Russian and American put options under exponential phase–type Lévy models

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Abstract. Consider the American put and Russian option [33, 34, 17] with the stock price modeled as an exponential Lévy process. We find an explicit expression for the price in the dense class of Lévy processes with phase–type jumps in both directions. The solution rests on the reduction to the first passage time problem for (reflected) Lévy processes and on an explicit solution of the latter in the phase–type case via martingale stopping and Wiener-Hopf factorisation. Also the first passage time problem is studied for a regime switching Lévy process with phase-type jumps. This is achieved by an embedding into a a semi-Markovian regime switching Brownian motion.

Key words: Lévy process, Markov additive process, first passage time, Wald martingale, Wiener-Hopf factorisation, Russian option.

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1 Introduction

Consider a model of a financial market with two assets, a savings account $B = \{B_t\}_{t \geq 0}$ and a asset $S = \{S_t\}_{t \geq 0}$. The evolution of B is deterministic, with

$$B_t = \exp(rt), \qquad r > 0,$$

and the asset price is random and evolves according to the exponential model

$$S_t = \exp(X_t),$$

where X is some Lévy process. If X has no jumps, it can be represented by $X_t = x + \sigma W_t + \mu t$, with $x, \mu \in \mathbf{R}$ and $W = \{W_t\}_{t \geq 0}$ a standard Wiener process; this is the classical Black-Scholes model. There has been considerable interest in replacing the classical Black-Scholes model by exponential Lévy models allowing also for jumps. This development is motivated by superior fits to the data and hence improved pricing formulas and hedging strategies, as well as by theoretical considerations outlined in [19].

The search for a special Lévy model to outperform the Black-Scholes model was initiated by Merton, with the jump-diffusion with Gaussian jumps, and continues nowadays in the work of Carr, Chang, Madan, Geman and Yor who propose the variance-gamma model [26, 14], of Eberlein who proposes the hyperbolic model [18], of Barndorff-Nielsen with the normal inverse Gaussian model [10] and of Kou who proposed a jump-diffusion with exponential jumps [24]. There are still many statistical issues which will need to be resolved before an appropriate replacement of the Black Scholes model can emerge. Our paper adresses only the issue of the analytical tractability of pricing certain perpetual American type options. We propose a jump-diffusion model, where the jumps are of *phase type* (e.g. [30, 5, 6], see further Section 2). On the one hand this model is rich enough, since phase type models are known to be dense in the class of all distributions, and on the other hand for many options the model is analytically tractable.

We illustrate this in the case of the American put option and the Russian option. The last one was originally introduced by Shepp and Shiryaev in the context of the Black-Scholes model [17, 33, 34, 21, 25]. The pricing of the Russian option rests on a well known reduction to the *first passage time problem* for a Lévy process reflected at its supremum, making it somewhat more difficult than the analogous problem for the unconstrained Lévy process (which is used to solve the pricing problem for barrier and perpetual

American options). We note that special solutions of this problem – see [9] and [29] – are currently available only under spectrally one sided Lévy models. The purpose of our note is to draw attention to the fact that under the phase–type assumption, easily implementable solutions for both the unconstrained and the reflected first passage time problems exist as well for *spectrally two sided* Lévy processes (and hence for the pricing of perpetual American put and Russian options). In fact, we show that the method employed – of obtaining barrier crossing probabilities via a martingale stopping approach – works equally for barrier problems under the much more general class of *regime switching* exponential Lévy models with phase–type jumps, or for the regime switching Brownian motion recommended for example by Guo [22]. Their analytical tractability suggests that this potentially very flexible class of models (which depart from the unrealistic assumption of independent increments of the Lévy models) deserves to be more fully investigated.

The rest of the paper is organised as follows. Section 2 presents the model, the problem and its reduction to the first passage time problems for (reflected) Lévy processes. The martingale stopping approach for reflected and nonreflected Lévy processes is reviewed in Section 3, including an application and explicit formulae for the pricing of the perpetual American put option. Finally, the solution of the first passage problem for reflected regime switching phase–type Lévy models via an embedding into a regime switching Brownian motion is presented in Section 4, with most proofs relegated to the Appendix.

2 Model and problem

We introduce now the model we consider.

2.1 Phase-type distributions

A distribution F on $(0, \infty)$ is *phase-type* if it is the distribution of the absorbtion time ζ in a finite continuous time Markov process $J = \{J_t\}_{t \geq 0}$ with one state Δ absorbing and the remaining ones $1, \dots, m$ transient. That is, $F(t) = \mathbb{P}(\zeta \leq t)$ where $\zeta = \inf\{s > 0 : J_s = \Delta\}$. The parameters are m, the restriction $\boldsymbol{T}$ of the full intensity matrix to the m transient states and the initial probability (row) vector $\boldsymbol{\alpha} = (\alpha_1 \dots \alpha_m)$ where $\alpha_i = \mathbb{P}(J_0 = i)$. For any $i = 1, \dots, m$, let t_i be the intensity of a transition $i \to \Delta$ and write $\boldsymbol{t} =$

$(t_1 \dots t_m)'$ for the (column) vector of such intensities. Note that $\boldsymbol{t} = -\boldsymbol{T}\mathbf{1}$, where $\mathbf{1}$ denotes a column vector of ones. It follows that the cumulative distribution F is given by:

$$1 - F(x) = \boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{T}x}\mathbf{1}, \tag{1}$$

the density is $f(x) = \boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{T}x}\boldsymbol{t}$ and the Laplace transform is $\hat{F}[s] = \int_0^\infty \mathrm{e}^{-sx}F(\mathrm{d}x) = \boldsymbol{\alpha}(s\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t}$. Note that $\hat{F}[s]$ can be extended to the complex plane except at a finite number of poles (the eigenvalues of $\boldsymbol{T}$).

Phase-type distributions include and generalize exponential distributions in series and/or parallel and form a dense class in the set of all distributions on $(0, \infty)$. They have found numerous applications in applied probability, see for example [5], [6] for surveys. Much of the applicability of the class comes from the probabilistic interpretation, in particular the fact that that the overshoot distributions $F(x + y)/(1 - F(x))$ belong to a finite vector space. More precisely, the overshoot distribution is again phase–type with the same m and $\boldsymbol{T}$ but α_i replaced by $\mathbb{P}(J_x = i|\zeta > x)$, which is reminiscent of the memoryless property of the exponential distribution ($m = 1$) and explains the availability of many matrix formulas which generalize the scalar exponential case.

2.2 Lévy phase-type models

Let $X = \{X_t\}_{t\geq 0}$ be a Lévy process defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ that satisfies the usual conditions. We consider X which can be represented as follows

$$X_t = x + \mu t + \sigma W_t + \sum_{k=1}^{N^{(+)}(t)} U_k^{(+)} - \sum_{\ell=1}^{N^{(-)}(t)} U_\ell^{(-)} \tag{2}$$

where $x \in \mathbf{R}$, W is standard Brownian motion, $N^{(\pm)}$ are Poisson processes with rates of arrival $\lambda^{(\pm)}$ and $U^{(\pm)}$ are i.i.d. random variables with respective jump size distributions $F^{(\pm)}$ of phase–type with parameters $m^{(\pm)}, \boldsymbol{\alpha}^{(\pm)}, \boldsymbol{T}^{(\pm)}$. All processses are assumed to be independent. Equivalently, for $s \in i\mathbf{R}$, the Lévy exponent κ of X, defined by $\kappa(s) = \log \mathbb{E}[\exp(s(X_1 - x))]$, is

$$\kappa(s) = s\mu + s^2\frac{\sigma^2}{2} + \lambda^{(+)}\big(\hat{F}^{(+)}[-s] - 1\big) + \lambda^{(-)}\big(\hat{F}^{(-)}[s] - 1\big) \tag{3}$$

where $\hat{F}^{(\pm)}[s] = \boldsymbol{\alpha}^{(\pm)}(s\boldsymbol{I} - \boldsymbol{T}^{(\pm)})^{-1}\boldsymbol{t}^{(\pm)}$. As above, $\kappa(s)$ can be extended to the complex plane except a finite number of poles; this extension will also be denoted by κ.

Any Lévy model may be approximated arbitrarily closely by processes of the form (2):

Proposition 1 *For any Lévy process X, there exists a sequence $X(n)$ of Lévy processes of the form (2) such that $X(n) \to X$ in $D[0,\infty)$.*

Proof Let d be some metric on D. Choose first $X'(n)$ as an independent sum of a linear drift, a Brownian component and a compound Poisson process such that $d(X, X'(n)) \leq 1/n$. Use next the denseness of phase–type distributions to find $X(n)$ of the form (2) with $d(X(n), X'(n)) \leq 1/n$. QED

Remark. The approximation in Proposition 1 is easy to carry out in practice: the compound approximation is obtained by just restricting the Lévy measure to $\{|x| > \epsilon\}$, and to get to phase–type jumps, the relevant methodology for fitting a phase–type distributions to a given distribution (or a set of data) is developed in [4] for traditional maximum likelihood and in [12] in a Bayesian setting.

In complete markets (with a unique risk-neutral martingale measure $\mathbb{P}^*$ under which $\mathbb{E}^*[\exp(X_t - x)] = \exp(rt)$ where r is the riskless discount rate), arbitrage free pricing is equivalent to computing expectations under this measure $\mathbb{P}^*$. Under the Lévy models we consider however, the market is incomplete, i.e. not all claims can be hedged against. In this case there are infinitely many equivalent martingale measures, and some choice must be made. We use here the so called Cramér-Esscher transform or exponential tilting proposed by Gerber and Shiu [20], which preserves the Lévy structure, and as shown in Chan [15], is indeed the solution to some of the most common criteria for selecting an equivalent martingale measure. Note that the Esscher transform preserves the phase-type structure of the log-price X (see Appendix A). From now on we assume that we are working *under the chosen equivalent martingale measure*. That is, we assume that the Lévy exponent κ satisfies under $\mathbb{P}$

$$\kappa(1) = \log \mathbb{E}[\exp(X_1 - x)] = r, \tag{EMM}$$

Remarks. - Many of the computations involving Lévy processes are based on finding the roots of the "Cramér-Lundberg equation (see [6] for terminology)

$$\kappa(s) = a \tag{4}$$

(for some a). From this perspective, working under the equivalent martingale measure means $s = 1$ is one of the roots of this equation when $a = r$.

- Using Appendix A, we can easily convert parameters of X under the real world measure into parameters under the Esscher transform and vice versa.

2.3 American put option

The α-discounted perpetual American put option with strike K gives the holder the right to exercise at any $\{\mathcal{F}_t\}$-stopping time τ yielding the pay-out

$$e^{-\alpha\tau}(K - S_\tau)^+, \qquad \alpha \geq 0, \tag{5}$$

where $c^+ = \max\{c, 0\}$. Recall that the process X satisfies (EMM). The arbitrage-free price under the martingale measure is given by

$$U^*(x) = \sup_\tau \mathbb{E}_x[e^{-(r+\alpha)\tau}(K - S_\tau)^+] \tag{6}$$

where the supremum runs over all $\{\mathcal{F}_t\}$-stopping times τ, $\mathbb{E}_x$ denotes the expectation with respect to the measure under which $\log S_0 = X_0 = x$.

