On exit and ergodicity of the spectrally one-sided Lévy process reflected at its infimum

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Abstract. Consider a spectrally one-sided Lévy process X and reflect it at its past infimum I. Call this process Y. For spectrally positive X, Avram et al. [2] found an explicit expression for the law of the first time that $Y = X - I$ crosses a finite positive level a. Here we determine the Laplace transform of this crossing time for Y, if X is spectrally negative. Subsequently, we find an expression for the resolvent measure for Y killed upon leaving $[0, a]$. We determine the exponential decay parameter ϱ for the transition probabilities of Y killed upon leaving $[0, a]$, prove that this killed process is ϱ-positive and specify the ϱ-invariant function and measure. Restricting ourselves to the case where X has absolutely continuous transition probabilities, we also find the quasi-stationary distribution of this killed process. We construct then the process Y confined to $[0, a]$ and proof some properties of this process.

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0 Introduction

A spectrally one-sided Lévy process is a real-valued stochastic process with stationary and independent increments, which has jumps of one sign. In this paper we will study such a Lévy process reflected at its past infimum, that is, the Lévy process minus its past infimum. In applied probability, these reflected processes frequently occur, for example in the study of the water level in a dam, the work load in a queue or the stock level (See e.g. [1, 8, 20] and references therein.) Moreover, the reflected Lévy process occurs in relation with a problem associated with mathematical finance. See [17, 22] and references therein.

The paper decomposes into three parts. In the first part, we study the level-crossing probabilities of the reflected Lévy process. For spectrally positive Lévy processes X, Avram et al. [2] found an explicit expression for the Laplace transform of the first exit-time of

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the reflected process from $[0,a]$. In Section 3.2 we complement this study by obtaining the Laplace transform of the exit time for the reflected process of the dual, a spectrally negative Lévy process. Subsequently, in Section 4, we solve for the resolvent measure of the transition probabilities of the reflected Lévy process killed upon leaving $[0,a]$.

Bertoin [6] investigated the exponential decay and ergodicity for completely asymmetric Lévy processes killed upon leaving a finite interval. The purpose of the second part is to extend Bertoins study to reflected Lévy processes killed upon up-crossing a finite level. We determine the exponential decay parameter ϱ of the semi-group, prove that the process is ϱ-positive in the classfication of Tuominen and Tweedie [29] and specify the ϱ-invariant function and measure. Restricting ourselves to Lévy processes of which the one dimesional distributions are absolutely continuous with respect to the Lebesgue measure, we also find the quasi-stationary distribution. Section 7 contains the main results in that direction. Section 6 contains a study of the transition probabilities of the reflected Lévy process with results that are preparatory for these results.

Important elements in the proof of the ergodic properties and the exponential decay are the special form of the earlier computed resolvent measure together with special properties of fluctuation theory of completely asymmetric Lévy processes, elementary properties of analytic functions and the R-theory developed by Tuominen and Tweedie [29] for a general irreducible Markov process.

The third part starts with the construction by h-transform of the reflected process conditioned to stay below the level a. We study then this process: we show positive recurrence and determine the stationary measure. If the one-dimesional distributions of the Lévy process are absolutely continuous, we observe that, as a direct consequence of the results of the second part mentioned above, the conditional probabilities of the reflected Lévy process conditioned on the fact that it exits $[0,a]$ after t, converge as t tends to infinity. The in this way constructed process conincides with the earlier mentioned h-transform. If X has unbounded variation, also the rate of convergence of the supremum of the reflected process to a is studied. The results obtained are analogous to the ones Lambert [18] achieves in his study of a completely asymmetric Lévy process confined in a finite interval. The results of this part can be found in section 8.

The function $Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)\mathrm{d}y$, where $W^{(q)}$ is the scale function of a spectrally negative Lévy process X that is killed at an independent exponential time with parameter q, plays an important role in our results. See for a precise definition the forthcoming section 1. The function $Z^{(q)}$ first occured, although implicitly, in [4, 6]. In [2] it is shown to be closely related to certain exit-problems of X and X reflected at its supremum. Just like the function $W^{(q)}$ is called a q-scale function since $\{\exp(-q(\widehat{T}\wedge t))W^{(q)}(X_{\widehat{T}\wedge t}), t \geq 0\}$ is a martingale for $\widehat{T}$ the first exit time of the positive real line, the function $Z^{(q)}$ has an analogous property, but now for the reflected process Y; indeed $\exp(-qt)Z^{(q)}(Y_t)$ is a martingale. Therefore we would propose to call $Z^{(q)}$ the *adjoint* q-scale function.

1 Preliminaries

This section reviews standard results on spectrally negative Lévy processes. For more background we refer to [7] or [5], Chapter VII.

Let $X = \{X_t, t \geq 0\}$ be a Lévy process without positive jumps defined on $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, a filtered probability space which satisfies the usual conditions. For all x the measure $\mathbb{P}_x$ will denote the translation of $\mathbb{P}$ under which $X_0 = x$. To avoid trivialities, we exclude the case where X has monotone paths. Since X has no positive jumps, the moment generating function $\mathbb{E}[\mathrm{e}^{\theta X_t}]$ exists for all $\theta \geq 0$. and is given by

$$\mathbb{E}[\exp(\theta X_t)] = \exp(t\ \psi(\theta))$$

for some function $\psi(\theta)$ which is well defined at least on the positive half axis, where it is convex with the property $\lim_{\theta\to\infty}\psi(\theta) = +\infty$. Let $\Phi(0)$ denote its largest root. On $[\Phi(0), \infty)$ the function ψ is stricly increasing and we denote its right inverse function by $\Phi : [0, \infty) \to [\Phi(0), \infty)$. It is well known, that the asymptotic behaviour of X can be determined from the sign of $\psi'_+(0)$, the right derivative of ψ in zero. Indeed, X drifts to $-\infty$, oscillates or drifts to $+\infty$ according to whether $\psi'_+(0)$ is negative, zero or positive.

We use the notations $c \vee d = \max\{c, d\}$ and $c \wedge d = \min\{c, d\}$. Denote by I and S the past infimum and supremum of X respectively, that is,

$$I_t = \inf_{0\leq s\leq t}(X_t \wedge 0), \qquad S_t = \sup_{0\leq s\leq t}(X_t \vee 0)$$

and introduce the notations $Y = X - I$ and $\widehat{Y} = \widehat{X} - \widehat{I} = S - X$ for the Lévy process X reflected at its past infimum I and its dual, the process X reflected at its supremum. Denote by $\eta(q)$ an exponential random variable with parameter $q > 0$ which is independent of X. The Wiener-Hopf factorization of X implies that $Y_{\eta(q)}$ has an exponential distribution with parameter $\Phi(q)$ and that

$$\mathbb{E}[\exp(-\theta \widehat{Y}_{\eta(q)})] = \frac{q}{q - \psi(\theta)} \cdot \frac{\Phi(q) - \theta}{\Phi(q)}. \tag{1}$$

2 Scale functions

As in e.g. [6, 2], a crucial role will be played by the function $W^{(q)}$ which is closely connected to the two-sided exit problem. To be precise we give a definition for $W^{(q)}$ and review some of its properties.

Definition 1 For $q \geq 0$ the *q-scale function* $W^{(q)} : (-\infty, \infty) \to [0, \infty)$ is the unique function whose restriction to $[0, \infty)$ is continuous and has Laplace transform

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x = (\psi(\theta) - q)^{-1}, \qquad \theta > \Phi(q)$$

and is defined to be identically zero for $x < 0$.

By taking $q = 0$ we get the 0-scale function which is usually called just "the scale function" in the literature [5]. For $q > 0$, $W^{(q)}$ can be regarded as "the scale function" of the original process X killed at an independent exponential time with parameter q. It is known that $W = W^{(0)}$ is increasing, when restricted to $(0, \infty)$. Moreover, the value of W is 0 and infinity is connected to certain global properties of X. Indeed, $W(0)$ is zero precisely if X has unbounded variation. Secondly, $W(\infty) = \lim_{x\to\infty} W(x)$ is finite, precisely if X drifts to ∞, which follows from a Tauberian theorem in conjunction with the earlier mentioned fact that $\psi'_+(0) > 0$ if and only if X drifts to ∞.

Inverting now the Laplace transform (1), we find that

$$\mathbb{P}(\widehat{Y}_{\eta(q)} \in \mathrm{d}y) = \frac{q}{\Phi(q)} W^{(q)}(\mathrm{d}y) - qW^{(q)}(y)\mathrm{d}y, \qquad y \geq 0, \tag{2}$$

where $W^{(q)}(\mathrm{d}y)$ denotes the Stieltjes measure associated with $W^{(q)}$ with mass $W^{(q)}(0)$ in zero.

For every fixed x, we can extend the mapping $q \mapsto W^{(q)}(x)$ to the complex plane by the identity

$$W^{(q)}(x) = \sum_{k\geq 0} q^k W^{\star k+1}(x) \tag{3}$$

where $W^{\star k}$ denoted the k-th convolution power of $W = W^{(0)}$. The convergence of this series is plain from the inequality

$$W^{\star k+1}(x) \leq x^k W(x)^{k+1}/k! \qquad x \geq 0, k \in \mathbb{N},$$

which follows from the monotonicity of W. From the expansion (3) and the properties of W, we see that the q-scale function is continuous except possibly at zero. and that it is for each $q \geq 0$ increasing on $(0, \infty)$.

Closely related to $W^{(q)}$ is the function $Z^{(q)}$. We recall the definition given in [2].

Definition 2 The *adjoint q-scale function* $Z^{(q)}$ is defined by

$$Z^{(q)}(x) = 1 + q\int_{-\infty}^{x} W^{(q)}(z)\mathrm{d}z. \tag{4}$$

Note that this function inherits some properties from $W^{(q)}(x)$. Specifically it is strictly increasing, differentiable and strictly convex on $(0, \infty)$ and is equal to the constant 1 for $x \leq 0$. Moreover, if X has unbounded variation, $Z^{(q)}$ is C^2 on $(0, \infty)$.

Example. A stable Lévy process X with index $\alpha \in (1, 2]$ has as cumulant $\psi(\theta) = \theta^\alpha$; its scale function and adjoint are respectively given by [4]

$$W^{(q)}(x) = \alpha x^{\alpha-1} E'_\alpha(qx^\alpha) \qquad Z^{(q)}(x) = E_\alpha(qx^\alpha)$$

where E_α is the Mittag-Leffler function with parameter α

$$E_\alpha(y) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(1+\alpha n)}, \qquad y \in \mathbb{R}. \tag{5}$$

In the case that $\alpha = 2$, the process $X/\sqrt{2}$ is a Brownian motion and $W^{(q)}, Z^{(q)}$ reduce to

$$W^{(q)}(x) = q^{-\frac{1}{2}} \sinh(x\sqrt{q}) \qquad Z^{(q)}(x) = \cosh(x\sqrt{q}). \tag{6}$$

Hence for a standard Brownian motion $W^{(q)}, Z^{(q)}$ are found by replacing (x, q) by $(2x, q/2)$ in (6).

For later reference, we give four lemmata with some properties of $W^{(q)}$ and $Z^{(q)}$ which we will need later on.

Lemma 1 *The function $x \mapsto W^{(q)}(x)$ is right- and left-differentiable on $(0,\infty)$. Moreover, if X has unbounded variation or its Lévy measure has no atoms, $W^{(q)}$ is continuously differentiable on $(0,\infty)$.*

By $W^{(q)\prime}_{\pm}(x)$, we will denote the right- and left-derivative of $W^{(q)}$ in x, respectively.
Proof In the proof of theorem VII.8 in [5] Bertoin shows that W satisfies for some constant K, $W(x) = K\exp(-\int_x^\infty \widehat{n}(h > t)\mathrm{d}t)$, where $\widehat{n}$ is the Itô excursion measure of $\widehat{Y} = S - X$ and h are excursion heights of excursions of $\widehat{Y}$ away from zero. From this representation, we deduce that

$$W'_+(x) = W(x)\widehat{n}(h > x) \qquad W'_-(x) = W(x)\widehat{n}(h \geq x). \tag{7}$$

It can be shown that the ditribution of h under excursion measure $\widehat{n}$ has no atoms if X has unbounded variation (see [18]) or if X has bounded variation but its Lévy measure Λ has no atoms (one way to see this is to invoke equation (20) to show that $\widehat{n}(h = x) = \mathtt{d}^{-1}\Lambda(\{-x\})W(0)/W(x))$. Hence, under these conditions, W restricted to $(0,\infty)$ is continuously differentiable. Using the expansion (3) and the monotonicity of W, it is not hard to prove that the properties of W carry over to $W^{(q)}$ (see [18, Prop. 5.1]). $\square$

The second lemma is immediate from the definition of $Z^{(q)}$ and $W^{(q)\prime}$.

Lemma 2 *The mapping $(x, q) \mapsto Z^{(q)}(x)$ is continuous on $[0,\infty) \times \mathbb{R}$ and, for every $x \geq 0$, $q \mapsto Z^{(q)}(x)$ and $q \mapsto W^{(q)\prime}_{\pm}(x)$ are analytic functions.*

Using the expansion (3), one can check the following convolution identity to be true:

Lemma 3 *For $q, r \in \mathbb{C}$ and $a > 0$ we have*

$$W^{(q)} \star W^{(r)}(a) = \frac{1}{r-q}(W^{(r)}(a) - W^{(q)}(a)),$$

where for $q = r$ the expression is to be understood in the limiting sense.

The following result concerns the asymptotic behaviour of $W^{(q)}$ and $Z^{(q)}$. We write $f \sim g$ if $\lim(f/g) = 1$.

Lemma 4 **(i)** *For $q > 0$, we have as $x \to \infty$*

$$W^{(q)}(x) \sim \mathrm{e}^{\Phi(q)x}/\psi'(\Phi(q)), \qquad Z^{(q)}(x) \sim q\mathrm{e}^{\Phi(q)x}/\left(\Phi(q)\psi'(\Phi(q))\right).$$

(ii) *As $x \downarrow 0$ the ratio $(W(x) - W(0)/x$ converges to a positive constant or to $+\infty$.*

Proof (i) We can straightforwardly check that the Laplace-Stieltjes transforms of the functions $U(x) := \mathrm{e}^{-\Phi(q)x}W^{(q)}(x)$ and $\widetilde{U}(x) := \mathrm{e}^{-\Phi(q)x}(Z^{(q)}(x) - 1)$ are given by

$$\int_0^\infty \mathrm{e}^{-\lambda x} U(\mathrm{d}x) = \frac{\lambda}{\psi(\lambda + \Phi(q)) - q} = \frac{\lambda}{\psi(\lambda + \Phi(q)) - \psi(\Phi(q))};$$
$$\int_0^\infty \mathrm{e}^{-\lambda x} \widetilde{U}(\mathrm{d}x) = \frac{q\lambda}{(\Phi(q) + \lambda)(\psi(\lambda + \Phi(q)) - q)},$$

where $\mathrm{d}U, \mathrm{d}\widetilde{U}$ denote the Stieltjes measure associated to $U, \widetilde{U}$ respectively, which respectively assign masses $W^{(q)}(0)$ and 0 to zero. Since $\psi'(\Phi(q)) > 0$, the statements follow using a Tauberian theorem (e.g. [5, p.10]).
(ii) Recall that the Laplace-Stieltjes transform of W is given by $\lambda/\psi(\lambda)$. If the Brownian coefficient $s := \lim_{\lambda\to\infty} \psi(\lambda)/\lambda^2$ is positive, the same Tauberian theorem implies that $W(x) \sim x/s$ for x tending to infinity. Set $d := \lim_{\lambda\to\infty} \lambda/\psi(\lambda)$. If $s = d^{-1} = 0$, that is X has no Brownian component and unbounded variation, we find, again using the Tauberian theorem, that $W(x)/x$ tends to infinity for $x \to \infty$. Similarly, if the Lévy measure of X has finite mass m, we can check $W(x) - W(0) \sim mx/d$, whereas for X with bounded variation but infinite mass of the Lévy measure we can verify that $(W(x) - W(0))/x$ tends to infinity as $x \to \infty$. $\square$

3 Exit problems

3.1 Two-sided exit

We now turn our attention to the two-sided exit problem and review the main results [6]. Denote the passage times $\widehat{T}_a, T_a$ for X and $-X$ above and below the level a by

$$T_a = \inf\{t \geq 0 : X_t > a\} \qquad \widehat{T}_a = \inf\{t \geq 0 : -X_t > a\}$$

The following result, the origins of which go back to Takács [27], expresses the (discounted) probabilities of exiting the interval $[0, a]$ above and below in terms of $W^{(q)}$ and $Z^{(q)}$.

