On doubly reflected completely asymmetric Lévy processes

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Abstract. Consider a completely asymmetric Lévy process X and let Z be X reflected at 0 and at $a > 0$. In applied probability (e.g. [12, 19]), the process Z turns up in the study of the virtual waiting time in a $M/G/1$-queue with finite buffer a or the water level in a finite dam of size a. We find an expression for the resolvent density of Z. We show Z is positive recurrent and determine the invariant measure. Using the regenerative property of Z, we determine the asymptotic law of $t^{-1}\int_0^t f(Z_s)\mathrm{d}s$ for an appropriate class of functions f. Finally, the long time average of the local time of Z in $x \in [0, a]$ is studied.

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0 Introduction

Consider the following model for the water level in a dam. Initially, the water level in the dam is $X_0 \geq 0$. At random times $T_1, T_2, \ldots$, for which the interarrival times $T_i - T_{i-1}$ (for $i \geq 1$, $T_0 = 0$) are independent and exponentially distributed with mean λ^{-1}, an amount of water $U_1, U_2, \ldots$ flows into the dam, where the U_i form a sequence of positive independent random variables with distribution function F. The water leaves the dam at constant rate one.

If the dam is of infinite depth, the water level at time t is given by $Y_t = X_t - \inf_{0\leq s\leq t} X_s \wedge 0$, where $X = (X_t, t \geq 0)$ is a compound Poisson process minus a unit drift

$$X_t = \sum_{i=1}^{N_t} U_i - t, \tag{1}$$

with $N = (N_t, t \geq 0)$ is a Poisson process with intensity $\lambda > 0$, which is independent of the U_i. See [1, 10, 16] for background on these class of models. Now we adapt this model by

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supposing the dam has a finite size $a > 0$: The amount of water overflowing the level a is then immediately lost. Denote the water level in this finite dam by Z. See Section 2.1 for a precise definition. Note that Z can also be interpreted as the workload of a $M(\lambda)/G/1$-queue with finite buffer size a and service time distribution F.

For this process Z, several questions may emerge: What is the stationary distribution of Z and what about the convergence to stationarity? Can we find an expression for the overflow probability or the part of the time the dam is empty in stationarity? What about the law of long time averages of functions of Z_t?

In the literature (e.g. Cohen [12] and more recently Zwart [19]) this model for a finite dam has been considered before. In this paper, however, we will follow a different approach by studying Z from the point of view of Lévy processes without negative jumps. A Lévy process X is a real-valued random process with stationary and independent increments. Clearly, processes of the form (1) are a subclass of this bigger class. We construct the corresponding process Z for this bigger class of Lévy processes X (Section 2.1), study its ergodic properties and provide the answers of the questions in the previous paragraph.

In Section 2.2, we use regenerative process theory in conjunction with the specific form of the resolvent measure of Z (found in Section 2.1) to show that Z is 0-positive in the classification of Tuomen and Tweedie [18], identify its 0-invariant measure and find that the transition probabilities of Z weakly converge to the invariant measure. Moreover, using the R-theory of irreducible Markov chains developed by Tuomen and Tweedie, we show this convergence to hold in total variation if the transition probabilities of X are absolutely continuous with respect to the Lebesgue measure. Finally, in Section 2.3 the asymptotic law of some time averages and the long time average of the local time in $x \in [0, a]$ of Z are determined.

1 Preliminaries

1.1 Setting

In this section we set some notation and review standard results on spectrally positive Lévy processes. For more background, we refer to [9] or [7], Chapter VII.

Let $X = \{X_t, t \geq 0\}$ be a Lévy process without negative jumps defined on $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, a filtered probability space which satisfies the usual conditions. For all x the measure $\mathbb{P}_x$ will denote the translation of $\mathbb{P}$ under which $X_0 = x$. To avoid trivialities, we exclude the case where X has monotone paths. Since X has no negative jumps, the moment generating function $\mathbb{E}[\mathrm{e}^{-\theta X_t}]$ exists for all $\theta \geq 0$. and is given by

$$\mathbb{E}[\exp(-\theta X_t)] = \exp(t\ \psi(\theta)) \tag{2}$$

for some function $\psi(\theta)$ which is well defined at least on the positive half axis, where it is convex with the property $\lim_{\theta\to\infty} \psi(\theta) = +\infty$. Let $\Phi(0)$ denote its largest root. On $[\Phi(0), \infty)$ the function ψ is stricly increasing and we denote its right inverse function by $\Phi : [0, \infty) \to [\Phi(0), \infty)$. It is well known, that the asymptotic behaviour of X can be determined from the sign of $\psi'_+(0)$, the right derivative of ψ in zero. Indeed, X drifts to ∞, oscillates or drifts to $-\infty$ according to whether $\psi'_+(0)$ is negative, zero or positive.

Denote by I and S the infimum and the supremum of X respectively, that is, $I_t = \inf_{0\le s\le t}(X_t \wedge 0)$ and $S_t = \sup_{0\le s\le t}(X_t \vee 0)$ where we used the notations $c \vee 0 = \max\{c, 0\}$ and $c \wedge 0 = \min\{c, 0\}$. By $Y = X - I$ and $\widehat{Y} = \widehat{X} - \widehat{I} = S - X$ we denote the Lévy process X reflected at its past infimum I and at its past supremum S, respectively.

For X and $\widehat{X} = -X$ the first entrance time of (a, ∞) are denoted by $\widehat{T}_a, T_a$ respectively,

$$T_a = \inf\{t \ge 0 : X_t > a\} \qquad \widehat{T}_a = \inf\{t \ge 0 : -X_t > a\}.$$

Similarly, τ_a and $\widehat{\tau}_a$ represent the first entrance time of (a, ∞) for Y and $\widehat{Y}$ respectively. We will use $\tau_0[\widehat{\tau}_0]$ to stand for the first time $Y[\widehat{Y}]$ hits 0.

