LOCAL EXTINCTION VERSUS LOCAL EXPONENTIAL GROWTH FOR SPATIAL BRANCHING PROCESSES

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[2] Let X be either the branching diffusion corresponding to the operator $Lu+\beta(u^2-u)$ on $D\subseteq\mathbb{R}^d$ (where $\beta(x)\geq 0$ and $\beta\not\equiv 0$ is bounded from above), or the superprocess corresponding to the operator $Lu+\beta u-\alpha u^2$ on $D\subseteq\mathbb{R}^d$ (with $\alpha>0$ and β is bounded from above but no restriction on its sign). Let λ_c denote the generalized principal eigenvalue for the operator $L+\beta$ on D. We prove the following dichotomy: either $\lambda_c\leq 0$ and X exhibits local extinction, or $\lambda_c>0$ and there is exponential growth of mass on compacts of D with rate λ_c. For superdiffusions, this completes the local extinction criterion obtained by Pinsky (1996) and a recent result on the local growth of mass under a spectral assumption given in Engländer and Turaev (2002). The proofs in the above two papers are based on PDE techniques, however the proofs we offer are probabilistically conceptual. For the most part they are based on 'spine' decompositions or 'immortal particle representations' (c.f. Lyons (1997), Evans (1993)) along with martingale convergence and the law of large numbers. Further they are generic in the sense that they work for both types of processes.

1. Introduction.

1.1. *Model.* Write $C^{i,\eta}(D)$ to denote the space of i times ($i=1,2$) continuously differentiable functions with all their ith order derivatives belonging to $C^\eta(D)$. [Here $C^\eta(D)$ denotes the usual Hölder space.] Let $D\subseteq\mathbb{R}^d$ be a domain and consider $Y=\{Y(t):t\geq 0\}$, the diffusion process with probabilities $\{\mathbb{P}_x,\ x\in D\}$ corresponding to the operator June 4, 2002

$$L=\frac{1}{2}\nabla\cdot a\nabla+b\cdot\nabla\quad\text{on }\mathbb{R}^d, \tag{1.1}$$

where the coefficients $a_{i,j}$ and b_i belong to $C^{1,\eta}$, $i,j=1,...,d$, for some η in $(0,1]$, and the symmetric matrix $a=\{a_{i,j}\}$ is positive definite for all $x\in D$. We do not assume that Y is conservative, i.e. Y may get killed at the Euclidean boundary of D or run out to infinity in finite time. Furthermore let $0\leq\beta\in C^\eta(D)$ be bounded from above on D and $\beta\not\equiv 0$. The (binary) $(L,\beta;D)$-*branching diffusion* is the Markov process with motion component Y and with spatially dependent rate β, replacing particles by precisely two offspring when branching. At each time $t>0$, the process consists of a point process X_t defined on Borel sets of D.

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Another closely related process is the $(L, \beta, \alpha; D)$-*superprocess*. Here L and D are as before and $\alpha, \beta \in C^\eta(D)$ with $\alpha > 0$ and β bounded above. This finite measure-valued process arises as a high density limit of certain, appropriately rescaled spatial branching processes (with random offspring numbers though). See Dawson (1993) or Etheridge (2000) for superprocesses in general and Engländer and Pinsky (1999) for this particular setting. The spatial dependence of β allows local (sub)criticality ($\beta \leq 0$) in certain regions and local supercriticality ($\beta > 0$) in others; α is related to the variance of the offspring distribution.

In the sequel and unless otherwise stated, X will denote both the $(L, \beta; D)$-branching diffusion and the $(L, \beta, \alpha; D)$-superprocess with probabilities and expectations P_μ, E_μ. Here the startng measure μ is a finite measure with support compactly embedded in D if X is a superprocess and given by a finite collection of points in D if X is a branching process. Also, unless otherwise stated, B will always denote an open set with $B \subset\subset D$ (meaning the closure of B is a bounded subset of D) having smooth boundary.

1.2. *Motivation.*

Definition 1 ((Weak) local extinction). Fix a finite μ with $\mathrm{supp}\mu \subset\subset D$.

(i) We say that X under P_μ *exhibits local extinction* if for every Borel set $B \subset\subset D$, there exists a random time τ_B such that

$$P_\mu(\tau_B < \infty) = 1 \text{ and } P_\mu(X_t(B) = 0 \text{ for all } t \geq \tau_B) = 1.$$

(ii) We say that X under P_μ *exhibits weak local extinction* if for every Borel set $B \subset\subset D$, $P_\mu\left(\lim_{t\uparrow\infty} X_t(B) = 0\right) = 1$.

If there is no weak local extinction, we shall say that X *exhibits re-charge.*

For the $(L, \beta; D)$-branching diffusion, note that local extinction and weak local extinction coincide.

For the $(L, \beta, \alpha; D)$ -superprocess, local extinction has been studied by Pinsky (1996). (Note that in Pinsky (1996) and Engländer and Pinsky (1999) the terminology is slightly different: it is said that the *support* of the superprocess exhibits local extinction.) To explain his result, let

$$\lambda_c = \lambda_c(L + \beta, D) := \inf\{\lambda \in \mathbb{R} \ : \ \exists u > 0 \text{ satisfying } (L + \beta - \lambda)u = 0 \text{ in } D\}$$

denote the *generalized principal eigenvalue* for $L + \beta$ on D (The boundedness of β ensures that $\lambda_c < \infty$ – see section 4.4 in Pinsky (1995)). Pinsky proved that the process exhibits local extinction if and only if $\lambda_c \leq 0$. Note in particular that local extinction does not depend on the coefficient α. His proof uses quite a bit of PDE machinery. To prove local extinction for $\lambda_c \leq 0$ turned out to be the harder part. The proof that there is no local extinction when $\lambda_c > 0$ is based on the proof of the existence of a nonzero stationary solution for the equation $Lu + \beta u - \alpha u^2 = 0$ on a large smooth domain $B \subset\subset D$ with Dirichlet boundary condition (the domain must be so large that $\lambda = \lambda_c(L + \beta, B)$, the principal Dirichlet eigenvalue on B is positive; this is possible because $\lambda_c > 0$). Using this, one can show that even the $(L, \beta, \alpha; B)$-superprocess survives and, with positive probability, its total mass does not tend to zero thus implying re-charge. The question about the behavior of the mass for *small* balls however is left open. On the other hand, it shows that *local extinction is in fact equivalent to weak local extinction for superprocesses* too.

REMARK 2. It turns out that using the technology developed in Engländer and Pinsky (1999), one can give an alternative proof of the re-charge part ($\lambda_c > 0$) for superprocesses, which we sketch here for completeness. Let $\phi \in C^{2,\eta}(B)$ be the eigenfunction corresponding to λ which is zero on ∂B (see Subsection 2.1). Then a (non-linear) h-transform (with $h = \phi$) takes the $(L, \beta, \alpha; B)$-superprocess $\widehat{X}$ into the $(L_0^{\phi}, \lambda, \alpha\phi; B)$-superprocess $\widehat{X}^{\phi} := \phi\widehat{X}$ (with a changed starting measure), where $L_0^{\phi} := L + a\frac{\nabla\phi}{\phi} \cdot \nabla$. It can be shown that L_0^{ϕ} corresponds to a conservative (positive recurrent) diffusion on B (see again Subsection 2.1) . Using this and the boundedness of $\alpha\phi$, it is easy to conclude that $\widehat{X}^{\phi}$ survives and, with positive probability, its total mass tends to infinity (see the proof of Proposition 8 in Section 3). Since ϕ is bounded, the same holds for $\widehat{X}$ too.

