredient will be that we embed the gradients of the continuous piecewise linear functions into a larger space for which we describe an orthonormal basis having some useful symmetry properties. Since gradients and rotations of standard finite element functions are in fact the rotation-free and divergence-free elements of Raviart-Thomas and Nédélec spaces in three dimensions, we expect our results to have applications also in those contexts.

Keywords: uniform partition, n-simplex, point symmetry, elliptic problems, linear splines, finite element method, superconvergence

AMS subject classification: 65N30

1 Introduction

In their famous 1969 paper, Oganesjan and Ruhovets [29] proved that the gradient ∇u_h of the continuous piecewise linear finite element approximation u_h of the solution u of the Poisson problem $-\Delta u = f$ with homogeneous Dirichlet boundary conditions on a rectangular two-dimensional domain using uniform triangular partitions, is closer to the gradient $\nabla L_h u$ of the nodal linear Lagrange interpolants $L_h u$ of u than to ∇u. In fact, using standard notation for Sobolev spaces and norms, they showed that

$$\|\nabla(u_h - L_h u)\|_0 \leq Ch^2 |u|_3, \tag{1}$$

whereas both ∇u_h and $\nabla L_h u$ approximate ∇u only with order $\mathcal{O}(h)$. This phenomenon, which has since the 1991 paper [27] by Lin et al. been called *supercloseness* to the nodal interpolant, can be exploited to develop *post-processing* schemes to improve the approximation order of ∇u_h from $\mathcal{O}(h)$ to $\mathcal{O}(h^2)$. This explains why [29] is often considered as the paper

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that initiated the research in the by now well-developed area of *superconvergence in finite element methods.*

We have found that in one space dimension, the supercloseness of the finite element solution was considered in the paper [33] by P. Tong 1969, but we suspect that in the engineering society, the same result has already been known much longer, although we could not find an earlier reference. The corresponding result in three space dimensions was firstly proved by Chen [10] in 1980. Unaware of the developments in China, Kantchev and Lazarov proved essentially the same result in 1986 [22]. Also in the paper [18] by Goodsell, the same supercloseness can be found. Interestingly, the proofs for one, two and three dimensions are all quite different in nature, and rather technical. In this paper, we will present just one key argument that proves the supercloseness (1) for all space dimensions $n \geq 1$ simultaneously. For dimensions four and up, we need to assume higher regularity of the solution u to guarantee that its nodal interpolant $L_h u$ is well-defined, but since this interpolant is not present anymore in Theorem 4.7, we can remove those additional regularity assumptions by a density argument, which results in Corollary 4.8.
Since the paper by Oganesjan and Ruhovets, a large number of superconvergence results have been derived by different authors in various contexts [19, 24, 26]. The supercloseness (1) to some local interpolant remained essential and of central importance in many of them. Indeed, apart from the fact that similar results were obtained for higher order triangular elements [1, 2, 7, 20] and for elliptic systems [21], also for elliptic equations in three space dimensions, one succeeded in proving (1) for tetrahedral partitions with a certain symmetry [10, 18, 22]. Typical difficulties with superconvergence in $\mathbb{R}^3$ are surveyed in [9].

1.1 Motivation

The value of this paper lies in three aspects. First, it presents a proof for supercloseness in dimensions one, two, and three that may be easier to understand than the proofs in the original papers [33, 29, 10]. Second, for dimensions four and up, the supercloseness is a new result. It might seem that in current practical applications of the finite element method there is no need for simplicial higher dimensional elements. Nevertheless, it is well-known that for example in areas like financial mathematics [35, 36], particle physics, statistical physics [13] and general relativity, higher dimensional PDEs need to be solved. As an example, we quote G.B. Cook and S.A. Teukolsky from the 1999 issue of Acta Numerica [12], in which they contemplated (page 11) on simplicial meshes in four dimensions:

"Given that general relativity is a four-dimensional theory, a natural approach for solving the equations might be to discretize the full four-dimensional domain into a collection of simplices and solve the equation somehow on this lattice. A discrete form of Einstein's equations based on this idea was developed by Regge [31, 34]. While considerable efforts have been made to implement numerical schemes based on this Regge calculus approach, they have not yet moved beyond test codes [4, 16]".

Finally, finite element discretizations based on tensor product partitions in three dimensions often lead to non-monotone matrices and a corresponding loss of the discrete maximum principle. Hence, the numerical solution might not preserve exactly those properties of the continuous system that are relevant. This may for instance result in a cash-flow opposite to the flow of the corresponding continuous problem or in matter-spitting black holes. Simplicial meshes seem to be the solution to this problem, as was proved for $n = 3$ in Korotov et al.

[23]. We expect that for the applications mentioned above, this is of vital importance. As a third motivation, as we will outline in more detail in Section 1.3, the supercloseness may be of use in the study of more complicated three-dimensional mixed finite element methods.

Our (for $n \geq 4$) supercloseness result (1), which follows from the *strengthened* Cauchy-Schwarz inequality (3), implies superconvergence for the standard finite element method. The superconvergence result itself may also be derived using an L^∞ estimate from the paper [32], whereas the individual bounds (1), (3), and (6) can not be found in [32].

1.2 The main inequality

Consider a face-to-face partition of a bounded polytopic domain $\Omega \subset \mathbb{R}^n$ into n-simplices. Denote the space of continuous piecewise linear polynomials with respect to this partition by V_h and let $V_{0h} = V_h \cap H_0^1(\Omega)$. Discretizing the Poisson equation $-\Delta u = f \in L^2(\Omega)$ with homogeneous Dirichlet boundary conditions using V_{0h} as approximating space, means finding $u_h \in V_{0h}$ such that $(\nabla u_h, \nabla v_h) = (f, v_h)$ for all $v_h \in V_{0h}$. This gives the Galerkin orthogonality relation $(\nabla(u - u_h), \nabla v_h) = 0$ for all $v_h \in V_{0h}$, and hence the difference $u_h - L_h u$ can be studied as follows. Assuming that u is smooth enough for $L_h u$ to be well-defined, we get that

$$\|\nabla(u_h - L_h u)\|_0^2 = (\nabla u_h - \nabla L_h u, \nabla(u_h - L_h u)) = (\nabla u - \nabla L_h u, \nabla(u_h - L_h u)). \quad (2)$$