Proposition 1 *For $q \geq 0$, the Laplace transform of the two-sided exit time $\widehat{T}_0 \wedge T_a$ on the part of the probability space where X starts at $x \in [0, a]$ and exits the interval $[0, a]$ above* *and below is respectively given by*

$$\mathbb{E}_x\left[\mathrm{e}^{-qT_a}I_{(\widehat{T}_0>T_a)}\right] = W^{(q)}(x)/W^{(q)}(a) \tag{8}$$

$$\mathbb{E}_x\left[\mathrm{e}^{-q\widehat{T}_0}I_{(\widehat{T}_0<T_a)}\right] = Z^{(q)}(x) - W^{(q)}(x)Z^{(q)}(a)/W^{(q)}(a) \tag{9}$$

Proof For $x \in (0,a)$, this result can be extracted directly out of existing litterature. See for example [5, Thm. VII.8] for a proof of (8) using excursion theory. Combining this with [6, Cor 1], we find equation (9). Note by a small typographic mistake in [6] $\int_0^x W^{(q)}(x)\mathrm{d}x$ is used instead of $q\int_0^x W^{(q)}(y)\mathrm{d}y = Z^{(q)}(x) - 1$. Since 0 is regular for $(0,\infty)$ for X, the identities hold for $x = a$. Similarly, they hold for $x = 0$ if X has unbounded variation. If X has bounded variation, 0 is irregular for $(-\infty, 0)$ and hence $\widehat{T}_0 > 0$ almost surely. Since $T_\epsilon \downarrow 0$ almost surely if $\epsilon \downarrow 0$, the strong Markov property implies that

$$\begin{aligned}\mathbb{E}_0\left[\mathrm{e}^{-qT_a}I_{(\widehat{T}_0>T_a)}\right] &= \lim_{\epsilon\downarrow 0}\mathbb{E}_0\left[\mathrm{e}^{-qT_\epsilon}I_{(\widehat{T}_0>T_\epsilon)}W^{(q)}(\epsilon)/W^{(q)}(a)\right]\\ &= W^{(q)}(0)/W^{(q)}(a),\end{aligned}$$

whence (8) is valid for $x = 0$ as well. Analogously (9) is shown to hold for $x = 0$. $\square$

Remark. Let n be the Itô-excursion measure associated to the excursions of $Y = X - I$ away from zero and let h, ζ denote the height and lifetime of the generic excursion respectively. In [5, Prop VII.15] Bertoin related n and the scale function W as follows

$$n(h > a) = W(a)^{-1}.$$

Using Propositions 14 and 15 in [5] combined with Proposition 1, we find the following link between n and $W^{(q)}$:

$$n(\mathrm{e}^{-q\zeta}, h > a) = \lim_{x\downarrow 0}\frac{\mathbb{E}_x(\mathrm{e}^{-qT_a}I_{(\widehat{T}_0>T_a)})}{W(x)} = W^{(q)}(a)^{-1},$$

since $W^{(q)}(0) = W(0)$.

3.2 Mixed exit

As a next step, we study exit problems of $[0,a]$ for the reflected Lévy processes Y and $\widehat{Y}$. The first passage time of a positive level $a > 0$ will be denoted by

$$\tau_a = \inf\{t \geq 0 : Y_t > a\} \qquad \text{and} \qquad \widehat{\tau}_a = \inf\{t \geq 0 : \widehat{Y}_t > a\},$$

where we will use τ_0 and $\widehat{\tau}_0$, respectively, to denote the first time that Y and $\widehat{Y}$ hit zero. The following result expresses the Laplace transforms of the exit times τ_a and $\widehat{\tau}_a$ in terms of the scale functions $W^{(q)}$ and $Z^{(q)}$. Note that $X_0 = x$ and hence $Y_0 = x$ under $\mathbb{P}_x$. Similarly, we see that $\widehat{Y}_0$ starts from x under the measure $\mathbb{P}_{-x}$.

Proposition 2 *Let $x \in [0,a]$ and $q \geq 0$. Then we have*

$$\begin{array}{ll} \textbf{(i)} & \mathbb{E}_x[\mathrm{e}^{-q\tau_a}] = Z^{(q)}(x)/Z^{(q)}(a). \\ \textbf{(ii)} & \mathbb{E}_{-x}[\mathrm{e}^{-q\widehat{\tau}_a}] = Z^{(q)}(a-x) - qW^{(q)}(a-x)W^{(q)}(a)/W_+^{(q)\prime}(a). \end{array}$$

By analyticity in q (Lemma 2) and monotone convergence, we find from Proposition 2 the following expressions for the expectations of the stopping times τ_a and $\widehat{\tau}_a$ for $x \in [0,a]$:

$$\mathbb{E}_x[\tau_a] = \overline{W}(a) - \overline{W}(x), \qquad \mathbb{E}_{-x}[\widehat{\tau}_a] = W(a-x)\frac{W(a)}{W_+'(a)} - \overline{W}(a-x), \tag{10}$$

where $\overline{W}(x) = \int_0^x W(y)\mathrm{d}y$. If X is a standard Brownian motion, we recall the form of the q-scale function given in the example in Section 2 and we find back the following well known identities:

$$\mathbb{E}_x[\mathrm{e}^{-q\tau_a}] = \cosh(x\sqrt{2q})/\cosh(a\sqrt{2q}), \qquad \mathbb{E}_x[\tau_a] = (a^2 - x^2)/2,$$

for $q \geq 0$ and $x \in [0,a]$.
Proof The second Laplace transform can be directly infered from Theorem 1 in [2]. To prove the form of the first Laplace transform, we use ideas developed in [2]. From the two-sided exit probability (8) we can extract that

$$M_t = \exp(-q(t \wedge \widehat{T}_0 \wedge T_a))W^{(q)}(X(t \wedge \widehat{T}_0 \wedge T_a)) \qquad t \geq 0,$$

is a martingale. Indeed, combining (8) and the fact that $W^{(q)}(X_{\widehat{T}_0 \wedge T_a})/W^{(q)}(a)$ is almost surely equal to the indicator of $\{\widehat{T}_0 > T_a\}$, we find for $x \in \mathbb{R}$

$$\mathbb{E}_x[\mathrm{e}^{-q(\widehat{T}_0 \wedge T_a)}W^{(q)}(X_{\widehat{T}_0 \wedge T_a})] = W^{(q)}(x).$$

Combined with the Markov property of X we see that

$$\begin{aligned} \mathbb{E}_x(\mathrm{e}^{-q(\widehat{T}_0 \wedge T_a)}W^{(q)}(X_{\widehat{T}_0 \wedge T_a})|\mathcal{F}_t) &= \mathrm{e}^{-qt}W^{(q)}(X_t)\mathbf{1}_{\{t < \widehat{T}_0 \wedge T_a\}} \\ &\quad + \mathrm{e}^{-q(\widehat{T}_0 \wedge T_a)}W^{(q)}(X_{\widehat{T}_0 \wedge T_a})\mathbf{1}_{\{t \geq \widehat{T}_0 \wedge T_a\}} \\ &= \mathrm{e}^{-q(t \wedge \widehat{T}_0 \wedge T_a)}W^{(q)}(X_{t \wedge \widehat{T}_0 \wedge T_a}), \end{aligned}$$

so that we have constance of expectation. Similarly, the martingale property follows. Exactly in the same vein, now using the exit probability in equation (9), we conclude that

$$\mathrm{e}^{-q(t \wedge \widehat{T}_0 \wedge T_a)}\left(Z^{(q)}(X_{t \wedge \widehat{T}_0 \wedge T_a}) - W^{(q)}(X_{t \wedge \widehat{T}_0 \wedge T_a})\frac{Z^{(q)}(a)}{W^{(q)}(a)}\right), \qquad t \geq 0$$

is a martingale. By taking a linear combination, we see that $\mathrm{e}^{-q(t \wedge \widehat{T}_0 \wedge T_a)}Z^{(q)}(X_{t \wedge \widehat{T}_0 \wedge T_a})$ is a martingale. Recall that $Z^{(q)}(\cdot)$ is once (twice) continuously differentiable on $(0,\infty)$ if X has (un)bounded variation, respectively. Applying Itô's lemma to $\mathrm{e}^{-qt}Z^{(q)}(X_t)$ (Theorema

II.31(32) in [23] in the case where X has (un)bounded variation) on the set $\{t \leq \widehat{T}_0\}$, we find that

$$\mathrm{e}^{-q(t\wedge\widehat{T}_0)} Z^{(q)}(X_{t\wedge\widehat{T}_0}) - \int_0^{t\wedge\widehat{T}_0} \mathrm{e}^{-qs}(\Gamma - q) Z^{(q)}(X_{s-})\mathrm{d}s$$

is a (local) martingale, where Γ is the infinitesimal generator of X. The martingale property of $\mathrm{e}^{-q(t\wedge\widehat{T}_0\wedge T_a)} Z^{(q)}(X_{t\wedge\widehat{T}_0\wedge T_a})$ implies now that

$$(\Gamma - q) Z^{(q)}(x) = 0, \qquad x \in (0, a). \tag{11}$$

Let I^c be the continuous part of I. By applying (the appropriate version of) Itô's lemma to $N_t = \exp(-qt) Z^{(q)}(Y_t)$ and using $Z^{(q)}(x) = 1$ for $x \leq 0$, one can verify that

$$N_t - \int_0^t \mathrm{e}^{-qt}(\Gamma - q) Z^{(q)}(Y_{s-})\mathrm{d}s + q\int_0^t W^{(q)}(Y_{s-})\mathrm{d}I_s^c$$

is a local martingale. Note that the last term in the previous display is identically zero. Indeed, if X has bounded variation $I^c \equiv 0$; otherwise we see that $\mathrm{d}I_s^c$ is negative if and only if $Y_{s^-} = 0$ and $W^{(q)}(0) = 0$ in this case. Noting that $N_{t\wedge\tau_a}$ is bounded by $Z^{(q)}(a)$ we deduce from equation (11) that $N_{t\wedge\tau_a}$ is a uniformly integrable martingale. Hence, as $t \to \infty$,

$$Z^{(q)}(x) = \mathbb{E}_x[N_{t\wedge\tau_a}] \to \mathbb{E}_x[N_{\tau_a}] = Z^{(q)}(a)\mathbb{E}_x[\mathrm{e}^{-q\tau_a}] \qquad x \in [0, a],$$

where we used that $\mathbb{P}_x$-almost surely $\tau_a < \infty$ and $Y_{\tau_a} = a$. □

3.3 Martingales

Another consequence of Proposition 2 is the following martingale property, which justifies the name *adjoint q-scale function* for $Z^{(q)}$.

Proposition 3 *For $q \geq 0$,*

$$(\mathrm{e}^{-q(t\wedge\widehat{T}_0)} W^{(q)}(X_{t\wedge\widehat{T}_0}), t \geq 0) \qquad \text{and} \qquad (\mathrm{e}^{-qt} Z^{(q)}(Y_t), t \geq 0)$$

are martingales.

Proof The first assertion follows by applying Lemma VII.11 in [5] to a spectrally negative Lévy process that is killed at an independent exponential time $\eta(q)$.

Recall that $N_t = \exp(-qtZ^{(q)}(Y_t)$. From the proof of Proposition 2, we know that $(N_{t\wedge\tau_a}, t \geq 0)$ is a martingale. We now claim that $N_{t\wedge\tau_a}$ converges in L^1 to N_t as a tends to ∞. Since for $s \leq t$

$$\mathbb{E}|\mathbb{E}(N_{t\wedge\tau_a}|\mathcal{F}_s) - \mathbb{E}(N_t|\mathcal{F}_s)| \leq \mathbb{E}|N_{t\wedge\tau_a} - N_t|,$$

the claim implies that N_t is a martingale. So we will be done after proving the claim. Write

$$N_{t\wedge\tau_a} = N_t \mathbf{1}_{\{t<\tau_a\}} + N_{\tau_a}\mathbf{1}_{\{t\geq\tau_a\}}.$$

Since $\tau_a < \infty$ a.s., monotone convergence implies that the first term on the right-hand side increases to N_t in L^1. For the second term we note that by the Cauchy-Schwarz-inequality

$$\mathbb{E}_x[\mathrm{e}^{-q\tau_a}\mathbf{1}_{\{t\geq\tau_a\}}] \leq (\mathbb{E}_x[\mathrm{e}^{-2q\tau_a}])^{1/2}\mathbb{P}_x(t \geq \tau_a)^{1/2}. \tag{12}$$

Recall that $\eta(r)$ denotes an independent exponential random variable with parameter r. Since $\tau_a \to \infty$ if $a \to \infty$ and we can check that, for $r > 0$,

$$\begin{aligned}\mathbb{P}_x(\tau_a \leq \eta(r)) = \mathbb{P}_x(\tau_a \leq t) + \mathbb{P}_x(\tau_a \in (t, \eta(r)], t < \eta(r))\\ - \mathbb{P}_x(\tau_a \in (\eta(r), t], \eta(r) < t),\end{aligned}$$

there exists an a_r large enough such that $\mathbb{P}_x(\tau_a \leq t)$ is bounded by $\mathbb{P}_x(\tau_a \leq \eta(r))$ for all $a \geq a_r$. Combining this property with equations (12) and Proposition 2(i), we find that

$$\mathbb{E}_x[N_{\tau_a}\mathbf{1}_{\tau_a\leq t}] \leq Z^{(2q)}(x)Z^{(q)}(a)/Z^{(2q)}(a).$$

By Lemma 4 (recalling that Φ is increasing), we conclude that the expectation in the previous display converges to zero, which finishes the proof. □

4 Resolvent measure

The Lévy process killed when it exits from $[0, a]$ has the strong Markov property; denote its transition probabilities by $(P^t, t \geq 0)$, that is, for a Borel set $A \subseteq [0, a]$ we have

$$P^t(x, A) = \mathbb{P}_x(X_t \in A, t < T_a \wedge \widehat{T}_0) \qquad \text{for } x \in [0, a].$$

and its q-resolvent kernel by

$$U^q(x, A) = \int_0^\infty P^t(x, A)\mathrm{e}^{-qt}\mathrm{d}t = \mathbb{E}_x\left(\int_0^{T\wedge\widehat{T}} \mathrm{e}^{-qt}\mathbf{1}_{\{X_t\in A\}}\mathrm{d}t\right), \qquad q \geq 0.$$

Since the Lévy process has an absolute continuous resolvent kernel, it follows from the Radon-Nikodym theorem that $U^q(x, \cdot)$ has a density with respect to the Lebesgue measure, which will be denoted by $u^q(x, \cdot)$. Suprun [26] showed that, for $x \in [0, a]$,

$$u^q(x, y) = \frac{W^{(q)}(x)W^{(q)}(a - y)}{W^{(q)}(a)} - W^{(q)}(x - y) \qquad y \in [0, a] \tag{13}$$

is a version of this density. Now we consider the Lévy processes Y and $\widehat{Y}$ killed upon leaving $[0, a)$. These killed processes still have the strong Markov property and we write $(Q^t, t \geq 0)$ and $(\widehat{Q}^t, t \geq 0)$ respectively to denote their transition probabilities. To be more precise, for Borel-sets $A \subseteq [0, a]$, we denote the transition probabilities of Y and $\widehat{Y}$ by

$$Q^t(x, A) = \mathbb{P}_x(Y_t \in A, t < \tau_a), \qquad \widehat{Q}^t(x, A) = \mathbb{P}_{-x}(\widehat{Y}_t \in A, t < \widehat{\tau}_a).$$

and the corresponding q-resolvent kernels by $R^q(x, A)$ and $\widehat{R}^q(x, A)$, respectively. We state the following result:

Theorem 1 **(i)** *The measure $R^q(x,\cdot)$ is absolutely continuous with respect to the Lebesgue measure and a version of its density is given by*

$$r^q(x,y) = \frac{Z^{(q)}(x)}{Z^{(q)}(a)} W^{(q)}(a-y) - W^{(q)}(x-y), \qquad x,y \in [0,a).$$

(ii) *Let $\widehat{r}^q(x,0) = W^{(q)}(a-x)W^{(q)}(0)/W_+^{(q)\prime}(a)$ for $x \geq 0$ and set*

$$\widehat{r}^q(x,y) = W^{(q)}(a-x)\frac{W_+^{(q)\prime}(y)}{W_+^{(q)\prime}(a)} - W^{(q)}(y-x) \qquad x,y \in [0,a], y \neq 0.$$

Then $\widehat{r}^q(x,0)\delta_0(\mathrm{d}y) + \widehat{r}^q(x,y)\mathrm{d}y$ is a version of the measure $\widehat{R}^q(x,\mathrm{d}y)$.