2.4 The Russian option

The Russian option is an American type option which gives the holder the right to exercise at any almost surely finite $\{\mathcal{F}_t\}$-stopping time τ yielding payouts

$$e^{-\alpha\tau} \max\left\{M_0, \sup_{0\leq u\leq\tau} S_u\right\}, \qquad M_0 \geq S_0, \alpha > 0.$$

The constant M_0 can be viewed as representing the "starting maximum" of the stock price (say, the maximum over some previous period $(-t_0, 0]$). The positive discount factor α is necessary in the perpetual version to guarantee that it is optimal to stop in an almost surely finite time and the value is finite. Since X satisfies (EMM), the arbitrage-free price of the Russian option for this martingale measure is given by

$$V^*(x, m) = \sup_\tau \mathbb{E}_x\left[e^{-(r+\alpha)\tau} \max\left\{e^m, \sup_{0\leq u\leq\tau} S_u\right\}\right] \tag{7}$$

where the supremum is taken over the set $\mathcal{T}$ of all almost surely finite $\{\mathcal{F}_t\}$-measurable stopping times and $m = \log(M_0)$.

Let $\overline{X}_t = \sup_{s \le t} X_s$ denote the supremum of the Lévy process and let $Y_t = \overline{X}_t \vee (m - x) - X_t$ denote the process reflected at its supremum level (started at $y = m - x$). The key simplification discovered by Shepp and Shiryaev (for the standard Black-Scholes model) is that the optimal stopping time must be of the form

$$\tau \ = \tau(k) = \ \inf\{t > 0 : Y_t \ge k\}, \tag{8}$$

i.e. τ must be the first time when the *reflected process* Y upcrosses a certain *constant* (positive) exercise level k^* (which may be found by solving a one dimensional optimisation problem). If X is given by a Lévy phase-type model (2), Theorem 1 below states that the optimal stopping time is still of the form (8). To be able to formulate the result we first introduce the first passage function

$$v_k(y) = v_k(y, a, b) = \mathbb{E}_y[\mathrm{e}^{-a\tau + b(Y_\tau - k)}] \tag{9}$$

of the crossing time (8), defined for $k, a \ge 0$ and with b such that $v_k(y, b)$ is finite. The subscript y in $\mathbb{E}_y$ indicates the fact that $Y_0 = y$.

Theorem 1 *The value function $V^*(x, m)$ of the* two dimensional stopping problem *(7) is given by:*

$$V^*(x, m) = \mathrm{e}^x v^*(m - x) \tag{10}$$

where $v^*(m - x)$ is the solution of the one dimensional stopping problem *of finding a function v^* and a $\tau^* \in \mathcal{T}$ such that*

$$v^*(y) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_y^{(1)}[\mathrm{e}^{-a\tau + Y_\tau}] = \mathbb{E}_y^{(1)}[\mathrm{e}^{-a\tau^* + Y_{\tau}^*}], \tag{11}$$

where $\mathbb{P}_y^{(1)}$ denotes the "tilted probability measure (for Y_t) given on $\mathcal{F}_t$ by $\mathrm{d}\mathbb{P}^{(1)}|_{\mathcal{F}_t} = \exp(X_t - x - \kappa(1)t)\mathrm{d}\mathbb{P}|_{\mathcal{F}_t}$*, with $Y_0 = y$.*

Furthermore, the optimal stopping time τ^* is the same in both problems, i.e. $\tau^* = \tau(k^*)$ with $k^* \ge 0$ given by

$$k^* = arg\ max_{k \ge 0} v_k(0)$$

Also, $\forall y, 0 \le \ y \le k^*$, we have: $k^* = arg\ max_{k \ge y} v_k(y)$.

In Appendix B we provide the proof. The same techniques can be used to prove that, if X is a general regime switching Lévy process satisfying some regularity conditions, the optimal stopping time is still a crossing time of the reflected process (where the level then may depend on the regime).

To explicitly solve our problem, the next goal will be the explicit evaluation of the first passage time function (9) required in (11). This may be achieved in principle by solving the corresponding Feynman-Kac integro-differential equation, which is tractable for this phase-type Lévy model and worked out in Appendix B. In the next section, however, we will follow a probabilistic approach, exploiting the probabilistic interpretation of phase-type distributions.

3 First passage time

3.1 First passage time for X

We first solve the first passage time problem for the Lévy process X. This problem consists in computing the joint moment generating function

$$u_k(x) = u_k(x, a, b) = \mathbb{E}_x[\mathrm{e}^{-aT+b(X_T-k)}] \tag{12}$$

of the crossing time

$$T = T(k) = \inf\{t > 0 : X_t \geq k\}$$

and of the overshoot $X_T - k$, with k, $a > 0$ and b such that $u_k(x)$ is finite. The subscript x in $\mathbb{E}_x$ refers to $X_0 = x$. At the crossing time $T(k)$, we must either have a upwards jump of X, or the component $\mu t + \sigma W_t$ must take the process X to the barrier k. Denote by G_0 the event that the last alternative occurs, by G_i, $i = 1, \ldots, m^{(+)}$, the event that the first occurs and the upcrossing of k occurs in phase i, i.e. that $J(k - X_{T(k)-}) = i$ where J is the underlying phase process for the jump causing the upcrossing, and by $M^{(\pm)}$ the set of all phases during which up- and downcrossing of a level may occur. Thus, calling the state where the Lévy process is moving continuously phase 0, $M^{(+)} = \{1, \ldots, m^{(+)}\}$ if the Brownian component is zero and if the drift points opposite to the barrier; otherwise, $M^{(+)} = \{0, \ldots, m^{(+)}\}$. Let $\pi_i = \mathbb{E}_x[\exp(-aT(k))G_i]$ denote the discounted probability of upcrossing in phase i, where $X_0 = x$. Moreover, let $\mathbf{1}_i$ denote a vector of zeros with a 1 on the ith position, $\boldsymbol{\pi} = (\pi_i, i \in M^{(+)})$, and let $\hat{\boldsymbol{f}}^{(+)}[-b]$ denote the vector (depending on the phase at the level crossing) of Laplace transforms at $-b$ of the overshoot $X_{T(k)} - k$. This vector can be analytically continuated to the complex plane except a finite number of poles (the eigenvalues of $-\boldsymbol{T}^{(+)}$). This analytic extension will also be denoted by $\hat{\boldsymbol{f}}^{(+)}$. Note that, if $0 \in M^{(+)}$, then the first component of $\hat{\boldsymbol{f}}^{(+)}[-b]$ is 1, and the other components are given by $(-b\boldsymbol{I} - \boldsymbol{T}^{(+)})^{-1}\boldsymbol{t}^{(+)}$ by the phase assumption and if $0 \notin M^{(+)}$, the first component is missing. Under the model (2) one can check that the function κ is the ratio of two polynomials of degree p and $m^{(+)} + m^{(-)}$ respectively, where $p = m^{(+)} + m^{(-)} + \epsilon$ with $\epsilon = 2, 1, 0$ according to whether $\sigma \neq 0$, $(\sigma = 0, \mu \neq 0)$ and $(\mu = \sigma = 0)$. Denote by $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_p)$ the roots of the Cramèr Lundberg equation

$$\kappa(\rho) = a \tag{13}$$

and let $\mathcal{I}^{\pm} = \{i : \pm\Re(\rho_i) > 0\}$ be the set of roots with positive or negative real part. We provide now a statement of the Wiener-Hopf factorization for our class of processes, together with a crucial consequence: the equality $\#M^{(\pm)} = \#\mathcal{I}^{\pm}$ between the "number of ways to cross an upper/lower boundary" and that of the number of roots with positive/negative real part of the Cramèr Lundberg equation (where a root is counted as many times as its multiplicity).

Let $M_a = \sup_{t \leq \eta(a)} X_t$ and $I_a = \inf_{t \leq \eta(a)} X_t$ be the supremum and infimum of X at an independent $\exp(a)$ random variable $\eta(a)$, respectively. Let for $\pm\Re(s) \geq 0$

$$\kappa_a^-(s) = \mathbb{E}[\exp(sI_a)], \qquad \kappa_a^+(s) = \mathbb{E}[\exp(sM_a)] \tag{14}$$

which are analytic for $\pm\Re(s) > 0$ respectively. Denoting their analytic continuation also by $\kappa^{\pm}$, they satisfy the Wiener-Hopf factorization $a/(a - \kappa(s)) = \kappa_a^+(s)\kappa_a^-(s)$ for all s (see e.g. [11, Thm. 1]). Thus, when $a > 0$ there are no roots of (13) with zero real part.

Lemma 1

1. *For processes (2) the Wiener-Hopf factors are given by:*

$$\kappa_a^{\pm}(s) = \frac{\prod_{i \in \mathcal{I}^{\pm}}(-\rho_i)}{\prod_{i \in \mathcal{I}^{\pm}}(s - \rho_i)} \cdot \frac{\det(\mp s\boldsymbol{I} - \boldsymbol{T}^{(\pm)})}{\det(-\boldsymbol{T}^{(\pm)})} \tag{15}$$

2. *Moreover,* $\# M^{(\pm)} = \# \mathcal{I}^{\pm}$.

3. *Supposing the roots of (13) with negative [positive] real part to be distinct,* $\mathbb{P}(-I_a \in \mathrm{d}x)$ $[\mathbb{P}(M_a \in \mathrm{d}x)]$ *is given by*

$$\sum_{j \in \mathcal{I}^-} A_j^-(-\rho_j)\mathrm{e}^{\rho_j x}\mathrm{d}x \qquad \left[\sum_{i \in \mathcal{I}^+} A_i^+ \rho_i \mathrm{e}^{-\rho_i x}\mathrm{d}x\right] \tag{16}$$

for $x > 0$ *where* $\boldsymbol{A}^{\pm} = (A_i^{\pm}, i \in \mathcal{I}^{\pm})$ *are the partial fractions coefficients of the expansions:*

$$\kappa_a^{\pm}(s) - \kappa_a^{\pm}(\mp\infty) = \textstyle\sum_{i \in \mathcal{I}^{\pm}} A_i^{\pm}\rho_i(\rho_i - s)^{-1}$$

and $\mathbb{P}(I_a = 0) = \kappa_a^-(\infty)$ $[\mathbb{P}(M_a = 0) = \kappa_a^+(-\infty)]$.

Remark. If the roots of (13) are distinct, the constants $\boldsymbol{A}^{\pm} = (A_i^{\pm}, i \in \mathcal{I}^{\pm})$ are given explicitly by

$$A_i^{\pm} = \frac{\prod_{j \in \mathcal{I}^{\pm}\setminus\{i\}}(-\rho_j)}{\prod_{j \in \mathcal{I}^{\pm}\setminus\{i\}}(\rho_i - \rho_j)} \frac{\det(\mp\rho_i \boldsymbol{I} - \boldsymbol{T}^{(\pm)})}{\det(-\boldsymbol{T}^{(\pm)})}, \qquad i \in \mathcal{I}^{\pm}. \tag{17}$$

In the case that the equation $\kappa(s) = a$ has multiple roots, let $n^{(\pm)}$ denote the number of *different* roots with positive/negative real part and $m^{(\pm,j)}$ the multiplicity of a root ρ_j with $j \in \mathcal{I}^{\pm}$. Then we find that for $k = 1, \ldots, m^{(\pm,j)}$ the coefficient $A_{j,k}^{\pm}$ of $(-\rho_j)^k/(s-\rho_j)^k$ in the partial fraction decomposition of $\kappa_a^{\pm}(s) - \kappa_a^{\pm}(\mp\infty)$ is given by

$$A_{j,k}^{\pm} = \frac{1}{(m-k)!}\frac{\mathrm{d}^{m-k}}{\mathrm{d}s^{m-k}}\frac{\kappa_a^{\pm}(s)(s-\rho_j)^m}{(-\rho_j)^k}\bigg|_{s=\rho_j} \qquad \text{with } m = m^{(\pm,j)}.$$

By straightforward Laplace inversion, we conclude that for $x > 0$ the measures $\mathbb{P}(-I_a \in \mathrm{d}x)$ and $\mathbb{P}(M_a \in \mathrm{d}x)$ are respectively given by

$$\sum_{j=1}^{n^{(-)}}\sum_{k=1}^{m^{(-,j)}} A_{j,k}^-(-\rho_j)\frac{(-\rho_j x)^{k-1}}{(k-1)!}\mathrm{e}^{\rho_j x}\mathrm{d}x \qquad \sum_{i=1}^{n^{(+)}}\sum_{k=1}^{m^{(+,i)}} A_{i,k}^+\rho_i\frac{(\rho_i x)^{k-1}}{(k-1)!}\mathrm{e}^{-\rho_i x}\mathrm{d}x.$$

Examples. For a spectrally negative Lévy process, (15) yields $\kappa_a^+(s) = \frac{\rho_+}{\rho_+ - s}$, where ρ_+ is the unique positive root of (13). For Kou's jump-diffusion with two-sided exponential jumps, (15) yields $\kappa_a^+(s) = \frac{\rho_1\rho_2}{(\rho_1-s)(\rho_2-s)}\frac{\mu_+-s}{\mu_+}$, where

ρ_1, ρ_2 are the positive roots and μ_+ is the rate of positive jumps. These explicit expressions are at the root of various explicit computations in the literature; see the example of the American put below.