Application of the Cauchy-Schwarz inequality at this point would only result in $\|\nabla(u_h - L_h u)\|_0$ being less than or equal to $\|\nabla u - \nabla L_h u\|_0$, which is not enough for our purposes of post-processing. Therefore, we will use an alternative analysis to show that, under additional assumptions on the mesh, and with $s = 3$ if $n \leq 5$ and $s > n/2$ if $n \geq 6$,

$$\forall u \in H^s(\Omega) \cap H_0^1(\Omega),\ \forall v_h \in V_{0h}, \quad |(\nabla u - \nabla L_h u, \nabla v_h)| \leq Ch^2 |u|_3 |\nabla v_h|_0. \quad (3)$$

According to the Sobolev Embedding Theorem, the condition $s > n/2$ assures that u has a representation as a continuous function, and hence that $L_h u$ is well-defined. The condition $s \geq 3$ comes from a Bramble-Hilbert argument in the proof of (3), and obviously $s = 3$ is necessary and sufficient to get the factor h^2 in (3) for all $n \leq 5$. This explains why no attention needed to be paid to this aspect in the existing proofs for space dimension up to three.

1.3 A more general setting for mixed finite elements

It is possible to formulate our objectives more generally. For this, we recall the notion of simplicial Nédélec [28] mixed finite elements in arbitrary dimension. These elements are used in the discretization of problems in which the unknowns are vector fields. Connected with the lowest order spaces $\mathbf{N}_h^0$ is an interpolation operator $\mathbf{\Pi}_h$ defined for vector fields $\mathbf{q} \in W \subset [H^2(\Omega)]^n$ having a certain smoothness. Denoting by $\tau(e)$ a unit vector along a given edge e, it is well-known that there exists a unique $\mathbf{\Pi}_h \mathbf{q} \in \mathbf{N}_h^0$ such that for all edges e

$$\int_e (\mathbf{q} - \mathbf{\Pi}_h \mathbf{q})^T \tau(e) \mathrm{d}e = 0. \quad (4)$$

For this interpolation operator, it typically holds that [28]

$$\forall \mathbf{q} \in W, \quad \|\mathbf{q} - \mathbf{\Pi}_h \mathbf{q}\|_0 \leq Ch|\mathbf{q}|_1. \quad (5)$$

An interesting fact is, that the subspace $\mathbf{Z}_h$ of curl-free elements of $\mathbf{N}_h^0$ satisfies $\nabla V_{0h} \subset \mathbf{Z}_h \subset \nabla V_h$. Equality to either V_h or V_{0h} arises when the equation to discretize has homogeneous boundary conditions of either Dirichlet or Neumann type, whereas mixed boundary conditions result in a $\mathbf{Z}_h$ strictly in between them. Moreover, if $\mathbf{q} = \nabla u$, then whenever both interpolants are well-defined, we have $\mathbf{\Pi}_h \mathbf{q} = \nabla L_h u$, which is expressed in the following commuting diagram:

$$\begin{array}{ccc} H^s(\Omega) & \xrightarrow{\quad\nabla\quad} & W \\ L_h \Big\downarrow & & \Big\downarrow \mathbf{\Pi}_h \\ V_{0h} & \xrightarrow[\quad\nabla\quad]{} & \mathbf{N}_h^0 \end{array}$$

Therefore, as a generalization of (3), we would like to show that

$$\forall \mathbf{q} \in W, \quad \forall v_h \in V_{0h}, \quad |(\mathbf{q} - \mathbf{\Pi}_h \mathbf{q}, \nabla v_h)| \leq C h^2 |\mathbf{q}|_2 |\nabla v_h|_0. \tag{6}$$

In this paper, we will prove (6) for all n and with the most general possible choice for $W \subset [H^2(\Omega)]^n$. For $n = 2$, (6) yields supercloseness results not only in triangular lowest order Nédélec elements, and hence through (3) for the standard finite element method. In [6, 7] it was moreover shown that also for lowest and one-but-lowest Raviart-Thomas triangular mixed finite elements for elliptic equations relation (6) yields superconvergence. Of central importance in that context is the property that curls are gradients that are rotated pointwise over $\pi/2$. In three space dimensions and up, this is not the case anymore. This complicates the analysis considerably. We refer to [17] for the definition and a summary of main results concerning the three-dimensional mixed elements. Denoting the lowest order $\mathbf{H}(\mathrm{div}\,;\Omega)$-conforming Raviart-Thomas space by $\mathbf{RT}_h^0$, in [17] it is moreover shown on page 275 that for all $\mathbf{q}_h \in \mathbf{RT}_h^0$, $\mathbf{q}_h$ is divergence-free if and only if $\mathbf{q}_h \in \mathbf{curl}\, \mathbf{N}_h^0$. So even though in three and higher dimensions, the relationship between Raviart-Thomas and Nédélec elements is more obscure, gradients and curls of continuous (vector-) piecewise linear functions are present, either implicitly or explicitly, as subspaces of physically relevant quantities. This motivates why we will pay special attention to them in this paper.

1.4 Outline

In Section 2, we will concentrate on simplicial partitions of n-dimensional polytopes and define the concept of uniformity. For $n = 2$ this will coincide with the usual definition of uniform triangulations, which is that any two triangles sharing an edge form a parallelogram. In Section 3, we derive results for continuous piecewise linear functions on such partitions. The main feature is that their gradients can be written as linear combinations of mutually orthogonal functions with small and point symmetric support. This point symmetry will be the basis of the proof of the strengthened Cauchy-Schwartz inequality (6). In Section 4 we will concentrate on post-processing, resulting in superconvergence. Finally, once the success of the post-processing on the gradient of the finite element solution has been shown, we remove the additional smoothness assumptions that were due to the definition of the interpolant by a density argument.

2 Simplicial partitions of polytopes

It is not standard terminology what is meant by a uniform simplicial partition of an n-dimensional polytope for $n > 3$, so we will concentrate on this in Section 2.1. In Section 2.2, we prove that for each n there exist n-simplicial partitions having the required properties.