1.2 Scale functions and exit problems

As in e.g. [8, 5, 15], a crucial role will be played by the function $W^{(q)}$ which is closely connected to the two-sided exit problem. We give a definition and review some of its properties.

Definition 1 For $q \ge 0$, the *q-scale function* $W^{(q)} : (-\infty, \infty) \to [0, \infty)$ is the unique function whose restriction to $[0, \infty)$ is continuous with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x = (\psi(\theta) - q)^{-1}, \qquad \theta > \Phi(q)$$

and is defined to be identically zero for $x < 0$.

By taking $q = 0$ we get the 0-scale function which is usually called just "the scale function" in the literature. It is well known that $W = W^{(0)}$ is right and left differentiable and increasing on $(0, \infty)$. We write $W'_\pm(x)$ to denote the right and left derivative of W in x respectively. The values of W in 0 and infinity are related to two global properties of X. Indeed, $W(0) = \lim_{\theta\to\infty} \theta/\psi(\theta)$ is zero precisely if X has unbounded variation. Secondly, $W(\infty) = \lim_{x\to\infty} W(x)$ is finite precisely if X drifts to $-\infty$, which follows from a Tauberian theorem in conjunction with the earlier mentioned fact that $\psi'_+(0) > 0$ if and only if X drifts to $-\infty$.

For every fixed $x \ge 0$, we can extend the mapping $q \mapsto W^{(q)}(x)$ to the complex plane by the identity

$$W^{(q)}(x) = \sum_{k\ge 0} q^k W^{\star k+1}(x) \tag{3}$$

where $W^{\star k}$ denoted the k-th convolution power of $W = W^{(0)}$. The convergence of this series is plain from the inequality

$$W^{\star k+1}(x) \le x^k W(x)^{k+1}/k! \qquad x \ge 0, k \in \mathbb{N},$$

which follows from the monotonicity of W. From the expansion (3) we find that on $(0,\infty)$ also $W^{(q)}(\cdot)$ is increasing and right and left differentiable. If X has unbounded variation or has a Lévy measure that is absolutely continuous with respect to the Lebsgue measure, $W^{(q)}$ restricted to $(0,\infty)$ is continuously differentiable (see [14]).

The function $W^{(q)}$ plays a key role in the solution of the two-sided exit problem as shown by the following identity for $x \in [0,a]$, (e.g. Bertoin [8] and references therein)

$$\mathbb{E}_x\left[\mathrm{e}^{-q\widehat{T}_0} I(\widehat{T}_0 < T_a)\right] = W^{(q)}(a-x)/W^{(q)}(a), \tag{4}$$

where $I(A)$ is the indicator of the event A.

We recall the definition, given in [5, 15], of the function $Z^{(q)}$, which is close relative of $W^{(q)}$.

Definition 2 The *adjoint q-scale function* $Z^{(q)}$ is defined by

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(z)\mathrm{d}z. \tag{5}$$

Indeed, as shown in [15], the Laplace transform of the first entrance time of (a,∞) for $\widehat{Y}$ can be expressed in terms of the function $Z^{(q)}$ as follows:

$$\mathbb{E}_{-x}\left[\mathrm{e}^{-q\widehat{\tau}_a}\right] = Z^{(q)}(x)/Z^{(q)}(a) \qquad x \in [0,a].$$

The reflected Lévy process $Y = X - I$ killed upon exiting $[0,a]$ has the strong Markov property; denote its transition probabilities and q-resolvent measure by $(P^t, t \geq 0)$ and $U^q(x,\cdot)$, respectively. Similarly, $(\widehat{P}^t, t \geq 0)$ and $\widehat{U}^q(x,\cdot)$ denote the transition probabilities and q-resolvent measure of $\widehat{Y}$ killed at the first exit of $[0,a]$. From [15] we have the following expressions for the resolvent measures $U^q(x,\cdot), \widehat{U}^q(x,\cdot)$

$$\widehat{U}^q(x,\mathrm{d}y) = \left(Z^{(q)}(x)W^{(q)}(a-y)/Z^{(q)}(a) - W^{(q)}(x-y)\right)\mathrm{d}y \tag{6}$$

$$U^q(x,\mathrm{d}y) = W^{(q)}(a-x)W^{(q)}(\mathrm{d}y)/W^{(q)\prime}_+(a) - W^{(q)}(y-x)\mathrm{d}y \tag{7}$$

where $x, y \in [0,a]$ and $W^{(q)}(\mathrm{d}y)$ denotes the Stieltjes measure associated to $W^{(q)}$ with mass $W^{(q)}(0)$ in zero. Note that $W^{(q)}(\mathrm{d}y)$ and $W(0)\delta_0(\mathrm{d}y) + W^{(q)\prime}_+(y)\mathrm{d}y$ (where δ_0 is the delta measure in zero) are versions of the same measure.