Proof of Lemma 1. 1. We note that by Cramér's rule the kth component of the vector $(s\boldsymbol{I} - \boldsymbol{T}^{(\pm)})^{-1}\boldsymbol{t}$ is given by $[\det(s\boldsymbol{I} - \boldsymbol{T}^{(\pm)})]^{-1} \cdot \det(S_k)$ where S_k is the matrix obtained from $(s\boldsymbol{I} - \boldsymbol{T}^{(\pm)})$ by replacing its kth column by $\boldsymbol{t}$. Hence, we deduce that $a/(a - \kappa(s)$ is given by

$$C \cdot \det(-s\boldsymbol{I} - \boldsymbol{T}^{(+)}) \det(s\boldsymbol{I} - \boldsymbol{T}^{(-)}) / \prod_{i \in \mathcal{I}^+ \cup \mathcal{I}^-} (s - \rho_i).$$

where $C = \prod_{i \in \mathcal{I}^+ \cup \mathcal{I}^-} (-\rho_i) / \det(\boldsymbol{T}^{(-)}\boldsymbol{T}^{(+)})$. Denoting the right-hand side of (15) by $\varphi_a^{\pm}$ respectively, it is readily checked that $\varphi^+(s)_a \varphi_a^-(s)/(a - \kappa(s))$. Moreover, $\varphi_a^{\pm}$ are seen to be analytic for $\pm\Re(s) < 0$ and continuous for $\pm\Re(s) \leq 0$. Since $\boldsymbol{T}^{(\pm)}$ is negative definite $\varphi_a^{\pm}(s)$ does not vanish on $\pm\Re(s) \leq 0$. Finally note that $\varphi_q^{\pm}(0) = 1$. Theorem 1 in [11] now implies that $\varphi_a^{\pm} = \kappa_a^{\pm}$.
2. From a Tauberian theorem we deduce from (14) that $\kappa_a^+(-\infty) = \mathbb{P}(M_a = 0)$ and $\kappa_a^-(\infty) = \mathbb{P}(I_a = 0)$. Since the jumps of X form a compound Poisson process, we see that $\mathbb{P}(M_a = 0)$ $[\mathbb{P}(I_a = 0)]$ is positive iff $\sigma = 0$ and $\mu \leq [\geq] 0$. Combining this with (15), we conclude that $\# M^{(\pm)} = \# \mathcal{I}^{\pm}$.
3. Follows from straightforward Laplace inversion of (15) QED

The next result gives an explicit expression for the moment-generating function $u_k(x)$ in terms of the roots with positive real part.

Proposition 2 *Subject to (2), and assuming all the roots of the equation (4) with positive real part to be distinct,*

1. *for any positive function f and $x < k$ we have:*

$$\mathbb{E}_x[\mathrm{e}^{-aT(k)} f(X_{T(k)} - k)] = \boldsymbol{\pi} \mathcal{G} f \tag{18}$$

where $\mathcal{G}f = (\int_0^\infty f(z) F_i^{(+)}(\mathrm{d}z), i \in M^{(+)})$ where $F_0^{(+)}(\mathrm{d}z) = \delta_0(\mathrm{d}z)$ and $1 - F_i^{(+)}(z) = \mathbf{1}_i \exp(\boldsymbol{T}^{(+)} z)\mathbf{1}$ and $\boldsymbol{\pi}$ is the unique solution of the system

$$\boldsymbol{\pi} \hat{\boldsymbol{f}}^{(+)}[-\rho_i] = \mathrm{e}^{\rho_i(x-k)}, \qquad \forall i \in \mathcal{I}^+. \tag{19}$$

2. *In particular, $u_k(x)$ defined in (12) for $x < k$ is given by*

$$u_k(x) = \kappa_a^+(b)^{-1} \sum_{j \in \mathcal{I}^+} A_j^+ \rho_j \mathrm{e}^{\rho_j(x-k)} / (\rho_j - b). \tag{20}$$

Remarks. - Taking Laplace transform of (20) in $k-x$, we recover a formula of [11] $\hat{u}_k(s) = (s+b)^{-1}\left(1 - \frac{\kappa_a^+(-s)}{\kappa_a^+(b)}\right)$

- Let $\boldsymbol{S}$ denote the matrix whose columns are $\hat{\boldsymbol{f}}^{(+)}[-\rho_i]$ for all i with $\Re(\rho_i) > 0$. The explicit formula $\boldsymbol{\pi} = \mathbf{1}'(\mathrm{e}^{\rho_i(x-k)})_{\mathrm{diag}}\boldsymbol{S}^{-1}$ for the crossing probabilities $\boldsymbol{\pi}$ (if the roots are distinct) has an interesting probabilistic interpretation. Indeed, rewriting this as

$$\boldsymbol{\pi} = \mathbf{1}'\boldsymbol{S}^{-1}\left(\boldsymbol{S}(\mathrm{e}^{\rho_i(x-k)})_{\mathrm{diag}}\boldsymbol{S}^{-1}\right) = \boldsymbol{\eta}\,\exp(\boldsymbol{Q}(x-k))$$

decomposes $\boldsymbol{\pi}$ as a product of the "downcrossing phase probabilities" $\boldsymbol{\eta}$ at the end of the first negative excursion and of the Markovian transition probabilities $\exp(\boldsymbol{Q}(x-k))$. In the case $\sigma = 0, \mu \leq 0$, the (sub)generator matrix $\boldsymbol{Q}$ has the structure $\boldsymbol{Q} = \boldsymbol{T}^{(+)} + \boldsymbol{t}^{(+)}\boldsymbol{\eta}$ (see [6]); if $\sigma \neq 0$ or $\mu > 0$, it is a matter of algebra to verify that $\boldsymbol{Q}$ is given by

$$\boldsymbol{Q} = \begin{pmatrix} -(\lambda+a)/\mu & \lambda^{(+)}\boldsymbol{\alpha}^{(+)}/\mu \\ \boldsymbol{t}^{(+)} & \boldsymbol{T}^{(+)} \end{pmatrix} + \boldsymbol{d}^{(+)}\boldsymbol{\eta}$$

where $\boldsymbol{d}^{(+)}$ is the column vector $\boldsymbol{d}^{(+)} = (\rho_i^2\frac{\sigma^2}{2\mu} + \widetilde{\kappa}(\rho_i), i \in \mathcal{I}^+)$ with $\widetilde{\kappa}(s) = \frac{\lambda^{(-)}}{\mu}(\boldsymbol{\alpha}^{(-)}(s\boldsymbol{I} - \boldsymbol{T}^{(-)})^{-1}\boldsymbol{t}^{(-)} - 1)$.

Proof of Proposition 2.
1. Splitting the probability space in $G_0, \ldots, G_{m^{(+)}}$ and using the fact that conditionally on the phase in which the upcrossing occurs, the time of overshoot $T(k)$ and the overshoot $X_{T(k)} - k$ are independent, yields the decomposition

$$\mathbb{E}_x[\mathrm{e}^{-aT}f(X_T - k)] = \mathbb{E}_x[\mathrm{e}^{-aT}; G_0] + \sum_{i=1}^{m^{(-)}} \mathbb{E}_x[\mathrm{e}^{-aT}; G_i]\mathbb{E}_i[f(X_T - k)]$$

where we wrote $T = T(k)$ and respectively used $\mathbb{E}_x, \mathbb{E}_i$ to denote the expectation under $\mathbb{P}$ conditioned on $\{X_0 = x\}$ and G_i. This yields (18).

The system (19) is derived by an optional stopping approach. By applying Ito's formula to the function $f(t, X_t) = \exp(-at + bX_t)$ for any a and $b \in i\mathbf{R}$ (which ensures that $\kappa(b)$ is well defined), we find that

$$\begin{aligned} M_t &= f(t, X_t) - f(0, X_0) - \int_0^t Gf(s, X_s)\mathrm{d}s \\ &= \exp(-at + bX_t) - \exp(bX_0) - (\kappa(b) - a)\int_0^t \exp(-as + bX_s)\mathrm{d}s, \end{aligned}$$

is a zero-mean martingale, where $G = \frac{\partial}{\partial t} + \Gamma$ with Γ the infinitesimal generator of $\{X_t, t \geq 0\}$ (note that $Gf(t, X_t) = (\kappa(b) - a)f(t, X_t))$. Applying for $a \geq 0$ Doob's optional stopping theorem with the stopping time $T(k) \wedge t$ and noting that $\sup_t |M_{T(k)\wedge t}|$ is bounded we find $\mathbb{E}_x[M_{T(k)}] = 0$. By a computation as above we can expand this for $x < k$ as

$$0 = \mathrm{e}^{bk}\boldsymbol{\pi}\hat{\boldsymbol{f}}^{(+)}[-b] - \mathrm{e}^{bx} - (\kappa(b) - a)\mathbb{E}_x\left[\int_0^{T(k)} \exp(-as + bX_s)\mathrm{d}s\right]. \quad (21)$$

By analytic continuation, the identity (21) can be extended to the half plane $\Re(b) > 0$ except finitely many poles (the eigenvalues of $-\boldsymbol{T}^{(+)}$, recall that $\boldsymbol{T}^{(\pm)}$ has negative eigenvalues). By choosing b with $\Re(b) > 0$ to be a root of the equation $\kappa(b) = a$, we find (19). By Lemma 1 the number of equations is equal to the number of unknowns. Finally, the distinct roots assumption implies the linear independence of $\hat{\boldsymbol{f}}^{(+)}[\rho_i]$, as proved in Appendix B. Hence the "Wald system" (19) is nonsingular, yielding $\boldsymbol{\pi}$.
2. Suppose first $b, a > 0$, and note that $u_k(x) = u_{k-x}(0)$. Define $A = \{T(k - x) < \eta(a)\}$. The strong Markov property of X applied at $T(k - x)$ implies that

$$\begin{aligned}\mathbb{E}[\exp(-bM_a)\mathbf{1}_A] &= \mathbb{E}[\exp(-bX_{T(k-x)})\mathbf{1}_A]\mathbb{E}[\exp\{-bM_a\}] \\ &= \mathbb{E}[\exp(-aT(k-x) - bX(T(k-x)))]\kappa_a^+(-b),\end{aligned}$$

where $\mathbf{1}_A$ denotes the indicator of the event A. Noting that $A = \{M > k - x\}$ and using (16) one finds the formula as stated. By analytic extension, the identity holds for all b for which the right-hand side of (20) is well defined.
QED

Remark. It can be shown that the law of M_a given in (15) and Proposition 2 remain valid if X is a general Lévy process with the only restrictions that the positive jumps are of phase type and X is not a subordinator. An outline of the proof of the Laplace transform of M_a is as follows. Since, if the roots ρ_i of (13) are distinct, A_i^+ is the coefficient of $\rho_i/(\rho_i - s)$ in the partial fraction decomposition of $\kappa_a^+(-s) - \kappa_a^+(-\infty)$, the Cayley-Hamilton theorem implies that we have the following matrix identity

$$\kappa_a^+(-\infty)\boldsymbol{I} = \textstyle\sum_{i\in\mathcal{I}^+} A_i^+\rho_i(-\rho_i\boldsymbol{I} - \boldsymbol{T}^{(+)})^{-1}. \quad (22)$$

Using (22), it is straightforward to check that $u_k(x, 0) = \mathbb{P}(M_a > k - x)$ satisfies

$$(\Gamma' - aI)u_k(x, 0) = 0 \qquad x < k, \quad (23)$$

where Γ' is the infinitesimal generator of X. In the case of mulitple roots, the identity (23) remains valid, which follows by approximation. The proof is completed by an application of the appropriate version of Itô's lemma to $\exp(-a(t\wedge T(k)))u_k(X_{t\wedge T(k)})$. See [28, 9] for similar more detailed reasonings.