2.1 Regular families of uniform partitions

We will be interested in *regular* families $(\Delta_h)_h$ of simplicial partitions of the polytope $\overline{\Omega} \subset I\!R^n$. By this we mean that $(\Delta_h)_h$ satisfies:

(R) There exists $\alpha > 0$ such that for all $\Delta_h \in (\Delta_h)_h$ and for all n-simplices $S \in \Delta_h$, the ratio of the volume of S and the volume of the circumscribed n-sphere S (see [15]) is larger than α.

Moreover, we assume that each partition $\Delta_h \in (\Delta_h)_h$ is *uniform*, which means that it satisfies the following *uniformity* conditions:

(U1) There exist n linearly independent unit vectors $\chi_1, \ldots, \chi_n$ such that for each $S \in \Delta_h$ and each $j \in \{1, \ldots, n\}$, the simplex S has an edge parallel to χ_j ($j = 1, \ldots, n$).

(U2) For each internal edge $e \not\subset \partial\Omega$ in one of the directions χ_j ($j = 1, \ldots, n$), the patch of simplices sharing e is point symmetric with respect to the midpoint of e.

Remark 2.1 In two space dimensions, it is clear that (U2) implies (U1), and after some puzzling one can show the same for $n = 3$. It is unclear, though not unlikely, that the same implication also holds for $n > 3$. The converse implication (U1) $\Rightarrow$ (U2) only holds in the trivial case $n = 1$.

2.1.1 Two-dimensional meshes

For $n = 2$, it can easily be checked that the so-called *three-directional* mesh satisfies both (U1) and (U2). Historically, this is the partition that is often called uniform, so this justifies our definition of uniformness for higher dimensions. The two-dimensional *chevron mesh* and the *criss-cross mesh* clearly satisfy (U1) whereas they do not satisfy (U2). See Figure 1 for the three mesh types just mentioned.

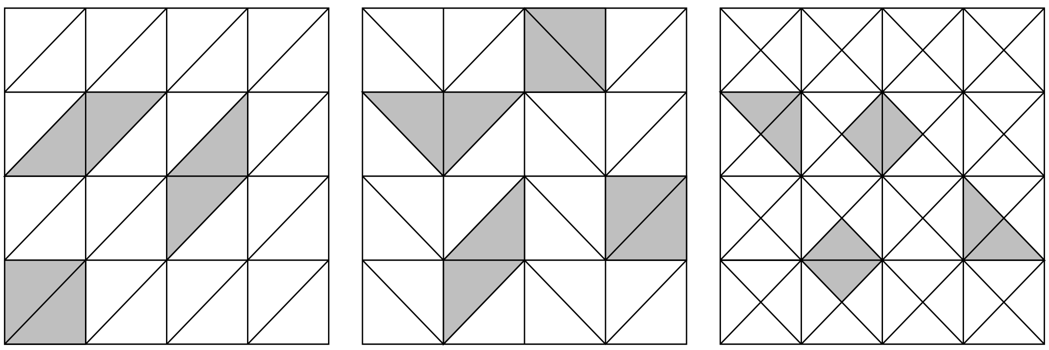


Figure 1. Three-directional mesh (left), chevron, and criss-cross mesh (right).

Remark 2.2 Note that the so-called *Union Jack* mesh, which is often seen [3] as another, differently structured mesh-type in two dimensions, is actually nothing else than the criss-cross mesh rotated about 45 degrees.

For the three-directional mesh, each choice of two independent directions χ_1, χ_2 out of the three directions present in the mesh, satisfies both (U1) and (U2). For the chevron mesh, there is only one possibility to choose χ_1 and χ_2 satisfying (U1), namely, the x- and y-direction. The patches corresponding to the y-direction are, however, not symmetric with respect to a point. For the criss-cross mesh, a similar observation holds for the two diagonal directions. This shows that (U1) and (U2) are not equivalent.

2.1.2 Three-dimensional meshes

One of the standard ways to construct a partition into tetrahedra in three space dimensions, is to first decompose the domain into parallelepipeds (with blocks and cubed as special cases) and then to continue according to Figure 2, in which we took the cube as example.

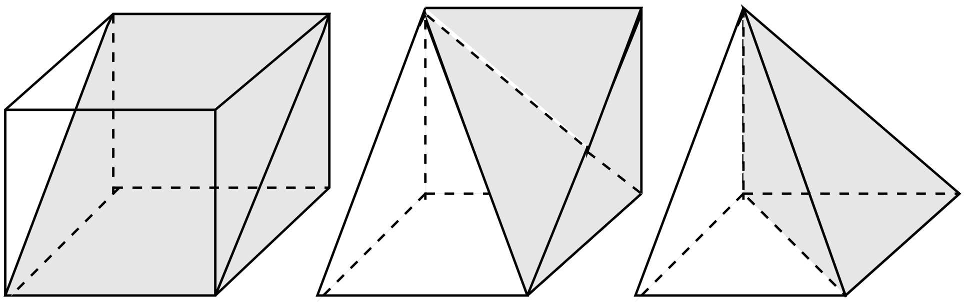


Figure 2. Partitioning of a cube into six tetrahedra.

This leads to a uniform partition, by choosing for χ_1, χ_2, χ_3 the three edge directions that define the parallelepipeds. Depicted in Figure 3 are the patches corresponding to different edges in the partition.