With some extra work one can show that *small* balls too are charged for arbitrarily large times with positive probability as follows. First, use a comparison argument and replace $\alpha\phi$ by its supremum (note that using PDE representations found in Engländer and Pinsky (1999) and the elliptic maximum principle, it can be shown that the new process has a smaller or equal probability of ever charging a given compactly embedded domain). Then use the result that for recurrent motion and constant supercritical branching, the process conditioned on survival charges all nonempty balls for arbitrarily large times almost surely (see Def 1.4, Prop 3.1 and Thrm 4.5(a) of Engländer and Pinsky (1999) for further elaboration.)

Note however that these arguments only work for superprocesses as they are based on the analytical tools of Engländer and Pinsky (1999), and that they do not give any information about the growth of mass on (small) compacts. (The previous reasoning for small balls for example does not rule out that the mass of the ball tends to zero a.s.) $\diamond$

For superprocesses, as far as the $\lambda_c > 0$ regime is concerned, *local exponential growth of mass in law* has been established in the recent paper Engländer and Turaev (2002); the rate is shown to be λ_c. Note, however, that in that paper only a particular class of superdiffusions is considered satisfying a spectral theoretical assumption.

In this paper we have three main goals.

- To prove that the same condition on λ_c holds for the $(L, \beta; D)$-branching diffusion with regard to local extinction versus re-charge.
- To give a proof which is *generic and conceptual.* That is, it works for both class of process and also provides an intuitive explanation of the result.
- To give new results on the growth rates of mass on small balls when $\lambda_c > 0$.

As far as the first goal is concerned, a natural idea is trying to use some kind of connection between the two types of processes along with the result for superprocesses. In particular, it may first seem to be easy to argue by a "Poissonization" argument and by exploiting Pinsky's result. That is, to use the well known fact that for fixed time, the distribution of a branching diffusion started from a Poisson number of particles at x is the same as that of a Poisson point process whose intensity is given by the superdiffusion. A second thought however shows that knowing the distributions for fixed times is not sufficient for investigating the large time behavior of the process. (Note that we do not know anything *quantitative* about those distributions for the superprocess, so for example it is not clear how to use Borel-Cantelli along a sequence of times.)

The other possibility is to express local extinction using PDE conditions and to try to compare those conditions for the two processes. It is indeed possible to follow this track by using certain stochastic representation theorems proven by Evans-O' Connell (1994) and Iscoe (1988). The proof is not too long but quite technical (using analytic and PDE tools) and does not give any insight into the origin of the criterion on λ_c. We will provide this proof in the appendix for completeness.

As far as the second goal is concerned, we will present a proof which uses a *"spine-decomposition"* for X. There are several similar spine (sometimes called backbone, immortal particle or immortal backbone) decomposition results in the literature which we shall discuss later. For our purposes we will need to add to this collection of decompositions with the proof of yet another theorem of that type. The novelty of this approach will be that it provides us with the following *intuitive picture*: for every nonempty bounded domain $B \subset\subset D$ with a smooth boundary there exists a change of probability such that under the new probability there is a particle (the spine or immortal particle) whose trajectory is that of an ergodic diffusion (different from $(Y, \mathbb{P}.)$) confined to B almost surely and along which copies of the original process under $P.$ immigrate at a certain rate. Then *the sign of λ_c determines whether or not this is a null-event under the original probability* for large B's. If not, since the spine visits every region of B for arbitrarily large times and since B can be chosen arbitrarily large, it follows that there is no local extinction. Regarding superprocesses, we will only prove weak local extinction for $\lambda_c \leq 0$.

As far as the third goal is concerned, we get new results on the *local growth rate* for the case when $\lambda_c > 0$ (for both processes).

Finally we mention that is also possible to find older but weaker results for a special class of Markov branching diffusions in Ogura (1983).

1.3. *Results.* In the sequel we will use the notation $\langle f, \mu \rangle := \int_D f(x)\, \mu(dx)$. Our main theorem is as follows.

THEOREM 3 (WEAK LOCAL EXTINCTION VS. LOCAL EXPONENTIAL GROWTH). *Let $0 \neq \mu$ be a finite measure with* $\operatorname{supp} \mu \subset\subset D$.

(i) X under P_μ exhibits weak local extinction if and only if there exists a function $h > 0$ satisfying $(L + \beta)h = 0$ on D, that is, if and only if $\lambda_c \leq 0$. In particular, the weak local extinction property does not depend on the starting measure μ.

(ii) When $\lambda_c > 0$*, there exist functions* $\exp\{-\rho_1\}$ *and* ρ_2 *in* $[0,1)$*, such that*

(1.2)
$$\log P_\mu\left(\lim_{t\uparrow\infty} X_t(B)=0\right) = \begin{cases} \langle -\rho_1,\mu\rangle & \text{if } X \text{ is the superprocess} \\ \langle \log\rho_2,\mu\rangle & \text{if } X \text{ is the branching diffusion,} \end{cases}$$

for all nonempty open $B \subset\subset D$*. Furthermore,* ρ_1 *solves* $L\rho+\beta\rho-\alpha\rho^2=0$ *and* ρ_2 *solves* $L\rho+\beta(\rho^2-\rho)=0$ *on* D *.*

(iii) When $\lambda_c > 0$*, for any* $\lambda<\lambda_c$ *and* $\emptyset \neq B \subset\subset D$*,*

$$P_\mu\left(\limsup_{t\uparrow\infty} e^{-\lambda t}X_t(B)=\infty\right)>0 \text{ and } P_\mu\left(\limsup_{t\uparrow\infty} e^{-\lambda_c t}X_t(B)<\infty\right)=1.$$

Remark 4 (Total mass). In Theorem 3 we are concerned with the *local* behaviour of the population size. When considering the *total mass* process $\|X_t\| := X_t(D)$, the growth rate may actually exceed λ_c. Indeed, take for example the $(L,\beta;D)$-branching diffusion with a conservative diffusion corresponding to L on D and with $\lambda_0 := \lambda_c(L,D)<0$, and let $\beta>0$ be constant. Then $\lambda_c(L+\beta,D) = \beta+\lambda_0<\beta$, but – since the branching rate is spatially constant – a classical theorem on Yule's processes tells us that $e^{-\beta t}\|X_t\|$ tends to a nontrivial random variable as $t\to\infty$, that is, that the growth rate of the total mass is $\beta>\lambda_c$. ◇

1.4. *Outline.* The rest of this article is organized as follows. Section two concerns certain spine or immortal particle decomposition theorems which are needed for our probabilistic proofs, while section three presents the proofs themselves. The results then are illustrated with examples in section four. Finally, the appendix provides the promised alternative proofs for part of the results along the lines of Iscoe (1988) and the original proofs in Pinsky (1996).

2. Martingales, spines and immortal particles

2.1. *A decomposition result* We begin this subsection by recalling some facts about changes of measure for both diffusions and Poisson processes. As before B will always denote a nonempty open set compactly embedded in D with a smooth boundary.

Girsanov change of measure. Let $\lambda=\lambda_c(L+\beta,B)$. It can be found in Pinsky (1995), for example, that there exists a $\phi\in C^{2,\eta}(B)$ such that

$$(L+\beta-\lambda)\phi=0 \text{ in } B \text{ with } \phi=0 \text{ on } \partial B.$$

Let $\tau^B=\inf\{t\geq 0: Y_t\notin B\}$ and assume that the diffusion $(Y,\mathbb{P}_x)$ is adapted to some filtration $\{\mathcal{G}_t : t\geq 0\}$. Then under the change of measure

$$\left.\frac{d\mathbb{P}_x^\phi}{d\mathbb{P}_x}\right|_{\mathcal{G}_t} = \frac{\phi\left(Y_{t\wedge\tau^B}\right)}{\phi(x)}\exp\left\{-\int_0^{t\wedge\tau^B}\lambda-\beta(Y_s)\,ds\right\}$$

the process $\left(Y,\mathbb{P}_x^{\phi}\right)$ corresponds to the h-transformed ($h=\phi$)generator $(L+\beta-\lambda)^{\phi}=L+a\phi^{-1}\nabla\phi\cdot\nabla$.