Example: Ruin. Note that from the (16) we immediately find an explicit expression for the ruin probability of the defined process X

$$\mathbb{P}_x(\exists t \leq \eta(a) : X_t < 0) = \mathbb{P}(-I_a > x) = \sum_{j\in\mathcal{I}^-} A_j^- \mathrm{e}^{\rho_j x}.$$

Example: The American put. Darling et al. [16] obtained the solution for the American optimal stopping problem (6) when X is a random walk. Extending this to continuous time, Mordecki [27] found that for a general Lévy process X, letting $I_\delta = \inf_{0\leq t\leq \eta(\delta)} X_t$ denote the infimum of X up to $\eta(\delta)$ with $\delta = r + \alpha$, the optimal stopping time in (6) is given by:

$$\widehat{T} = \widehat{T}(k^*) = \inf\{t \geq 0 : X_t \leq k^*\}$$

where $\exp(k^*) = K\ \mathbb{E}[\mathrm{e}^{I_\delta}] = K\ \kappa_\delta^-(1)$. The important application here is with the parameter $\delta = r + (T - t)^{-1}$, where t, T denote the current and expiration time of a finite expiration option. Recalling that $\kappa(1) = r$ we see that the optimal exercise level $k^* = k^*(t, T)$ is given by

$$\exp(k^*) = K\frac{\delta}{\delta - \kappa(1)}\ \frac{1}{\kappa_\delta^+(1)} = K(r(T-t)+1)\ \frac{1}{\kappa_\delta^+(1)}.$$

As noticed in [8], k^* yields a time dependent approximation for the optimal excercise boundary of an American put with expiration time T, which may be checked to be asymptotically exact when $t \to -\infty$ and also when $t \to T$.

Under the model (2), the value of the American put option for $e^x > e^{k^*} = K\kappa_\delta^-(1)$ can be checked to be given by

$$\begin{aligned} U^*(x) &= K\mathbb{E}_x[\mathrm{e}^{-\delta\widehat{T}(k^*)}] - \mathrm{e}^x\mathbb{E}_x[\mathrm{e}^{-\delta\widehat{T}(k^*)+X_{\widehat{T}(k^*)}}] \\ &= K\sum\nolimits_{j\in\mathcal{I}^-} \mathrm{e}^{\rho_j(x-k^*)}A_j^-/(1-\rho_j), \end{aligned}$$

where $\rho_j = \rho_j(\delta)$ denote the roots of $\kappa(\rho) = \delta$ for $\delta = r + \alpha$ (just insert the expressions for k^* and the joint moment-generating function of $\widehat{T}$ and $X_{\widehat{T}} - k^*$ which follow from above Proposition 2 applied to $-X$).

We can now also obtain the value of an American put on a stock paying proportional dividends. Indeed, the value of an American put option with payoff (5) on a stock paying dividends at rate $q \geq 0$ can be found by choosing $\mathbb{P}$ such that $\kappa(1) = r - q$ (instead of r) and by replacing everywhere in above display (r, α) by $(r - q, \alpha + q)$.

Using the approach of [13], in [32] explicit formulae are developed for a sequence of functions that point wise converges to the price of the American put with finite time of expiration, extending spectrally negative results in [8, 9].

3.2 First passage time for Y

We now consider the first passage time problem for Y, which, analogously, consists in computing the joint moment generating function

$$v_k(y) = v_k(y, a, b) = \mathbb{E}_y[\mathrm{e}^{-a\tau + b(Y_\tau - k)}] \tag{24}$$

of the crossing time

$$\tau = \tau(k) = \inf\{t > 0 : Y_t \geq k\}$$

and of the overshoot $Y_\tau - k$, with k, $a \geq 0$, and where b is such that $v_k(y)$ finite.

Analogously to the last section, we note that at the crossing time $\tau(k)$, we must either have a downward jump of X, or the component $\mu t + \sigma W_t$ must take the process Y to the barrier k. Denote by $M^{(-)}$ the set of all phases during which upcrossing may occur (again calling the non-jumping time phase 0). Let $\widetilde{\pi}_i = \mathbb{E}_y[\mathrm{e}^{-a\tau}; G_i]$ denote the (discounted) probability of upcrossing in phase i, where under $\mathbb{P}_y$ the process Y starts in y. As before we let $\widetilde{\boldsymbol{\pi}} = (\widetilde{\pi}_i, i \in M^{(-)})$, and let $\hat{\boldsymbol{f}}^{(-)}[b]$ denote the vector (depending on the initial starting state) of Laplace transforms at b of the overshoot $Y_{\tau(k)} - k$. Let $L_t = \sup_{0 \leq s \leq t} X_s \vee y$ be the running supremum of X, with L_t^c the continuous part of L and $\Delta L_t = L_t - L_{t-}$ the jump of L at time t. Introduce the dummy-variables $\delta_0 = \mathbb{E}_y[\int_0^{\tau(k)} \exp(-as)\mathrm{d}L_s^c]$ and

$$\delta_i = \mathbb{E}_y[\sum_{0 < s \leq \tau(k)} \exp(-as) I(\Delta L_s > 0, H_j)], \; j = 1, \dots, m^{(+)}$$

where H_j is the event of crossing the supremum in phase j.

Proposition 3 *Subject to (2), the joint moment generating function $v_k(y)$ defined in (24) is for $y \in [0,k)$ given by*

$$v_k(y) = \widetilde{\boldsymbol{\pi}} \hat{\boldsymbol{f}}^{(-)}[-b].$$

Assuming all roots ρ_i of $\kappa(\rho) = a$ to be distinct, the numbers $\pi_0, \ldots, \pi_{m^{(-)}}$ and $\delta_0, \ldots, \delta_{m^{(+)}}$, are the unique solution of the system of p equations

$$\mathrm{e}^{-\rho_i y} = \mathrm{e}^{-\rho_i k} \widetilde{\boldsymbol{\pi}} \hat{\boldsymbol{f}}^{(-)}[\rho_i] - \rho_i \delta_0 + \sum_{j=1}^{m^{(+)}} \delta_j (1 - \hat{\boldsymbol{f}}^{(+)}[-\rho_i]_j). \qquad i = 1, \ldots, p \tag{25}$$

Proof. The proof of the first part is analogous to the proof of the second part of Proposition 2 and left to the reader. To compute the vector $\widetilde{\boldsymbol{\pi}}$, we apply the optimal stopping approach to the reflected process Y, using the martingale introduced by Kella and Whitt [23]. Note that L^c and $\Delta L_t = L_t - L_{t-}$ have finite expected variation resp. finite number of jumps in each finite time interval. From Kella and Whitt [23] we find then that for $a > 0, b \in i\mathbf{R}$

$$\begin{aligned} N_t = (\kappa(b) - a) \int_0^t (-as - bY_s) \mathrm{d}s + \exp(-bY_0) - \exp(-at - bY_t) \\ - b \int_0^t \exp(-as) \mathrm{d}L_s^c + \sum_{0 < s \leq t} \exp(-as)[1 - \exp(b \Delta L_s)] \end{aligned}$$

is a zero mean martingale (where we used that if ΔL_s or $\mathrm{d}L_s$ is positive then $Y_s = 0$). Applying, as before, Doob's optional stopping theorem with the stopping time $\tau(k) \wedge t$ and straightforwardly checking that $|N_{\tau(k) \wedge t}|$ can be dominated by an integrable function, we find $\mathbb{E}_y[N_{\tau(k)}] = 0$. Then, expanding $\mathbb{E}_y[N_{\tau(k)}] = 0$ for $y < k$ leads to

$$\begin{aligned} 0 = (\kappa(b) - a) \mathbb{E}_y \left[\int_0^{\tau(k)} \exp(-as - bY_s) \mathrm{d}s \right] + \mathrm{e}^{-by} - \mathrm{e}^{-bk} \widetilde{\boldsymbol{\pi}} \hat{\boldsymbol{f}}^{(-)}[b] \\ - b\delta_0 + \sum_{i=1}^{m^{(+)}} \delta_i (1 - \hat{\boldsymbol{f}}^{(+)}[-b]_i) \end{aligned} \tag{26}$$

By analytic continuation, the identity (26) can be extended to hold for b in the complex plane except finitely many poles (the eigenvalues of $\boldsymbol{T}^{(-)}$, $-\boldsymbol{T}^{(+)}$)

Letting ρ_j to be a root of $\kappa(b) = a$, we find the system (25). If the roots ρ_i are distinct, then there are in all cases exactly enough equations to determine $(\widetilde{\pi}_i)_{i=0}^{m^{(-)}}$ and $(\delta_i)_{i=0}^{m^{(+)}}$, since $\widetilde{\pi}_0 = 0$ iff $\sigma = 0, \mu \geq 0$ and $\delta_0 = 0$ iff $\sigma = 0, \mu \leq 0$. The linear independence will be dealt with in Appendix B. QED

Remark. The "Canadized" Russian option is the Russian option with an independent exponential random variable $\eta(\lambda)$ as expiration and can be considered as a first approximation to the Russian option with finite expiration $1/\lambda$. See [13]. The value of the Canadized Russian option is given by $V_c^*(x, m) = \mathrm{e}^x v_c^*(m - x)$, where v_c^* is the value function of the optimal stopping problem

$$v_c^*(y) = \sup \mathbb{E}_y^{(1)}[\mathrm{e}^{-a(\tau\wedge\eta(\lambda))+Y_{\tau\wedge\eta(\lambda)}}].$$

where the supremum runs over τ in $\mathcal{T}$. From Theorem 1 we check that again the optimal stopping time is of the form (8). The quantities $\widetilde{\boldsymbol{\pi}}$ and $\boldsymbol{\delta}$ are now understood to be taken under the measure $\mathbb{P}_y^{(1)}$. Then we can read off from equation (26) that for $y < k$ and with $\gamma = \lambda/(a + \lambda)$, we have that

$$\mathbb{E}_y^{(1)}[\mathrm{e}^{-a(\tau_k\wedge\eta(\lambda))+Y_{\tau_k\wedge\eta(\lambda)}-k}] = \widetilde{\boldsymbol{\pi}}\hat{\boldsymbol{f}}^{(-)}[-1](1-\gamma) + \gamma(\mathrm{e}^y + \delta_0 + \sum_{i=1}^{m^{(+)}} \delta_i(1 - \hat{\boldsymbol{f}}^{(+)}[1])).$$

By optimisation of this expression over all levels $k \geq 0$, we find $v_c^*(y)$.