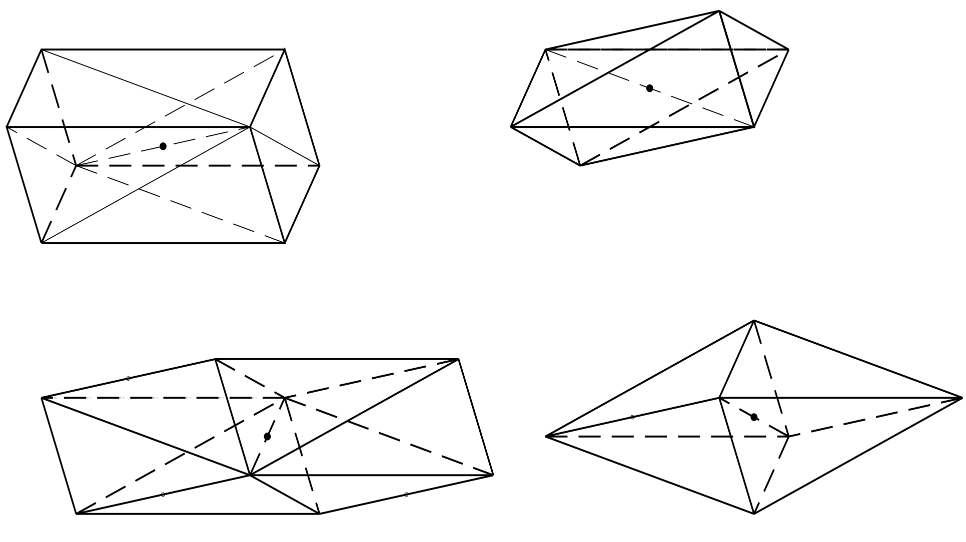


Figure 3. Point symmetric patches around edges.

Each patch is symmetric with respect to a point. Note that patches corresponding to the block-directions χ_1, χ_2 and χ_3 and to the direction of the longest diagonal are space-fillers. As opposed to the two-dimensional case, there exist directions in the mesh that are not present in all tetrahedra. The patches corresponding to these directions do not fill space.

2.2 Construction of uniform n-simplectic partitions

We will now construct partitions of $\overline{\Omega} \subset \mathbb{R}^n$ into n-simplices that satisfy conditions (U1)-(U2). At the basis lies the generalization of the following description of the partition of the unit square into two congruent triangles S_{12} and S_{21},

$$S_{12} = \{x \in \mathbb{R}^2 \mid 0 \le x_1 \le x_2 \le 1\} \quad \text{and} \quad S_{21} = \{x \in \mathbb{R}^2 \mid 0 \le x_2 \le x_1 \le 1\}. \tag{7}$$

By Σ^n we denote the group of all $n!$ permutations of the numbers $1, \ldots, n$. We will write $\sigma(j) \in \{1, \ldots, n\}$ for the j-th component of $\sigma \in \Sigma^n$. Recall [5] that a path simplex is a simplex of which the mutually orthogonal edges form a path.

Lemma 2.3 *The unit n-hypercube $K^n = [0,1]^n$ can be decomposed into $n!$ path simplices of dimension n such that n of the $\frac{1}{2}n(n+1)$ edges of each simplex coincide with n orthonormal edges of K^n.*

Proof. It is well-known that K^n can be decomposed into n-simplices S_σ ($\sigma \in \Sigma^n$) where S_σ is elegantly described as

$$S_\sigma = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \le x_{\sigma(1)} \le \cdots \le x_{\sigma(n)} \le 1\}. \tag{8}$$

Now, let $\sigma \in \Sigma^n$ and $j \in \{1, \ldots, n\}$ be given. According to (8), $x_{\sigma(j)}$ can, within S_σ, range from zero to one if (and actually, only if) $x_{\sigma(i)} = 0$ for all $i < j$ and $x_{\sigma(i)} = 1$ for all $i > j$. This is exactly at an edge of K^n. The path starting at the origin and consisting of edges in the $x_{\sigma(n)}, x_{\sigma(n-1)}, \ldots, x_1$ directions consecutively, and ending at $(1, 1, \ldots, 1)$, lies in S_σ. This proves that it is a path simplex. QED

A more descriptive characterization of the simplices that build up the hypercube is the following. First note, that each of the simplices contains the origin $\mathcal{O}$ as well as the point $\mathcal{P}$ furthest away from the origin with all coordinates equal to one. Hence, the longest diagonal of length $\sqrt{n}$ is shared by all simplices. The path simplex S_σ is then defined by walking from $\mathcal{O}$ to $\mathcal{P}$ along n edges of the cube: first in the direction $x_{\sigma(n)}$, then in the direction $x_{\sigma(n-1)}$, and so on, as illustrated in Figure 4.

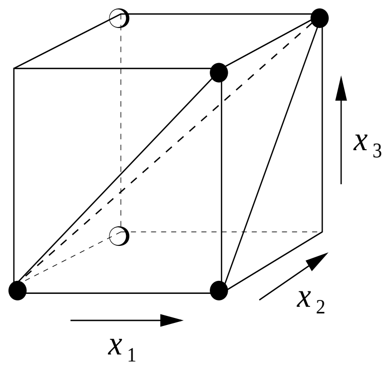


Figure 4. The simplex defined by $\sigma = (2,3,1)$ is the one starting at the origin and following the path of orthogonal edges of the cube in respective directions x_1, x_3 and x_2. Its nodes are black bullets in the picture. Starting at the origin and walking in the reverse order, i.e., along edges in the directions x_2, x_3 and x_1 consecutively, defines the simplex that lies point symmetrically with respect to the center to the original one. Its two missing nodes are open bullets.

Lemma 2.4 *Let $\Delta = \{S_\sigma \mid \sigma \in \Sigma^n\}$ be the partition of K^n into n-simplices according to (8). Define $\rho \in \Sigma^n$ by $\rho(j) = n + 1 - j$ $(j = 1, \ldots, n)$ and let $R : K^n \to K^n$, $x \mapsto 2M x$ be the reflection with respect to the center M of K^n. Then,*

$$S_{\sigma \circ \rho} = R(S_\sigma). \tag{9}$$

Proof. Let $s = (s_1, \ldots, s_n) \in S_\sigma$. Then $0 \le s_{\sigma(1)} \le \ldots \le s_{\sigma(n)} \le 1$. The permutation that orders the coordinates of $R(s) = (1 - s_1, \ldots, 1 - s_n)$ is the reverse ordering of σ, which is $\sigma(n), \ldots, \sigma(1)$, and this is exactly ρ. QED

The point symmetry property is clearly visible from (7) for $n = 2$. For $n = 3$, it is depicted in Figure 4. An important consequence of Lemmas 2.3 and 2.4 are the properties (U1)-(U2) for the partition Δ_h of $\mathbb{R}^n$ that is defined by translation of the partitioned hypercube over all $v \in \mathbb{Z}^n$.