For further reference we point out that the process $\left(Y,\mathbb{P}_x^{\phi}\right)$ is *ergodic* on B (i.e. it is positive recurrent). This follows from the following three facts (see Pinsky (1995)). A diffusion is positive recurrent if and only if it corresponds to a so-called "product L^1-critical operator"; this latter property of operators is invariant under h-transforms and finally, the operator $L+\beta-\lambda$ on B possesses this property.

Change of measure for Poisson point processes. Suppose now that given a nonnegative bounded continuous function $g(t)$, $t\geq 0$, the Poisson process $(n,\mathbb{L}^g)$ where $n=\{\{\sigma_i : i=1,...,n_t\} : t\geq 0\}$ has instantaneous rate $g(t)$. Further, assume that n is adapted to $\{\mathcal{G}_t : t\geq 0\}$. Then under the change of measure

$$\left.\frac{d\mathbb{L}^{2g}}{d\mathbb{L}^{g}}\right|_{\mathcal{G}_t}=2^{n_t}\exp\left\{-\int_0^t g(s)ds\right\}$$

the process $(n,\mathbb{L}^{2g})$ is also a Poisson process with rate $2g$ (cf. Chapter 3, Jacod and Shiryayev (1987)).

Let us assume for simplicity that μ is finite and $\operatorname{supp}\mu\subset B$. Let $\{\mathcal{F}_t : t\geq 0\}$ denote the natural filtration up to time t and let $X^{t,B}$ denote the exit measure from $B\times[0,t)$ – note that the exit measure is defined for both types of processes (see Dynkin (2001) for the definition of an exit measure). Let $\widehat{X}_t$ denote the $(L,\beta;B)$ branching process, respectively the $(L,\beta,\alpha;B)$-superprocess. Recall from the definition in the introduction that by changing the domain from D to B it is implicit that mass is killed when it meets the boundary ∂B on account of the restriction of L to B.

On account of the fact that ϕ solves $(L+\beta-\lambda)\phi=0$ on B, one can apply Kolmogorov's backwards equations to show that $\left\{e^{-\lambda t}\langle\phi,\widehat{X}_t\rangle : t\geq 0\right\}$, is a martingale; cf. Dynkin (1993, Thrm 3.1). With a small abuse of notation we will write $\left\{e^{-\lambda t}\langle\phi,X^{t,B}\rangle : t\geq 0\right\}$ instead. Fundamental to the calculations associated with the next theorem are the following facts ("log-Laplace equations") which we recall for superprocess and branching processes. Let $g\in C_b^+$, where C_b^+ denotes the cone of nonnegative, bounded continuous functions. When X is a superprocess, we have $E_\mu\left(\exp\{-\langle g,X_t\rangle\}\right)=\exp\{-\langle u_g(\cdot,t),\mu\rangle\}$ where u_g is the minimal non-negative solution to $\dot{v}=Lv+\beta v-\alpha v^2$ in D with $v\left(x,0\right)=g\left(x\right)$. When X is a branching process, $E_\mu\left(\exp\{-\langle g,X_t\rangle\}\right)=\exp\{-\langle\log u_g(\cdot,t),\mu\rangle\}$ where u_g is the minimal non-negative solution to $\dot{v}=Lv+\beta\left(v^2-v\right)$ in D with $v\left(x,0\right)=\exp\{-g(x)\}$.

THEOREM 5. *Suppose that μ is finite measure with* $\operatorname{supp}\mu\subset B$. *Define $\widetilde{P}_\mu$ by the martingale change of measure*

$$\left.\frac{d\widetilde{P}_\mu}{dP_\mu}\right|_{\mathcal{F}_t}=M_t^{\phi}:=e^{-\lambda t}\frac{\langle\phi,X^{t,B}\rangle}{\langle\phi,\mu\rangle}.$$

Define

$$\mathbb{P}_{\phi\mu}^{\phi}=\int_B\frac{\phi\left(x\right)\mu\left(dx\right)}{\langle\phi,\mu\rangle}\mathbb{P}_x^{\phi},$$

that is the measure under which Y *has a randomized starting position. Note in particular that when* $\mu = \delta_x$, $\mathbb{P}^{\phi}_{\phi\mu} = \mathbb{P}^{\phi}_{x}$.

(i) *Suppose that* $g \in C_b^+(D)$ *and* u_g *is as specified for either a superprocess or branching process above. Then*

(2.3)

$$\widetilde{E}_\mu\left(e^{-\langle g, X_t\rangle}\right) = \begin{cases} \mathbb{E}^{\phi}_{\phi\mu}\left(\exp\left\{-\int_0^t ds\, 2\alpha\left(Y_s\right) u_g\left(Y_s, t-s\right)\right\}\right) E_\mu\left(e^{-\langle g, X_t\rangle}\right) \\ \mathbb{E}^{\phi}_{\phi\mu}\mathbb{L}^{2\beta(Y)}\left(e^{-g(Y_t)}\prod_{i=1}^{n_t} u_g\left(Y_{\sigma_i}, t-\sigma_i\right)\right), \end{cases}$$

where the first equation applies for the case when X *is a superprocess, and the second for the case when* X *is a branching process.*

(ii) *With the same order of cases, we have*

$$\langle\phi, \mu\rangle \widetilde{E}_\mu\left(M_t^\phi\right) = \begin{cases} \langle\phi, \mu\rangle + \mathbb{E}^{\phi}_{\phi\mu}\left(\int_0^t ds\, e^{-\lambda s} 2\alpha\left(Y_s\right)\phi\left(Y_s\right)\right) \\ \mathbb{E}^{\phi}_{\phi\mu}\mathbb{L}^{2\beta(Y)}\left(e^{-\lambda t}\phi\left(Y_t\right) + \sum_{i=1}^{n_t} e^{-\lambda\sigma_i}\phi\left(Y_{\sigma_i}\right)\right). \end{cases}$$

2.2. *Discussion* For superprocesses, the decompositions suggests that $(X, \widetilde{P}_\mu)$ is equal in law to the sum of two independent processes. The first is a copy of (X, P_μ) and the second is a process of immigration; at every time $t > 0$, a copy of the $(L, \beta, \alpha; D)$-superprocess is initiated at Y_t, where $Y = \{Y_t : t \geq 0\}$ is a copy of $(Y, \mathbb{P}^{\phi}_{\phi\mu})$ and further, the "rate" of immigration is $2\alpha(Y_t)$. For branching processes we have a very similar construction. The decompositions suggest that $(X, \widetilde{P}_\mu)$ is the result of a single particle, undergoing a $(Y, \mathbb{P}^{\phi}_{\phi\mu})$-motion, along which $(L, \beta; D)$-branching diffusions immigrate at space time points $\{(\sigma_i, Y_{\sigma_i}) : i = 1, ..., n_t\}$ where $(n, \mathbb{L}^{2\beta(Y)})$ is a Poisson process. Both cases relate to a spectrum of similar results that exist for both super and branching processes.

For two different classes of critical superprocesses, Evans (1993) and Etheridge and Williams (2000) consider a change of measure based on the martingale associated with the total mass process $\|X\|$. They find a decomposition consisting of a copy of the original process together with an independent immigration process along the path of a $(Y, \mathbb{P}^{\phi}_{\phi\mu})$ diffusion. One sees (cf. Roelly and Rouault (1989)) that their change of measure is equivalent to conditioning the process on survival. This decomposition is known as the *Evans immortal particle picture.* For supercritical superprocess, Evans and O'Connell (1994) and Engländer and Pinsky (1999) have demonstrated a decomposition in which a given superprocess is equal in law to a process consisting of two independent components: the first being a copy of the process conditioned on extinction, the second is an immigration process, where immigration is initiated along the trajectory of a "backbone" branching Markov diffusion that starts with a random set of points.