1.3 Examples

1. A stable Lévy process X with index $\alpha \in (1,2]$ has as cumulant $\psi(\theta) = \theta^\alpha$; in [6], Bertoin showed that the Mittag-Leffler function plays an important role in the two-sided exit problem for X. To be more precise, he found that q-scale and adjoint q-scale function of X are respectively given by

$$W^{(q)}(x) = \alpha x^{\alpha-1}E'_\alpha(qx^\alpha) \qquad Z^{(q)}(x) = E_\alpha(qx^\alpha)$$

where E_α is the Mittag-Leffler function with parameter α

$$E_\alpha(y) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(1+\alpha n)}, \qquad y \in \mathbb{R}. \tag{8}$$

In the case that $\alpha = 2$, the process $X/\sqrt{2}$ is a Brownian motion and $W^{(q)}, Z^{(q)}$ reduce to

$$W^{(q)}(x) = q^{-\frac{1}{2}} \sinh(x\sqrt{q}) \qquad Z^{(q)}(x) = \cosh(x\sqrt{q}). \tag{9}$$

Replacing (x, q) by $(2x, q/2)$ in (9), we find $W^{(q)}, Z^{(q)}$ for a standard Brownian motion.
2. Let $X_t = J_t - \mu t$ where J is a compound Poisson process where the jump distribution F is of *phase type* $(m, \boldsymbol{\alpha}, \boldsymbol{T})$ (see [2, 3])

$$1 - F(x) = \boldsymbol{\alpha} \exp(\boldsymbol{T}x)\mathbf{1}, \qquad x \geq 0$$

where $\mathbf{1}$ is a m-column vector of ones, $\boldsymbol{\alpha}$ is a m-probability vector and $\boldsymbol{T}$ is a m times m matrix with negative elements on the diagonal and non-negative ones off-diagonal, such that the row sums are non-positive. From [4] we find that the q-scale function of X is given by

$$W^{(q)}(x) = \left(\exp\left(\Phi(q)x\right) - \boldsymbol{\alpha}^* \exp\left((\boldsymbol{T} + \boldsymbol{t}\boldsymbol{\alpha}_*)x\right)\mathbf{t}\right) / \psi'(\Phi(q))$$

where $\boldsymbol{t} = -\boldsymbol{T}\mathbf{1}$, $\boldsymbol{\alpha}_* = \frac{\lambda}{\mu}\mathbf{p}(\Phi(q)\boldsymbol{I} - \boldsymbol{T})^{-1}$, $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}_*(\Phi(q)\boldsymbol{I} - \boldsymbol{T})^{-1}$ and $\psi'(\Phi(q)) = \mu - \boldsymbol{\alpha}^*\boldsymbol{t}$.
3. If $X_t = J_t - \mu t$ is a standard Poisson process J minus a drift $\mu > 0$, it has as cumulant $\psi(\theta) = \mu\theta + \mathrm{e}^{-\theta} - 1$. The process X has as (0-)scale function

$$W^{(0)}(x) = W(x) = \sum_{n \geq 1} F^{\star n}(x)/\mu^n$$

where $F = F^{\star 1}$ is the cumulative distribution function of a uniform$([0, 1])$-random variable and $F^{\star n+1}(x) = \int F^{\star n}(x - y)\mathrm{d}F(y)$ for $n \geq 1$.

2 The doubly reflected process

2.1 Construction and resolvent

We now show how to construct the path of a Lévy process which is is reflected at 0 and at the level a. This stochastic process moves, while it is in $(0, a)$ as a Lévy process. Each time it attempts to cross 0 or a, it is “regulated” or “perturbed” to keep it inside the interval $[0, a]$. Denote by $D[0, \infty)$ the space of cadlag functions $\omega : [0, \infty) \to \mathbb{R}$. Define the map $g_a : D[0, \infty) \to D[0, \infty)$ piecewise by

$$g_a(\omega)(t) = \begin{cases} (\omega(t) \vee 0) \wedge a & \text{if } \omega(y) \in (0, a)(0 \leq y < t) \text{ or } t = 0 \\ (\omega(t) - \inf_{s \leq y \leq t} \omega(y)) \wedge a & \text{if } g_a(\omega)(s) = 0,\ g_a(\omega)(y) < a\ (s < y < t) \\ (a + \omega(t) - \sup_{s \leq y \leq t} \omega(y)) \vee 0 & \text{if } g_a(\omega)(s) = a,\ g_a(\omega)(y) > 0\ (s < y < t). \end{cases}$$

From the path ω of a Lévy process, the map g_a constructs uniquely the path $g_a(\omega)$ of a doubly reflected Lévy process. For a spectrally positive Lévy process X, we denote this pathwise constructed stochastic process by $Z = \{Z_t = g_a(X)(t), t \geq 0\}$. From the construction, we see that the process Z satisfies the strong Markov property. Denote its transition probabilities and q-resolvent respectively by $\tilde{P}_t(x, A) = \mathbb{P}_x(Z_t \in A)$ and

$$\tilde{U}^q(x, A) = \int_0^\infty \mathrm{e}^{-qt} \tilde{P}_t(x, A)\mathrm{d}t,$$

where $A \subseteq [0, a]$ is a Borel set. Using the results from the previous section, we are able to express the resolvent $\tilde{U}$ in terms of the scale functions $W^{(q)}$ and $Z^{(q)}$.

Theorem 1 *For any $x \in [0, a]$, set $\tilde{u}^q(x, 0) = Z^{(q)}(a - x)W^{(q)}(0)/qW^{(q)}(a)$ and*

$$\tilde{u}^q(x, y) = \frac{Z^{(q)}(a - x)W_+^{(q)\prime}(y)}{qW^{(q)}(a)} - W^{(q)}(y - x) \qquad x, y \in [0, a], y \neq 0. \tag{10}$$

Then $\tilde{u}^q(x, 0)\delta_0(\mathrm{d}y) + u^q(x, y)\mathrm{d}y$ is a version of the measure $\tilde{U}^q(x, \mathrm{d}y)$.