Remark. If $\sigma \neq 0$, the solution of the system (25) is in matrix notation form:

$$\boxed{(\widetilde{\boldsymbol{\pi}} \ - \boldsymbol{\delta}) = (\mathrm{e}^{-\rho_1 y} \ \ldots \ \mathrm{e}^{-\rho_p y})\widetilde{\boldsymbol{S}}^{-1}} \tag{27}$$

where $\widetilde{\boldsymbol{S}} = \begin{pmatrix} \widetilde{\boldsymbol{S}}_1 \\ \hline \widetilde{\boldsymbol{S}}_2 \end{pmatrix}$ is a $p \times p$ matrix whose first $m^{(-)} + 1$ rows $\widetilde{\boldsymbol{S}}_1$ are columnwise given by

$$\widetilde{\boldsymbol{k}}_1^{(j)} = \mathrm{e}^{-\rho_j k} \begin{pmatrix} 1 \\ (\rho_j \boldsymbol{I} - \boldsymbol{T}^{(-)})^{-1}\boldsymbol{t}^{(-)} \end{pmatrix}$$

and whose last $m^{(+)} + 1$ rows $\widetilde{\boldsymbol{S}}_2$ are columnwise given by

$$\widetilde{\boldsymbol{k}}_2^{(j)} = \rho_j \begin{pmatrix} 1 \\ (-\rho_j \boldsymbol{I} - \boldsymbol{T}^{(+)})^{-1}\boldsymbol{1} \end{pmatrix},$$

where $\mathbf{1}$ is a vector of ones. From Proposition 3 we conclude now that, if $\sigma \neq 0$,

$$v_k(y) = (\mathrm{e}^{-\rho_1 y} \ldots \mathrm{e}^{-\rho_p y})\widetilde{\boldsymbol{S}}^{-1}\hat{\boldsymbol{f}}_o{}^{(-)}[-1]$$

where $\hat{\boldsymbol{f}}_o{}^{(-)}[-1]$ denotes the column vector of Laplace transforms of the overshoots over k prolonged by 0's. Therefore, $v_k(y) = \sum_{i=1}^{p} \mathrm{e}^{-\rho_i y} A_i$ is a linear combination of the exponentials, with the vector $\boldsymbol{A}$ satisfying the linear system

$$\widetilde{\boldsymbol{S}}\boldsymbol{A} = \hat{\boldsymbol{f}}_o{}^{(-)}[-1]. \tag{28}$$

For the other cases we find similar expressions.

To connect to other results in the literature, we reformulate now the system (28) for $\boldsymbol{A}$ in terms of the eigenvalues of the matrices $-\boldsymbol{T}^{(\pm)}$, allowing at the same time for a general Jordan structure. Let $\eta^{(\pm,j)}$, where $j = 1, \ldots, n^{(\pm)}$, denote the eigenvalues of $-\boldsymbol{T}^{(\pm)}$ with respective multiplicities $m^{(\pm,j)}$. Note the multiplicities satisfy $m^{(\pm)} = \sum_{j=1}^{n^{\pm}} m^{(\pm,j)}$. (w.l.o.g. we may assume that $\boldsymbol{T}^{(\pm)}$ is a phase-type representation of minimal dimension, therefore the *geometric* multiplicity of each eigenvalue $\eta^{(\pm,j)}$ is one, that is, we only have *one* Jordan block with each eigenvalue.) Then, the system (28) becomes:

Proposition 4 *Assuming the roots ρ_i of $\kappa(\rho) = a$ to be distinct, we have*

$$v_k(y) = \sum_{i=1}^{p} A_i \mathrm{e}^{-\rho_i y}, \qquad y \in [0, k)$$

where $A_1, \ldots, A_p$ solve the $m^{(-)} + m^{(+)}$ equations

$$\sum_{i=1}^{p} \frac{A_i \mathrm{e}^{-\rho_i k}}{(\rho_i - \eta^{(-,j)})^l} = \frac{1}{(-b - \eta^{(-,j)})^l} \tag{*}$$

$$\sum_{i=1}^{p} \frac{A_i \rho_i}{(-\rho_i + \eta^{(+,j)})^l} = 0, \tag{**}$$

where in the second and third line $l = 1, \ldots, m^{(\mp,j)}$, $j = 1, \ldots, n^{(\mp)}$ respectively. If $\sigma \neq 0$ or $\mu > 0$ [$\sigma \neq 0$ or $\mu > 0$] the A_i solve (*)[(**)] for $l = 0$.

Example. Let X be given by a jump-diffusion where the jumps have a negative hyper-exponential distribution. In the general setting we choose $\sigma > 0$, $\lambda^{(+)} = 0$, $-\boldsymbol{T}^{(-)} = \mathrm{diag}(\beta_1, \dots, \beta_n)$, β_i different, and $\boldsymbol{\alpha}^{(-)} = (A_1, \dots, A_n)$. From Appendix A, we find that the parameters of X under $\mathbb{P}^{(1)}$ are given by

$$\widetilde{\mu} = \mu + \sigma^2, \qquad \widetilde{\lambda}^{(+)} = 0, \qquad -\widetilde{\boldsymbol{T}}^{(-)} = \mathrm{diag}(1+\beta_1, \dots, 1+\beta_n)$$

$$\widetilde{\lambda}^{(-)} = \lambda^{(-)}\boldsymbol{\alpha}^{(-)}(\boldsymbol{I} - \boldsymbol{T}^{(-)})^{-1}\boldsymbol{t}^{(-)} = \lambda^{(-)}\sum_{i=1}^{n} A_i \frac{\beta_i}{\beta_i + 1}$$

$$\widetilde{\boldsymbol{\alpha}}^{(-)} = \boldsymbol{\alpha}^{(-)}\mathrm{diag}(k_1, \dots, k_n)/\hat{F}^{(-)}[1] = \frac{1}{\sum_{i=1}^{n}\frac{A_i\beta_i}{\beta_i+1}}\left(\frac{A_1\beta_1}{\beta_1+1}, \dots, \frac{A_n\beta_n}{\beta_n+1}\right)$$

where $\boldsymbol{k} = (\boldsymbol{I} - \boldsymbol{T}^{(-)})^{-1}\boldsymbol{t}^{(-)}$. Let ρ_i be the roots of $\kappa_1(s) = a$, which are all distinct. Then the price $V(x,m)$ of the Russian option is given by $V(x,m) = \mathrm{e}^x v(m-x)$ where

$$v(y) = \begin{cases} \mathrm{e}^{k^*}\sum_{i=0}^{n+1} A_i \mathrm{e}^{-\rho_i y} & 0 \le y < k^* \\ \mathrm{e}^y & y \ge k^* \end{cases} \tag{29}$$

where the A_i and k^* are determined by

$$\sum_{i=0}^{n+1} A_i \mathrm{e}^{-\rho_i k^*} = 1 \qquad \sum_{i=0}^{n+1} A_i\rho_i = 0 \qquad \sum_{i=0}^{n+1} A_i\rho_i \mathrm{e}^{-\rho_i k^*} = -1$$

$$\sum_{i=0}^{n+1} A_i\rho_i \mathrm{e}^{-\rho_i k^*}\frac{1}{1+\beta_j+\rho_i} = \frac{1}{1+\beta_j-1} \qquad (j = 1, \dots, n)$$

where the first equation in the second line is the so called "smooth" fit condition which determines k^* (See also Appendix B.1. Write now $C_i = A_i \mathrm{e}^{-\rho_i k}$ then we can rewrite the previous system as

$$1 = \sum_{i=0}^{n+1} C_i = -\sum_{i=0}^{n+1} C_i\rho_i = \sum_{i=0}^{n+1} C_i \frac{\beta_j}{1+\beta_j+\rho_i} \quad (j = 1, \dots, n)$$

to find the B_i and then to find the k^* the equation $\sum_{i=0}^{n+1} C_i\rho_i \mathrm{e}^{\rho_i k^*} = 0$. By a partial fraction argument based on the rational function

$$\frac{\prod_{j=0}^{n+1}(\rho_j+1)}{\prod_{j=1}^{n}(-\beta_j)}\frac{\prod_{j=1}^{n}(s-\beta_j)}{\prod_{j=0}^{n+1}(s+\rho_j+1)},$$

we see that

$$A_i e^{-\rho_i k} = C_i = \frac{\prod_{j=0}^{n+1}(\rho_j + 1)}{\prod_{j=0}^{n+1}(\rho_j - \rho_i)} \frac{\prod_{j=1}^{n}(1 + \rho_i + \beta_j)}{\prod_{j=1}^{n}\beta_j}.$$

The found formula for the value of the Russian option conincides with [29].

4 Regime-switching Lévy processes.

4.1 Introduction

In this section, we study a certain class of regime-switching Lévy processes following an approach based on embedding first the Lévy model into a *continuous regime switching Brownian motion*, as proposed in [3] (see also [5], [6]).

Definition. A regime switching phase–type Lévy process X is a semi-markov process to which is associated an ergodic finite state space Markov process J such that, conditional on $J_t = j$, X_t is a Lévy model of the form (2) with parameters depending on j. In the case of no jumps the process is called a regime switching Brownian motion.

The trick of passing from a phase–type regime switching Lévy process to a regime switching Brownian motion is to level out the positive jumps to sample path segments with slope $+1$ and the negative jumps to sample path segments with slope -1, and add an extra phase, say 0, for the "regular time" when the process evolves continuously. This embeds a process with phase-type jumps X in a continuous Markov additive process (J, X'), or regime switching Brownian motion, where the Markov component J_t is in phase 0 at a regular time and gives the current phase of the jump otherwise.

For a general regime switching Lévy process Z, let us denote by $\boldsymbol{F}_t[s]$ the $p \times p$ matrix with ijth element $\mathbb{E}_i[e^{sZ_t}; J_t = j]$. Then ([6] p. 41) $\boldsymbol{F}_t[s] = e^{t\boldsymbol{K}[s]}$ where

$$\boldsymbol{K}[s] \ = \boldsymbol{Q} + \{\kappa^{(j)}(s)\}_{\text{diag}} \qquad (30)$$

and $\kappa^{(j)}(s)$ is the Lévy exponent in phase j. Many of the computations involving regime switching Lévy processes reduce to finding the eigenstructure of the matrix $\boldsymbol{K}[s]$. For example, Asmussen & Kella [7] solved the first passage time problem for reflected regime switching Brownian motion by introducing the (row) vector martingale

$$\mathrm{e}^{bY_t-at}\mathbf{1}_{J_t}-\mathrm{e}^{by}\mathbf{1}_{J_0}-b\int_0^t \mathrm{e}^{-au}\mathbf{1}_{J_u}dL_u-\int_0^t \mathrm{e}^{bY_u-au}\mathbf{1}_{J_u}du\ \ \boldsymbol{K}[b]$$

where $\mathbf{1}_i$ denotes a (row) vector with a 1 in the ith coordinate and 0's everywhere else and L represents the local time at 0. To use the vector martingale, one forms first scalar martingales obtained by choosing $b=\rho_j$ such that $\boldsymbol{K}[b]$ is singular and by multiplying the vector martingale by the right eigenvectors $\boldsymbol{h}^{(j)}$ of $\boldsymbol{K}[\rho_j]$, with the effect that the last term falls down, yielding the family of scalar martingales

$$M_t^{(j)}\ =\mathrm{e}^{-at+\rho_j Y_t}\boldsymbol{h}_{J_t}^{(j)}-\ \mathrm{e}^{\rho_j y}\boldsymbol{h}_0^{(j)}-b\int_0^t \mathrm{e}^{-as}\boldsymbol{h}_{J_s}^{(j)}\,\mathrm{d}L_s,$$

to which one may apply the optional stopping theorem.