Example. If X is a standard Brownian motion, a famous result of Lévy states that $|X| = Y$, where the equality is in law. Let τ' be the first exit time of $|X|$ from $[0,a]$ and let as before $\eta(q)$ is an independent exponential random variable with parameter $q > 0$. Recalling from the example in Section 2 the form of the functions $W^{(q)}, Z^{(q)}$ for a Brownian motion and substituting in Theorem 1, we find, after some algebra,

$$\mathbb{P}_x(|X|_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau')/\mathrm{d}y = \frac{\sqrt{q}}{\sqrt{2}} \cdot \frac{\sinh((a-|y-x|)\sqrt{2q}) + \sinh((a-x-y)\sqrt{2q})}{\cosh(a\sqrt{2q})}$$

for $0 \leq x, y \leq a$. This formula is well known in the literature (e.g. [9, 3.1.1.6]).

Proof of part (i) Pick $x, y \in [0,a]$ arbitrary and let $q > 0$. By applying the strong Markov property of $Y = X - I$ at the stopping time τ_x and using the memoryless property of the exponential distribution, we find

$$\begin{aligned}(14) \qquad \mathbb{P}_0(Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_a) &= \mathbb{P}_0(Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_x) \\ &\quad + \mathbb{E}_0[\mathrm{e}^{-q\tau_x}]\mathbb{P}_x(Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_a)\end{aligned}$$

Analogously, the probability in (14) admits as second decomposition

$$(15) \qquad \mathbb{P}_0(Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_a) = \mathbb{P}_0(Y_{\eta(q)} \in \mathrm{d}y) - \mathbb{E}_0[\mathrm{e}^{-q\tau_a}]\mathbb{P}_a(Y_{\eta(q)} \in \mathrm{d}y).$$

Combining the two decompositions (14) and (15) we find

$$(16) \qquad \mathbb{P}_x(Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_a) = \mathbb{P}_a(Y_{\eta(q)} \in \mathrm{d}y) - \frac{\mathbb{E}_0[\mathrm{e}^{-q\tau_a}]}{\mathbb{E}_0[\mathrm{e}^{-q\tau_x}]}\mathbb{P}_x(Y_{\eta(q)} \in \mathrm{d}y).$$

Our next step is to evaluate the probability $\mathbb{P}_x(Y_{\eta(q)} \in \mathrm{d}y)$. Applying as before the strong Markov property at the stopping time τ_0, we find the decomposition

$$\begin{aligned}(17) \qquad \mathbb{P}_x(Y_{\eta(q)} \in \mathrm{d}y) &= \mathbb{P}_x(Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_0) + \mathbb{E}_x[\mathrm{e}^{-q\tau_0}]\mathbb{P}_0(Y_{\eta(q)} \in \mathrm{d}y) \\ &= \mathbb{P}_x(X_{\eta(q)} \in \mathrm{d}y, \eta(q) < \widehat{T}_0) + \mathbb{E}_x[\mathrm{e}^{-q\widehat{T}_0}]\mathbb{P}_0(Y_{\eta(q)} \in \mathrm{d}y)\end{aligned}$$

where in the second line we used that $(Y_t, t \leq \tau_0)$ has the same law as $(X_t, t \leq \widehat{T}_0)$. Suprun [26] showed that a version of the resolvent density of the process X killed upon entering the negative half-line is given by

$$q^{-1}\mathbb{P}_x(X_{\eta(q)} \in \mathrm{d}y, \eta(q) < \widehat{T}_0)/\mathrm{d}y = \mathrm{e}^{-\Phi(q)y}W^{(q)}(x) - W^{(q)}(x-y). \tag{18}$$

By integrating this resolvent density over y, we find the Laplace transform of $\widehat{T}_0$ to be equal to

$$\mathbb{E}_x[\mathrm{e}^{-q\widehat{T}_0}] = Z^{(q)}(x) - q\Phi(q)^{-1}W^{(q)}(x). \tag{19}$$

Substituting (19) and (18) into (17) and recalling that $Y_{\eta(q)}$ has an exponential distribution with parameter $\Phi(q)$ we end up with

$$\mathbb{P}_x(Y_{\eta(q)} \in \mathrm{d}y)/\mathrm{d}y = Z^{(q)}(x)\Phi(q)\mathrm{e}^{-\Phi(q)y} - qW^{(q)}(x-y).$$

Substituting this into equation (16) and recalling from Theorem 2 that $\mathbb{P}(\tau_x > \eta(q))$ is given by $Z^{(q)}(x)^{-1}$, we get the formula as stated in the Theorem for $q > 0$. For $q = 0$, the result follows by letting $q \downarrow 0$. $\square$

Proof of part (ii) Let $x, y \in [0, a]$ and let $q > 0$. Since $((S-X)_t; t < \widehat{\tau}_0)$ has the same law as $(-X_t; t < \widehat{T}_0)$, the strong Markov property of $\widehat{Y} = S - X$ enables us to write

$$\begin{aligned}\mathbb{P}_x((S-X)_{\eta(q)} \in \mathrm{d}y, \eta(q) < \widehat{\tau}(a)) &= \mathbb{P}_x(-X_{\eta(q)} \in \mathrm{d}y, \eta(q) < \widehat{T}_0 \wedge T_a) \\ &\quad + \frac{W^{(q)}(a-x)}{W^{(q)}(a)}\mathbb{P}_0((S-X)_{\eta(q)} \in \mathrm{d}y, \eta(q) < \widehat{\tau}(a)),\end{aligned}$$

where we used the two-sided exit probability (8). The first quantity on the right-hand side is seen to be equal to $qu^q(a-x, a-y)\mathrm{d}y$, where u^q is given in (13). To evaluate the probability in the second term on the right-hand side we are going to make use of the Master formula of excursion theory (e.g. [5, Cor. IV.11]). We shall use standard notation (see Bertoin [5, Ch. IV]). To this end, we introduce the excursion process $\widehat{e} = (\widehat{e}_t, t \geq 0)$ of $\widehat{Y}$, which takes values in the space of excursions

$$\mathcal{E} = \{f \in D[0,\infty) : f \geq 0,\ \exists\ \zeta = \zeta(f) \text{ such that} f(x) = f(0) = 0 \text{ for all } x \geq \zeta\}.$$

of càdlàg functions f with a generic life time $\zeta = \zeta(f)$ and is given by

$$\widehat{e}_t = (\widehat{Y}_s, L^{-1}(t^-) \leq s < L^{-1}(t)) \qquad \text{if } L^{-1}(t^-) < L^{-1}(t)$$

where L^{-1} is the right-inverse of a local time L of $\widehat{Y}$ at zero; else $\widehat{e}_t = \partial$, some isolated point. We take the running supremum S to be this local time L (cf. [5, Ch, VII]). The space $\mathcal{E}$ is endowed with the Itô excursion measure $\widehat{n}$. A famous theorem of Itô states that $\widehat{e}$ is a Poisson point process with characteristic measure $\widehat{n}$, if $\widehat{Y}$ is recurrent; otherwise $(\widehat{e}_t, t \leq L(\infty))$ is a Poisson point process stopped at the first excursion of infinite lifetime. For an excursion

$\epsilon \in \mathcal{E}$ its supremum is denoted by $\overline{\epsilon}$. By $\epsilon_g = (\widehat{Y}_{g+t}, t \leq \zeta_g)$ we denote the excursion of $\widehat{Y}$ with left-end point g, where ζ_g and $\overline{\epsilon}_g$ denote its lifetime and supremum respectively.

Letting $T_a(\epsilon) = \inf\{t \geq 0 : \epsilon(t) \geq a\}$ an application of the compensation formula yields for $y > 0$

$$
\begin{aligned}
&\mathbb{P}_0((S-X)_{\eta(q)} \in \mathrm{d}y, \eta(q) < \widehat{\tau}(a)) \\
&\qquad = \mathbb{E}\left[\sum_g I\left(\epsilon_g(\eta(q)) \in \mathrm{d}y, g < \eta(q) < g + \zeta_g, \eta(q) < \widehat{\tau}_a, \sup_{h<g} \overline{\epsilon}_h \leq a\right)\right] \\
&\qquad = \mathbb{E}\left[\int \mathrm{e}^{-qs} I\left(\sup_{h<s} \overline{\epsilon}_h \leq a\right) \mathrm{d}S_s\right] \widehat{n}(\epsilon(\eta(q)) \in \mathrm{d}y, \eta(q) < \zeta \wedge T_a(\epsilon)).
\end{aligned}
$$

The first factor can be infered from [2] to be equal to $W^{(q)}(a)/W_+^{(q)\prime}(a)$. For the second factor, we distinguish between the case that X has bounded or unbounded variation.

If X has bounded variation, it is well known (e.g. [24] or [19] for a more recent reference) that an excursion starts with a jump almost surely. Denote by $\mathtt{d}, \Lambda(\mathrm{d}x)$ the drift and Lévy measure of X, respectively. Note that in this case the time up to time t that the process $\widehat{Y}$ has spent in zero is equal to local time S_t divided by the drift $\mathtt{d}$. By the Markov property, under $\widehat{n}$, the excursion of $\widehat{Y}$, once in $(0, \infty)$, evolves as $-X$ killed at time T_0. Furthermore, the entrance law of an excursion of $\widehat{Y}$ under $\widehat{n}$ is given by $\Lambda/\mathtt{d}$. Indeed, letting $F : \mathcal{E} \to [0, \infty)$ be any bounded measurable functional on the space of excursions, we find that

$$
\begin{aligned}
\int F(\epsilon)\widehat{n}(\mathrm{d}\epsilon) &= \mathbb{E}\left[\sum_{0 \leq s \leq 1} F(e_s)\right] \\
&= \mathbb{E}\left[\sum_{0 \leq t < \infty} I(S_t \leq 1, X_{t^-} = S_t, \Delta X_t < 0) F(\{-X_{s+t} + X_{t^-}, s \leq \widehat{\tau}_0\})\right] \\
&= \mathbb{E}\left[\int_0^\infty I(S_t \leq 1, X_{t^-} = S_t)\mathrm{d}t \int_{-\infty}^0 F(\{-X_s - x, s \leq T_{-x}\})\Lambda(\mathrm{d}x)\right] \\
&= \frac{1}{\mathtt{d}} \int_{-\infty}^0 \mathbb{E}_x\left[F(\{-X_s, s \leq T_0\})\right] \Lambda(\mathrm{d}x), \qquad (20)
\end{aligned}
$$

where on the first line we used as before the Master formula of excursion theory followed in the third line by an application of the compensation formula applied to the Poisson point process $(\Delta X_t, t \geq 0)$ with characteristic measure $\Lambda(\mathrm{d}x)$ combined with the independent increments property of X. Applying this identity to $F(\epsilon) = I(\epsilon(t) \in \mathrm{d}y, t < \zeta \wedge T_a(\epsilon))$, taking the Laplace transform in t and using (13), we find that

$$
\frac{\widehat{n}(\epsilon(\eta(q)) \in \mathrm{d}y, \eta(q) < \zeta \wedge T_a)}{\mathrm{d}y} = \frac{q}{\mathtt{d}} \int_{-\infty}^0 \left(\frac{W^{(q)}(a+x)W^{(q)}(y) - W^{(q)}(a)W^{(q)}(y+x)}{W^{(q)}(a)}\right) \Lambda(\mathrm{d}x).
$$

We claim that the following identity holds true for all $a > 0$:

$$
\mathtt{d}W_+^{(q)\prime}(a) = \int_0^\infty \left(W^{(q)}(a) - W^{(q)}(a-x)\right) \Lambda(\mathrm{d}x) + qW^{(q)}(a). \qquad (21)
$$

To see this, first note that the right and left hand side of (21) have the same Laplace transform in a. Moreover, equation (7) and the decomposition (3) imply that $W_+^{(q)\prime}$ is bounded on any compact interval in $(0,\infty)$. It follows that the right hand side of (21) is right-continuous in $a>0$, as is certainly the left hand side of (21). This continuity combined with the almost sure unicity of the Laplace transform shows the claim is true for $a>0$.

After some algebra, we find that

$$\widehat{n}(\epsilon(\eta(q)\in \mathrm{d}y, \eta(q)<\zeta\wedge T_a)/\mathrm{d}y = q\left(W_+^{(q)\prime}(y) - \frac{W_+^{(q)\prime}(a)}{W^{(q)}(a)}W^{(q)}(y)\right). \tag{22}$$

Substituting back the expression (22), we find the stated form of the density for $y>0$. Noting that

$$1-\mathbb{E}_{-x}[\mathrm{e}^{-q\widehat{\tau}_a}] = \int_{0+}^a q\widehat{r}^q(x,y)\mathrm{d}y + \mathbb{P}_{-x}(\widehat{Y}_{\eta(q)}=0, \eta(q)<\widehat{\tau}_a),$$

we can verify, by combining Proposition 2 with the just found density, that $\widehat{R}^q(x,0) = W^{(q)}(a-x)W^{(q)}(0)/W_+^{(q)\prime}(a)$ which finishes the proof in the bounded variation case.