In the branching particle process literature, conditioning the process on survival, its equivalence with the martingale associated with total mass and a representation of a single randomly chosen genealogical line of descent with size biasing of the offspring distribution along it is the result of Lyons *et al. (1995)* for Galton-Watson processes. Changes of measure using martingales of an innerproduct form for supercritical spatial and typed branching processes have also been considered by Athreya (2000), Lyons (1997) and Biggins and Kyprianou (2001) for example. These authors found that the change of measure again induced a randomly chosen genealogical line of decent, known as a *spine*, along which the spatial reproductive distribution is size biased. The size biased behaviour along the spine is analogous to Evan's immortal particle picture. It is worth noting that whilst in the superprocess literature the decompositions are mostly proved in law, in the branching process literature the results are established in the stronger context of path-wise constructions.

Some other decompositions are as follows. Geiger and Kersting (1999) produce a 'spine-like' decomposition for finite Galton-Watson trees (different from those of the previous paragraph) by conditioning on the height of the tree being at generation n. By taking limits as n tends to infinity they produce a Galton-Watson process conditioned on survival. Overbeck (1993) considers changing measure of critical super-Brownian motion using martingales constructed from innerproducts of the process with space-time harmonic functions; again, a decomposition appears in which the immortal particle has an h-transformed Brownian motion. Salisbury and Verzani (1999) show for critical super-Brownian motion that a backbone decomposition appears when conditioning the process to hit n specified points when exiting a bounded smooth domain. See also the citations contained in all the aforementioned publications for yet more examples. The fundamental concepts for all these decompositions can be routed back to ideas found in the paper Kallenberg (1977).

Theorem 5 offers decompositions in law with the new feature that the immortal particle or spine is represented by a diffusion conditioned to stay in the compactly embedded domain B.

2.3. *Proof of Theorem 5* We will only prove part (i), the proof of part (ii) follows by similar reasoning.

Superprocesses. In this proof we will work with "backward" solutions of the log-Laplace equation for convenience. Thus we will work with the function $u_g^t := u_g(\cdot, t-\cdot)$ where t is fixed instead of u_g. Also, we will need a stronger form of the log-Laplace equation as follows. Recall that $X^{t,B}$ is a measure with support on $\partial[B\times[0,t)]$. For any function $f:\partial[B\times[0,t)]\to\mathbb{R}^+$ which is bounded and continuous, and for $X^{0,B}=\mu$, we have $E_\mu(e^{-\langle f,X^{t,B}\rangle})=e^{-\langle v_f(\cdot,0),\mu\rangle}$ where v_f solves $-\dot v=Lv+\beta v-\alpha v^2$ in $B\times(0,t)$ with $v=f$ on $\partial[B\times[0,t)]$ (Dynkin (2001)). Starting with the left hand side of (2.3), and applying the Markov property of exit measures (Dynkin (2001), or Dynkin (1993), Theorem 1.3.), one has

(2.4)
$$\widetilde{E}_\mu\left(e^{-\langle g,X_t\rangle}\right)=E_\mu\left(\frac{\langle\phi,X^{t,B}\rangle}{\langle\phi,\mu\rangle}e^{-\lambda t-\langle g,X_t\rangle}\right)=\frac{1}{\langle\phi,\mu\rangle}E_\mu\left(\langle\phi,X^{t,B}\rangle e^{-\lambda t-\langle u_g^t,X^{t,B}\rangle}\right).$$

It now follows that

$$\widetilde{E}_\mu\left(e^{-\langle g,X_t\rangle}\right)=-\frac{1}{\langle\phi,\mu\rangle}\frac{\partial}{\partial\theta}E_\mu\left(e^{-\lambda t-\langle u_g^t+\theta\phi,X^{t,B}\rangle}\right)\bigg|_{\theta=0}.$$

(The differentiation and the expectation has been interchanged using bounded convergence). Note however that $E_\mu\left(e^{-\langle u_g^t+\theta\phi, X^{t,B}\rangle}\right) = e^{-\langle v^\theta, X^{0,B}\rangle}$ where v^θ is the unique solution to the system, $-\dot{v}^\theta = Lv^\theta + \beta v^\theta - \alpha\left(v^\theta\right)^2$ in $B \times (0,t)$ and $v^\theta = u_g^t + \theta\phi$ on $\partial[B\times[0,t)]$. (Here we suppressed the dependence of v^θ on t). Using that $v^0 = u_g^t$, $X^{0,B} = \mu$ and hence that $\langle u_g^t, X_0\rangle = \langle u_g(\cdot,t),\mu\rangle$ we obtain using the log-Laplace equation that

$$\widetilde{E}_\mu\left(e^{-\langle g, X_t\rangle}\right) = E_\mu\left(e^{-\langle g, X_t\rangle}\right)\frac{1}{\langle\phi,\mu\rangle}e^{-\lambda t}\left.\frac{\partial}{\partial\theta}\langle v^\theta,\mu\rangle\right|_{\theta=0}, \tag{2.5}$$

thus leaving us the job of proving that $-\langle\phi,\mu\rangle^{-1}e^{-\lambda t}\partial\langle v^\theta,\mu\rangle/\partial\theta|_{\theta=0}$ is equal to the first factor on the right hand side of (2.3). A simple calculation, using again the fact that $v^0 = u_g^t$, shows that $\eta := \partial v^\theta/\partial\theta|_{\theta=0}$ is the unique solution to the system $-\dot{\eta} = L\eta + \beta\eta - 2\alpha u_g^t\eta$ in $B\times(0,t)$ and $\eta = \phi$ on $\partial[B\times[0,t)]$. Note that the boundary condition (that is the smoothness of η up to the boundary $\partial[B\times[0,t)]$) follows by comparing the righthand sides of (2.4) and (2.5). Taking the aforementioned first factor, use the Girsanov density for $\mathbb{P}_x^\phi$ to write it as

$$\begin{aligned}
&\int\frac{1}{\langle\phi,\mu\rangle}\mathbb{E}_x^\phi\left(e^{-\int_0^t ds\, 2\alpha(Y_s)u_g^t(Y_s,s)}\right)\phi(x)\,\mu(dx)\\
&= e^{-\lambda t}\int\frac{1}{\langle\phi,\mu\rangle}\mathbb{E}_x\left(\phi\left(Y_{t\wedge\tau^B}\right)e^{-\int_0^{t\wedge\tau^B}ds\,\left[2\alpha(Y_s)u_g^t(Y_s,s)-\beta(Y_s)\right]}\right)\mu(dx)\,.
\end{aligned}\tag{2.6}$$

Consider now the righthand side of (2.6) and observe that the expectation is precisely the "Feynman-Kac type" probabilistic representation for η. This observation together with (2.5) yield (2.3).