4.2 First passage for regime switching Lévy processes

Let now X be a regime-switching Lévy process with two regimes, where the regimes of X switch from 1 to 2 and vice versa at rates η_1 and η_2 respectively. We denote by $J\in\{1,2\}$ the corresponding Markov-process indicating the current regime of X. If $J_t=i\in\{1,2\}$, $X=X^i$ is of the form (2) with parameters μ_i, σ_i, $\boldsymbol{T}_i^{(\pm)}$ and $\boldsymbol{\alpha}_i^{(\pm)}$. We study the first passage problem for $Y=\overline{X}-X$, X reflected at its supremum. Analogously to what we did before, we compute the joint moment generating function

$$v_k^{(i,j)}(y)=v_k^{(i,j)}(y,a,b)=\mathbb{E}_{y,i}[\exp(-a\tau+b(Y_\tau-k)I(J_\tau=j))]$$

of the crossing time

$$\tau=\inf\{t\geq 0:Y_t\geq k\}$$

and the overshoot $Y_\tau-k$. Here $i,j\in\{1,2\}$, $a\geq 0$ and b such that $v_k^{(i,j)}$ is finite. $\mathbb{E}_{i,y}$ denotes the measure under which $\{Y_0=y,J_0=i\}$. By the Markov property, we find as before that the moment-generating function $v_k^{(i,j)}$ is given by

$$v_k^{(i,j)}(y)=\boldsymbol{\pi}^{(i,j)}\hat{\boldsymbol{f}}^{j(-)}[-b]$$

where

$$\boldsymbol{\pi}^{(i,j)}=(\mathbb{E}_{i,y}[\mathrm{e}^{-a\tau}I(J_\tau=j,\text{level }k\text{ crossed in phase }j')],j'\in M^{j(-)}),$$

with $M^{j(-)}$ denoting all phases in which Y^j, Y being in regime $j \in \{1,2\}$, can upcross a level, and where $\hat{\boldsymbol{f}}^{j(-)}[-b]$ is the corresponding Laplace-transform of overshoots. We embed now the regime-switching Lévy process X into a fluid process X' by leveling out positive jumps of X to sample path segments of X' with slope $+1$, and negative jumps of X to sample path segments of X' with slope -1. More precisely, the phase process $J' = (J, \widetilde{J})$ is defined as follows. The first component $J(t) = i \in \{1,2\}$, indicates that the regime-switching Lévy process X is at time t in regime i. The second component $\widetilde{J}$ takes value $\widetilde{J}(t) = j \in \{1, \ldots, m_i^{(+)}\}$ if, at time t, X' is in one of the segments with slope $+1$ (such that the phase of the corresponding upward jump of X^i is j), and value $j \in \{-1, \ldots, -m_i^{(-)}\}$ if, at time t, X' is in one of the segments with slope -1 (such that the phase of the corresponding downward jump of X^i is j); when at time t the X'–process operates according to the Lévy exponent $s\mu_i + s^2\sigma_i^2/2$, we let $\widetilde{J}(t) = 0$. The resulting process is a particular case of a regime switching Brownian motion.

Let $\boldsymbol{K}_a[s]$ be the moment generating matrix of X' killed at rate a while $\widetilde{J}(t) = 0$ (note that then the crossing probabilities coincide with those of the original model). Then, from [6] p. 41, we find that $\boldsymbol{K}_a[s]$ is, in obvious block–partitioned notation, given by

$$\boldsymbol{K}_a[s] = \begin{pmatrix} \boldsymbol{K}_a^{(1)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_a^{(2)} \end{pmatrix} + \begin{pmatrix} -\boldsymbol{\Lambda}^{(1)} & \boldsymbol{\Lambda}^{(1)} \\ \boldsymbol{\Lambda}^{(2)} & -\boldsymbol{\Lambda}^{(2)} \end{pmatrix} \tag{31}$$

where

$$\boldsymbol{K}_a^{(i)}[s] \;=\; \begin{pmatrix} -\lambda_i - a + s\mu_i + s^2\sigma_i^2/2 & \lambda_i^{(-)}\boldsymbol{\alpha}_i^{(-)} & \lambda_i^{(+)}\boldsymbol{\alpha}_i^{(+)} \\ \boldsymbol{t}_i^{(-)} & \boldsymbol{T}_i^{(-)} - s\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{t}_i^{(+)} & \boldsymbol{0} & \boldsymbol{T}_i^{(+)} + s\boldsymbol{I} \end{pmatrix} \tag{32}$$

and $\boldsymbol{\Lambda}^{(i)}$ is a matrix of the size of $\boldsymbol{K}_a^{(i)}$ with $\eta^{(i)}$ on position $(1,1)$ and zeros for the rest.

We determine now the eigenstructure of $\boldsymbol{K}_a[s]$. We note first from (3) that under the model (2), κ_i, the Lévy exponent of X^i, is the ratio between two polynomials of degree p_i, resp. $m_i^{(+)} + m_i^{(-)}$ where $p_i = \epsilon_i + m_i^{(+)} + m_i^{(-)}$ and $\epsilon_i = 2,1,0$ if $\sigma_i \neq 0$, $(\sigma_i = 0, \mu_i \neq 0)$ and $(\mu_i = \sigma_i = 0)$, respectively. Hence the equation

$$\eta_1\eta_2 = (\kappa_1(s) - a - \eta_1)(\kappa_2(s) - a - \eta_2) \tag{33}$$

has p_1+p_2 roots which we denote by $\varrho_1, \dots \varrho_{p_1+p_2}$. For each $r = 1, \dots, p_1+p_2$ define

$$\boldsymbol{h}^{(r)} = \begin{pmatrix} \gamma_r \boldsymbol{k}_1^{(r)} \\ -\boldsymbol{k}_2^{(r)} \end{pmatrix} \text{ where } \boldsymbol{k}_i^{(r)} = \begin{pmatrix} 1 \\ \left(\varrho_r \boldsymbol{I} - \boldsymbol{T}_i^{(-)}\right)^{-1} \boldsymbol{t}_i^{(-)} \\ \left(-\varrho_r \boldsymbol{I} - \boldsymbol{T}_i^{(+)}\right)^{-1} \boldsymbol{t}_i^{(+)} \end{pmatrix} \tag{34}$$

and $\gamma_r = (\kappa_2(\varrho_r) - a - \eta_2)/\eta_2$. By straightforward algebra we can check:

Lemma 2 *For $j = 1, \dots, p_1 + p_2$, $\boldsymbol{K}_a[\varrho_j]\boldsymbol{h}^{(j)} = \boldsymbol{0}$.*

We adapt now the semi-Markov generalization of the Kella-Whitt martingale introduced by Asmussen and Kella [7]. First, we introduce some more notation. By Y' we will denote the process X' reflected in its supremum, that is, $Y' = \{Y'_t, t \geq 0\}$ with

$$Y'_t = \sup_{0 \leq s \leq t} X'_s \vee Y'_0 - X'_t.$$

By $L' = \{L'_t, t \geq 0\}$ we will denote the supremum of X', $L'_t = \sup_{s \leq t} X'_s \vee Y'_0$. Finally, we introduce the time spent by Y' in *phase 0* (which is the time of the original regime switching Lévy process) up to time t by

$$T'_0(t) = \int_0^t I(\widetilde{J}(s) = 0)\mathrm{d}s.$$

Let $\mathbb{P}_{(i,l),y}$ refer to the case $J_0 = (i,l), Y'_0 = y$ and $\tau' = \tau'_k = \inf\{t > 0 : J_0 = j, Y'_t = k\}$. It is immediate by a sample path comparison that $\tau = T'_0(\tau')$ and $\pi_{j'}^{(i,j)} = \mathbb{E}_{(i,0),y}[\mathrm{e}^{-aT'_0(\tau')}; J'_{\tau'} = (j,j')]$ for $i,j \in \{1,2\}$ and $j' \in M^{j(-)}$. Finally, let

$$\delta_\ell^{(i,j)} = \mathbb{E}_{(i,0),y}\left[\int_0^{\tau'} \mathrm{e}^{-aT'_0(t)} I(J'_t = (j,\ell))\mathrm{d}L'_t\right], \qquad j \geq 0.$$

By $\boldsymbol{1}_{J'_t} = \boldsymbol{1}_{(r,s)}$, we denote a row-vector of the length of $\boldsymbol{K}_a$ with all zeros but a one on position $(r-1)(m_1^{(+)} + m_1^{(-)} + 1) + s + 1$, which corresponds with phase s in regime r.

The theorem below identifies a vector martingale (35), a set of $p+1$ scalar martingales (36) and an "optional stopping system (37).

Theorem 2

1. *The process*

$$\mathrm{e}^{-aT_0'(t)+bY_t'}\mathbf{1}_{J_t'} - \mathrm{e}^{bY_0'}\mathbf{1}_{J_0'} + b\int_0^t \mathrm{e}^{-aT_0'(u)}\mathbf{1}_{J_u'}dL_u' - \int_0^t \mathrm{e}^{-aT_0'(u)+bY_u'}\mathbf{1}_{J_u'}du \;\; \boldsymbol{K}_a[-b] \quad (35)$$

is a mean zero (vector) $\mathbb{P}$*–martingale.*

2. *Let* ϱ_r *denote any root of the equation (33). Then*

$$M_t = \mathrm{e}^{-aT_0'(t)-\varrho_r Y_t'}h^{(j)}_{J_t'} - \mathrm{e}^{-\varrho_r y}h^{(r)}_{J_0'} - \varrho_r\int_0^t \mathrm{e}^{-aT_0'(s)}h^{(j)}_{J_s'}\,\mathrm{d}L_s' \quad (36)$$

are mean zero (scalar) martingales for each $j = 1,\ldots,p_1+p_2$.

3. *Let* $i \in \{1,2\}$ *and* $y \in [0,k)$. *If the roots* ϱ_r *are distinct, then the numbers* $\pi_0^{(i,j)},\ldots,\pi^{(i,j)}_{m_j^{(-)}}$, *and* $\delta_0^{(i,j)},\ldots,\delta^{(i,j)}_{m_j^{(+)}}$ $(j=1,2)$ *are the unique solution of the* $p = p_1+p_2$ *linear equations*

$$\begin{aligned}
\mathrm{e}^{-\varrho_1 y}h^{(1)}_{(i,0)} &= \sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(-)}}\pi_\ell^{(i,j)}\mathrm{e}^{-\varrho_1 k}h^{(1)}_{j,\ell} - \varrho_1\sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(+)}}\delta_\ell^{(i,j)}h^{(1)}_{j,\ell},\\
\mathrm{e}^{-\varrho_2 y}h^{(2)}_{(i,0)} &= \sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(-)}}\pi_\ell^{(i,j)}\mathrm{e}^{-\varrho_2 k}h^{(2)}_{j,\ell} - \varrho_2\sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(+)}}\delta_\ell^{(i,j)}h^{(2)}_{j,\ell},\\
&\;\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (37)\\
\mathrm{e}^{-\varrho_p y}h^{(p)}_{(i,0)} &= \sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(-)}}\pi_\ell^{(i,j)}\mathrm{e}^{-\varrho_p k}h^{(p)}_{j,\ell} - \varrho_p\sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(+)}}\delta_\ell^{(i,j)}h^{(p)}_{j,\ell}.
\end{aligned}$$

where $h^{(r)}_{j,\ell}$ *is the coordinate of* $\boldsymbol{h}^{(r)}$ *corresponding to regime* j *and phase* ℓ.