Suppose now X has unbounded variation. Let $g(\widehat{\tau}_a)$ and $d(\widehat{\tau}_a)$ be the last time before and first time after $\widehat{\tau}_a$ that $\widehat{Y}$ visits zero. Consider now the excursion straddling $\widehat{\tau}_a$, $\{-\widehat{Y}_t, t\in[g(\widehat{\tau}_a), d(\widehat{\tau}_a)\}$, and denote its law by $Q^{(a)}$. Since we are in the case of unbounded variation, all excursions of $-\widehat{Y}$ away from zero leave continuously. In canonical notation, $T(x)=\inf\{t\geq 0: X_t\geq x\}$ tends to zero almost surely under $Q^{(a)}$ as $x\downarrow 0$. We can verify that the same holds for $T'(x)=\inf\{t\geq 0: X_t = x\}$. By right-continuity of the paths, $X\circ\theta_{T'(x)}$ (with θ the shift-operator) converges to X under $Q^{(a)}$ in the Skorohod topology as x tends to zero. The strong Markov property implies that under $Q^{(a)}(\cdot|T'(x)<T(a))$ the shifted process $X\circ\theta_{T'(x)}$ has the same law as $\widehat{X}=-X$ under $\mathbb{P}$, starting at x and conditioned to exit $[0,a]$ at a. Using (13) and (8), we find for $A\in\mathcal{F}_t$

$$\begin{aligned} Q_x^{t,(a)}(A) &:= Q^{(a)}(A\circ\theta_{T'(x)}, t<T(a)|T'(x)<T(a)) \\ &= \int_A \frac{W(a)-W(a-y)}{W(a)-W(a-x)}\mathbb{P}_{-x}(-X_t\in \mathrm{d}y, t<\widehat{T}_a\wedge T_0) \end{aligned}$$

Combining with the foregoing, we note that $Q_x^{t,(a)}$ converges weakly to $Q^{(a)}(\cdot, t<T(a))$ as $x\downarrow 0$. From (13) it follows that

$$\lim_{x\downarrow 0}\mathbb{P}_{-x}(-X_{\eta(q)}\in\mathrm{d}y, \eta(q)<\widehat{T}_a|\widehat{T}_a<T_0) = \frac{W(a)-W(a-y)}{W'(a)}f(y,a)\mathrm{d}y$$

where the limit is in the sense of weak convergence and $f(y,a)$ is equal to the right-hand side of (22). We deduce that

$$Q^{(a)}(X_{\eta(q)}\in\mathrm{d}y, \eta(q)<T(a)) = \frac{W(a)-W(a-y)}{W'(a)}f(y,a)\mathrm{d}y. \tag{23}$$

By a computation based on the compensation formula for excursion theory (cf. proof of Theorem 4 in [10]), one can verify that

$$\mathbb{E}\left[\int_0^\infty I(S_t \le x) q\mathrm{e}^{-q(t-g_t)} I(\widehat{Y}_{t-g_t} \in \mathrm{d}y, t - g_t < \widehat{\tau}_a)\right] = x\widehat{n}(\epsilon(\eta(q)) \in \mathrm{d}y, \eta(q) < \zeta \wedge T_a(\epsilon)),$$

where $g_t = \sup\{s \le t : \widehat{Y}_s = 0\}$. Thus, we find

$$\begin{aligned}
\widehat{n}(\epsilon(\eta(q)) \in \mathrm{d}y,&\eta(q) < \zeta \wedge T_a(\epsilon)) \\
&= \lim_{x\downarrow 0} \frac{1}{x}\mathbb{E}_{-x}\left[\int_0^\infty q\mathrm{e}^{-q(t-g_t)} I(\widehat{Y}_{t-g_t} \in \mathrm{d}y, t - g_t < \widehat{\tau}_a \wedge \widehat{\tau}_0)\right] \\
&= \lim_{x\downarrow 0} \frac{W(a) - W(a-x)}{x} \times \\
&\times \lim_{x\downarrow 0} \frac{1}{W(a) - W(a-x)} \int_0^\infty q\mathrm{e}^{-qt}\mathbb{E}_{-x}[I(X_{t-g_t} \in \mathrm{d}y, t - g_t < \widehat{T}_a \wedge T_0)\mathrm{d}t \\
&= W'(a)Q^{(a)}\left[\frac{1}{W(a) - W(a - X_{\eta(q)})} I(X_{\eta(q)} \in \mathrm{d}y, \eta(q) < T(a))\right] \mathrm{d}t. \qquad (24)
\end{aligned}$$

Combining (24) and (23) we deduce that (22) is also valid in this case. □

Remark. If X drifts to $-\infty$, we can relate the conditionings in the proof of the theorem to those in the literature on spectrally negative Lévy processes conditioned to stay in a half line. Recall that, since X drifts to $-\infty$, we have $\Phi(0) > 0$ and $\psi'(\Phi(0)) > 0$. We write $W(x) = \mathrm{e}^{\Phi(0)x}W^{\#}(x)$ where $W^{\#}$ is the scale function of X under the measure $\mathbb{P}^{\#}$ which is for $A \in \mathcal{F}_t$ given by $\mathbb{P}^{\#}(A) = \mathbb{E}[\exp(\Phi(0)X_t)I_A]$. Since $\psi^{\#\prime}(0) = \psi'(\Phi(0)) > 0$, X drifts to $+\infty$ under $\mathbb{P}^{\#}$ and $W^{\#}$ is bounded. Then it follows from Proposition 1 that the probability $\mathbb{P}_{-x}(\widehat{T}_a < T_0)$ converges to $1 - \exp(-\Phi(0)x)$ as $a \to \infty$. By bounded convergence we then find for $A_t \in \mathcal{F}_t$

$$\begin{aligned}
\mathbb{P}_{-x}(A_t|\widehat{T}_a < T_0) &= (W(a) - W(a-x))^{-1}\mathbb{P}_{-x}((W(a) - W(a + X_t))A_t) \\
&\to \mathbb{P}^{\downarrow}_{-x}(A_t) := \mathbb{E}_{-x}\left(\tfrac{1-\mathrm{e}^{\Phi(0)X_t}}{1-\mathrm{e}^{-\Phi(0)x}} I(A_t)\right) \qquad \text{as } a \to \infty.
\end{aligned}$$

Note that $\mathbb{P}^{\downarrow}_{-x}(A_t)$ is also equal to $\mathbb{P}_{-x}(A_t|S_\infty < 0)$. Hence the notation $\mathbb{P}^{\downarrow}_{-x}$ is justified since under this measure the process always stay below zero with probability one. As $x \downarrow 0$ the measures $\mathbb{P}^{\downarrow}_{-x}$ converge weakly (in the Skorohod topology) to a measure $\mathbb{P}^{\downarrow}$. For an analysis of this case, see [3].

5 Analytic continuation

In this subsection, we show that the resolvent measure $R^q(x, \cdot)$ and $\widehat{R}^q(x, \cdot)$ can be extended to some negative values of q. Let us define ϱ and $\widehat{\varrho}$ by

$$\varrho = \inf\{q \ge 0 : Z^{(-q)}(a) = 0\} \qquad \widehat{\varrho} = \inf\{q \ge 0 : W_+^{(-q)\prime}(a) = 0\}. \qquad (25)$$

Continuity of $q \mapsto Z^{(q)}(a)$ (Lemma 2) combined with the fact $Z^{(0)}(a) \equiv 1$ implies that ϱ is positive. Similarly, we claim $\widehat{\varrho}$ is positive. Indeed, since W is increasing, $W'_+(a)$ is positive for all $a > 0$ except possibly at countably many, where it could be zero. However, any zero of W' would lead to a contradiction in view of the expectation of $\widehat{\tau}_a$ as stated in (10). The claim then follows by continuity of $q \mapsto W_+^{(q)\prime}(a)$.

Proposition 4 *Let $x \in [0, a]$ and A a Borel subset of $[0, a)$. We have for $q < \varrho$*

$$\int_0^\infty \mathrm{e}^{qt} Q^t(x, A)\mathrm{d}t = \int_A \left\{ \frac{Z^{(-q)}(x)}{Z^{(-q)}(a)} W^{(-q)}(a - y) - W^{(-q)}(x - y) \right\} \mathrm{d}y$$

and for $q < \widehat{\varrho}$

$$\int_0^\infty \mathrm{e}^{qt} \widehat{Q}^t(x, A)\mathrm{d}t = \int_A \frac{W^{(-q)}(a - x)}{W_+^{(-q)\prime}(a)} W^{(-q)}(\mathrm{d}y) - \int_A W^{(-q)}(y - x)\mathrm{d}y.$$

Proof For $q \leq 0$, the statement (i) rephrases Theorem 1. By Lemma 2 and the properties of $q \mapsto W^{(q)}(x)$ as listed in [6, Lemma 4], we can extend the right hand side for $q < \varrho$. The coefficient c_n of q^n in the corresponding expansion as a power series in zero is given in terms of the left-derivative of the left-hand side,

$$c_n = \textstyle\int_0^\infty t^n Q^t(x, A)\mathrm{d}t/n!.$$

We know that the series $\sum_n c_n q^n$ converges for $|q| < \varrho$. The statement follows. The proof of (ii) is similar and left to the reader. □

6 Irreducibility and Continuity

Let μ denote any σ-finite measure on $([0, a), \mathcal{B}_{[0,a)})$, the interval $[0, a)$ endowed with the Borel σ-algebra $\mathcal{B}_{[0,a)}$. Examples are the Lebesgue measure λ and the dirac measure in $x \in [0, a)$, δ_x. One says that transition probabilities $(P^t, t \geq 0)$ are μ-irreducible if, for every $A \in \mathcal{B}_{[0,a)}$ with $\mu(A) > 0$, its potential $U(x, A)$ of A is positive for every $x \in [0, a)$. Before we formulate the result, we set the condition (R) by

$$\text{(R)} \qquad \left\{ \begin{array}{c} X \text{ has jumps of absolute size smaller than } a \\ \text{or the Brownian coefficient } s = \lim_{\lambda \to \infty} \lambda^{-2} \psi(\lambda) \text{ is positive} \end{array} \right\}.$$

Proposition 5 *Q^t is λ-irreducible and under condition (R) $\widehat{Q}^t$ is $(\lambda + W(0)\delta_0)$-irreducible.*

Proof The first statement follows since, by Theorem 1,

$$r^0(x, y) = W(a - y) - W(x - y) > 0 \qquad \text{for all } x, y \in [0, a),$$

as W is increasing. For the second statement, we note that $\widehat{Q}^t(x, \mathrm{d}y) \geq P^t(a-x, \mathrm{d}(a-y))$. Thus $\widehat{r}^0(x,y) \geq u^0(a-x, a-y)$, where $u^q(a-x,a-y) > 0$ for all $x, y \in (0,a)$ by Corollary 3 in [6]. If $x = 0$, we see from (7) that for $y < a$

$$\widehat{r}^0(0,y)/\left(W(a)W(y)\right) = \frac{W'_+(y)}{W(y)} - \frac{W'_+(a)}{W(a)} = \widehat{n}(h > y) - \widehat{n}(h > a) > 0.$$

Finally, note that $\widehat{r}^q(x,0) > 0$ for $x \in [0,a)$, if X has bounded variation. $\square$
Proposition 5 implies the following property of $Z^{(q)}$.

Corollary 1 *For every $q < \varrho$ and $x \in [0,a)$, we have $Z^{(-q)}(x) > 0$.*

Proof We know from Lemma 2 that $Z^{(-q)}(x) > 0$ if x is sufficiently small. Let x_0 be the smallest zero of $Z^{(-q)}(x) = 0$. If we had $x_0 < a$, then we would have $\int_0^\infty \mathrm{e}^{qt} Q^t(x_0, (x_0, a))\mathrm{d}t = 0$ by Proposition 4, which conflicts with the fact that Q^t is Lebesgue irreducible. $\square$
In order to be able to prove continuity in space and time of the transition probabilities $(Q^t, t \geq 0)$ and $(\widehat{Q}^t, t \geq 0)$, we restrict ourselves to Lévy processes X of which the one-dimensional distributions are absolutely continuous with respect to the Lebesgue measure, that is,

$$\text{(AC)} \qquad \mathbb{P}_0(X_t \in \mathrm{d}y) \ll \mathrm{d}y \qquad \text{for all } t > 0.$$

It is known that (AC) holds whenever the Brownian coefficient is positive or when the mass of the absolutely continuous part of the Lévy measure is infinite (see Tucker [28]).We use the standard notation $Q^t f(x) = \int_{[0,a]} f(y) Q^t(x, \mathrm{d}y)$. Recall that the semi-group Q has the strong Feller property if for every Borel bounded function f, $Q^t f(\cdot)$ is a continuous function on $[0,a]$ for all $t > 0$. If a semigroup has the Feller as well as the strong Feller property it is called doubly Feller.

Proposition 6 *Assume (AC) is satisfied. Then the following hold true:*

(i) For every $x \in [0,a]$ and Borel set $A \subseteq [0,a]$ the mappings $t \mapsto Q^t(x,A)$ and $t \mapsto \widehat{Q}^t(x,A)$ are continuous on $(0,\infty)$.

(ii) For every $t > 0$, Q^t and $\widehat{Q}^t$ have the strong Feller property.

To prove Proposition 6 we need the following auxiliary results.

Lemma 5 *Assume (AC) holds.*

(i) The one dimensional distributions of the reflected Lévy process Y are absolutely continuous, that is,

$$\mathbb{P}_x(Y_t \in \mathrm{d}y) \ll \mathrm{d}y \qquad \textit{for every } t > 0, x \geq 0.$$

(ii) For any $t > 0, x \geq 0$, the measure $\mathbb{P}_x(\widehat{Y}_t \in \mathrm{d}y)$ is absolutely continuous on $(0,\infty)$. If X has (un)bounded variation, $\mathbb{P}_x(\widehat{Y}_t = 0) > (=) 0$.

Proof (ii) Let $N \subset (0,\infty)$ be an arbitrary Borel set of measure zero and fix $t > 0$. The form of the law of $\widehat{Y}_{\eta(q)}$ given in (2) combined with the absolute continuity of of $W^{(q)}(\mathrm{d}x)$ for $x > 0$ implies that

$$\mathbb{P}_0(\widehat{Y}_t \in N) = 0 \qquad \text{for Lebesgue-almost all } t > 0. \tag{26}$$

Next we note that $\mathbb{P}_x(\widehat{\tau}_0 \in \mathrm{d}t)$ has no atoms for $x > 0$. Indeed, since the sample paths of a Lévy process are continuous at each fixed time a.s. we see that under (AC)

$$\mathbb{P}_x(\widehat{\tau}_0 = t) = \mathbb{P}_{-x}(T_0 = t) \le \mathbb{P}_{-x}(X_t = 0) = 0 \qquad x > 0. \tag{27}$$

Applying the Markov property at $\widehat{\tau}_0$ yields that

$$\mathbb{P}_x(\widehat{Y}_t \in N) = \mathbb{P}_x(\widehat{Y}_t \in N, t < \widehat{\tau}_0) + \int_0^t \mathbb{P}_0(\widehat{Y}_{t-s} \in N)\mathbb{P}_x(\widehat{\tau}_0 \in \mathrm{d}s). \tag{28}$$

Noting that $\mathbb{P}_x(\widehat{Y}_t \in A, t < \widehat{\tau}_0)$ is dominated by $\mathbb{P}_{-x}(-X_t \in A)$ and invoking (26) and (27), we deduce from (28) that $\mathbb{P}_x(\widehat{Y}_t \in A)$ is zero under (AC) for all $t, x > 0$. By an application of the Markov property at time $s \notin N$, we can now remove the "almost" in (26) and the first assertion follows. Recalling that $\widehat{Y}_t$ has the same law as $-I_t$ and using (2), we see that $\mathbb{P}_0(\widehat{Y}_t = 0)$ is zero for all $t > 0$ if and only if X has unbounded variation. The proof of (ii) is complete. The proof of (i) is similar as (ii) and left to the reader. ☐

Lemma 6 *For $a > 0$, the distribution of $\widehat{\tau}_a$ has no atom, that is,*

$$\mathbb{P}_x(\widehat{\tau}_a = t) = 0 \qquad \textit{for every } x \in [0, a) \textit{ and } t \ge 0.$$

Under (AC), the same holds for the distribution of τ_a.