Branching processes. We shall only consider the case $\mu = \delta_x$ where $x\in B$. The adjustments that are necessary for the case that μ is finite can be seen in (2.6) of the above proof for superprocesses. Write $\xi(x,t)$ for the left hand side of (2.3) and note by conditioning on the first time of fission we have after a routine application of the Markov property

$$\begin{aligned}
\xi(x,t) &= \mathbb{E}_x\mathbb{L}^{\beta(Y)}\left(\mathbf{1}_{(t\wedge\tau^B<\sigma_1)}\frac{\phi\left(Y_{t\wedge\tau^B}\right)}{\phi(x)}e^{-\lambda t-g(Y_t)}\right.\\
&\qquad\left.+\,\mathbf{1}_{(t\wedge\tau^B\geq\sigma_1)}\frac{\phi\left(Y_{\sigma_1}\right)}{\phi(x)}e^{-\lambda\sigma_1}\xi\left(Y_{\sigma_1},t-\sigma_1\right)u_g\left(Y_{\sigma_1},t-\sigma_1\right)\right)\\
&= \mathbb{E}_x\left(e^{-\int_0^{t\wedge\tau^B}\beta(Y_s)ds}\frac{\phi\left(Y_{t\wedge\tau^B}\right)}{\phi(x)}e^{-\lambda t-g(Y_t)}\right.\\
&\qquad\left.+\int_0^{t\wedge\tau^B}\beta(Y_s)\,e^{-\int_0^s\beta(Y_u)du}\frac{\phi(Y_s)}{\phi(x)}e^{-\lambda s}\xi(Y_s,t-s)\,u_g(Y_s,t-s)\,ds\right)
\end{aligned}\tag{2.7}$$

Now let $\rho(x,t)$ be the right hand side of (2.3). Condition on σ_1 to produce

$$\rho(x,t) = \mathbb{E}_x^\phi\mathbb{L}^{2\beta(Y)}\left(\mathbf{1}_{(\sigma_1>t)}e^{-g(Y_t)} + \mathbf{1}_{(\sigma_1\leq t)}u_g\left(Y_{\sigma_1},t-\sigma_1\right)\rho\left(Y_{\sigma_1},t-\sigma_1\right)\right).$$

By changing measure back to $\mathbb{P}_x$ in the last expression, one produces a second solution to the functional equation (2.7). Let $\Pi := |\xi - \rho|$. Our goal is to show that $\Pi \equiv 0$. Suppose to the contrary that $\Pi(x_0, t_0) > 0$ for some $x_0 \in B$ and $t_0 > 0$. On account of continuity on $[0,t] \times \overline{B}$, there exist $\varepsilon, a > 0$ such that $f(t) := \Pi(x_0, t) \geq a$ on $I := [t_0, t_0 + \varepsilon]$. Then, by the boundedness of β, ϕ and Π, $f(t) \leq C \int_0^t f(s)\, ds$ on I, where C is an appropriate positive constant. Gronwall's inequality now implies that $f \equiv 0$; contradiction. Therefore $\Pi \equiv 0$ must hold. □

2.4. *Mean convergence for the martingale* Before finishing this section, we make one immediate application of the previous theorem to the martingale limit M_∞^ϕ.

LEMMA 6. *Suppose that μ is finite and* $\operatorname{supp}\mu \subset B$ *and* $\lambda = \lambda_c (L + \beta, B) > 0$ *then M_t^ϕ converges to its almost sure limit M_∞^ϕ in $L^1(P_\mu)$.*

Proof. We shall make use of the fundamental measure theoretic result (cf Durrett (1995) p242) which says that $E_\mu(M_\infty^\phi) = 1$ if and only if $\widetilde{P}_\mu \ll P_\mu$ if and only if $\limsup_{t\uparrow\infty} M_t^\phi < \infty$ $\widetilde{P}_\mu$-almost surely. It is clear from part (ii) of Theorem 5 together with Fatou's Lemma that because ϕ, β and α are all bounded on B, $\lambda > 0$, and Y is $\mathbb{P}_{\phi\mu}^\phi$-ergodic,

$$\widetilde{E}_\mu \left(\liminf_{t\uparrow\infty} M_t^\phi \right) \leq \liminf_{t\uparrow\infty} \widetilde{E}_\mu \left(M_t^\phi \right) < \infty.$$

It follows that $\liminf_{t\uparrow\infty} M_t^\phi < \infty$ $\widetilde{P}_\mu$-almost surely. Since $(M_t^\phi)^{-1}$ is a $\widetilde{P}_\mu$-martingale, it has a $\widetilde{P}_\mu$-limit and hence $\limsup_{t\uparrow\infty} M_t^\phi < \infty$ $\widetilde{P}_\mu$-almost surely. The statement now follows. □

3. Proof of Theorem 3

3.1. *Two preparatory propositions*

PROPOSITION 7 (BRANCHING PROCESS). *For any Borel set $B \subset\subset D$ and finite μ,*

$$P_\mu \left(\limsup_{t\uparrow\infty} X_t(B) \in \{0, \infty\} \right) = 1.$$

Proof: It suffices to prove the proposition for the case $\mu = \delta_x$. Let Ω_0 denote the event that $\limsup_{t\uparrow\infty} X_t(B) > 0$. Let $\tau_0 = \inf\{t > 0 : X_t(B) \geq 1\}$ and $\tau_{n+1} = \inf\{t > \tau_n + 1 : X_t(B) \geq 1\}$. Fix $K > 0$, let $A_n := \{X_{\tau_n+1}(B) \geq K\}$ and let $\Omega_1 := \{\omega : \omega \in A_n$ infinitely often$\}$. Using elementary properties of the diffusion Y along with the fact that β is assumed to be bounded away from zero in some region, it is straightforward to prove that $\varepsilon(K, B) := \inf_{x\in B} P_{\delta_x}(X_1(B) \geq K) > 0$. Thus, by the strong Markov property ,

$$\sum_{n=1}^{n} P(A_n \mid X_{\tau_n}, ..., X_{\tau_1}) = \infty \text{ a.s. on } \Omega_0.$$

By the extended Borel-Cantelli lemma (see Corollary 5.29 in Breiman (1992)), almost every $\omega \in \Omega_0$ belongs to Ω_1. Therefore $\limsup_{t\uparrow\infty} X_t(B) \geq K$ a.s. on Ω_0, and since K can be arbitrarily large, the result follows. $\square$

PROPOSITION 8 (SUPERPROCESS). *Let $\emptyset \neq B' \subset\subset B \subset\subset D$ and let μ be a finite measure. Then*

$$P_\mu\left(\{\limsup_{t\uparrow\infty} X_t(B) = \infty\} \cup \{\lim_{t\uparrow\infty} X_t(B') = 0\}\right) = 1. \tag{3.8}$$

Proof: Let $0 < \varepsilon$ and let $\mathcal{M}_\varepsilon$ denote the following set of measures: $\mu \in \mathcal{M}_\varepsilon \Leftrightarrow$ supp $\mu \subset B'$ and $\varepsilon < \|\mu\| < \infty$. The proof essentially requires us to show that all $K > 0$,

$$\inf_{\mu\in\mathcal{M}_\varepsilon} P_\mu(X_1(B) \geq K) > 0. \tag{3.9}$$

For once we are in possesion of (3.9), a very similar argument to the one given for the branching process in Proposition 7 yields the statement of the proposition.

We continue then with the proof of (3.9). By comparison, it is enough to prove it when X replaced by $\widehat{X}$ corresponding to the quadruple $(L, \beta, \alpha; B)$. Let λ and ϕ denote the principal Dirichlet eigenvalue of $L + \beta$ on B and the corresponding (positive) Dirichlet eigenfunction, respectively. (The sign of λ plays no role in the following argument.) Since ϕ is only determined up to constant multiples, we may assume that $\phi \leq 1$ on B. Then

$$P_\mu(\|\widehat{X}_1\| \geq K) > P_\mu(\langle \widehat{X}_1, \phi\rangle \geq K) = P_{\phi\mu}(\|\widehat{X}_1^\phi\| \geq K),$$

where $\widehat{X}^\phi = \phi\widehat{X}$ denotes the superprocess corresponding to the quadruple $(L_0^\phi, \lambda, \alpha\phi; B)$. [Here $L_0^\phi = L + a\frac{\nabla\phi}{\phi}\cdot\nabla$.] Let X^* be the superprocess corresponding to the quadruple $(L_0^\phi, \lambda, c; B)$, where $c := \sup_{x\in B} \alpha\phi$. Then

$$P_{\phi\mu}(\|\widehat{X}_1^\phi\| \geq K) \geq P_{\phi\mu}(\|X_1^*\| \geq K), \tag{3.10}$$

because the log-Laplace equation and the parabolic maximum principle (see Engländer and Pinsky (1999)) imply that for $g \in C_b^+(B)$,

$$E_{\phi\mu}\exp\langle -g, \widehat{X}_1^\phi\rangle \leq E_{\phi\mu}\exp\langle -g, X_1^*\rangle,$$

and consequently $\widehat{X}_1^\phi$ is stochastically larger than X_1^*.