Proof Let $x, y \in [0, a]$ and write $T'_x = \inf\{t \geq 0 : Z_t = x\}$. By $\eta(q)$ we denote an independent exponential random variable with parameter $q > 0$. To compute the probability $\mathbb{P}_x(Z_{\eta(q)} \in \mathrm{d}y)$, we use the strong Markov property of Z to write

$$\mathbb{P}_x(Z_{\eta(q)} \in \mathrm{d}y) = \mathbb{P}_x(Z_{\eta(q)} \in \mathrm{d}y, \eta(q) < T'_a) + \mathbb{E}_x[\mathrm{e}^{-qT'_a}]\mathbb{P}_a(Z_{\eta(q)} \in \mathrm{d}y). \tag{11}$$

From the construction of Z we see that $\{Z_t, t < T'_a\}$ has the same law as $\{Y_t, t < \tau_a\}$. Invoking now (7), we find that the first probability and the Laplace transform on the right-hand side of (11) are given by

$$q\frac{W^{(q)}(a - x)W(0)}{W_+^{(q)\prime}(a)}\delta_0(\mathrm{d}y) + q\left(\frac{W^{(q)}(a - x)W_+^{(q)\prime}(y)}{W_+^{(q)\prime}(a)} - W^{(q)}(y - x)\right)\mathrm{d}y$$

and $Z^{(q)}(a - x) - qW^{(q)}(a)W^{(q)}(a - x)/W_+^{(q)\prime}(a)$ respectively. For the second probability on the right-hand side the strong Markov property implies that

$$\begin{aligned}
\mathbb{P}_a(Z_{\eta(q)} \in \mathrm{d}y) &= \mathbb{E}_a[\mathrm{e}^{-qT'_x}]\mathbb{P}_x(Z_{\eta(q)} \in \mathrm{d}y) + \mathbb{P}_a(Z_{\eta(q)} \in \mathrm{d}y, \eta(q) < T'_x) \\
&= \frac{\mathbb{P}_x(Z_{\eta(q)} \in \mathrm{d}y) + qW^{(q)}(y - x)\mathrm{d}y}{Z^{(q)}(a - x)} - qW^{(q)}(y - a)\mathrm{d}y \qquad (12)
\end{aligned}$$

where in the second line we used that $\{a - Z_t, Z_0 = a, t < T'_x\}$ has the same law as $\{\widehat{Y}_t, \widehat{Y}_0 = 0, t < \widehat{\tau}_{a-x}\}$ in conjunction with the the resolvent (6). Substituting everything yields a linear equation for $\mathbb{P}_x(Z_{\eta(q)} \in \mathrm{d}y)$, which has the stated expression as solution. $\square$

Restricting ourselves to Lévy processes without negative jumps that have absolutely continuous transition probabilities,

$$\mathbb{P}(X_t \in \mathrm{d}x) \ll \mathrm{d}x \qquad \text{for all } t > 0, \tag{AC}$$

we can prove continuity in the time and space variable of the transition probabilities $(\tilde{P}^t, t \geq 0)$. It is known that (AC) holds whenever the Brownian coefficient is positive or when the mass of the absolutely continuous part of the Lévy measure is infinite (see [17]). We use the standard notation $\tilde{P}^t f(x) = \int f(y)\tilde{P}(x, \mathrm{d}y)$.

Proposition 1 *Let f be any Borel bounded function on $[0, a]$. Supposing (AC) holds, we have:*

(i) For every $x \in [0, a]$ the mapping $t \mapsto \tilde{P}^t f(x)$ is continuous on $(0, \infty)$;

(ii) For every $t > 0$ the mapping $x \mapsto \tilde{P}^t f(x)$ is continuous on $[0, a]$.

If a semi-group has property (ii), one says that it has the strong Feller property.
Proof (i) Let $A \subseteq [0, a]$ be any Borel set. By the Markov property as in (11) and by the fact that $\{Z_t, t < T'_a\}$ has the same law as $\{Y_t, t < \tau_a\}$, we find for $x, y \in [0, a]$

$$\mathbb{P}_x(Z_t \in A) = \mathbb{P}_x(Y_t \in A, t < \tau_a) + \int_0^t \mathbb{P}_a(Z_{t-s} \in A)\mathbb{P}_x(\tau_a \in \mathrm{d}s). \tag{13}$$

Under (AC), Proposition 5 in [15] states that the first term in (13) is continuous in t. Moreover, by Lemma 6 the distribution of τ_a has no atoms, that is, $\mathbb{P}_x(\tau_a = t)$ is zero for all $x \in [0, a]$ and $t > 0$. Thus, the measure $\mathbb{P}_x(\tau_a \in \mathrm{d}s)$ is absolutely continuous with respect to the Lebesgue measure and therefore the convolution in t in (13) is t-continuous (e.g. [13]). The assertion (i) is proved for indicator functions I_A. To extend this to any bounded Borel function f, we approximate f by a sequence of step-functions $f_n = \sum_k \frac{k}{2^n} I_{A_k}$ with $A_k = \{y \in [0, a] : f(y) \in\in [\frac{k}{2^n}, \frac{k+1}{2^n})$. Write h_n and g respectively for the integral of f_n and f against $\mathbb{P}_x(Z_t \in \mathrm{d}y)$ and note the h_n are continuous in t on $(0, \infty)$. Continuity in t of g follows now since it is readily verified that h_n converges to g uniformly on compact subsets of $(0, \infty)$.