Proof Since a Lévy process (and also a reflected Lévy process) is almost surely continuous at time t, we have

$$\mathbb{P}_x(\tau_y = t) \le \mathbb{P}_x(Y_t = y) \text{ and } \mathbb{P}_x(\widehat{\tau}_y = t) \le \mathbb{P}_x(\widehat{Y}_t = y)$$

which are both zero under (AC) by the first and second part of the Lemma 5 respectively. Suppose now (AC) is not satisfied; X is then a drift minus pure jump process of bounded variation. Hence $\widehat{Y}$ can cross the level $a > 0$ only by a jump. However, the probability is zero that the Poisson point process $(\Delta X_t, t \ge 0)$ jumps at time t. ☐

Lemma 7 *Assume (AC) holds and let $A \subseteq \mathbb{R}$ be an arbitrary Borel set.*

(i) For every $t > 0$, $\mathbb{P}_x(Y_t \in \cdot)$ and $\mathbb{P}_x(\widehat{Y}_t \in \cdot)$ have the strong Feller property.

(ii) For every $x \ge 0$, $t \mapsto \mathbb{P}_x(Y_t \in A)$ and $t \mapsto \mathbb{P}_x(\widehat{Y}_t \in A)$ are continuous on $(0,\infty)$.

Proof (i) By the strong Markov property, we may write

$$\mathbb{P}_x(Y_t \in A) = \mathbb{P}_x(Y_t \in A, t < \tau_0) + \int_0^t \mathbb{P}_x(\tau_0 \in \mathrm{d}s)\mathbb{P}_0(Y_{t-s} \in A).$$

The first term on the right-hand side is equal to $\mathbb{P}_x(X_t \in A, t < \widehat{T}_0)$. From Hawkes [14] we know that under (AC) X is doubly Feller, from Chung [11] we know that a doubly Feller process remains doubly Feller if killed upon hitting an open set. For the second term, we note that $\mathbb{P}_x(\tau_0 \in \mathrm{d}s) = \mathbb{P}(\widehat{T}_{-x} \in \mathrm{d}s)$. Since $(\widehat{T}_{-x}, x \geq 0)$ is a subordinator with absolute continuous transition probabilities under (AC) and since $s \mapsto \mathbb{P}_0(Y_{t-s} \in A)$ is bounded Borel measurable, we conclude from Lemma 2 in [6] that the second term of the previous display is continuous on $(0, \infty)$ as a function of x. The proof for $\widehat{Y}$ is similar.
(ii) From the proof of Theorem 2.2 in [14] and Lemma 5, we can deduce, following an analogous line of reasoning, that there exists a version $(t, x, y) \mapsto q(t, x, y)$ of the density of the one-dimensional distributions of $X - I$, such that for all Borel bounded f and for all $x \geq 0$

$$\mathbb{E}_x[f(Y_t)] = \int f(y)q_t(x, y)\mathrm{d}y$$

and $\int q_t(\cdot, z)q_s(z, \cdot)\mathrm{d}z = q_{t+s}(\cdot, \cdot)$ for all $s, t > 0$. Moreover, by part (i) $x \mapsto \mathbb{E}_x[f(Y_t)]$ is continuous. Since $q_\epsilon(x, z)\mathrm{d}z$ converges weakly to the dirac point measure at x as $\epsilon \downarrow 0$, we see that, for all $x \geq 0$,

$$\int f(y)q_{t+\epsilon}(x, y)\mathrm{d}y = \int q_\epsilon(x, z) \int f(y)q_t(z, y)\mathrm{d}y\mathrm{d}z \to \int f(y)q_t(x, y)\mathrm{d}y$$

as $\epsilon \downarrow 0$. This proves the right-continuity in the statement. To establish the left-continuity, take $0 < \eta < \epsilon < t$ and write

$$\int f(y)q_{t-\eta}(x, y)\mathrm{d}y = \int q_{\epsilon-\eta}(x, z) \int f(y)q_{t-\epsilon}(z, y)\mathrm{d}y\mathrm{d}z. \tag{29}$$

By almost sure sample path continuity of $X - I$ at time ϵ, $q_{\epsilon-\eta}(x, z)\mathrm{d}z$ converges weakly to $q_\epsilon(x, z)\mathrm{d}z$ as $\eta \downarrow 0$. Hence the right-hand side of (29) converges for all $x \geq 0$ to

$$\int\int f(y)q_\epsilon(x, z)q_{t-\epsilon}(z, y)\mathrm{d}y\mathrm{d}z = \int f(y)q_t(x, y)\mathrm{d}y$$

which establishes left-continuity. To prove the second statement of (ii), we repeat above proof where everywhere the Lebesgue measure $\mathrm{d}y$ is replaced by the measure $\mathrm{d}y + \delta_0(y)$, the Lebesgue measure $\mathrm{d}y$ with an atom of size one in $y = 0$. □
Proof of Proposition 6 We only prove the statements for Y, the proofs for $\widehat{Y}$ are similar.
(i) By the strong Markov property of Y applied at τ_a, we find that

$$\mathbb{P}_x(Y_t \in A) = Q^t(x, A) + \int_0^t \mathbb{P}_x(\tau_a \in \mathrm{d}s)\mathbb{P}_a(Y_{t-s} \in A).$$

The left-hand side is continuous in t on $(0,\infty)$ by Lemma 7. The same holds for the integral on the right-hand side, as the distribution of τ_a has no atom. Hence $t \mapsto Q^t(x,A)$ is continuous.
(ii) Proposition VI.1 in [5] states that Y has the Feller property. Combining this with Lemma 7, we see that Y is doubly Feller. From Chung [11], we know that a doubly Feller process killed upon hitting an open set remains doubly Feller. □

7 Ergodicity and exponential decay

Under the assumption (AC) Bertoin [6] identifies the decay parameter of the transition probabilities $(P^t, t \geq 0)$ of X killed upon leaving $[0,a]$ as $\rho = \rho(a)$ where

$$\rho(a) = \inf\{q \geq 0 : W^{(-q)}(a) = 0\}.$$

The following result concerns the ergodic properties of the transition probabilites Q^t and $\widehat{Q}^t$ and identifies their decay parameters as ϱ and $\widehat{\varrho}$ respectively. The proof uses the R-theory of irreducible Markov processes developed by Tuominen and Tweedie [29].

Theorem 2 **(A)** *We have that $\varrho \in (0,\infty)$ and ϱ is a simple root of $q \mapsto Z^{(-q)}(a)$ and the following are true:*

(i) Q^t is ϱ-recurrent and, more precisely, ϱ-positive.

(ii) $x \mapsto Z^{(-\varrho)}(x)$ is positive on $[0,a)$ and ϱ-invariant for Q^t; that is,

$$Q^t Z^{(-\varrho)}(x) = \mathrm{e}^{-\varrho t} Z^{(-\varrho)}(x) \qquad \text{for all } x \in [0,a). \tag{30}$$

(iii) $x \mapsto W^{(-\varrho)}(a-x)$ is positive almost everywhere on $(0,a)$ and the measure $\Pi(\mathrm{d}x) = W^{(-\varrho)}(a-x)\mathrm{d}x$ on $[0,a)$ is ϱ-invariant for Q^t, that is,

$$\Pi Q^t = \mathrm{e}^{-\varrho t}\Pi. \tag{31}$$

(iv) Assume (AC) is satisfied. Then for every $x \in [0,a]$ we have

$$\lim_{t\to\infty} \mathrm{e}^{\varrho t} Q^t(x,\cdot) = c^{-1} Z^{(-\varrho)}(x)\Pi(\cdot) \tag{32}$$

in the sense of weak convergence where $c = \frac{\mathrm{d}}{\mathrm{d}q} Z^{(q)}(a)|_{q=-\varrho} > 0$.

(B) *Suppose X satisfies (R). Then $\widehat{\varrho} \in (0,\infty)$ and $\widehat{\varrho}$ is a simple root of $q \mapsto W_+^{(-q)\prime}(a)$ and the following hold:*

(i) $\widehat{Q}^t$ is $\widehat{\varrho}$-recurrent and, more precisely, $\widehat{\varrho}$-positive;

(ii) $x \mapsto W^{(-\widehat{\varrho})}(a-x)$ is positive on $(0,a)$ and $\widehat{\varrho}$-invariant for $\widehat{Q}^t$;

(iii) $x \mapsto W_+^{(-\widehat{\varrho})\prime}(x)$ *is a.s. positive on* $(0,a)$ *and the measure* $\widehat{\Pi}(\mathrm{d}x) = W^{(-\widehat{\varrho})}(\mathrm{d}x)$ *on* $[0,a)$ *is* $\widehat{\varrho}$*-invariant for* Q^t*;*

(iv) Assume (AC) is satisfied. Then for every $x \in [0,a]$ *we have*

$$\lim_{t\to\infty} \mathrm{e}^{\widehat{\varrho}t}\widehat{Q}^t(x,\cdot) = \widehat{c}^{-1}W^{(-\widehat{\varrho})}(a-x)\widehat{\Pi}(\cdot)$$

in the sense of weak convergence where $\widehat{c} = \frac{\mathrm{d}}{\mathrm{d}q}W_+^{(q)\prime}(a)|_{q=-\widehat{\varrho}} > 0$.

Remarks.

(i) Specialising Theorem 2(A,iv) and (B,iv) we get the following asymptotic identities for $t \to \infty$ and $x, y \in [0,a]$

$$\begin{aligned} \mathbb{P}_x(\tau_a > t) &\sim c' Z^{(-\varrho)}(x)\mathrm{e}^{-\varrho t}, \qquad \text{for a constant } c' > 0;\\ \mathbb{P}_x(\widehat{\tau}_a > t) &\sim \tilde{c} W^{(-\widehat{\varrho})}(a-x)\mathrm{e}^{-\widehat{\varrho}t}, \qquad \text{for a constant } \tilde{c} > 0;\\ \mathbb{P}_x(X_t \in A | \tau_a > t) &\sim \Pi(A)/\Pi([0,a)), \qquad \text{for Borel sets } A \subseteq [0,a). \end{aligned}$$

(ii) Take $\alpha \in (1,2]$. In the case X is stable process of index α we recall from the example in Section 2 that $Z^{(q)}(x) = E_\alpha(qx^\alpha)$. The root introduced in (25) is hence given by $\varrho = a^{-\alpha}r(\alpha)$ where $-r(\alpha)$ is the first negative root of E_α and $\widehat{\varrho} = a^{-\alpha}\tilde{r}(\alpha)$ where $-\tilde{r}(\alpha)$ is the first negative root of

$$\sum_{n=0}^{\infty} \frac{y^n}{\Gamma(\alpha - 1 + n\alpha)}.$$

In the special case $\alpha = 2$, $X/\sqrt{2}$ is a standard Brownian motion and $E_2(-x) = \cos\sqrt{x}$ for $x > 0$. In particular, $r(2) = \pi^2/4$ and

$$\varrho = \pi^2/(4a^2) \qquad\qquad Z^{(-\varrho)}(x) = \cos\left(\frac{\pi}{2a}x\right).$$

Since in this case $W^{(-q)\prime}(x) = Z^{(-q)}(x) = \cos(x\sqrt{q})$, we see that $\widehat{\varrho} = \varrho$, as it should be.

(iii) By the decomposition (3) we find that

$$\begin{aligned} \frac{\partial}{\partial q}Z^{(q)}(a)|_{q=-\varrho} &= W^{(-\varrho)} \star Z^{(-\varrho)}(a)\\ \frac{\partial}{\partial q}W_+^{(q)\prime}(a)|_{q=-\widehat{\varrho}} &= \int_0^a W^{(-\widehat{\varrho})}(a-x)W^{(-\widehat{\varrho})}(\mathrm{d}x). \end{aligned}$$

Hence the constants $c, \widehat{c}$ in the Theorem make $\mu(\mathrm{d}x) = c^{-1}Z^{(-\varrho)}(x)\Pi(\mathrm{d}x)$ and $\widehat{\mu}(\mathrm{d}x) = \widehat{c}^{-1}W^{(-\widehat{\varrho})}(a-x)\widehat{\Pi}(\mathrm{d}x)$ into probability measures.

(iv) By letting a run through $(0,\infty)$ we now consider equation (25) to define mappings $\varrho = \varrho(a)$ and $\widehat{\varrho} = \widehat{\varrho}(a)$ from $(0,\infty)$ to $(0,\infty)$. Much in the same vein as [18], we can prove from Theorem 2 and Lemma 2 that ϱ is a decreasing continuously differentiable mapping on $(0,\infty)$ with derivative

$$\varrho'(a) = \tfrac{\partial}{\partial x} Z^{(-\varrho)}(x)|_{x=a} / \tfrac{\partial}{\partial q} Z^{(q)}(a)|_{q=-\varrho(a)}. \tag{33}$$

Similarly, if we assume that W (and hence $W^{(-\varrho)}$) is twice continuously differentiable, we can show that $\widehat{\varrho}$ is decreasing and continuously differentiable with derivative

$$\widehat{\varrho}'(a) = \tfrac{\partial^2}{\partial x^2} W^{(-\widehat{\varrho})}(x)|_{x=a} / \tfrac{\partial}{\partial q} W^{(q)\prime}(a)|_{q=-\widehat{\varrho}(a)}.$$

Introduce the set $\mathcal{D}_\varrho = \{a > 0 : \varrho'(a) \neq 0\}$. Note that it is open and its complement $\mathbb{R}_+ \backslash \mathcal{D}_\varrho$ is a countable set,as ϱ is decreasing. We have now the following reinforcement of the relation between ρ on the one hand and $\widehat{\varrho}$, ϱ on the other hand.

Corollary 2 *Suppose X has a Brownian component or jumps of size smaller than a. Then:*

(i) $\widehat{\varrho}(a) < \rho(a)$

(ii) $\varrho(a) < [=]\rho(a)$ *if and only if* $a \in [\notin]\mathcal{D}_\varrho$. *If* $a \in \mathcal{D}_\varrho$, $W^{(-\varrho)}(x) > 0$ *for* $x \in (0,a)$.

Proof of Corollary 2 (i) Since $\widehat{Q}^t > P^t$, we see that $\widehat{\varrho}$ is bounded from above by ρ. From Theorem 2(B,iii) combined with the right-continuity of $x \mapsto W_+^{(-\widehat{\varrho})\prime}(x)$, we see that $x \mapsto W_+^{(-\widehat{\varrho})\prime}(x)$ is nonnegative on $(0,a)$ and zero on a null-subset of $(0,a)$. Since $W^{(-\widehat{\varrho})}(0) = W(0)$ is nonnegative, it follows that $W^{(\widehat{\varrho})}(a) > 0$, which implies by definition of ρ, that $\widehat{\varrho} < \rho$.