In light of (3.10), the lower bound (3.9) will immediately follow if we show that

$$\inf_{\mu\in\mathcal{M}_\varepsilon} P_{\phi\mu}(\|X_1^*\| \geq K) > 0. \tag{3.11}$$

Recall from Subsection 2.1 that L_0^ϕ corresponds to a conservative diffusion on B. Since X^* has conservative motion part and spatially constant branching, it is well known that $\|X^*\|$ is a non-degenerate diffusion process on $[0,\infty)$ ("Feller's diffusion"). On the left hand side of (3.11), the starting point for this one-dimensional diffusion process is $\|\phi\mu\|$ and thus (3.11) follows from the fact that $\|\phi\mu\|$ is uniformly bounded from below by $\varepsilon \inf_{B'} \phi$ for $\mu \in \mathcal{M}_\varepsilon$. $\square$

3.2. *Proof of Theorem 3 (i)* Assume that $\lambda_c \leq 0$; then there exists a $h > 0$ solving $(L+\beta)h = 0$. Kolmogorov's backwards equation (cf. Dynkin 1993) implies that $\langle h, X_t \rangle$ is a positive local martingale, and hence a supermartingale, for all $\mathrm{supp}\mu \subset\subset D$. It follows that

$$\limsup_{t\uparrow\infty} X_t\,(B) \leq C \limsup_{t\uparrow\infty} \langle h, X_t \rangle < \infty \tag{3.12}$$

P_μ-almost surely where C is a constant. For branching processes by Proposition 7 it follows that $\limsup_{t\uparrow\infty} X_t\,(B) = 0$ P_μ-almost surely. If X is the superprocess, then let $\emptyset \neq B' \subset\subset B$. Proposition 8 then implies "weak local extinction on B'". Since B and B' can be chosen arbitrarily, weak local extinction follows.

Assume now that $\lambda_c > 0$. Since it is assumed $\mathrm{supp}\mu \subset\subset D$, we can choose a large enough B for which $\mathrm{supp}\mu \subset B$ and $\lambda = \lambda_c\,(L+\beta, B) > 0$. Change measure using M_t^ϕ as in the previous section and choose $g \in C_c^+$ so that $g \leq \mathbf{1}_B$. From Theorem 5 and that for $\lambda > 0$, $\widetilde{P}_\mu \ll P_\mu$ (Lemma 6) we have that $P_\mu\left(\limsup_{t\uparrow\infty} X_t(B) > 0\right) > 0$ if $P_\mu\left(\limsup_{t\uparrow\infty} \langle g, X_t \rangle > 0\right) > 0$ which happens if $E_\mu(\mathbf{1}_{(\limsup_{t\uparrow\infty}\langle g, X_t\rangle > 0)} M_\infty^\phi) > 0$ which happens if and only if $\widetilde{P}_\mu\left(\limsup_{t\uparrow\infty} \langle g, X_t \rangle > 0\right) > 0$ which, once again, happens if and only if $\widetilde{E}_\mu\left(\exp\{-\limsup_{t\uparrow\infty}\langle g, X_t\rangle\}\right) < 1$. Note now that for sufficiently small $\varepsilon > 0$ we have (for the two classes of process)

$$\begin{aligned}
\widetilde{E}_\mu\left(e^{-\limsup_{t\uparrow\infty}\langle g, X_t\rangle}\right) &\leq \liminf_{t\uparrow\infty} \widetilde{E}_\mu\left(e^{-\langle g, X_t\rangle}\right) \\
&\leq \liminf_{t\uparrow\infty} \begin{cases} \mathbb{E}_{\phi\mu}^\phi\left(\exp\left\{-\int_{t-\varepsilon}^t ds\, 2\alpha\,(Y_s)\, u_g\,(Y_s, t-s)\right\}\right) \\ \mathbb{E}_{\phi\mu}^\phi\left(e^{-g(Y_t)}\right) \end{cases}
\end{aligned}$$

by Fatou's Lemma and the fact that $u_g \leq 1$. Ergodicity of $(Y, \mathbb{P}_{\phi\mu}^\phi)$ and strict positivity of u_g on $(0,\varepsilon) \times B$ implies that the right hand side of the inequality is strictly less than one. Intuitively speaking, the spine or immortal particle visits every part of B infinitely often because it is an ergodic diffusion. This forces the process itself to do the same. □

3.3. *Proof of Theorem 3 (ii)* We prove the case for superprocesses, the case for branching processes is virtually identical. For any Borel $B \subset\subset D$ note that the strong Markov property implies that for any finite μ such that $\mathrm{supp}\mu \subset\subset D$ and $t \geq 0$,

$$P_\mu\left(\limsup_{u\uparrow\infty} X_u\,(B) = 0\right) = E_\mu P_{X_t}\left(\limsup_{u\uparrow\infty} X_u\,(B) = 0\right).$$

With $\rho_1\,(x) = -\log P_{\delta_x}\left(\limsup_{t\uparrow\infty} X_t\,(B) = 0\right) < 0$ it follows that $\exp\{-\langle \rho_1, X_t\rangle\}$ is a martingale. An application of Kolmogorov's backwards equations (cf. Dynkin (1993)) together with part (i) shows that ρ_1 is a non-trivial solution to the prescribed equation on D. □

3.4. *Proof of Theorem 3 (iii)* We may assume that $\lambda > 0$. By standard theory, there exists a $B^* \subset\subset D$ with a smooth boundary so that $\lambda^* := \lambda_c\,(L+\beta, B^*) > \lambda$.

We first claim that if $\Omega_0 := \{\lim_{t\uparrow\infty} e^{-\lambda t} X_t\,(B^*) = \infty\}$, then $P_\mu\,(\Omega_0) > 0$. Indeed, if $\widehat{X}$ is defined as the $(L,\beta;B^*)$ branching process or the $(L,\beta,\alpha;B^*)$ superprocess respectively then

$$
\begin{aligned}
P_\mu\,(\Omega_0) &\geq P_\mu\left(\liminf_{t\uparrow\infty} e^{-\lambda^* t} X_t\,(B^*) > 0\right) \geq P_\mu\left(\liminf_{t\uparrow\infty} e^{-\lambda^* t} \|\widehat{X}_t\| > 0\right) \\
&\geq P_\mu\left(\lim_{t\uparrow\infty} e^{-\lambda^* t} \langle \phi^*, \widehat{X}_t\rangle > 0\right), \qquad (3.13)
\end{aligned}
$$

where $(L+\beta-\lambda^*)\phi^* = 0$ in B^* and $\phi^* = 0$ on ∂B^* . Since $\lambda^* > 0$, Lemma 6 implies that the last term in (3.13) is positive.

Now let $\emptyset \neq B \subset\subset D$. From this point we consider two cases separately. Let X be the branching diffusion first. Let $p := \inf_{x\in B^*} p(1,x,B) > 0$, where $\{p(t,.,dy) : t > 0\}$ is the transition measure for $(Y,\mathbb{P})$. Let $0 < q < p$ and $A_n := \{X_{n+1}(B) \geq qX_n(B^*)\}$, and let $\Omega_1 := \{\omega :\ \omega \in A_n \text{ infinitely often}\}$. It follows from the law of large numbers and the Markov property that on Ω_0, $\lim_{n\uparrow\infty} P(A_n \mid X_n,...,X_1) = 1$. Using the extended Borel-Cantelli lemma just like in the proof of Proposition 7, it follows that $\limsup_{t\uparrow\infty} e^{-\lambda t} X_t(B) = \infty$ a.s. on Ω_0.