Theorem 2.5 *Let Δ_I be the partition of $\mathbb{R}^n$ into n-simplices defined by translation over all $v \in \mathbb{Z}^n$ of the unit n-hypercube K^n partitioned according to Lemma 2.3. Then Δ_I is face-to-face and satisfies properties (U1)-(U2) with $(\chi_j)_{j \le n}$ the canonical basis of $\mathbb{R}^n$.*

Proof. (i) Note that each two faces of K^n that are point symmetrically opposite to each other with respect to the center M, are themselves point symmetrically partitioned with respect to their centers. Therefore, translation over unit distance in the direction orthogonal to those faces results in a face-to-face contact. Now, according to Lemma 2.3, the partition Δ of K^n satisfies (U1) with $(\chi_j)_{j \le n}$ the canonical basis of $\mathbb{R}^n$. Simplices that are formed through translation of K^n in the canonical directions clearly inherit this property.

(ii) According to Lemma 2.4, the partition Δ of K^n is point symmetric with respect to its center M. This means that also each neighboring hypercube $\tilde{K}^n$ that is obtained through translation over $0 \ne v \in \mathbb{Z}^n$ with $v(j) \in \{-1, 0, 1\}$, is point symmetrically partitioned with respect to its center $\tilde{M}$. But then the union of K^n and $\tilde{K}^n$ is also point symmetric with respect to $(M + \tilde{M})/2$, which is a midpoint of an edge. So each simplex $S \subset K^n$ satisfying (8) has an image $\tilde{S} \subset \tilde{K}^n$ reflected in $(M + \tilde{M})/2$. This holds in particular for any simplex containing the point $(M + \tilde{M})/2$, proving (U2). QED

Clearly, not only the partition Δ_I from Theorem 2.5 satisfies the uniformity conditions (U1)-(U2), but also transformations and subsets of it in the following sense.

Corollary 2.6 *Each nonsingular $n \times n$ matrix A induces a uniform simplectic partition of $\mathbb{R}^n$ by the formula*

$$\Delta_A = \{AS \mid S \subset \mathbb{R}^n, S \in \Delta_I\}, \tag{10}$$

where the normalized columns of A form the directions χ_j from (U1). Given a polytope $\Omega \subset \mathbb{R}^n$,

$$\Delta_{A,\Omega} = \{S \in \Delta_A \mid S \subset \overline{\Omega}\} \tag{11}$$

is a uniform simplectic partition of Ω if each $S \in \Delta_A$ is either inside or essentially outside $\overline{\Omega}$.

Several techniques that tile $\mathbb{R}^n$ into congruent n-simplices are described in [14].

3 Linear splines and their derivatives

Now that we have defined the partitions on which to define continuous linear splines, we can concentrate on some special properties of directional derivatives of such linear splines which follow from the uniformity of the partition. First, we define $V_{0h} = V_h \cap H_0^1(\Omega)$, where

$$V_h = \{v \in H^1(\Omega) \mid \forall S \in \Delta_{A,\Omega}, v_{|S} \text{ is linear}\}. \tag{12}$$

So we implicitly assume that we are dealing with a polytope $\overline{\Omega} \subset I\!R^n$ that can be uniformly partitioned into n-simplices, using the columns of some nonsingular $n \times n$ matrix A to define the directions χ_j from property (U1). These directions give rise to a corresponding set of directional derivatives $\partial\chi_j$ as follows,

$$\nabla_A = A^T\nabla = \begin{pmatrix} \chi_1^T\nabla \\ \vdots \\ \chi_n^T\nabla \end{pmatrix} = \begin{pmatrix} \partial_{\chi_1} \\ \vdots \\ \partial_{\chi_n} \end{pmatrix}. \tag{13}$$

It is easy to see that for all $v \in H^1(\Omega)$,

$$\sigma_1 \|\nabla v\|_0 \leq \|\nabla_A v\|_0 \leq \sigma_n \|\nabla v\|_0, \tag{14}$$

where σ_1 and σ_n are the smallest and largest singular values of A, respectively. Finally, we will from now on assume that Ω is convex. This implies that the Poisson problem is H^2-regular on Ω.

3.1 Constant directional derivatives on patches

The main observation that we wish to point out is not very difficult, and can easily be understood by looking at the two-dimensional case. For convenience, we first introduce some notations.

Definition 3.1 Let $\Delta_{A,\Omega}$ be a uniform partition of $\overline{\Omega}$. Given $j = 1, \ldots, n$, let E_j be the set of all edges of $\Delta_{A,\Omega}$ parallel to χ_j. For $e \in E_j$, let P_e be the union of all $S \in \Delta_{A,\Omega}$ for which e is an edge. We will call P_e a *patch*. Let M_e be the midpoint of e and ϕ_e the characteristic function of P_e.

Lemma 3.2 *Let $v_h \in V_h$ and $j \in \{1, \ldots, n\}$. Then for each $e \in E_j$, $\partial_{\chi_j} v_h$ is constant on P_e. The patches P_e form a partition of $\overline{\Omega}$. If $e \not\subset \partial\Omega$, then P_e is point symmetric with respect to M_e. If $e \subset \partial\Omega$ and $v_h \in V_{0h}$, then $\partial_{\chi_j} v_h$ is zero on P_e.*

Proof. Given $e \in E_j$, the derivative $\partial_\chi v_h$ is for each $S \subset P_e$ determined by two different values of v_h on e, and hence constant on P_e. By property (U1), each $S \in \Delta_{A,\Omega}$ has exactly one edge $e \in E_j$ so the union of the patches covers the domain. If v_h is zero on $e \subset \partial\Omega$, any derivative tangential to the boundary vanishes. QED

Corollary 3.3 *Let $\Delta_{A,\Omega}$ be a uniform partition of $\overline{\Omega}$. Then each $v_h \in V_h$ has an expansion*

$$\partial_{\chi_j} v_h = \sum_{e \in E_j} \alpha_e \phi_e, \tag{15}$$

with coefficients $\alpha_e \in I\!R$. Moreover, if $v_h \in V_{0h}$ and if $e \subset \partial\Omega$ then $\alpha_e = 0$.