The proof is provided in Appendix B.

Remark. We can also determine the joint Laplace transform

$$\overline{v}_i(y) = \overline{v}_i(y,\boldsymbol{k}) = \mathbb{E}_{(y,i)}[\mathrm{e}^{-a\tau+b(Y_\tau-k_{J(\tau)})}] \qquad i=1,2.$$

of the level dependent crossing time

$$\tau = \tau(\boldsymbol{k}) = \tau(k_1, k_2) = \inf\{t \geq 0 : Y_t \geq k_j, J_t = j, j = 1, 2\}$$

and the overshoot $Y_\tau - k_{J(\tau)}$. By optimizing over the levels (or by smooth fit/value matching) the optimal level $\boldsymbol{k}^* = (k_1^*, k_2^*)$ can then be found and hence the value function of the optimal stopping problem (7) for the regime-switching Lévy process X. We will give an outline how to find the joint moment generating function of τ. Assume $k_1 < k_2$. Denote by $\boldsymbol{\Lambda}, \hat{\boldsymbol{\Lambda}}$ the matrices $\text{diag}(\rho_1, \ldots, \rho_{p_2})$ and $\text{diag}(\varrho_1, \ldots, \varrho_{p_1+p_2})$, respectively. Here the ρ_i, ϱ_i denote the roots of $\kappa_2(s) = a$ and those of (33), respectively. Then, we claim that for some vectors $\boldsymbol{b}_1, \boldsymbol{b}_2 \in \mathbf{R}^{p_1+p_2}$ and $\boldsymbol{c} \in \mathbf{R}^{p_2}$ we have that

$$\overline{v}_1(y) = \boldsymbol{b}_1 \exp(-\hat{\boldsymbol{\Lambda}} y)\mathbf{1}, \qquad y \in [0, k_1) \tag{38}$$

$$\overline{v}_2(y) = \begin{cases} \boldsymbol{b}_2 \exp(-\hat{\boldsymbol{\Lambda}} y)\mathbf{1} & y \in [0, k_1); \\ \boldsymbol{c} \exp(-\boldsymbol{\Lambda} y)\mathbf{1} & y \in [k_1, k_2). \end{cases} \tag{39}$$

Before we give the argument, we set some notation. By $\mu_y^{(i)}(\mathrm{d}x)$ we denote the overshoot distribution of Y over the level k_1, conditioned on Y starting in y and in regime i and on Y crossing k_1 in regime 2,

$$\mu_y^{(i)}(\mathrm{d}x) = \boldsymbol{\pi}^{(i,2)}(y, k_1) \exp(\boldsymbol{T}^{(2)-} x)\boldsymbol{t}^{(2)-}\mathrm{d}x \qquad i = 1, 2.$$

Let v_{k_2} refer to (3) where the underlying Lévy process is given by X^2 and the level to cross by k_2 and write $\mu_y(\mathrm{d}x) = \mathbb{P}_y(Y^2_{\eta(a+\eta_2)} \in \mathrm{d}x)$ to denote the distribution of Y^2, X^2 reflected at its supremum, at an exponential time $\eta(a+\eta_2)$. By Proposition 4 $v_{k_2}(y)$, restricted to $y < k_2$, is a function in the span of $\{\mathrm{e}^{-\rho_i y}\}$. Moreover, it turns out (see [32]) that also $\mu_y(\mathrm{d}x)/\mathrm{d}x$ is linear combination of $\exp(-\rho_i y)$.

Then we have the following recursion:

$$\overline{v}_2(y) = \begin{cases} v_{k_1}^{(2,1)}(y) + \int_0^\infty \overline{v}_2(k_1 + x)\hat{\mu}_y(\mathrm{d}x) & y < k_1; \\ \frac{a}{a+\eta_2} v_{k_2}(y) + \frac{\eta_2}{a+\eta_2} \int_0^\infty \overline{v}_1(x)\mu_y(\mathrm{d}x) & y \geq k_1; \end{cases} \tag{40}$$

$$\overline{v}_1(y) = v_{k_1}^{(1,1)}(y) + \int_0^\infty \overline{v}_2(k_1 + x)\tilde{\mu}_y(\mathrm{d}x) \qquad y < k_1. \tag{41}$$

The second line of (40) follows by splitting the probability space according to whether the killing at rate a or the switching of regime takes place first.

The other two lines follow by splitting according to the regime (1 or 2) in which Y crosses the level k_1.

From the identity (40)–(41), above remarks and Theorem 2, the claim follows. Moreover, the recursion (40)–(41) can be used to determine explicitly the coefficients $\boldsymbol{b}_1, \boldsymbol{b}_2$ and $\boldsymbol{c}$.

Appendix

A Exponential tilting of X

Consider the probability measure $\mathbb{P}^{(u)}$ given by $\mathbb{P}^{(u)}(A) = \mathbb{E}[e^{uX_t - t\kappa(u)}; A]$, $A \in \mathcal{F}_t$. It is standard (e.g. [6] p. 38) that X is again a Lévy process w.r.t. $\mathbb{P}_s$, with Lévy exponent given by $\kappa_u(s) = \kappa(u+s) - \kappa(u)$ corresponding to the following change of parameters:

$\mathbb{P}$	μ	σ^2	$\lambda^{(+)}$	$F^{(+)}$	$\lambda^{(-)}$	$F^{(-)}$
$\mathbb{P}^{(u)}$	$\mu + u\sigma^2$	σ^2	$\lambda^{(+)}\hat{F}^{(+)}[-u]$	$F_u^{(+)}$	$\lambda^{(-)}\hat{F}^{(-)}[u]$	$F_{-u}^{(-)}$

where $F_u^{(+)}(dx) = e^{ux}F^{(+)}(dx)/\hat{F}^{(+)}[-u]$, $F_{-u}^{(-)}(dx) = e^{-ux}F^{(-)}(dx)/\hat{F}^{(-)}[u]$. These distributions are again phase–type, as follows by the following result from [1]:

Lemma 3 *Let F be phase–type with parameters $(\boldsymbol{\alpha}, \boldsymbol{T})$ and let $F_u(dx) = e^{ux}F(dx)/\hat{F}[-u]$. Define $\boldsymbol{k} = (-u\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t}$ and let $\boldsymbol{\Delta}$ be the diagonal matrix with the k_i on the diagonal. Then F_u is phase–type with parameters*

$$\boldsymbol{\alpha}_u = \boldsymbol{\alpha}\boldsymbol{\Delta}/\hat{F}^{(+)}[-u], \quad \boldsymbol{T}_u = \boldsymbol{\Delta}^{-1}\boldsymbol{T}\boldsymbol{\Delta} + u\boldsymbol{I}.$$

Further, $\boldsymbol{t}_u = \boldsymbol{\Delta}^{-1}\boldsymbol{t}$.

B Proofs

B.1 Proof of Theorem 1

We start with a lemma which explores properties of v^*:

Lemma 4 *If $v^*(0) > 1$, then there exists a unique k^* such that*

$$\begin{cases} \exp(x) < v^*(x) < \exp(k^*) & 0 \le x < k*, \\ \exp(x) = v^*(x) & k^* \le x; \end{cases}$$

If $v^*(0) = 1$, *then* $v^* = \exp$.

Proof The assertions follow from the following three observations: (1) $v^*(x) \geq e^x$, which follows by choosing $\tau = 0$; (2) $x \mapsto e^{-x}v^*(x)$ is convex and v^* is non-decreasing, which can be seen as follows. For each fixed τ and ω the functions $x \mapsto \exp(-a\tau(\omega) + \overline{X}_{\tau(\omega)}(\omega) \vee x - X_{\tau(\omega)}(\omega) - x)$ and $x \mapsto \exp(-a\tau(\omega) + \overline{X}_{\tau(\omega)}(\omega) \vee x - X_{\tau(\omega)}(\omega))$ are convex and non-decreasing respectively. Integration over ω and taking the supremum over τ preserves these properties. and finally (3) $v^*(x) = \exp(x)$ for x large enough: Indeed, for τ arbitrary we can write using the strong Markov property of Y

$$\mathbb{E}_y^{(1)}[e^{-a\tau+Y_\tau}] = \mathbb{E}_y^{(1)}[e^{-a\tau_0}\mathbf{1}_{\{\tau>\tau_0\}}]\mathbb{E}_0^{(1)}[e^{-a\tau+Y_\tau}] + \mathbb{E}^{(1)}[e^{-a\tau+y-X_\tau}\mathbf{1}_{\{\tau<\tau_0\}}].$$

From the Wiener-Hopf-factorisation, we can infer that $\sup_\tau \mathbb{E}_0^{(1)}[e^{-a\tau+Y_\tau}] < \infty$ (since $\kappa(1) = r$). Since $\tau < \infty$ a.s. and $\tau_0 \to \infty$ as $y \to \infty$, the first term on the right-hand side converges to zero as y tends to infinity. The second term is equal to $\exp(y)$ times $\mathbb{E}_y[e^{-(a+r)\tau}\mathbf{1}_{\{\tau<\tau_0\}}]$. Hence we can check that $v^*(y) = \exp(y)$ for y large enough. QED

Proof of Theorem 1 Let $f_t = \exp(-at + \sup_{0\leq s\leq t} X_s \vee m)$ denote the system of pay-off functions belonging to the problem (7). Note that f_t has no negative jumps and $\{e^{-r\tau}f_\tau : \tau \in \mathcal{T}\}$ is uniformly integrable with respect to $\mathbb{P}$. Theorem 2 in Shiryaev et al. [35] now implies that the optimal stopping time in (7) is given by

$$\begin{aligned}
\tau^* &= \inf\{t \geq 0 : \operatorname*{esssup}_{\tau\in\mathcal{T},\tau\geq t} \mathbb{E}[e^{-r(\tau-t)}f_\tau|\mathcal{F}_t] \leq f_t\} \\
&= \inf\{t \geq 0 : \sup_{\tau\in\mathcal{T}} \mathbb{E}_{X_t,\overline{X}_t\vee m}[e^{-r\tau}f_\tau] \leq e^{at}f_t\} \\
&= \inf\{t \geq 0 : V^*(X_t, \overline{X}_t \vee m) = e^{X_t}v^*(\overline{X}_t \vee m - X_t) \leq e^{\overline{X}_t\vee m}\}
\end{aligned}$$

where in the second line, we used the Markov property of $(X_t, \overline{X}_t \vee m)$ and $\mathbb{P}_{x,z}$, $z \geq m$, denotes the probability measure under which the process $(X_t, \overline{X}_t \vee m)$ starts in (x, z). From the final line of the previous display combined with Lemma 4, we conclude that the optimal stopping time is a crossing time τ_{k^*} of Y, where the optimal level k^* can be found by optimisation. QED

Remark 1 (Smooth fit, value matching) If $\sigma = 0, \mu \geq 0$, the optimal k can also be found as the first nonnegative k for which $v_k(k^-)$ equals 1; if $\sigma \neq 0$

or $\sigma = 0, \mu < 0$, the optimal k can be found as the first nonnegative k for which $v_k'(k^-) = 1$, where the derivative is with respect to y. A proof of this can be based on the fact that for this choice of k the process $\mathrm{e}^{-at} v_k(Y_{t\wedge\tau})$ is a supermartingale for each stopping time τ and it is a martingale for $\tau = \tau(k)$, which follows from applying Itô's lemma to $\mathrm{e}^{-at} v_k(Y_{t\wedge\tau})$. See for more details of this line of reasoning [29, 9].