If X is the superprocess, the proof goes through with minor modifications as follows. Use the branching property (i.e. the property that $P_{\mu+\nu}$ is the convolution of P_μ and P_ν) and split the mass in B^* into unit masses (with some possible leftover). Then, to imitate the previous proof, one only needs to know that for some $\varepsilon > 0$,

$$
\inf_{\mu:\ \mathrm{supp}\,\mu\subset B^*, \|\mu\|\geq 1} P_\mu(X_1(B) > \varepsilon) > 0. \qquad (3.14)
$$

Indeed, replace $\mathbf{1}_B$ by a nonnegative smooth continuous function g, such that $g \leq \mathbf{1}_B$. Recall the log-Laplace equation: $E_\mu\,(\exp\{-\langle g, X_t\rangle\}) = \exp\{-\langle u_g, \mu\rangle\}$ where u_g is the minimal non-negative solution to $v_t = Lv + \beta v - \alpha v^2$ in D with $v\,(x,0) = g\,(x)$. Note that $0 < \inf_{B^*} u_g(\cdot,1) =: 2\varepsilon$. Thus (3.14) follows from the Markov-inequality:

$$
P_\mu(\langle g, X_t\rangle \leq \varepsilon) = P_\mu\left(\exp\{-\langle g, X_t\rangle\} \geq e^{-\varepsilon}\right) \leq \exp\{-\langle u_g, \mu\rangle\} e^{\varepsilon} \leq e^{-\varepsilon}.
$$

This completes the proof of the first statement.

For the remaining statement, note that there exists an $h > 0$ such that $\exp(-\lambda_c t)\langle h, X_t\rangle$ is a supermartingale converging almost surely. Since $X_t(B)$ is bounded above by a constant times $\langle h, X_t\rangle$, a.s. finiteness of the lim sup follows. □

4. Examples In this section we will present four examples for branching diffusions which will illustrate the general results of this paper.

4.1. *Branching Brownian motion (with drift).* Let $L = 1/2\,(\mathrm{d}^2/\mathrm{d}x^2) + \varepsilon\,(\mathrm{d}/\mathrm{d}x)$ and let β be a positive constant. Then, for a small enough ε, the reproduction "wins" against the transient motion, where ε being small is expressed by the condition $\lambda_c > 0$ of Theorem 3.

Indeed, a standard computation shows that $\lambda_c = \beta - (1/2)\,\varepsilon^2$, which is positive if and only if $|\varepsilon| < \sqrt{2\beta}$. According to Theorem 3 (ii), $\rho_2 \in [0,1]$ satisfies

$$\frac{1}{2}\frac{d^2\rho}{dx^2} + \varepsilon\frac{d\rho}{dx} + \beta\left(\rho^2 - \rho\right) = 0.$$

Kolmogorov et al. (1937) proved that there are no non-trivial solutions bounded in $[0,1]$ to this, the travelling wave K-P-P equation, for $|\varepsilon| < \sqrt{2\beta}$ and otherwise there is a unique non-trivial solution. We see that the probability that balls with positive radius become empty is either zero or one (i.e. $\rho_2 \equiv 0$ or $\rho_2 \equiv 1$) according to whether $\lambda_c > 0$ or $\lambda_c \leq 0$ respectively.

4.2. *Transient L and compactly supported β* Let L correspond to a transient diffusion on $D \subseteq \mathbb{R}^d$ and let β be a smooth nonnegative compactly supported function. Since the generalized principal eigenvalue coincides with the classical principal eigenvalue for smooth bounded domains, it follows that for any nonempty ball $B \subset\subset D$ one can pick such a β with $\overline{B} = \text{supp}(\beta)$ and so that $L+\beta$ is supercritical on D, that is $\lambda_c > 0$ (all one has to do is to ensure that the infimum of β on a somewhat smaller ball $B' \subset\subset B$ is larger then the absolute value of the principal eigenvalue on B'. Then, a fortiori, $L+\beta$ is supercritical on D as well). On the other hand, by the transience assumption, it is clear that the initial L-particle wanders out to infinity (or gets killed at the Euclidean boundary) with positive probability without ever visiting B (and thus without ever branching), when starting from a point in $D \setminus B$. This now shows that *there exists a non-trivial travelling wave solution* to $Lu + \beta\left(u^2 - u\right) = 0$ for such an L and β. To the best of our knowledge, this is a new result concerning generalized K-P-P travelling wave equations.

4.3. *Branching Ornstein-Uhlenbeck process and generalization* Let $L = \frac{1}{2}\Delta - kx\cdot\nabla$ on $\mathbb{R}^d$, $d \geq 1$, where $k > 0$. Then L corresponds to the d-dimensional Ornstein-Uhlenbeck process with drift parameter k. Note that it is a (positive) recurrent process. Furthermore let β be a positive constant. Consider now the $(L, \beta; \mathbb{R}^d)$-branching diffusion X. We call X a *branching Ornstein-Uhlenbeck process*. By recurrence it follows that L is a critical operator, and thus $\lambda_c = \lambda_c(L, \mathbb{R}^d) = 0$. Consequently $\lambda_c(L+\beta, \mathbb{R}^d) = \beta$. Obviously (by comparison with a single L-particle), the process does not exhibit local extinction. By Theorem 3(iii), X exhibits local exponential growth with rate β. In fact, as Theorem 4.6.3(i) in Pinsky (1996) shows, $\lambda_c(L+\beta, D) > 0$, whenever L corresponds to a recurrent diffusion on D and the branching rate $\beta \geq 0$ is not identically zero. Therefore, X *exhibits local exponential growth for any recurrent motion and any not identically zero branching rate.*

4.4. *Branching outward Ornstein-Uhlenbeck process* Let $L = \frac{1}{2}\Delta + kx\cdot\nabla$ on $\mathbb{R}^d$, $d \geq 1$, where $k > 0$. Then L corresponds to the d-dimensional "outward" Ornstein-Uhlenbeck process with drift parameter k. This process is transient. Furthermore let β be a positive constant, and consider the $(L, \beta; \mathbb{R}^d)$-branching diffusion X. Following Example 2 in Pinsky (1996), we have that $\lambda_c(L+\beta, \mathbb{R}^d) = \beta - kd$. From Theorem 3(i) we conclude that if $\beta > kd$ then X exhibits local exponential growth (with rate $\beta - kd$). However if $\beta \leq kd$ then X exhibits local extinction.

It is easy to see that $h(x) = \exp\{-k|x|^2\}$ satisfies $(L + \beta - \lambda_c)h = 0$, and that making an h-transform with this h, $L + \beta - \lambda_c$ transforms into

$$(L + \beta - \lambda)^h = \frac{1}{2}\Delta - kx \cdot \nabla. \tag{4.15}$$

Now the operator in (4.15) corresponds to an (inward) Ornstein-Uhlenbeck process which is (positive) recurrent.

Using the associated inner-product martingale (which can again be shown to be a martingale exploiting Kolmogorov's backwards equations together with boundedness) we can follow the arguments of the proof of Theorem 5 to produce (under a changed probability measure) a spine with a doubled rate of reproduction. This spine is precisely the Ornstein-Uhlenbeck process corresponding to the operator (4.15). In order to transfer statements of local survival back to the process under the original measure, we would need mean convergence of the inner-product martingale, or equivalently, the condition $\beta - kd > 0$.