(ii) Following the same line of reasoning as in the proof of Theorem 2, one can show that Theorem 2 (i)-(iv) in [6] continue to hold if assumption (AC) is replaced by the assumption of the Corollary. Since $Q^t > P^t$ we see that ϱ is bounded from above by ρ. For $a \in \mathcal{D}_\varrho$, the derivative $\varrho'(a)$ is negative, which combined with (33), Theorem 2(A,iv) and above note 3 implies that $W^{(-\varrho)}(a) > 0$. The statement now follows from the definition of ρ in conjuntion with Theorem 2(iv). □

Proof of Theorem 2 By a close reading of the proofs of Theorema 2 and 3 in [29] one notes that these are valid under the requirement of irreducibility (instead of simultaneous irreducibility). By Proposition 5, we can thus use Theorema 2 and 3 from [29]. Recall from Section 5 that ϱ is positive. We also see that $\varrho < \infty$, since otherwise $\int_0^\infty \mathrm{e}^{qt} Q^t(x,A)\mathrm{d}t$ would be finite for all $x \in [0,a)$ and $q > 0$, which would not agree with Theorem 2 in [29]. We identify ϱ as the decay parameter and show Q^t is ϱ recurrent. From Lemma 2 combined with Lemma 5 in [6] we know we can find a $\delta \in (0, a/2)$ such that $Z^{(-\varrho)}(x) > 1/2$ and $W^{(-\varrho)}(x) > 0$ if $x \in (0,\delta)$; Since $q \mapsto Z^{(q)}(x)$ and $q \mapsto W^{(q)}(x)$ are continuous as stated in Lemma 2 and [6, Lemma 4(i)] respectively, we find that, for every $x < \delta, y \in (a-\delta, a)$,

$$\lim_{q \uparrow \varrho} \frac{Z^{(-q)}(x) W^{(-q)}(a-y)}{Z^{(-q)}(a)} = \infty. \tag{34}$$

Let $A \subseteq (a-\delta, a]$ be any Borel set with positive Lebesgue measure. By Proposition 4, monotone convergence and Fatou Lemma we deduce that

$$\int_0^\infty \mathrm{e}^{\varrho t} Q^t(x, A)\mathrm{d}t = \infty \qquad \text{for every } x \in (0, \delta). \tag{35}$$

Theorem 2 in [29] now implies that ϱ coincides with the decay parameter and Q^t is ϱ-recurrent. In particular, it implies that (35) holds for all $x \in [0, a]$ and non-null Borel-sets $A \subseteq [0, a]$. Hence, we deduce that $Z^{(-\varrho)}(x)$ and $W^{(-\varrho)}$ are positive for all respectively Lebesgue almost all $x \in (0, a)$. By note 3 above the Theorem, we now see that $\frac{\partial}{\partial q} Z^{(-\varrho)}(a) > 0$ thus ϱ is a simple root.

Using the identity of Lemma 3 combined with the observation $Z^{(q)}(x) - 1 = q(1 \star W^{(q)}(x))$ and the form of the resolvent given in Theorem 1, one finds, after some algebra, that $\int \mathrm{e}^{-qt} Q^t Z^{(-\varrho)}(x)\mathrm{d}t = Z^{(-\varrho)}(x)/(q+\varrho)$. By unicity of the Laplace transform, we find that there exists a null set N such that (30) holds for $t \notin N$. Since N is a Lebesgue null-set, for any $t \in N$, there exists an $s \notin N$ such that $(t-s) \notin N$. Applying the Markov property at s, we see

$$Q^t Z^{(-\varrho)}(x) = Q^{(t-s)}(Q^s Z^{(-\varrho)}(x)) = \mathrm{e}^{-\varrho s} Q^{(t-s)} Z^{(-\varrho)}(x) = \mathrm{e}^{-\varrho t} Z^{(-\varrho)}(x),$$

from which we see that (30) holds for all $t > 0$ and hence $Z^{(-\varrho)}$ is the ϱ-invariant function for Q^t (unicity from Theorem 3 in [29]). Analogously, one can prove that $W^{(-\varrho)}(a-x)\mathrm{d}x$ is the ϱ-invariant measure for Q^t.

(iv) By Proposition 6 and Theorem 1 from [29], we are allowed to apply Theorem 5 and 7 in [29]. Note that the ϱ-invariant measure Π has a finite mass and $Q^t 1(x)$ converges to zero for all x as t tends to ∞. Following the argument given [6], the proof of Theorem 2(v), we find from Theorem 5(i) and 7 that (32) is valid for almost all $x \in [0, a]$, where the Markov property and the absolute continuity of Q^t enable us to show the last statement is valid for all $x \in [0, a]$. This completes the proof of part (A).

Part (B) follows along the same lines as the part (A). By Proposition 5, we can again use Theorem 2 and 3 of [29], where the role of the measure m is now played by the Lebesgue measure with an atom of size one in zero. We invoke Lemma 5 in [6] to find a $\delta \in (0, a/2)$ such that $W^{(-\widehat{\varrho})}(y) > 0$ for $y \in (0, \delta)$.

By the expansion (3) we see that $W_+^{(-\widehat{\varrho})\prime}(0) = W_+'(0) - \widehat{\varrho} W(0)^2$. Lemma 4 [and monotoncty of $\widehat{\varrho}(\cdot)$ in the compound Poisson case] implies $W_+^{(-\widehat{\varrho})\prime}(0)$ is positive or infinite By right-continuity (Lemma 2) of $x \mapsto W_+^{(-\widehat{\varrho})\prime}(x)$ we can then find a δ' such that $W^{(-\widehat{\varrho})\prime}(y) > 0$ for $y \in (0, \delta')$. Analogously as in (A), we can then prove the $\widehat{\varrho}$-recurrence and -positivity of $\widehat{Q}^t$ and the stated properties of $W_+^{(-\widehat{\varrho})\prime}(x), W^{(-\widehat{\varrho})}(x)$.

To identify the $\widehat{\varrho}$-invariant function and measure we follow an analogous line reasoning using the following identity which follows by taking the right-derivative with respect to a of the identity in Lemma 3

$$\int_0^a W^{(r)}(a-x) W^{(q)}(\mathrm{d}x) = \frac{1}{r-q}(W_+^{(r)\prime}(a) - W_+^{(q)\prime}(a)).$$

The proof of (iv) goes along the same lines as above. □

8 The processes Y and $\widehat{Y}$ conditioned to stay below a

We study the process Y and $\widehat{Y}$ conditioned to stay below a fixed level $a > 0$. We introduce the measures $\mathbb{P}^\diamond$ and $\widehat{\mathbb{P}}^\diamond$ by

$$\mathrm{d}\mathbb{P}^\diamond_{x|\mathcal{F}_t} = H_t \mathrm{d}\mathbb{P}_{x|\mathcal{F}_t} \qquad \text{and} \qquad \mathrm{d}\widehat{\mathbb{P}}^\diamond_{x|\mathcal{F}_t} = \widehat{H}_t \mathrm{d}\mathbb{P}_{-x|\mathcal{F}_t}$$

where

$$H_t = \mathrm{e}^{\varrho t}\mathbf{1}_{\{t<\tau_a\}}\frac{Z^{(-\varrho)}(Y_t)}{Z^{(-\varrho)}(x)} \qquad \text{and} \qquad \widehat{H}_t = \mathrm{e}^{\widehat{\varrho} t}\mathbf{1}_{\{t<\widehat{\tau}_a\}}\frac{W^{(-\widehat{\varrho})}(a-\widehat{Y}_t)}{W^{(-\widehat{\varrho})}(a-x)}.$$

Theorem 2 implies that $\mathbb{P}^\diamond_x$ and $\widehat{\mathbb{P}}^\diamond_x$ are h-transforms of $\mathbb{P}_x$ and $\mathbb{P}_{-x}$ respectively. Indeed, by the Markov property of Y under the probability measure $\mathbb{P}$:

$$\begin{aligned}\mathbb{E}_x(H_{t+s}|\mathcal{F}_t) &= \frac{\mathrm{e}^{\varrho(t+s)}}{Z^{(-\varrho)}(x)}\mathbb{E}_x(\mathbf{1}_{\{t+s<\tau_a\}}Z^{(-\varrho)}(Y_{t+s})|\mathcal{F}_t)\\ &= \frac{\mathrm{e}^{\varrho(t+s)}}{Z^{(-\varrho)}(x)}\mathbf{1}_{\{t<\tau_a\}}\mathbb{E}_{Y_t}(\mathbf{1}_{\{s<\tau_a\}}Z^{(-\varrho)}(Y_s))\end{aligned}$$

and the martingale property of H follows from Theorem 2(A,iii). Similarly, using Theorem 2(B,iii) we can verify that $\widehat{H}$ is a martingale under $\mathbb{P}_{-x}$. The next result proves properties of the constructed processes and shows that, if (AC) holds, the h-transforms are equal to the limit as t tends to infinity of the conditional probabilities of Y (resp. $\widehat{Y}$) exiting $[0, a]$ after t. Recall the measures μ and $\widehat{\mu}$ given in note 3 after Theorem 2.

Theorem 3 *Let $x \in [0, a)$. The following are true:*

(i) Under $\mathbb{P}^\diamond$, Y has the strong Markov property and is positive recurrent with stationary probability measure μ. Moreover, we have in the sense of weak convergence

$$\lim_{t\to\infty}\mathbb{P}^\diamond_x(Y_t \in \cdot) = \mu. \tag{36}$$

(ii) If X satisfies (R), (i) continues to hold if we replace the triple $(Y, \mathbb{P}^\diamond, \mu)$ by $(\widehat{Y}, \widehat{\mathbb{P}}^\diamond, \widehat{\mu})$.

(iii) Suppose (AC) holds. Then the convergence in (i) and (ii) holds in total variation norm. Moreover, for any $s \ge 0$ and $A \in \mathcal{F}_s$, the condional laws converge as $t \to \infty$

$$\mathbb{P}_x(A|\tau_a > t) \to \mathbb{P}^\diamond_x(A) \qquad \textit{and} \qquad \mathbb{P}_{-x}(A|\widehat{\tau}_a > t) \to \widehat{\mathbb{P}}^\diamond_x(A).$$

Example. Let X be a standard Brownian motion. Theorem 3 implies that the process Y conditioned to stay below a has generator L on $(0, a)$ given by

$$Lf(x) = \frac{1}{2}f''(x) - \frac{\pi}{2a}\tan\left(\frac{\pi x}{2a}\right)f'(x) \tag{37}$$

for all functions $f \in D$, where $D = \{f \in C^2(0,a) : f'(0) = 0\}$ is the domain of the operator. By a famous theorem of Lévy, the process Y is in law equal to the process $|X|$. Hence, by symmetry and $\tan(x) = -\tan(-x)$, we find, as in [16], that the generator of Brownian motion conditioned to stay in $(-a,a)$, is given on $(-a,a)$ by (37) for all functions f in $C^2(-a,a)$. This conditioned Brownian motion is called the Brownian Taboo process with taboo states $\{-a,a\}$ in the nomenclature of [16].

Proof We only proof the part of the theorem involving Y, leaving the rest to the reader.
(i) It is well known that under $\mathbb{P}^\diamond_x$ and $\widehat{\mathbb{P}}^\diamond_x$ the strong Markov property is preserved [12, Thm. XVI.28 p. 329] and the process has as the semigroup $P_t^\diamond(x,\mathrm{d}y) = Q_t(x,\mathrm{d}y)\mathrm{e}^{\varrho t}\frac{Z^{(-\varrho)}(y)}{Z^{(-\varrho)}(x)}$. The positive recurrence of and the invariance of μ for $P_t^\diamond$ are immediate from Theorem 2(A;i,iii) combined with the form of the resolvents of the process under $\mathbb{P}^\diamond_x$, which follows now directly from Theorem 1. The form of the constant follows from note 3 after Theorem 2. To prove the convergence, we will use the regenerative property of Y under $\mathbb{P}^\diamond$. To be more precise, under $\mathbb{P}^\diamond$, Y is a delayed regenrative process, where the delay is the time to reach zero and a cycle starts at zero and ends again at the first return to zero after a crossing of the level $a/2$. Denoting by T^* the cyle length, we see from the forthcoming Proposition 7 that T^* has a finite mean. Note that T^* has the same distribution as $\widehat{\tau}_0 \circ \theta_{\tau_{a/2}}$ under $\mathbb{P}^\diamond_0$. ¿From Lemma 6 we see that the distribution of $\widehat{\tau}_0 \circ \theta_{\tau_{a/2}}$ under $\mathbb{P}^\diamond_0$ has no atoms. In particular, T^* is not concentrated on a lattice. Theorem V.1.2 from Asmussen [1] now implies the weak convergence (36).

(iii) As in the proof of Theorem 2(A; iv), under (AC), we invoke Theorem 5(i) of [29] to find that (36) holds in total variation norm for Π-a.e. $x \in [0,a)$. Combining the Markov property with the absolute continuity of the transition probabilities $P_t^\diamond$ of Y under $\mathbb{P}^\diamond$ under (AC), we find

$$\|P_t^\diamond(x,\cdot) - \mu\| \le P_s^\diamond(x, \|P_{t-s}^\diamond(Y_s,\cdot) - \mu\|)$$

where $\|\cdot\|$ denotes the total variation norm. By bounded convergence the right-hand side converges to zero as t tends to infinity.

To prove the convergence of the conditional laws, pick $s,t > 0$. From the notes after Theorem 2, we see that the variables

$$H_{t,s} = \frac{\mathbb{P}_x(\tau_a > t+s|\mathcal{F}_t)}{\mathbb{P}_x(\tau_a > t+s)} = \mathbf{1}_{\{\tau_a > t\}}\frac{\mathbb{P}_{Y_t}(\tau_a > s)}{\mathbb{P}_x(\tau_a > t+s)}$$

converge to H_t a.s. as $s \to \infty$. Since $\mathbb{E}_x(H_{t,s}) = 1 = \mathbb{E}_x(H_t)$, it follows from Scheffe's lemma that the preceding convergence holds in L^1. We deduce that for every $Y \in L^\infty(\mathcal{F}_t)$ $\mathbb{E}_x(YH_{t,s})$ converges to $\mathbb{E}_x(YH_t)$. By the Markov property this means:

$$\lim_{s\to\infty} \mathbb{E}_x(Y|\tau_a > t+s) = \mathbb{E}_x(YH_t) = \mathbb{E}^\diamond_x(Y).$$

□

In the sequel, we will frequently use the fact (from the optional stopping theorem) that for every finite stopping time S and $Y \in L_+(\mathcal{F}_S)$

$$\mathbb{P}^\diamond_x(Y) = \mathbb{E}_x(YH_S), \qquad \widehat{\mathbb{P}}^\diamond_x(Y) = \mathbb{E}_{-x}(Y\widehat{H}_S).$$

We now collect some Laplace transforms of hitting times under $\mathbb{P}^\diamond$.