(ii) We will use now the decomposition (13) to prove the continuity of $x \mapsto \tilde{P} f(x)$. Invoking Proposition 5, we find that $P^t f$, f integrated against the first term on the right-hand side of (13), is continuous on $[0, a]$. In particular, we see that $x \mapsto \mathbb{P}_x(\tau_a > t)$ is continuous and then (by absence af atoms) also $x \mapsto \mathbb{P}_x(\tau_a \in F)$ for all open and closed intervals F. Since by part (i) $g : s \mapsto \int_0^a f(y)\mathbb{P}_a(Z_s \in \mathrm{d}y)$ is continuous on $(0, t)$, the set $F_k := \{s : g(s) \in [\frac{k}{2^n}, \frac{k+1}{2^n})\}$ is the union of an open and a closed set. Set the stepfunctions $g_n(s) = \sum_k \frac{k}{2^n} I_{F_k}(s)$ and note that $x \mapsto \int_0^t g_n(t - s)\mathbb{P}_x(\tau_a \in \mathrm{d}s)$ is continuous on $[0, a]$ for each n. Since

$$\sup_{x \in [0,a]} \left| \int_0^t (g_n(t - s) - g(t - s))\, \mathbb{P}_x(\tau_a \in \mathrm{d}s)) \right| \leq 2^{-n},$$

we conclude that the second term of (13) is continuous in x and the proof is done. □

2.2 Ergodicity

We now turn our attention to the ergodic properties of the transition probabilities $\tilde{P}^t$. The next theorem contains the results in that direction. For the terminology in the theorem, we refer to Tuominen and Tweedie [18].

Theorem 2 *The following are true:*

(i) $\tilde{P}^t$ *is* 0*-recurrent and, more precisely,* 0*-positive;*

(ii) The function 1 *and measure* $\tilde{\Pi}_a(\mathrm{d}x) = W(\mathrm{d}x)/W(a)$ *are invariant for* $\tilde{P}^t$*, that is*

$$\tilde{P}^t 1 \equiv 1 \qquad \text{and} \qquad \tilde{\Pi}_a \tilde{P}^t = \tilde{\Pi}_a \qquad\qquad \text{for all } t > 0.$$

(iii) For every $x \in [0, a]$*, we have*

$$\lim_{t\to\infty} \tilde{P}^t(x, \cdot) = \tilde{\Pi}_a(\cdot), \tag{14}$$

where the limit is in the sense of weak convergence.

(iv) Suppose in addition (AC) holds. Then, for every $x \in [0, a]$*, the convergence in (14) holds in total variation norm.*

Example. If X is a Brownian motion with drift μ, the invariant measure for Z is given by $2\mu \mathrm{e}^{-2\mu x}\mathrm{d}x/(1 - \mathrm{e}^{-2\mu a})$. For a stable process with index $\alpha \in (1, 2]$, the corresponding process Z has as invariant measure $(\alpha - 1)/a \cdot (x/a)^{\alpha-2}\mathrm{d}x$ on $[0, a]$. In particular, if X is a standard Brownian motion, we see the invariant measure for Z is given by the uniform measure on $[0, a]$.

If X drifts to $+\infty$, we recall that $W(\infty)$ is finite and readily check that $\tilde{\Pi}_a(\mathrm{d}x)$ weakly converges to $\tilde{\Pi}_\infty(\mathrm{d}x) = W(\mathrm{d}x)/W(\infty)$ as a tends to infinity. Since $\tilde{\Pi}_\infty$ is the invariant distribution of $Y = X - I$, we see that the *proportionality* $\tilde{\Pi}_a(\mathrm{d}x) = \tilde{\Pi}_\infty(\mathrm{d}x)/\tilde{\Pi}_\infty([0, a])$ holds, an identity which is well known in queueing theory (see [19] and references therein).

Remark. Consider workload process Z of the $M(\lambda)/G/1$-queue with finite buffer a as described in the introduction. Let w_n be the actual waiting time of the nth customer in this queue. Then $w_n = Z_{\tau(n)-}$, where $\tau(n)$ is the nth jump time of X in (1). By the PASTA property (Poisson Arrivals See Time Averages, e.g. Theorem 3.3 in Asmussen [1]) $(w_n)_n$ has the same stationary distribution as Z, namely $W(\mathrm{d}x)/W(a)$. In particular, in stationarity an arriving customer is served immediately with probability $W(0)/W(a)$.

The probability p_a that part of the work U offered by a customer (entering the queue in its stationary regime Z_∞) goes lost is given by

$$p_a = \mathbb{P}(Z_\infty + U > a) = \int_0^\infty \left(1 - \frac{W(a - x)}{W(a)}\right) F(\mathrm{d}x) = \frac{1}{\lambda} \cdot \frac{W'_+(a)}{W(a)},$$

where the third equality follows by verifying that that the the functions $a \mapsto W'_+(a)/\lambda$ and $a \mapsto \int_0^\infty (W(a) - W(a - x))F(\mathrm{d}x)$ are right-continuous and their Laplace transforms are both equal to

$$\theta/\psi(\theta) - 1 = \int (1 - \mathrm{e}^{-\theta x})F(\mathrm{d}x)/\psi(\theta).$$

Proof of Theorem 2 (i) & (ii) Note, from the expansion (3), that $Z^{(q)}(x)$,$W^{(q)}(x)$ and $W_+^{(q)\prime}(x)$ are continuous as function of q. Moreover, $Z^{(0)} \equiv 1$ and $W(x)$, $W'_+(x)$ are positive on $(0, \infty)$ (cf. [15]). Take now any $x \in [0, a]$ and let A be any non-null Borel set then by Theorem 1, Fatou lemma and monotonic convergence we deduce that

$$\infty \leq \lim_{q \downarrow 0} \left\{ \frac{Z^{(q)}(a-x)}{qW^{(q)}(a)} \int_A W^{(q)}(\mathrm{d}y) - \int_A W^{(q)}(y-x)\mathrm{d}y \right\} = \int \tilde{P}^t(x, A)\mathrm{d}t. \tag{15}$$

We see that $\tilde{P}^t$ is 0-recurrent in the classification of Tuomen and Tweedie [18].