5. Appendix A: Local extinction criterion: analytical arguments In this section we present an *analytical* proof of the local extinction for the $(L, \beta; D)$-branching diffusion. As far as the proof of the condition for local extinction is concerned, we will show how to derive this from the result of Pinsky (discussed in the introduction of this paper) using a comparison argument between branching diffusions and superdiffusions. Our proof of the condition for local non-extinction will be essentially the same as his proof for superdiffusions.

Regarding the comparison mentioned above, it is likely that the deeper reason for it is hidden in the Evans-O'Connell (1994) "immigration picture" (see the comments after Theorem 5). For the rigorous proof we will utilize a result on the "weighted occupation time" for branching particle systems obtained by Evans and O'Connell (1994) (also used for proving the immigration picture in the same paper). In this section Z will denote the $(L, \beta; D)$-branching diffusion.

Proof of the criterion on local extinction.

(i) Assume that $\lambda_c \le 0$. Let $(x, s) \mapsto \psi(s, x)$ be jointly measurable in (x, s) and let $\psi(s) = \psi(s, \cdot)$ be nonnegative and bounded for each $s \ge 0$. By Evans and O'Connell (1994, Theorem 2.2), $E_x\left[\exp\left(-\int_0^t \langle \psi(s), Z_s \rangle ds\right)\right] = u(t, x)$, where u is the so-called *mild solution* of the evolution equation

$$\begin{aligned} &\dot{u}(s) = Lu(s) - \beta u(s) + \beta u^2(s) - \psi(t-s)u(s), \ \ 0 < s \le t, \\ &\lim_{s \downarrow 0} u(s) = 1. \end{aligned} \tag{5.16}$$

[Here we used the notation $u(s) = u(s, \cdot)$ and $\dot{u}$ denotes the time-derivative of u]. Pick a $\psi \in C_c^+(D)$ satisfying $\psi(x) > 0$, for $x \in B$ and $\psi(x) = 0$, for $x \in D \setminus B$. Let $u = u_{t,\theta}^{(T)}$ be the mild solution of the evolution equation

$$\begin{aligned} &\dot{u}(s) = Lu(s) - \beta u(s) + \beta u^2(s) - \theta\psi 1_{[t,\infty)}(T-s)u(s), \ \ 0 < s \le T, \\ &\lim_{s \downarrow 0} u(s) = 1. \end{aligned} \tag{5.17}$$

For the rest of the proof of part (i), let the starting point $x \in D$ be fixed. Using the argument given in Iscoe (1988, p.207), we have that Z exhibits local extinction if and only if

$$\lim_{t\uparrow\infty}\lim_{\theta\uparrow\infty}\lim_{T\uparrow\infty} u_{t,\theta}^{(T)}(T,x)=1. \tag{5.18}$$

Let X be the (L,β,β,D)-superdiffusion and let $U=U_{t,\theta}^{(T)}$ be the mild solution of the evolution equation

$$\begin{aligned} &\dot U(s)=LU(s)+\beta U(s)-\beta U^2(s)+\theta\psi 1_{[t,\infty)}(T-s),\ 0<s\le T,\\ &\lim_{s\downarrow 0}U(s)=0. \end{aligned}\tag{5.19}$$

Again, the argument given in Iscoe (1988,p. 207) shows that X exhibits local extinction if and only if

$$\lim_{t\uparrow\infty}\lim_{\theta\uparrow\infty}\lim_{T\uparrow\infty} U_{t,\theta}^{(T)}(T,x)=0. \tag{5.20}$$

In light of Pinsky's result, (5.20) follows from $\lambda_c\le 0$. We now show that (5.20) implies (5.18), which will complete the proof of this part. Making the substitution $v:=1-u$, we have that v is the mild solution of the evolution equation

$$\begin{aligned} &\dot v(s)=\\ &Lv(s)+\beta v(s)-\beta v^2(s)+\theta\psi 1_{[t,\infty)}(T-s)(1-v(s)),\ 0<s\le T,\\ &\lim_{s\downarrow 0}v(s)=0. \end{aligned}\tag{5.21}$$

By Iscoe (1988, pp. 204), U and v (with t,θ fixed) have the following probabilistic representations:

$$\begin{aligned} &U(T,x)=-\log E_x\exp\left(-\int_0^T \mathrm{d}s\,\langle\theta\psi 1_{[t,\infty)}(s),X_s\rangle\right),\\ &v(T,x)=-\log E_x\exp\left(-\textstyle\int_0^T \mathrm{d}s\,\langle\theta\psi 1_{[t,\infty)}(s)(1-v(T-s)),X_s\rangle\right). \end{aligned}\tag{5.22}$$

From (5.22) it is clear that $v\le U$. Hence $\lim_{t\uparrow\infty}\lim_{\theta\uparrow\infty}\lim_{T\uparrow\infty} v_{t,\theta}^{(T)}(T,x)=0$.

(ii) Assume now that $\lambda_c>0$. The proof of this part is almost the same as the proof of the analogous statement for superdiffusions Pinsky (1996, p.262-263). In that proof it is shown that the assumption $\lambda_c>0$ guarantees the existence of a (large) subdomain $D_0\subset\subset D$, and a function $v\ge 0$ defined on D_0 which is not identically zero and which satisfies

$$\begin{aligned} &Lv+\beta v-\beta v^2=0 \text{ in } D_0\\ &\lim_{x\to\partial D_0}v(x)=0,\\ &v>0 \text{ in } D_0. \end{aligned}\tag{5.23}$$

(The proof of the existence of such a v relies on finding so-called lower and upper solutions for (5.23). The assumption $\lambda_c>0$ enters the stage when a positive lower solution is constructed.) Since $f\equiv 1$ also solves $Lf+\beta f-\beta f^2=0$ *in* D_0, the elliptic maximum principle (see Pinsky (1996, Proposition 3) and Engländer and Pinsky (1999, Proposition 7.1)) implies that $v\le 1$. Let $w:=1-v$. Then $w\ge 0$ and furthermore w satisfies

$$\begin{aligned} &Lw-\beta w+\beta w^2=0 \text{ in } D_0\\ &\lim_{x\to\partial D_0}w(x)=1,\\ &w<1 \text{ in } D_0. \end{aligned}\tag{5.24}$$

Let $\widehat{P}$ denote the probability for the branching diffusion $\widehat{Z}$ obtained from Z by killing the particles upon exiting ∂D_0. Obviously $\widehat{P}_x(\widehat{Z}$ survives$) \le P_x(Z(t, D_0) > 0$ for arbitrary large t's), and thus, it is enough to show that

$$0 < \widehat{P}_x(\widehat{Z} \text{ survives}). \tag{5.25}$$

We now need the fact that $w > 0$ on D_0. This follows from the equation

$$(L - \beta(1-w))w = 0 \text{ in } D_0$$

and the strong maximum principle (Theorem 3.2.6 in Pinsky (1995)) applied to the linear operator $L - \beta(1-w)$. (Indeed, w is a nonnegative harmonic function for the operator, and thus, by the strong maximum principle it must be either everywhere zero (i.e. $v \equiv 1$) or everywhere positive; however the first case is ruled out by the second equation of (5.23).)

Since w is a positive solution to the elliptic equation and is one at the boundary, it is standard to prove that $\exp\{-\langle \log w, \widehat{Z}\rangle\}$ is a martingale and thus

$$\widehat{E}_x\left(e^{-\langle \log w, \widehat{Z}\rangle}\right) = w(x), \ t \ge 0. \tag{5.26}$$

Suppose that (5.25) is not true. Then the left-hand side of (5.26) converges to 1 as $t \uparrow \infty$. On the other hand, the right-hand side of (5.26) is independent of t and is smaller than 1, which is a contradiction. Consequently, (5.25) is true. □

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