Now we are able to extend the supercloseness result (1) to arbitrary space dimensions by giving a proof for (6).

Theorem 3.4 *Let $(\Delta_h)_h$ be a regular family of uniform partitions of the polytope $\overline{\Omega}$. Let $\mathbf{q} \in [H^2(\Omega)]^n$ be such that $\mathbf{\Pi}_h\mathbf{q}$ is well-defined. Then for all $v_h \in V_{0h}$,*

$$|(\mathbf{q} - \mathbf{\Pi}_h\mathbf{q}, \nabla v_h)| \leq Ch^2|\mathbf{q}|_2|\nabla v_h|_0. \tag{16}$$

Proof. By definition of ∇_A, and writing $\mathbf{e}_j$ for the j-th column of the $n\times n$ identity matrix, we have

$$|(\mathbf{q} - \mathbf{\Pi}_h\mathbf{q}, \nabla v_h)| = |(A^{-1}(\mathbf{q} - \mathbf{\Pi}_h\mathbf{q}), \nabla_A v_h)| = |\sum_{j=1}^{n}(\mathbf{e}_j^T A^{-1}(\mathbf{q} - \mathbf{\Pi}_h\mathbf{q}), \partial_{\chi_j} v_h)|.$$

Let $j \in \{1, \ldots, n\}$ be given. Use decomposition (15) for $\partial_{\chi_j} v_h$ and consider for some internal $e \in E_j$ the single term

$$F(\mathbf{q}) = (\mathbf{e}_j^T A^{-1}(\mathbf{q} - \mathbf{\Pi}_h\mathbf{q}), \phi_e). \tag{17}$$

Using the Cauchy-Schwarz inequality and a priori bound (5), we find that

$$|F(\mathbf{q})| \leq Ch|\mathbf{q}|_{1,P_e}|\phi_e|_{0,P_e}. \tag{18}$$

For any constant vector field $\mathbf{r}$ on P_e we have that $F(\mathbf{r}) = 0$ because $\mathbf{r} = \mathbf{\Pi}_h\mathbf{r}$. However, also for linear vector fields $\mathbf{r}$ we get $F(\mathbf{r}) = 0$ because we may assume that $\mathbf{r}$ is odd at the center of gravity of P_e. In that case, its interpolant $\mathbf{\Pi}_h\mathbf{r}$ is odd too, with zero mean as a result. Following a Bramble-Hilbert like argument, we then find, using (18), that

$$|F(\mathbf{q})| = |F(\mathbf{q} - \mathbf{r})| \leq Ch|\mathbf{q} - \mathbf{r}|_{1,P_e}|\phi_e|_{0,P_e}, \tag{19}$$

for all linear vector fields $\mathbf{r}$. Then, standard approximation theory yields that

$$|F(\mathbf{q})| \leq Ch^2|\mathbf{q}|_{2,P_e}|\phi_e|_{0,P_e}. \tag{20}$$

By mutual orthogonality of all ϕ_e, this finally results in

$$\begin{aligned} |(A^{-1}(\mathbf{q} - \mathbf{\Pi}_h\mathbf{q}), \nabla_A v_h)| &\leq Ch^2\sum_{j=1}^{n}\sum_{e\in E_j}|\mathbf{q}|_{2,P_e}|\alpha_e\phi_e|_{0,P_e} \leq Ch^2|\mathbf{q}|_2\left(\sum_{e\in E}|\alpha_e\phi_e|_{0,P_e}^2\right)^{\frac{1}{2}} \\ &\leq Ch^2|\mathbf{q}|_2|\nabla_A v_h|_0 \leq Ch^2|\mathbf{q}|_2|\nabla v_h|_0, \end{aligned} \tag{21}$$

the latter inequality is due to (14). This proves the statement. QED

The proof of this theorem serves as a basis for the following result, which can be used to prove superconvergence for elliptic problems with variable coefficients.

Theorem 3.5 *Let B be an $n\times n$ matrix-valued function on Ω with bounded total derivative D_B. Then, under the conditions of Theorem 3.4,*

$$|(B(\mathbf{q} - \mathbf{\Pi}_h\mathbf{q}), \nabla v_h)| \leq Ch^2\left(|\mathbf{q}|_1 + |\mathbf{q}|_2\right)|\nabla v_h|_0. \tag{22}$$

Proof. Let P_e be a patch. Then, by the Mean Value Theorem, for each $x \in P_e$ there exists a convex combination ξ of x and M_e such that $B(x) = B(M_e) + D_B(\xi(x))(x - M_e)$. Then, (17) changes into

$$|(\mathbf{e}_j^T BA^{-1}(\mathbf{q} - \mathbf{\Pi}_h\mathbf{q}), \phi_e)| \leq |(\mathbf{e}_j^T B(M_e)A^{-1}(\mathbf{q} - \mathbf{\Pi}_h\mathbf{q}), \phi_e)| + Ch^2\|D_B\|_\infty|\mathbf{q}|_{1,P_e}|\phi_e|_{0,P_e}. \tag{23}$$

The first term in the right-hand side can be treated similarly as in the proof of Theorem 3.4 because $B(M_e)$ is just a constant factor, and the result follows easily. QED

4 Post-processing

Post-processing by an operator K_h is based on the property that K_h applied to the gradient of a quadratic function, locally recovers this gradient exactly. Together with an application of a Bramble-Hilbert-like argument, this results in an increase of the approximation order. The procedure, of which the computational costs are negligible compared to the computation of u_h, is a fairly standard idea from the superconvergence community (see, for example [25, 26]), but now extended to higher dimensional problems.