Note that $\tilde{P}^t 1 = 1$ or 1 is an invariant function for $\tilde{P}^t$. Next we identify the invariant measure. After some algebra, one can check from the decomposition (3) that $qW \star W^{(q)}(a) = W^{(q)}(a) - W(a)$ and $q\int_0^a W^{(q)}(a-x)W(\mathrm{d}x) = W_+^{(q)\prime}(a) - W'_+(a)$. Moreover, by partial integration we find

$$\int_0^a Z^{(q)}(a-x)W(\mathrm{d}x) = W(a) + \int_0^a W^{(q)}(a-x)W(x)\mathrm{d}x.$$

Combining these three identities with Theorem 1, we find that

$$\int \mathrm{e}^{-qt} \tilde{\Pi}_a \tilde{P}^t(x, \mathrm{d}y)\mathrm{d}t = q^{-1}\tilde{\Pi}_a(\mathrm{d}y).$$

Unicity of the Laplace transform in conjunction with the previous display implies that $\tilde{\Pi}_a \tilde{P}^t(x, \cdot) = \tilde{\Pi}_a(\cdot)$ for all t outside some Lebesgue null-set N. Take any $t \in N$. Then we can find a $s \notin N$ such that $t - s \notin N$. An application of the Markov property yields then

$$\tilde{\Pi}_a \tilde{P}^t(\cdot, \mathrm{d}y) = \tilde{\Pi}_a \left(\int \tilde{P}^s(z, \mathrm{d}y)\tilde{P}^{t-s}(\cdot, \mathrm{d}z) \right) = \tilde{\Pi}_a(\tilde{P}^s(\cdot, \mathrm{d}y)) = \tilde{\Pi}_a(\mathrm{d}y).$$

Thus, we conclude that the measure $\tilde{\Pi}_a$ is invariant for $\tilde{P}^t$ and, as the integral of the function 1 under $\tilde{\Pi}_a$ is finite, $\tilde{P}^t$ is 0-positive.

To prove (iii) we show that Z is a delayed regenerative process. The delay is the time till Z reaches zero, the regeneration points are then the visits to zero which take place after a visit to a. Denote by C the cycle length. Then we can verify that the distribution of C is not concentrated on a lattice. Indeed, recalling that T'_c denotes the first time Z hits the set $\{c\}$, we find that

$$\mathbb{P}_0(C = t) = \int_0^t \mathbb{P}_0(T'_a = t - s)\mathbb{P}_a(T'_0 \in \mathrm{d}s)$$

which is zero by the fact that, under $\mathbb{P}_0$, T'_a and τ_a have the same law and the distribution of the latter has no atoms (Lemma 6 in [15]). Moreover, C has finite mean (see the forthcoming Lemma 1(i) for an explicit computation of $\mathbb{E}[C]$). Theorem V.1.2 in [1] then implies $\mathbb{P}(Z_t \in A) \to \mathbb{P}^e(Z_\infty \in A) = \mathbb{E}[\int_0^C I(Z_t \in A)\mathrm{d}t]/\mathbb{E}[C]$. By bounded convergence combined with the invariance of $\tilde{\Pi}_a$, we identify $\mathbb{P}^e(Z_\infty \in \cdot)$ as $\tilde{\Pi}_a$.

(iv) Assume now (AC) holds. From equation (15), we see that the transition probabilities $\tilde{P}^t$ are Lebesgue irreducible (i.e. for any Borel set $A \subseteq [0, a]$ and for any $x \in [0, a]$ the potential $U(x, A) \in (0, \infty])$. By Proposition 1 in conjunction with Theorem 1 in [18], we can now invoke Theorem 5 in [18] to conclude that (iv) holds for $\tilde{\Pi}_a$-a.e. $x \in [0, a]$. Note that for any $x \in [0, a]$, the measure $\tilde{U}^q(x, \cdot)$ is absolutely continuous w.r.t. $\tilde{\Pi}_a$. Proposition 1(i) implies that for all $t > 0$ the same holds for the measure $\tilde{P}^t(x, \cdot)$. Take now any $x \in [0, a]$ and let $\|\cdot\|$ denote the total variation norm on $[0, a]$ equipped with the Borel-σ-algebra, then by the Markov property of Z and the absolute continuity of $\tilde{P}^t(x, \cdot)$

$$\|\tilde{P}^t(x, \cdot) - \tilde{\Pi}_a(\cdot)\| \leq \int_0^a \|\tilde{P}^{t-s}(y, \cdot) - \tilde{\Pi}_a(\cdot)\| \tilde{P}^s(x, \mathrm{d}y),$$

which converges to zero as $t \to \infty$ (by bounded convergence). □

2.3 Long time averages

In the proof of Theorem 2 we used the fact that Z is a delayed regenerative process where the delay is the time for Z to reach zero and the cycles are the first visits to zero after a visit to a. Since the cycles of Z are i.i.d., the theory of regenerative processes enables us to derive a strong law of large numbers and a central limit theorem for $t^{-1} \int_0^t f(Z_s)\mathrm{d}s$ as $t \to \infty$. For a proof we refer to Theorema 3.1 and 3.2 in [1]. Recall we denote the cycle length by C.