Remark 2 (Regime switching) In the case where we have switching regimes, we solve the optimal stopping problem analogously as above. Indeed, let X now be a general regime-switching Lévy process with characteristics given by the (generating) matrix $\boldsymbol{K}[s]$ as defined before in (30). Assume that the Lévy processes in the different regimes have Lévy measures ν_i such that $\int_{-1}^{1} \exp(y)|y|\nu_i(\mathrm{d}y) < \infty$ and assume that $\{\exp(-(\alpha+r)\tau+\overline{X}_\tau); \tau \in \mathcal{T}\}$ is uniformly integrable. Denote as before by J the underlying finite state Markov process which determines the regime and by Y the process X reflected in its supremum, $Y = \overline{X} \vee m - X$. Assume that the Perron-Frobenius eigenvalue of $\boldsymbol{K}[1]$ is given by r and let $h = h(1)$ be the corresponding eigenvector. Then (see e.g. [31, p. 17]) $M_t = \exp(X_t - rt)\frac{h_{J_t}}{h_{J_0}}$ is a mean one martingale, which can be used as equivalent change-of-measure. Let $\mathbb{P}$ denote the measure of the process X then we define the tilted measure $\mathbb{P}^{(1)}$ on $\mathcal{F}_t$ by $\mathrm{d}\mathbb{P}^{(1)}|_{\mathcal{F}_t} = M_t \mathrm{d}\mathbb{P}|_{\mathcal{F}_t}$. The corresponding optimal stopping problem reads as

$$\begin{aligned} V^*(j,x,m) &= \sup \mathbb{E}_j[\mathrm{e}^{-(r+a)\tau+(\overline{X}_\tau+x)\vee m}] \\ &= \mathrm{e}^x \sup \mathbb{E}^{(1)}_{j,m-x}[\mathrm{e}^{-a\tau+Y_\tau}\tfrac{h_j}{h_{J_\tau}}] := v_j^*(m-x)\cdot \mathrm{e}^x h_j \end{aligned}$$

where the supremum is taken over all $\tau \in \mathcal{T}$ and where under the measures $\mathbb{P}_j, \mathbb{P}_{j,y}$ the process J and (J,Y) start in j and (j,y) respectively.

As above we can derive that the optimal stopping time is given by

$$\begin{aligned} \tau^* &= \inf\{t \geq 0 : V^*(J_t, X_t, \overline{X}_t \vee m) \leq \mathrm{e}^{\overline{X}_t \vee m}\} \\ &= \inf\{t \geq 0 : v_j^*(\overline{X}_t \vee m - X_t) \leq \mathrm{e}^{\overline{X}_t \vee m - X_t}/h_j, J_t = j\} \end{aligned}$$

By a similar argument as before, we see that τ^* can be reformulated as

$$\tau^* = \inf\{t \geq 0 : Y_t \leq k_j^*, J_t = j\},$$

where the optimal levels k_j^* can be found by optimisation. Note this agrees with earlier results in the literature on pricing of Russian options for a regime switching Brownian motion [22].

B.2 Proof of linear independence

Without loss of generality, we may assume that κ_a^+ the quotient of two polynomials which have no factor in common.

• $\sigma = 0$ and $\mu \leq 0$. It is a matter of algebra to verify that the linear independence of the vectors $\hat{\boldsymbol{f}}^{(+)}[-\rho_i]$ for $i \in \mathcal{I}^+$ is equivalent with the invertibility of the matrix $\boldsymbol{M}$ with rows $\sum_{k=1}^{j-1} m^{(+,k)} + 1$ till $\sum_{k=1}^{j} m^{(+,k)}$ given by

$$(\rho_i/(\rho_i + \eta^{(+,j)})^\ell, i \in \mathcal{I}^+), \qquad \ell = 1, \ldots, m^{(+,j)} \tag{42}$$

where $\eta^{(+,j)}$ are the eigenvalues of $-\boldsymbol{T}^{(+)}$ with multiplicitites $m^{(+,j)}$. We consider now the system $\boldsymbol{M}\boldsymbol{c} = -_a(\boldsymbol{v}, \infty)$ where $\boldsymbol{v}$ is the vector with κ^+

$1, m^{(+,1)}+1, m^{(+,2)}+1, \ldots$ and the rest zeros. Recall we restricted

the cases where the roots of $\kappa(s) = a$ with positive real part are

hen we can check that any solution $\boldsymbol{c}$ of this system gives rise

fraction decomposition of $\kappa_a^+(-s) - \kappa_a^+(-\infty)$. Indeed, writing

$(s)/q(s)$ and taking $\boldsymbol{c}$ to be a solution of above system we have

$$p(s) = \left(\kappa_a^+(-\infty) + \sum_i c_i\rho_i/(\rho_i + s)\right) q(s)$$

ides of the equation have the same zeros with same multiplicities

$\infty) = \kappa_a^+(-\infty) > 0$. By unicity of this partial decompositon, we

$\boldsymbol{M}$ is invertible.

or $\mu > 0$. Again, it is not hard to verify that the linear indepen-

ctors $\hat{\boldsymbol{f}}^{(+)}[\rho_i]$ for $i \in \mathcal{I}^+$ is equivalent to the invertibility of the

with the final $\mathcal{I}^+ - 1$ rows given by (42) and the first row of ones.

at any solution of $\widetilde{\boldsymbol{M}}\widetilde{c} = \mathbf{1}_1$, where $\mathbf{1}_1$ is a vector of zeros with as

nate a 1, gives rise to a partial fraction decomposition of κ^+

B.3 Spectral proof of Proposition 4

Combining the Markov property of Y with the constance of expectation

$$v_k(y) = \mathbb{E}_y[\mathrm{e}^{-a\tau_k + b(Y_{\tau_k} - k)}] = \mathbb{E}_y[\mathrm{e}^{-a\tau_k} v_k(Y_{\tau_k})]$$

one readily verifies that $\{\exp(-a(t \wedge \tau_k) v_k(Y_{t\wedge\tau_k}), t \geq 0\}$ is a martingale. Let $T_u(\widehat{T}_u)$ be the first time X is larger (smaller) than u. By the strong Markov property and the fact that $\{Y_0 = y > 0, Y_t, t \leq \tau_0\}$ has the same law as $\{X_0 = -y, -X_t, t \leq T_0\}$ we can write

$$v_k(y) = \mathbb{E}_{-y}[\mathrm{e}^{-a\widehat{T}_k - bX_{\widehat{T}_k}} I(\widehat{T}_k < T_0)] + v_k(0)\mathbb{E}_{-y}[\mathrm{e}^{-aT_0} I(T_0 < \widehat{T}_k)].$$

From the Desiree-André-equation together with the fact that the Lévy exponent κ is a quotient of two polynomials, we deduce that the expectations on the right-hand side of above display (and hence $v_k(y)$ itself for $y < k$) is a linear combination of exponentials $\exp(-\rho_i y)$ where ρ_i a root of $\kappa(\rho) = a$. Using Itô's lemma in conjunction with above mentioned martingale property, it is not hard to prove that $v_k'(0^+) = 0$ (if $\sigma \neq 0$ or $\mu > 0$) and and $\Gamma v_k(x) = a v_k(x)$ for $x \in (0, k)$ where Γ acts on $f \in C^2(0, k)$ as

$$\begin{aligned}\Gamma f(x) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} f(x) - \mu \frac{\partial}{\partial x} f(x) + \lambda^{(-)} \int_0^\infty \big(f(x+z) - f(x)\big) F^{(-)}(\mathrm{d}z) \\ + \lambda^{(+)} \int_0^\infty \big(f((x-z)^+) - f(x)\big) F^{(+)}(\mathrm{d}z),\end{aligned}$$

for $x \in (0, k)$. Moreover, we see that $v_k(k^-) = 1$ (if $\sigma \neq 0$ or $\mu < 0$). It is now a matter of algebra to verify that the A_i solve the system as stated in the proposition. QED

B.4 Proof of Theorem 2

Let the process $Z = \{Z_t, t \geq 0\}$ be given by

$$Z_t = Y_t' - \frac{a}{b} T_0'(t) = -X_t' + L_t' - \frac{a}{b} T_0'(t).$$

Since Z has continuous sample paths, applying Theorem 2.1 d) of [7]), we find that – without restrictions on b, $M = \{M_t, t \geq 0\}$ with

$$\begin{aligned}M_t = &\int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}\mathrm{d}s\boldsymbol{K}_0[-b] + \mathrm{e}^{by}\mathbf{1}_{J_0} - \mathrm{e}^{-bZ_t}\mathbf{1}_{J_t} + b\int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}\mathrm{d}L'_s \\ &- a\int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}I(J_s=0)\mathrm{d}s \\ = &\int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}\mathrm{d}s\boldsymbol{K}_a[-b] + \mathrm{e}^{by}\mathbf{1}_{J_0} - \mathrm{e}^{-bZ_t}\mathbf{1}_{J_t} + b\int_0^t \mathrm{e}^{-aT'_0(s)}\mathbf{1}_{J_s}\mathrm{d}L'_s,\end{aligned}$$

is a zero mean $\mathbb{P}_{0,y}$ (row) martingale. We used that L'_t can increase only if X'_t is equal to its current supremum or $Y'_t = 0$. Moreover $\int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}I(J_s = 0)\mathrm{d}s = \int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}\mathrm{d}s\boldsymbol{\Delta}$ with $\boldsymbol{\Delta}$ a diagonal matrix with a 1 on positions 1 and p_1+1 and the rest zeros. Choosing $-b$ to be a root of $\kappa(s) = a$ and multiplying by the zero-eigenvectors of $K_a[-b]$ (using Lemma 2) completes the proof of 1 and 2.

Since $M_{t\wedge\tau'}$ is bounded for all t, for each j, can we apply optional stopping theorem to M at $\tau' = \tau'_k$, i.e. $\mathbb{E}_{(i,0),y}[M_{\tau'}] = \mathbb{E}_{(i,0),y}[M_0] = 0$. Since $\sup_{s\leq t} X'_s$ can increase only when $Y'_t = 0$ and $J_t \geq 0$, we find

$$\begin{aligned}&\mathbb{E}_{(i,0),y}\left[\int_0^{\tau'} \mathrm{e}^{-aT'_0(s)}h^{(r)}_{J_s}\mathrm{d}L'_s\right] \\ &\qquad = \sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(+)}} h^{(r)}_{j,\ell}\mathbb{E}_{(i,0),y}\left[\int_0^{\tau'} \mathrm{e}^{-aT'_0(s)}I(J_s=(j,\ell))\,dL'_s\right]\end{aligned}$$

which is equal to $\sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(+)}} \delta_k^{(i,j)}h^{(r)}_{j,\ell}$. Similarly, we must have $J_{\tau'} \leq 0$ so that

$$\begin{aligned}\mathbb{E}_{(i,0),y}\left[\mathrm{e}^{-\varrho_r Z_{\tau'}}h^{(r)}_{J_{\tau'}}\right] &= \mathbb{E}_{(i,0),y}\left[\mathrm{e}^{-\varrho_r k_{J_{\tau'}} - aT'_0(\tau')}h^{(r)}_{J_{\tau'}}\right] \\ &= \sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(-)}} \pi_\ell^{(i,j)}\mathrm{e}^{-\varrho_r k_j}h^{(r)}_{j,\ell}.\end{aligned}$$

Thus the rth equation is the same as $\mathbb{E}_{0,y}[M_{\tau'}] = 0$. If the roots ϱ_r are different, the equations are linearly independent, which can be proved as in Appendix B.2. QED

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