4.1 Main idea and implementation of the post-processing

Let $S \in \Delta_{A,\Omega}$ be given, and q a quadratic polynomial on S. Then for each j we have

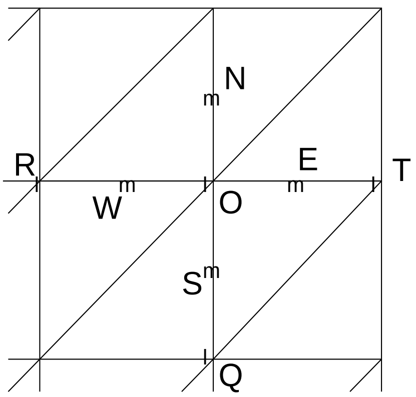
$$\partial_{\chi_j}(L_h q)(M_e) = \partial_{\chi_j} q(M_e) \tag{24}$$

at the midpoint M_e of the edge of S in the direction χ_j. Basically, this states that the top of a parabola is situated at the average of its zeros. This means that if $\nabla L_h q$ is given on a patch of elements that is large enough, it is possible to recover ∇q. We will give an example how this reconstruction may take place.

Illustration of the reconstruction process on a two-dimensional domain

In Figure 2, we see a two-dimensional partition with mesh size h. Suppose the piecewise constant gradient of the interpolant $L_h q$ of some unknown quadratic polynomial q is given. We will now recover the linear vector field ∇q at the vertices O, Q and T.

Figure 5. Post-processing of the gradient of the continuous piecewise linear interpolant of a quadratic function.



Linear interpolation between those three vectors will then result in ∇q. We will start with finding the exact value of ∇q at the point O. Around O, we see four midpoints of edges, denoted by N, E, S, and W. Since q is quadratic, by (24) we have

$$\frac{\partial}{\partial x}(L_h q)(E) = \frac{\partial}{\partial x} q(E) \quad \text{and} \quad \frac{\partial}{\partial x}(L_h q)(W) = \frac{\partial}{\partial x} q(W). \tag{25}$$

Since $\frac{\partial}{\partial x} q$ is linear on the line going through E and W, its value at O is simply the average of the values at E and W, so

$$\frac{\partial}{\partial x} q(O) = \frac{1}{2}\left(\frac{\partial}{\partial x}(L_h q)(E) + \frac{\partial}{\partial x}(L_h q)(W)\right). \tag{26}$$

Note that we can write this accordingly as

$$\frac{\partial}{\partial x}q(O) = \frac{(L_h q)(T) - (L_h q)(R)}{2h}, \tag{27}$$

which is in fact a reconstruction in the form of a so-called *long difference quotient*. Clearly, the y-derivative at the point O can be computed similarly, using the values of the y-derivative of $L_h q$ at the points N and S in the form (26), or the closest-by nodal values of $L_h q$ in the y-direction in the form (27). Finally, by doing the same at the points Q and T, we obtain the six scalar values that are necessary to find ∇q on the element OQT. Note that in case of non-orthogonal directional derivatives, the gradient can be easily recovered by the formula $\nabla v = A^{-T}\nabla_A$.

Remark 4.1 In case a node is at the boundary, we compute the exact value by extrapolation from the nearest suitable points. For example, supposing that in Figure 5 the point T is at the boundary, we compute $\frac{\partial}{\partial x}q(T)$ by extrapolating the x-derivatives of $L_h q$ at the points W and E (cf. [21]).

Definition 4.2 Let $v_h \in V_{0h}$. Then K_h denotes the linear operator that maps ∇v_h onto the continuous piecewise linear vector function $K_h \nabla v_h$ obtained by means of the reconstruction process which is the n-dimensional analogue of the one explained above for $n = 2$. For each n-simplex S, we denote the convex hull of the patch of elements that is needed to obtain the reconstructed function on S by $P(S)$.

The reconstruction process above can be applied to gradients of arbitrary functions $v_h \in V_{0h}$. If $v_h = \nabla L_h w$ with w smooth enough, then $K_h \nabla L_h w$ is an improved approximation of ∇w.

4.2 Post-processing of the interpolant

First we study the reconstruction process applied to gradients of interpolants. The following technical lemma will be needed. It is presented in an L^∞ setting, whereas the corresponding result in L^2 is formulated in Lemma 4.6.

Lemma 4.3 *Let $v_h \in V_{0h}$ be given. Then for all $S \in \Delta_{A,\Omega}$,*

$$||K_h \nabla v_h||_{\infty,S} \leq 2||\nabla v_h||_{\infty,P(S)}. \tag{28}$$

Proof. First, assume that S is an element whose nodes are not on $\partial\Omega$. Then the nodal values of $K_h \nabla v_h$ on S are convex combinations of values of partial derivatives of v_h on $P(S)$, therefore,

$$||K_h \nabla v_h||_{\infty,S} \leq ||\nabla v_h||_{\infty,P(S)}, \tag{29}$$

since the maximum of a linear function over an element is attained at one of the nodes. Second, if S is an element such that one of its nodes lies on $\partial\Omega$, then the reconstructed value is obtained by linear extrapolation, as mentioned in Remark 4.1. Since the extrapolation does not go over a longer distance than half the edge length (i.e., the length between the points E and T in Figure 2), we have

$$||K_h \nabla v_h||_{\infty,S} \leq 2||\nabla v_h||_{\infty,P(S)}. \tag{30}$$

This proves the statement. QED

A direct corollary is the following result for interpolants. If $n \leq 5$, we will set $s = 3$, whereas if $n \geq 6$, it will denote a number larger than $n/2$.