Proposition 2 *(i) For all functions f which $\mathbb{E}[\sup_{0 \leq t < C} |\int_0^t f(Z_s)\mathrm{d}s|] < \infty$ or that are nonnegative on $[0, a]$, it holds that*

$$\frac{1}{t} \int_0^t f(Z_s)\mathrm{d}s \to \tilde{\Pi}_a(f) \qquad \text{a.s. as } t \to \infty.$$

(ii) If $\mathbb{V}\mathrm{ar}(\int_0^C f(Z_s)\mathrm{d}s) < \infty$, the limiting distribution of

$$\left(\int_0^t f(Z_s)\mathrm{d}s - t\tilde{\Pi}_a(f)\right) / \sqrt{t}$$

is normal with mean 0 and variance $\sigma^2/\mathbb{E}_0[C]$ where

$$\sigma^2 = \mathbb{V}\mathrm{ar}(U) + \tilde{\Pi}_a(f)^2 \mathbb{V}\mathrm{ar}(C) - 2\tilde{\Pi}_a(f)\mathbb{C}\mathrm{ov}(U, C)$$

where $U = \int_0^C f(Z_s)\mathrm{d}s$ and where $\mathbb{V}\mathrm{ar}[\mathbb{C}\mathrm{ov}]$ denotes the [co]variance under $\mathbb{P}_0$.

The variance $\sigma^2/\mathbb{E}_0[C]$ can be expressed in terms of the scale function W and its right-derivative, as shown in the following lemma.

Lemma 1 *(i) For all $a > 0$, $W'_+(a) > 0$ and the mean of C and U are given by $\mathbb{E}_0[C] = W(a)^2/W'_+(a)$ and $\mathbb{E}_0[U] = W(a)^2\tilde{\Pi}_a(f)/W'_+(a)$.*

(ii) The expectations $\mathbb{E}_0[C^2]$*,* $\mathbb{E}_0[CU]$ *and* $\mathbb{E}_0[U^2]$ *can be computed by integrating the functions* $2, yf(z) + zf(y)$ *and* $2f(z)f(y)$*, respectively, with respect to the measure*

$$\mathbb{E}_0\left[\int_{(0,\infty)^2} I(X_t \in \mathrm{d}y, X_s \in \mathrm{d}z, 0 < t < s < C)\mathrm{d}t\mathrm{d}s\right]$$
$$= \frac{W(a)W(a-y)}{W'_+(a)^2}W(\mathrm{d}y)W(\mathrm{d}z) - \frac{W(y)W(a-y)}{W'_+(a)}W(\mathrm{d}z)\mathrm{d}y$$
$$+ (W(z) - W(z-y))\frac{W(a)}{W'_+(a)}W(\mathrm{d}y)\mathrm{d}z.$$

Proof (i) Recalling that $\mathbb{E}_0[U] = \mathbb{E}_0[C]\tilde{\Pi}_a(f)$, we concentrate on the computation of the expectation of C. Note that the cycle length C has the same distribution as $(T'_0 \circ \theta_{T'_a}, \mathbb{P}_0)$, the first time *after* T'_a that Z reaches the level zero. Furthermore, as in the proof of Theorem 1 we see that T'_a and τ_a have the same law under $\mathbb{P}_0$ and T'_0 has under $\mathbb{P}_a$ the same law as $\widehat{\tau}_a$ under $\mathbb{P}_0$. Using the resolvents (6) and (7), one checks that $\mathbb{E}_0[\widehat{\tau}_a] = \overline{W}(a)$ and $\mathbb{E}_a[\tau_0] = W(a)^2/W'_+(a) - \overline{W}(a)$, where $\overline{W}(x) = \int_0^x W(y)\mathrm{d}y$. Putting all bits together leads to the stated result Since W is increasing and right-differentiable, it follows that $W'_+(a) > 0$ for all but countably many $a > 0$. Since $\mathbb{E}_r[\tau_0] \le \mathbb{E}_s[\tau_0]$ for $r \le s$, we then deduce $W'_+(a) > 0$ for all $a > 0$.

(ii) Let $\eta(q), \eta(r)$ be two independent exponential random variables with parameters q, r respectively. Recall from the proof of Theorem 1 the definition of T'_x. Following the same methodology as in the proof of Theorem 1, we split the probability space according to whether $\eta(q), \eta(r)$ are before or after T'_a and apply the Markov property of Z. The equalities in law in the proof of Theorem 1 imply then that $\mathbb{P}(Z_{\eta(q)} \in \mathrm{d}y, Z_{\eta(r)} \in \mathrm{d}z, \eta(q) < \eta(r) < T')$ is equal to the sum of the three terms

$$\mathbb{P}(Z_{\eta(q)\wedge\eta(r)} \in \mathrm{d}y, \eta(q) < T'_a, \eta(q) < \eta(r))\mathbb{P}_y(Z_{\eta(r)} \in \mathrm{d}z, \eta(r) < T'_a)$$
$$= q(q+r)^{-1}\mathbb{P}(Y_{\eta(q+r)} \in \mathrm{d}y, \eta(q+r) < \tau_a)\mathbb{P}_y(Y_{\eta(r)} \in \mathrm{d}z, \eta(r) < \tau_a),$$

where we used $\mathbb{P}(\eta(q) < \eta(r)) = q/(q+r)$, and

$$\mathbb{P}(Z_{\eta(q)\wedge\eta(r)} \in \mathrm{d}y, T'_a < \eta(q), \eta(q) < \eta(r))\mathbb{P}_y(Z_{\eta(r)} \in \mathrm{d}z, \eta(r) < T'_0)$$
$$= q(q+r)^{-1}\mathbb{E}[\mathrm{e}^{-(q+r)\tau_a}]\mathbb{P}_0(\widehat{Y}_{\eta(q+r)} \in \mathrm{d}(a-y), \eta(q+r) < \widehat{\tau}_a)\mathbb{P}_{a-y}(\widehat{Y}_{\eta(r)} \in \mathrm{d}(a-z), \eta(r) < \widehat{\tau}_a)$$