Proposition 7 *For any $0 \le b < x < c < a, q \ge 0$ the following hold:*

(i) Two sided exit problem under $\mathbb{P}^\diamond$: if $T' = \inf\{t \ge 0 : Y_t = X_t - I_t \notin (b,c)\}$,

$$\mathbb{E}^\diamond_x(\mathrm{e}^{-qT'}\mathbf{1}_{\{Y_{T'}=c\}}) = \frac{Z^{(-\varrho)}(c)}{Z^{(-\varrho)}(x)}\frac{W^{(q-\varrho)}(x-b)}{W^{(q-\varrho)}(c-b)}.$$

(ii) Passage at an upper level:

$$\mathbb{E}^\diamond_x(\exp(-q\tau_c)) = \frac{Z^{(-\varrho)}(c)}{Z^{(-\varrho)}(x)}\frac{Z^{(q-\varrho)}(x)}{Z^{(q-\varrho)}(c)}$$

(iii) Passage time below a lower level: if $T'' = \inf\{t \ge 0 : Y_t = X_t - I_t \notin (b,a]\}$,

$$\begin{aligned}&\mathbb{E}^\diamond_x(\exp(-qT'')\mathbf{1}_{\{Y_{T''-}\in\mathrm{d}y\}}\mathbf{1}_{\{\Delta Y_{T''}\in\mathrm{d}z\}})\\ &\quad= \frac{Z^{(-\varrho)}(y+z)}{Z^{(-\varrho)}(x)}\left(\frac{W^{(q-\varrho)}(x-b)W^{(q-\varrho)}(a-y)}{W^{(q-\varrho)}(a-b)} - W^{(q-\varrho)}(x-y)\right)\mathrm{d}y\Lambda(\mathrm{d}z).\end{aligned}$$

Proof For $q \ge \varrho$, (i) and (ii) follow readily from (8) and the remark preceding the proposition. The identity can then be extended by an argument analogous to the proof of Proposition 4. (iii) We now set $0 \le b < x < a$,$0 \le b < y < a$, $0 < y+z < a$ and compute by the compensentation formula applied to the Poisson point process of jumps, the following quantity:

$$\begin{aligned}&\mathbb{E}^\diamond_x(\exp(-qT'')\mathbf{1}_{\{Y_{T''}\in\mathrm{d}y\}}\mathbf{1}_{\{\Delta Y_{T''}\in\mathrm{d}z\}})\\ &= \mathbb{E}_x(\frac{Z^{(-\varrho)}(y+z)}{Z^{(-\varrho)}(x)}\mathrm{e}^{(\varrho-q)T''}\mathbf{1}_{\{T''<\tau_a\}}\mathbf{1}_{\{Y_{T''}\in\mathrm{d}y\}}\mathbf{1}_{\{\Delta Y_{T''}\in\mathrm{d}z\}})\\ &= \frac{Z^{(-\varrho)}(y+z)}{Z^{(-\varrho)}(x)}\mathbb{E}_x(\sum_{t\ge 0}\mathrm{e}^{(\varrho-q)t}\mathbf{1}_{\{Y_s\in(b,a)\forall s<t\}}\mathbf{1}_{\{Y_t\in\mathrm{d}y\}}\mathbf{1}_{\{\Delta Y_t\in\mathrm{d}z\}})\\ &= \frac{Z^{(-\varrho)}(y+z)}{Z^{(-\varrho)}(x)}\mathbb{E}_x(\int_0^\infty \mathrm{d}t\Lambda(\mathrm{d}z)\mathrm{e}^{(\varrho-q)t}\mathbf{1}_{\{X_s\in(b,a)\forall s\le t\}}\mathbf{1}_{\{X_t\in\mathrm{d}y\}})\\ &= \Lambda(\mathrm{d}z)u^{(\varrho-q)}_{(a-b)}(x-b,\mathrm{d}y-b),\end{aligned}$$

where $u^q_{(a-b)}$ denotes the resolvent density of the process X, killed as it exits from $[0,a-b]$. The result follows by applying (13). In the one but last line we used that, under $\mathbb{P}_x$, $(Y_{T''}, T'' < \tau_a)$ is in law equal to $(X_{\widehat{T}_{-b}}, \widehat{T}_{-b} < T_a)$. □

Similarly, we state some Laplace transforms of hitting times under $\widehat{\mathbb{P}}^\diamond$.

Proposition 8 *For any $0 < b < x < c < a$ the following hold:*

(i) Two sided exit problem under $\mathbb{P}^\diamond$: if $T' = \inf\{t \geq 0 : \widehat{Y} = S_t - X_t \notin (b,c)\}$,

$$\widehat{\mathbb{E}}^\diamond_x(\mathrm{e}^{-qT'}\mathbf{1}_{\{X_{T'}=b\}}) = \frac{W^{(-\widehat{\varrho})}(a-b)}{W^{(-\widehat{\varrho})}(a-x)}\frac{W^{(q-\widehat{\varrho})}(c-x)}{W^{(q-\widehat{\varrho})}(c-b)}.$$

(ii) Passage at a lower level:

$$\widehat{\mathbb{E}}^\diamond_x(\exp(-qT_b)) = \frac{W^{(-\widehat{\varrho})}(a-b)}{W^{(-\widehat{\varrho})}(a-x)}\frac{W^{(q-\widehat{\varrho})}(a-x)}{W^{(q-\widehat{\varrho})}(a-b)}$$

(iii) Passage time above an upper level: if $T'' = \inf\{t \geq 0 : \widehat{Y}_t = S_t - X_t \notin [0,c)\}$,

$$\begin{aligned}&\widehat{\mathbb{E}}^\diamond_x(\exp(-qT'')\mathbf{1}_{\{X_{T''-}\in \mathrm{d}y\}}\mathbf{1}_{\{\Delta X_{T''\in \mathrm{d}z}\}})\\ &\quad = \frac{W^{(-\widehat{\varrho})}(a-y-z)}{W^{(-\widehat{\varrho})}(a-x)}\left(\frac{W^{(q-\widehat{\varrho})}(c-x)W^{(q-\widehat{\varrho})}(\mathrm{d}y)}{W_+^{(q-\widehat{\varrho})\prime}(c)} - W^{(q-\widehat{\varrho})}(y-x)\mathrm{d}y\right)\Lambda(\mathrm{d}z).\end{aligned}$$

8.1 Excursion measure away from a point under $\mathbb{P}^\diamond$

Recall that a point $x \in [0,a)$ is said to be regular (for itself) under $\mathbb{P}^\diamond$ if

$$\mathbb{P}^\diamond_x(\inf\{s > 0 : Y_s = x\}) = 1.$$

Obviously, $x > 0$ in regular under $\mathbb{P}^\diamond$ if and only if $x > 0$ is regular under $\mathbb{P}$, hence if and only if X has unbounded variation under $\mathbb{P}$. We assume this throughout from now on. The local time at level at x, denoted by L^x is defined as the occupation density

$$L^x_t = \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^t \mathbf{1}_{\{|Y_s - x|<\epsilon\}}\mathrm{d}s.$$

Let σ_s be its right-continuous inverse:

$$\sigma_s = \inf\{t > 0 : L^x_t > s\}, \qquad s \geq 0.$$

Analogously to what we did in the proof of Theorem 1, we now consider the excursion process $e = (e_s, s \geq 0)$ of Y away from $\{x\}$ where

$$e_s = (Y_u, \sigma_{s^-} \leq u < \sigma_s) \qquad \text{if } \sigma_{s^-} < \sigma_s$$

and else e_s takes the value ∂ where ∂ is an additional isolated point. Again, a famous theorem of Itô states that e is a Poisson point process valued in the space $\mathcal{E}$. Its characteristic measure is denoted by n_x under $\mathbb{P}$ (and $n^\diamond_x$ under $\mathbb{P}^\diamond$) and is called the excursion measure away from $\{x\}$.

In this section we present some useful formulas involving the local time L^x and the excursion measure $n^\diamond_x$. For every excursion of $X - I$ away from $\{x\}$, we denote its height by $m = m(\epsilon)$:

$$m(\epsilon) = \sup_{u\leq\zeta(\epsilon)}(\epsilon_u - \epsilon_0) = \sup_{u\leq\zeta(\epsilon)}\epsilon_u - x.$$

Recall that σ stands for the inverse of the local time L^x. As well known, σ is a subordinator. Define Laplace exponent $\Phi_x^\diamond$ by

$$\mathbb{E}_x^\diamond(\mathrm{e}^{-\lambda\sigma_t}) = \exp(-t\Phi_x^\diamond(\lambda)), \qquad \lambda \geq 0.$$

Proposition 9 *For any nonnegative λ and any $\eta \in [0, a-x]$,*

$$n_x^\diamond(1 - \mathbf{1}_{\{m<\eta\}}\mathrm{e}^{-\lambda\zeta}) = \frac{Z^{(\lambda-\varrho)}(x+\eta)}{Z^{(\lambda-\varrho)}(x)W^{(\lambda-\varrho)}(\eta)}.$$

In particular, for any nonnegative λ,

$$\Phi_x^\diamond(\lambda) = \frac{Z^{(\lambda-\varrho)}(a)}{Z^{(\lambda-\varrho)}(x)W^{(\lambda-\varrho)}(a-x)}$$

and for any $\eta \in [0, a-x]$

$$n_x^\diamond(m > \eta) = \frac{Z^{(-\varrho)}(x+\eta)}{Z^{(-\varrho)}(x)W^{(-\varrho)}(\eta)}.$$

The proof is defered to the Appendix. The previous proposition enables us to specify the asymptotic behaviour of the local time. Recall from the Theorem 3 that the stationary measure μ of the conditioned process is absolutely continuous with density p, say.

Corollary 3 *If $x \in (0, a)$ or $x = 0$ and $a \in \mathcal{D}_\varrho$, we have a.s.*

$$\lim_{t\to\infty} L_t^x/t = \mu(\mathrm{d}x)/\mathrm{d}x = p(x).$$

Proof We deduce from Proposition 9 that $\Phi_x^\diamond$ has a right-derivative in 0 equal to

$$\frac{C(a)}{Z^{(-\varrho)}(x)W^{(-\varrho)}(a-x)} = p(x)^{-1}.$$

Hence using $\mathbb{E}_x^\diamond(\sigma_t) = t/p(x)$ and that $\{\sigma_t : t \geq 0\}$ is a Lévy process, the law of large numbers entails that $L_t^x/t = t/\sigma_t$ converges a.s. to $p(x)$ as t tends to infinity. $\square$

8.2 Excursion measure away from 0 under $\widehat{\mathbb{P}}^\diamond$

We consider now the process $S - X = \{(S-X)_t, t \geq 0\}$ under the measure $\widehat{\mathbb{P}}^\diamond$. Note that, since X has no positive jumps, the supremum S is a local time in 0 for $S - X$ (Bertoin [5]). Furthermore,

$$T_u = \inf\{s \geq 0 : S_s \geq u\} = \inf\{s \geq 0 : X_s \geq u\}$$

is a subordinator under $\widehat{\mathbb{P}}^\diamond$. In this section we consider again the excursion process $\widehat{e} = \{\widehat{e}_s, s \geq 0\}$ of $S - X$ away from zero as defined in equation (). As earlier saw in the proof of Theorem 1(ii), a famous theorem by Itô implies that the process $\{\widehat{e}_s, s \geq 0\}$ is a Poisson point process valued in the space $\mathcal{E}$. By $\widehat{h}$ we denote the height of the excursion. Denote its characteristic measure by $\widehat{n}$ and $\widehat{n}^\diamond$ under $\mathbb{P}$ and $\widehat{\mathbb{P}}^\diamond$ respectively.

Proposition 10 *For any $\eta \in (0,a), \lambda \geq 0$ we have*

$$\widehat{n}^{\diamond}(1 - \mathbf{1}_{\{\widehat{h}<\eta\}}\mathrm{e}^{-\lambda\zeta}) = \frac{W^{(\lambda-\widehat{\varrho})\prime}(\eta)}{W^{(\lambda-\widehat{\varrho})}(\eta)}$$

In particular for any $\eta \in (0,a), \lambda \geq 0$,

$$\phi^{\diamond}(\lambda) = \frac{W^{(\lambda-\widehat{\varrho})\prime}(a)}{W^{(\lambda-\widehat{\varrho})}(a)}, \qquad \widehat{n}^{\diamond}(\widehat{h} > \eta) = \frac{W^{(-\widehat{\varrho})\prime}(\eta)}{W^{(-\widehat{\varrho})}(\eta)}.$$

Proof The second and the third formula follow directly from the first one (taking $\eta = a$, $\lambda = 0$ respectively). As in Proposition 9, an application of the exponential formula yields that

$$\widehat{n}^{\diamond}(1 - \mathbf{1}_{\{\widehat{h}<\eta\}}\mathrm{e}^{-\lambda\zeta})) = \left[\mathbb{E}^{\diamond}\left(\int_0^{\widehat{\tau}_\eta} \mathrm{e}^{-\lambda t}\mathrm{d}S_t\right)\right]^{-1}.$$

Suppose first that $\lambda \geq \widehat{\varrho}$. Then we find for the right-hand side of previous display that

$$\begin{aligned}\mathbb{E}^{\diamond}\left(\int_0^{\widehat{\tau}_\eta} \mathrm{e}^{-\lambda t}\mathrm{d}S_t\right) &= \int_0^\infty \mathbb{E}^{\diamond}\left(\mathrm{e}^{-\lambda T_s}\mathbf{1}_{\{T_s<\widehat{\tau}_\eta\}}\right)\mathrm{d}s = \int_0^\infty \mathbb{E}\left(\mathrm{e}^{(\widehat{\varrho}-\lambda)T_s}\mathbf{1}_{\{T_s<\widehat{\tau}_\eta\}}\right)\mathrm{d}s \\ &= \int_0^\infty \mathrm{e}^{-\Phi(\lambda-\widehat{\varrho})s}\mathbb{E}\left(\mathrm{e}^{\Phi(\lambda-\widehat{\varrho})X_{T_s}+(\widehat{\varrho}-\lambda)T_s}\mathbf{1}_{\{T_s<\widehat{\tau}_\eta\}}\right)\mathrm{d}s.\end{aligned}$$

Changing the measure and using the Poisson point character of the excursion process, one finds that the expectation in the last display is equal to $W^{(\lambda-\widehat{\varrho})}(\eta)/W^{(\lambda-\widehat{\varrho})\prime}(\eta)$. See [2] for a similar (and more detailed) computation. By analytic continuation we can extend the formula to hold for all $\lambda \geq 0$ and $\eta \in (0,a)$. □

Similarly as we proved Corollary 3, we can prove the following result:

Corollary 4 $\lim_{t\to\infty} S_t/t = \lim_{t\to\infty} t/T_t = W^{(-\widehat{\varrho})}(a)/\frac{\partial}{\partial_q}|_{q=-\widehat{\varrho}}W^{(q)\prime}(a)$ *a.s.*

8.3 Convergence of the supremum

The fact that the conditioned processes is recurrent implies that under the measures $\mathbb{P}^{\diamond}$, $\widehat{\mathbb{P}}^{\diamond}$ the suprema of Y and $\widehat{Y}$, $M_t = \sup\{Y_s; s \in [0,t]\}$ and $\widehat{M_t} = \sup\{\widehat{Y}_s; s \in [0,t]\}$ respectively, converge to a; our purpose is to investigate the rate of convergence. We still assume that X has unbounded variation.

Let $f : [0,\infty) \to (0,\infty)$ be a decreasing function and write

$$l_f = \liminf_{t\to\infty} \frac{a - M_t}{f(t)}, \qquad L_f = \limsup_{t\to\infty} \frac{a - M_t}{f(t)}.$$

and $\widehat{l}_f, \widehat{L}_f$ for the corresponding quantities involving $\widehat{M}$. Recall that a real funtion is said to be slowly varying at infinity if for any $\lambda > 0$,

$$\lim_{t\to\infty} \frac{g(\lambda t)}{g(t)} = 1.$$

Finally we recall the notations $\mathcal{D}_\varrho = \{a > 0 : \varrho'(a) \neq 0\}$ and $\mathcal{D}_{\widehat{\varrho}} = \{a > 0 : \widehat{\varrho}'(a) \neq 0\}$.

Theorem 4 *Assume $a \in \mathcal{D}_\varrho$ for the statements involving M_t. Assume that $W^{(-\widehat{\varrho})}(\cdot)$ is twice continuously differentiable and let $a \in \mathcal{D}_{\widehat{\varrho}}$ for the statements involving $\widehat{M_t}$. Then following three assertions hold.*

(i) The random variables $t(a - M_t)$ and $t(a - \widehat{M_t})$ converge in distribution as $t \to \infty$ to exponential random variables with parameters $|\varrho'(a)|$ and $|\widehat{\varrho}'(a)|$ respectively.

(ii) The random variable l_f and $\widehat{l}_f$ are 0 or ∞ almost surely according to whether $\int_1^\infty f(s)\mathrm{d}s$ converges or diverges.

(iii) Assume further that $t \mapsto tf(t)$ is increasing and slowly varying at infinity and let

$$\gamma_f = \inf\{\gamma > 0 : \int_1^\infty f(t)\mathrm{e}^{-\gamma t f(t)}\mathrm{d}t < \infty\},$$

with the convention $\inf \emptyset = +\infty$. Then $L_f = |\varrho'(a)|^{-1}\gamma_f$ and $\widehat{L}_f = |\widehat{\varrho}'(a)|^{-1}\gamma_f$ almost surely.

Remarks

(i) Let $= \int^\infty \mathrm{d}t f(t)\mathrm{e}^{-\gamma t f(t)}$. One easily sees that if $\gamma_f < \infty$, I_f is finite (and decreasing) on (γ_f, ∞) and that if $\gamma_f > 0, I_f = \infty$ on $[0, \gamma_f)$.