Corollary 4.4 *Let $w \in H^s(\Omega)$. Then for all $S \in \Delta_{A,\Omega}$,*

$$\|K_h \nabla L_h w\|_{\infty,S} \leq 2\|\nabla w\|_{\infty,P(S)}. \tag{31}$$

Proof. Since $w \in H^s(\Omega)$, by the Mean Value Theorem we have that

$$\|\nabla L_h w\|_{\infty,P(S)} \leq \|\nabla w\|_{\infty,P(S)}, \tag{32}$$

hence, the corollary follows directly from Lemma 4.3. QED

Theorem 4.5 *Let $(\Delta_h)_h$ be a regular family of uniform partitions of $\overline{\Omega}$. Then there exists a constant $C > 0$ such that for each $S \in \Delta_h$ and for all $w \in H^s(\Omega)$,*

$$\|\nabla w - K_h \nabla L_h w\|_{0,S} \leq Ch^2 |w|_{3,P(S)}. \tag{33}$$

Proof. Switching from the L^2-norm to the supremum-norm gives, using a crude triangle inequality, that for arbitrary $w \in H^s(\Omega)$,

$$\|\nabla w - K_h \nabla L_h w\|_{0,S} \leq Ch^{n/2}\|\nabla w - K_h \nabla L_h w\|_{\infty,S}$$

$$\leq Ch^{n/2}\left(\|\nabla w\|_{\infty,S} + \|K_h \nabla L_h w\|_{\infty,S}\right) \leq Ch^{n/2}\|\nabla w\|_{\infty,P(S)}, \tag{34}$$

where in the latter bound we have used Corollary 4.4. Since the constant C in (34) does not depend on w, the following holds for all polynomials q that are quadratic on $P(S)$, since on S we have that $\nabla q = K_h \nabla L_h q$ by construction, and thus

$$\|\nabla w - K_h \nabla L_h w\|_{0,S} = \|\nabla(w-q) - K_h \nabla L_h (w-q)\|_{0,S} \leq Ch^{n/2}\|\nabla(w-q)\|_{\infty,P(S)}. \tag{35}$$

Interpolation theory in Sobolev spaces ([11], p. 124) yields that by choosing for q the best approximation for w on $P(S)$ in the $W^{1,\infty}$ sense,

$$\|\nabla(w-q)\|_{\infty,P(S)} \leq Ch^{-n/2}h^2|w|_{3,P(S)}. \tag{36}$$

Combining (35) and (36), this Bramble-Hilbert approach proves the theorem. QED.

4.3 Post-processing of the finite element solution

We are now able to prove that our post-processing operator K_h is also successfully applicable to the gradient of the finite element approximation.

Lemma 4.6 *Let $(\Delta_h)_h$ be a regular family of uniform partitions of $\overline{\Omega}$. Then there exists a constant $C > 0$ such that for each $S \in \Delta_h$ and for all $v_h \in V_{0h}$,*

$$\|K_h \nabla v_h\|_{0,S} \leq C\|\nabla v_h\|_{0,P(S)}. \tag{37}$$

Proof. Working through the supremum-norm gives

$$\|K_h \nabla v_h\|_{0,S} \leq Ch^{n/2}\|K_h \nabla v_h\|_{\infty,S} \leq Ch^{n/2}\|\nabla v_h\|_{\infty,P(S)}, \tag{38}$$

using Lemma 4.3. Consider the discrete inverse inequality ([11], p. 142) for continuous piecewise linear finite element functions, which states that

$$\|\nabla v_h\|_{\infty,S} \leq Ch^{-n/2}\|\nabla v_h\|_{0,S}. \tag{39}$$

Applying it to (38) proves the statement. QED

Theorem 4.7 *Let $(\Delta_h)_h$ be a regular family of uniform partitions of $\overline{\Omega}$, and suppose that the solution u is in $H^s(\Omega)$. Then we have*

$$\|\nabla u - K_h \nabla u_h\|_{0,\Omega} \leq Ch^2 |u|_{3,\Omega}. \tag{40}$$

Proof. We start with a simple triangle inequality,

$$\|\nabla u - K_h \nabla u_h\|_{0,\Omega} \leq \|\nabla u - K_h \nabla L_h u\|_{0,\Omega} + \|K_h \nabla (L_h u - u_h)\|_{0,\Omega}. \tag{41}$$

The first term in the right-hand side of (41) can be bounded by splitting it into contributions over each element, and applying Theorem 4.5. The second term in the right-hand side of (41) can be related to (1) by applying Lemma 4.6 and Theorem 3.4 which results in

$$\|K_h \nabla (L_h u - u_h)\|_{0,\Omega} \leq C \|\nabla (L_h u - u_h)\|_{0,\Omega} \leq Ch^2 |u|_{3,\Omega}. \tag{42}$$

By adding the two contributions, we arrive at the bound (40). QED

4.4 Density argument

So far, in order to have a well-defined nodal interpolant, we have assumed that the solution $u \in H^s(\Omega)$ with $s = 3$ if $n \leq 5$ and $s > n/2$ if $n \geq 6$. The latter assumption is, however, an artifact of the proof. Indeed, the formulation of Theorem 4.7 above does not involve the nodal interpolant, but still suffers from the assumption $u \in H^s(\Omega)$. We will now use a density argument to prove (40) for all $u \in H^3(\Omega)$.

Let $F_h : H_0^1(\Omega) \to V_{0h}$ be the Galerkin projection, defined as usual by $(\nabla F_h w, \nabla v_h) = (\nabla w, \nabla v_h)$ for all $v_h \in V_{0h}$. Then the choice $v_h = F_h w$ shows that

$$\|\nabla F_h w\|_0 \leq \|\nabla w\|_0 \tag{43}$$

for all $w \in H_0^1(\Omega)$. Now, let $u \in H^3(\Omega)$ and $\varepsilon > 0$ be given. Then $L_h u = \nabla u - K_h \nabla F_h u$ is well-defined, and by density of $H^s(\Omega)$ in $H^3(\Omega)$ there exists a function $w \in H^s(\Omega)$ such that $\|u - w\|_3 \leq \varepsilon$. Moreover,

$$\|L_h(u - w)\|_0 \leq \|\nabla(u - w)\|_0 + \|K_h \nabla F_h (u - w)\|_0 \leq (1 + C)\|\nabla(u - w)\|_0 \leq C\varepsilon, \tag{44}$$

where we have used Lemma 4.6 globally to get rid of K_h, and (43) to get rid of F_h. This results in

$$\|L_h u\|_0 \leq \|L_h(u - w)\|_0 + \|L_h w\|_0 \leq C\varepsilon + Ch^2 |w|_3. \tag{45}$$

Corollary 4.8 *Suppose that a regular family of uniform partitions is used in the discretization and that the solution u is in $H^3(\Omega)$. Then,*

$$\|\nabla u - K_h \nabla u_h\|_{0,\Omega} \leq Ch^2 |u|_{3,\Omega}. \tag{46}$$

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