and finally

$$\mathbb{P}(Z_{\eta(q)\wedge\eta(r)} \in \mathrm{d}y, \eta(q) < T'_a, \eta(q) < \eta(r))\mathbb{P}_y(Z_{\eta(r)} \in \mathrm{d}z, T'_a < \eta(r) < T'_0 \circ \theta_{T'_a})$$
$$= q(q+r)^{-1}\mathbb{P}(Y_{\eta(q+r)} \in \mathrm{d}y, \eta(q+r) < \tau_a)\mathbb{E}_y[\mathrm{e}^{-(q+r)\tau_a}]\mathbb{P}_0(\widehat{Y}_{\eta(r)} \in \mathrm{d}(a-z), \eta(r) < \widehat{\tau}_a).$$

Substituting the expressions from the resolvents (6) and (7) and letting q, r tend to zero, we find after some algebra the stated formula. $\square$

If X has bounded variation, we see from previous proposition, that Z spends on average $W(0)/W(a) > 0$ part of the time in 0. In any $x \in (0, a]$ (and also in $x = 0$ if X has unbounded variation) Z spends on average no time. Therefore we look at the local time at x. Define the local time at x as an occupation density as follows

$$L_t^x = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{|Z_t - x| < \epsilon\}} \mathrm{d}t.$$

Then we have the following result:

Proposition 3 *For $x \in (0, a)$ and $y \in [0, a]$ we have*

$$\lim_{t\to\infty} \mathbb{E}_y[L_t^x]/t = c \qquad \textit{and} \qquad \lim_{t\to\infty} L_t^x/t = c \qquad \mathbb{P}_y\textit{–almost surely}$$

where

$$c = \frac{1}{2} \cdot \frac{W'_+(x) + W'_-(x)}{W(a)}. \tag{16}$$

For $x = a$ (and for $x = 0$ if X has unbounded variation) the proposition remains valid if we set $c = W'_-(a)/W(a)$ (and $c = W'_+(0)/W(a)$ respectively).
Proof Following the same line of reasoning as in the first lines of the proof of [7, Prop V.5], we can prove

$$\frac{1}{2\epsilon} \int_0^{\eta(q)} \mathbf{1}_{\{|Z_t - x| < \epsilon\}} \mathrm{d}t \to L_{\eta(q)}^x$$

in $L^2(\mathbb{P})$ as $\epsilon \downarrow 0$. Hence, we note that if X has unbounded variation

$$\begin{aligned} \mathbb{E}_y\left[\int_0^\infty \mathrm{e}^{-qt} \mathrm{d}L_t^x\right] &= \lim_{\epsilon\downarrow 0} \frac{1}{2\epsilon} \mathbb{E}_y\left[\int_0^\infty \mathrm{e}^{-qt} \mathbf{1}_{\{|Z_t - x| < \epsilon\}} \mathrm{d}t\right] \\ &= \lim_{\epsilon\downarrow 0} \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} \tilde{u}^q(y, z)\mathrm{d}z = \tilde{u}^q(y, x), \end{aligned}$$

where for the final equality we used that $x \mapsto u^q(y, x)$ is continuous on $(0, a)$ (see (10) and recall that we have $W(0) = 0$). By partial integration and Fubini's theorem $\mathbb{E}_y[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L_t^x] = \int_0^\infty \mathrm{e}^{-qt}\mathrm{d}\mathbb{E}_y[L_t^x]$. A Tauberian theorem (e.g. [7, p.10]) in conjunction with the form of the density $\tilde{u}^q(y, x)$ and the continuity in q of $W^{(q)\prime}$ and $Z^{(q)}$ implies then that for $y \in [0, a]$

$$\mathbb{E}_y[L_t^x]/t \to W'(x)/W(a) \qquad \text{as } t \to \infty, \tag{17}$$

recalling that W is continuously differentiable in this case.

If X has bounded variation, W is not necessarily continuously differentiable and $W(0) > 0$. However, it is not hard to verify that the limit (17) remains valid if we replace $W'(x)$ by $(W'_+(x) + W'_-(x))/2$.

Denote by $\sigma_s^x = \inf\{t > 0 : L_t^x > s\}$ the right-continuous inverse of L^x. Note that $(\sigma_s^x, s \geq 0)$ is a subordinator. Then we see that the process $\lceil L_t^x \rceil$ defined by $\lceil L_t^x \rceil = \inf\{n \in$

$\mathbb{N} : \sigma_n^x > t\}$ is a renewal process under $\mathbb{P}_x$ (resp. delayed renewal process under $\mathbb{P}_y$ with $y \in [0,a], y \neq x$). A renewal theorem implies then that for $y \in [0,a]$

$$\mathbb{E}_y[\lceil L_t^x \rceil]/t \to \mathbb{E}_x[\sigma_1^x]^{-1} \qquad \text{as } t \to \infty. \tag{18}$$

Since $\lceil L_t^x \rceil - 1 < L_t^x \leq \lceil L_t^x \rceil$, the convergence (18) continues to hold if we replace $\lceil L_t^x \rceil$ by L_t^x. By comparing limits we see that $\mathbb{E}_x[\sigma_1^x]^{-1}$ is equal to (16).

Finally, by the strong law of large numbers combined with the independent increments property of σ^x we find

$$\lim_{t\to\infty} L_t^x/t = \lim_{t\to\infty} t/\sigma_t^x = \mathbb{E}_x[\sigma_1^x]^{-1} \qquad \mathbb{P}_y\text{-almost surely, } y \in [0,a],$$

which completes the proof. □

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