(ii) If $\log_k$ denotes the k-th iterate of the logarithm, then for

$$\begin{array}{lll} f(t) = t^{-1}\log t, & L_f = 0, & l_f = 0; \\ f(t) = t^{-1}\log_2 t, & L_f = |\varrho'(a)|^{-1}, & l_f = 0; \\ f(t) = t^{-1}\log_3 t, & L_f = \infty, & l_f = 0. \end{array}$$

(iii) Recall that for $\alpha \in (1, 2]$, $-r(\alpha)$ denotes the first negative root of E_α, where E_α is the Mittag-Leffler function of parameter α. In the case X is a stable process of index $\alpha \in (1, 2]$,

$$\limsup_{t\to\infty} \frac{t(a - M_t)}{\log_2 t} = \frac{a^{\alpha+1}}{\alpha r(\alpha)} \qquad \text{a.s.,}$$

which yields in the case X is a Brownian motion

$$\limsup_{t\to\infty} \frac{t(a - M_t)}{\log_2 t} = \frac{2a^3}{\pi^2} \qquad \text{a.s..}$$

The proof can be found in the Appendix. Following the lead of Lambert [18], the idea is to exploit the Poisson point character of the excursions away from $\{x\}$ under $\mathbb{P}^\diamond$ and of the excursions of $\widehat{Y}$ away from zero under $\widehat{\mathbb{P}}^\diamond$,

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Appendix

A Proof of Proposition 9

Proof The last two assertions follow easily from the first (by taking $\eta = a - x$ and $\lambda = 0$ respectively). To prove the first assertion we start with the following identity:

$$n_x^\diamond(1 - \mathbf{1}_{\{m<\eta\}}\mathrm{e}^{-\lambda\zeta}) = \left[\mathbb{E}_x^\diamond\left(\int_0^{\tau_{x+\eta}} \mathrm{e}^{-\lambda t}\mathrm{d}L_t^x\right)\right]^{-1}. \tag{38}$$

Indeed we have

$$\mathbb{E}^\diamond_x\left(\int_0^{\tau_{x+\eta}} \mathrm{e}^{-\lambda t}\mathrm{d}L^x_t\right) = \mathbb{E}^\diamond_x\left(\int_0^\infty \mathrm{e}^{-\lambda\sigma_s}\mathbf{1}_{\{\sigma_s<\tau_{x+\eta}\}}\mathrm{d}s\right)$$
$$= \int_0^\infty \mathrm{d}s\mathbb{E}^\diamond_x\left(\exp(-\sum_{0\le u\le s}\lambda(\tau_u-\tau_{u-})\chi_{\{m(e_u)<\eta\}})\right),$$

where we wrote $\chi_A(\omega) = 1[\infty]$ if $\omega \in [\notin]A$. By the exponential formula, the foregoing quantity is thus equal to

$$\int_0^\infty \mathrm{d}s \exp\left(-sn^\diamond_x(1-\exp(-\lambda\zeta\chi_{\{m<\eta\}}))\right) = [n^\diamond_x(1-\mathbf{1}_{\{m<\eta\}}\mathrm{e}^{-\lambda\zeta})]^{-1}$$

which establishes (38). The next step consists in proving the following identity

$$\mathbb{E}^\diamond_y(\int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{d}L^x_t) = u^\diamond_\lambda(y,x) \tag{39}$$

Indeed, following a line of reasoning similar to the first lines of the proof of [5, Proposition V.2], we see that if $\eta(q)$ is an independent exponential random variable with parameter $q>0$ that

$$(2\epsilon)^{-1}\int_0^{\eta(q)}\mathbf{1}_{\{|Y_s-x|<\epsilon\}}\mathrm{d}s \overset{L^2(\mathbb{P})}{\longrightarrow} L^x_{\eta(q)} \qquad \text{as } \epsilon\downarrow 0.$$

As a consequence, provided $q>\varrho$, the convergence also holds in $L^2(\mathbb{P}^\diamond)$ and

$$\mathbb{E}^\diamond_y[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L^x_t] = \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\mathbb{E}^\diamond_y[\int_0^\infty \mathrm{e}^{-qt}\mathbf{1}_{\{|Y_t-x|<\epsilon\}}\mathrm{d}t]$$
$$= \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} u^\diamond_q(y,u)\mathrm{d}u = u^\diamond_q(y,x)$$

(recall that we assumed that the Lévy process has unbounded variation which ensures $W^{(q-\varrho)}(0)=0$, from where we see that $u^\diamond_q(y,\cdot)$ is continuous.) This proves (39) for $q>\varrho$, which can be analytically extended as before.

We are now able to complete the proof of our statement. The identity (39) and the Markov property enable us to write

$$\mathbb{E}^\diamond_y[\int_0^\infty \mathrm{e}^{-qs}\mathrm{d}L^x_s|\mathcal{F}_t] = \int_0^t \mathrm{e}^{-qs}\mathrm{d}L^x_s + \mathrm{e}^{-qt}u^\diamond_q(Y_t,x)$$

which entails by an application of the optional sampling theorem at $\tau_{x+\eta}$, that

$$\mathbb{E}^\diamond_y[\int_0^{\tau_{x+\eta}} \mathrm{e}^{-qs}\mathrm{d}L^x_s] = u^\diamond_q(x,x) - u^\diamond_q(x+\eta,x)\mathbb{E}^\diamond_x[\mathrm{e}^{-q\tau_{x+\eta}}].$$

Proposition 7(ii), jointly with the expression for $u^\diamond_q$ in given in Theorem 3(iii) yield the result for any $q>\varrho$, and then by an analytical continuation argument, for any $q>0$. □

B Proof of Theorem 4

Proof We only proof the statements involving $\widehat{M_t}$, the proof of the statements involving M_t runs analogously (using then excursions away from $\{x\}$ as in [18]) and is skipped. We split the proof into three parts.
(i) This part uses the following elementary lemma.

Lemma 8 *Fix $y \in (0, \infty]$ and let R be an increasing function. Next consider a Poisson point process $(Z_s, s \geq 0)$ on $[0, y)$ with characteristic measure* $\mathrm{d}R$. *For every $t > 0$, set $i_t = \inf_{0 \leq s \leq t} Z_s$, where we agree that* $\inf \emptyset = y$. *Then*

$$\mathbb{E}(\mathrm{e}^{-\lambda i_t}) = 1 - \lambda \int_0^y \mathrm{e}^{-\lambda u} \mathrm{e}^{-tR(u)} \mathrm{d}u, \qquad \lambda > 0.$$

The lemma follows immediately from the identity

$$\mathbb{P}(Z_s > x \text{ for all } s \in [0, t]) = \exp(-tR(x)), \qquad x \in (0, y).$$

Recall the notation involving the excursions of $\widehat{Y}$ away from 0. Our argument relies on the elementary observation that $\widehat{M_{T_t}}$ is the maximum of the excursion heights $(\widehat{h}(\epsilon_s), s \leq t)$. Recall from Proposition 9 that

$$\widehat{R}(u) = \frac{W^{(-\widehat{\varrho})\prime}(a-u)}{W^{(-\varrho)}(a-u)} \qquad u \in [0, a),$$

is the distribution function of the measure $\widehat{n}^{\diamond}(a - \widehat{h} \in \cdot)$. The properties of $W^{(q)}$ and $W^{(q)\prime}$ imply that the function $\widehat{R}$ is of class C^1 and its derivative at 0 is positive. The point process $(K_s, s \geq 0)$ defined by

$$K_s = \begin{cases} a - \widehat{h}(e_s) & \text{if } T_{s-} < T_s \\ \infty & \text{otherwise,} \end{cases}$$

is a Poisson point process with characteristic measure $\mathrm{d}\widehat{R}$. We deduce from the previous lemma that

$$\widehat{\mathbb{E}}^{\diamond}(\mathrm{e}^{-\lambda t(a - \widehat{M}_{T_t})}) = 1 - \lambda \int_0^{ta} \mathrm{d}v \mathrm{e}^{-\lambda v} \mathrm{e}^{-t\widehat{R}(v/t)},$$

hence by dominated convergence,

$$\lim_{t \to \infty} \widehat{\mathbb{E}}^{\diamond}(\mathrm{e}^{-\lambda t(a - \widehat{M}_{T_t})}) = 1 - \lambda \int_0^{\infty} \mathrm{d}v \mathrm{e}^{-\lambda v} \mathrm{e}^{-\widehat{R}'(0)} = \frac{\widehat{R}'(0)}{\lambda + \widehat{R}'(0)}.$$

Let $\widehat{p}_0$ denote the constant $W^{(-\widehat{\varrho})}(a) / \frac{\partial}{\partial_q}|_{q=-\widehat{\varrho}} W^{(q)\prime}(a)$ Now let $0 < \delta < \widehat{p}_0$. According to Corollary 3, one has $T_{\delta t} < t$ for suffiently large t, so that

$$\liminf_{t \to \infty} \widehat{\mathbb{E}}_x^{\diamond}(\mathrm{e}^{-\lambda t(a - \widehat{M}_t)}) \geq \frac{\widehat{R}'(0)}{\lambda \delta^{-1} + \widehat{R}'(0)}.$$

Letting $\delta \to \widehat{p}_0$, we get that

$$\liminf_{t\to\infty} \widehat{\mathbb{E}}^{\diamond}(\mathrm{e}^{-\lambda t(a-\widehat{M}_t)}) \geq \frac{\widehat{R}'(0)\widehat{p}_0}{\lambda + \widehat{R}'(0)\widehat{p}_0}.$$

From Proposition 10,the form of $\widehat{p}_0$ and note 4 after Theorem 2 we find that $\widehat{p}_0\widehat{R}'(0)$ equals $-\widehat{\varrho}'(a)$. Proceeding similarly with $\delta > \widehat{p}_0$, we get

$$\lim_{t\to\infty} \widehat{\mathbb{E}}^{\diamond}(\mathrm{e}^{-\lambda t(a-\widehat{M}_t)}) = \frac{\widehat{R}'(0)\widehat{p}_0}{\lambda + \widehat{R}'(0)\widehat{p}_0} = \frac{|\widehat{\varrho}'(a)|}{\lambda + |\widehat{\varrho}'(a)|}.$$

(ii) We claim that $\widehat{\mathbb{P}}^{\diamond}(\widehat{M_t} > a - f(t)$ i.o. as $t \to \infty)$ is 0 or 1 according to whether the integral $\int_1^{\infty} W^{(-\widehat{\varrho})\prime}(a - f(s))\mathrm{d}s$ converges or diverges. Since the function $W^{(-\widehat{\varrho})\prime}$ is of class C^1, $\int^{\infty} f(s)\mathrm{d}s < \infty$ implies $\int^{\infty} W^{(-\widehat{\varrho})\prime}(a-f(s))\mathrm{d}s < \infty$. Replacing f by λf and then letting $\lambda \to \infty$, we get the result. The same method applies to the converse assertion now letting $\lambda \to 0$: indeed, the second derivative at a of $W^{(-\widehat{\varrho})}$ is negative, hence $\int^{\infty} f(s)\mathrm{d}s = \infty$ implies that $\int^{\infty} W^{(-\widehat{\varrho})\prime}(a - f(s))\mathrm{d}s = \infty$.
We now prove the claim. Let t_0 be such that $f(s) < a/2$ for $s \geq t_0$ and define

$$N_t = \sum_{t_0 \leq s \leq t} \mathbf{1}_{\{\widehat{h}(e_s) > a - f(s)\}} \qquad t_0 < t$$

to be the number of points of the excursion process in the time interval $[t_0, t]$ whose absolute maximum exceeds $a - f(T)$. We know that N_t is a Poisson variable of parameter

$$\int_{t_x}^{t} \widehat{n}^{\diamond}(\widehat{h} > a - f(s))\mathrm{d}s = \int_{t_x}^{t} \frac{W^{(-\widehat{\varrho})\prime}(a - f(s))}{W^{(-\widehat{\varrho})}(a - f(s))}\mathrm{d}s.$$

Therefore, N_∞ is infinite a.s., that is $\widehat{M_{T_t}} > a - f(t)$ i.o. as $t \to \infty$ if $\int^{\infty} W^{(-\widehat{\varrho})\prime}(a-f(s))\mathrm{d}s = \infty$. Otherwise the last event is evanescent.

To get the results for $\widehat{M_t}$ recall from Corollary 4 that as $t \to \infty$, $T_t \sim t/\widehat{p}_0$ a.s. for some positive finite constant $\widehat{p}_0$. Observing that the integral criterion of the theorem remains unchanged when replacing f by $t \mapsto f(\lambda t)$, one easily deduces that the events $\{\widehat{M_t} > a - f(t)$ i.o. as $t \to \infty\}$ and $\{\widehat{M_{T_t}} > a - f(t)$ i.o. as $t \to \infty\}$ have the same probability.
(iii) Consider the previously defined Poisson point process K with characteristic measure $\mathrm{d}\widehat{R}$. Recall that the function R is C^1 and has a positive derivative in some interval $[0, \delta)$. As $a - M_{T_t} = \min\{K_s : s \leq t\}$, we find that, for t sufficiently large, say $t > t_1$ (hence $a - M_{T_t}$ suffiently close to 0), the path $(\widehat{R}(a - M_{T_t}), t \geq t_1)$ coincides with that of $(u_t, t \geq t_1)$, where u_t is the minimum on $[0, t]$ of a Poisson point process with characteristic measure the uniform distribution on $(0, 1)$.

The key step in the proof is now to appeal to the extremal process $(v_t, t > 0)$ defined in [21]. Such a process starting from $v_0 > 0$ has the same law as $((v_0 \wedge u_t), t \geq 0)$. For functions f as in the statement, Theorem 3 in [21] states that $v_t > f(t)$ infinitely often as $t \to \infty$ with probability 0 or 1 according whether $I_f(1)$ converges or diverges. Suppose that $I_f(1) = \infty$ then

$$\limsup_{t\to\infty} \frac{\widehat{R}(a - M_{T_t})}{f(t)} \geq 1 \qquad \text{a.s.}$$

Now pick $\lambda > 1$. Then we have

$$\begin{aligned}\limsup_{t\to\infty} \frac{\widehat{R}(a - M_{T_t})}{\lambda f(t)} &= \limsup_{t\to\infty} \frac{\widehat{R}(a - M_{T_{\lambda \widehat{p}_0 t}})}{f(t)} = \limsup_{t\to\infty} \frac{\widehat{R}'(0)(a - \widehat{M}_t)}{f(\lambda \widehat{p}_0 t)} \\ &= \limsup_{t\to\infty} \lambda \widehat{p}_0 \widehat{R}'(0) \frac{(a - \widehat{M}_t)}{f(t)},\end{aligned}$$

where the first inequality is due to Corollary 4 and the last equality to the slow variation of $t \mapsto tf(t)$. Recalling that $\widehat{p}_0 \widehat{R}'(0) = |\widehat{\varrho}'(a)|$ and letting $\lambda \to 1$, we obtain that $L_f \geq |\widehat{\varrho}'(a)|^{-1}$ a.s.

In the case where $I_f(1) < \infty$, one can prove the opposite inequality in the same way.

Pick $\lambda > \gamma_f$ (if $\gamma_f = \infty$ there is nothing to prove) and apply the foregoing result to λf: $I_{\lambda f}(1)$ converges so $L_f = \lambda L_{\lambda f}$ is bounded above a.s. by $\lambda |\widehat{\varrho}'(a)|^{-1}$. Letting $\lambda \to \gamma_f$, we get $L_f \leq \gamma_f / |\widehat{\varrho}'(a)|$ a.s. With $\lambda < \gamma_f$ (for $\gamma_f > 0$, otherwise there is nothing to prove), one can prove the opposite inequality. $\square$