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ON THE COMPRESSIBILITY OF OPERATORS IN WAVELET COORDINATES

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ABSTRACT. In [CDD00], Cohen, Dahmen and DeVore proposed an adaptive wavelet algorithm for solving operator equations. Assuming that the operator defines a boundedly invertible mapping between a Hilbert space and its dual, and that a Riesz basis of wavelet type for this Hilbert space is available, the operator equation can be transformed into an equivalent well-posed infinite matrix-vector system. This system is solved by an iterative method, where each application of the infinite stiffness matrix is replaced by an adaptive approximation. Assuming that the stiffness matrix is sufficiently compressible, i.e., that it can be sufficiently well approximated by sparse matrices, it was proven that the adaptive method has optimal computational complexity in the sense that it converges with the same rate as the best N-term approximations for the solution assuming it would be explicitly available. With the available results concerning compressibility however, this optimality was actually restricted to solutions with limited Besov regularity. In this paper we derive improved results concerning compressibility, which imply that with wavelets that have sufficiently many vanishing moments and that are sufficiently smooth, the adaptive wavelet method has optimal computational complexity independent of the regularity of the solution.

1. INTRODUCTION

As has been first observed in [BCR91], the stiffness matrix resulting from a Galerkin discretization of a singular integral operator, which using standard single scale bases is densely populated, turns out to be close to a sparse matrix when wavelet bases are exploited. Responsible for this phenomenon is that the kernel of such an operator is increasingly smooth away from the diagonal, and that the wavelets have vanishing moments. Quantitative analyses in [Sch98, DHS02, vPS97] show that with wavelets that have a sufficient number of vanishing moments, the stiffness matrix can be compressed to a sparse one, whose application requires only $\mathcal{O}(n)$ operations with n being the number of unknowns, whereas the order of convergence is maintained.

Compared to alternative approaches for compression as panel clustering ([HN89]) and multipole expansions ([GR87]), the wavelet approach has the additional advantage that properly scaled wavelets generate Riesz bases for a range of Sobolev spaces. So, in any case for strongly elliptic equations, if the operator defines a boundedly invertible mapping

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between a Sobolev space in this range and its dual, then the stiffness matrices with respect to the wavelet bases are well-conditioned uniformly in their sizes, allowing for a fast iterative solution. In summary, with suitable wavelets the discretization error accuracy can be realized in $\mathcal{O}(n)$ operations.

This Riesz basis property inspired Cohen, Dahmen en DeVore in [CDD01, CDD00] to go one step further. Instead of first discretizing the problem, i.e. replacing the underlying infinite dimensional space by some fixed finite dimensional one, and then solving the resulting finite dimensional system by some iterative method, they transformed the original problem into an equivalent well-posed infinite matrix-vector system. This system can be solved iteratively, where in each iteration the application of the infinite matrix has to be approximated. The main advantage of their approach is that in the course of the iteration the spaces in which the approximations are sought, which are always spanned by a finite linear combination of wavelets, will adapt to the solution in an optimal way. Because of this adaptivity, the method is attractive for both integral- and differential equations in variational form.

In the following we assume that the problem to be solved has the form

$$Lu = g,$$

where for some closed subspace $\mathcal{H} \subset H^t$, being a Sobolev space of order $t \in I\!R$, the linear operator $L : \mathcal{H} \to \mathcal{H}'$ is boundedly invertible, the right-hand side $g \in \mathcal{H}'$, and thus the unknown solution $u \in \mathcal{H}$. With $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$ being a Riesz basis for $\mathcal{H}$ of wavelet type, the equivalent infinite matrix-vector problem reads as

$$\mathbf{M}\mathbf{u} = \mathbf{g}, \tag{1.1}$$

where $\mathbf{M} := \langle \Psi, L\Psi \rangle : \ell_2(\Lambda) \to \ell_2(\Lambda)$ is boundedly invertible, $\mathbf{g} := \langle \Psi, g \rangle \in \ell_2(\Lambda)$, with $\langle\, ,\, \rangle$ denoting the duality product on $\mathcal{H} \times \mathcal{H}'$, and $u = \mathbf{u}^T \Psi$.

In [CDD01, CDD00], the quality of the proposed iterative method is assessed by comparing the $\ell_2(\Lambda)$-error of the current approximation having N coefficients with that of a *best N-term approximation* for $\mathbf{u}$, i.e., a vector $\mathbf{u}_N$ with at most N non-zero coefficients that has distance to $\mathbf{u}$ less or equal to that of any vector with a support of that size. Recall that since Ψ is a Riesz basis, the sizes of the error measured in $\ell_2(\Lambda)$ or H^t-metric differ at most a constant factor.

In any case for wavelets that are sufficiently smooth, the theory of non-linear approximation ([DeV98, Coh00]) learns that if *both*

$$0 < s < \tfrac{d-t}{n},$$

where n is the space dimension and d is the order of the wavelets, *and* u is in the Besov space $B^{sn+t}_\tau(L_\tau)$ with $\tau = (\frac{1}{2} + s)^{-1}$, then

$$\sup_{N \in I\!N} N^s \|\mathbf{u} - \mathbf{u}_N\| < \infty. \tag{1.2}$$

The potential here lies in the fact the condition involving Besov regularity is much milder that the corresponding condition $u \in H^{sn+t}$ involving Sobolev regularity that would be needed to guarantee the same rate of convergence with linear approximation in the spaces spanned by N wavelets on the coarsest scales. Indeed, assuming a sufficiently smooth right-hand side, for several boundary value problems it has been proved that the solution has a much higher Besov- than Sobolev regularity ([DD97, Dah99]). Note that a rate higher than $\frac{d-t}{n}$ can never be expected with wavelets of order d, except when the solution u happens to be a finite linear combination of wavelets.

Returning to the adaptive wavelet algorithm from [CDD00], besides a clean-up step that is applied after every K iterations to remove small coefficients in order to control the vector length, the other crucial ingredient is the adaptive way in which in each iteration the application of the infinite matrix $\mathbf{M}$ to a finitely supported vector is approximated. Given such a vector, before multiplication each column of $\mathbf{M}$ that corresponds to a non-zero entry in this vector is replaced by a finitely supported approximation within a tolerance that decreases as function of the size of this coefficient. To be able to prove results about complexity, information is needed about the number of entries that are necessary to approximate any column within some given tolerance. We recall the following definition from [CDD00]:

Definition 1.1. $\mathbf{M}$ is called *s^*-compressible*, when for each $j \in I\!N$ there exists an infinite matrix $\tilde{\mathbf{M}}_j$ with at most $\alpha_j 2^j$ non-zero entries in each row and column with $\sum_{j\in I\!N} \alpha_j < \infty$, such that for any $s < s^*$, $\sum_{j\in I\!N} 2^{js}\|\mathbf{M} - \tilde{\mathbf{M}}_j\| < \infty$.

An equivalent definition is obtained by requiring that for any $s < s^*$ and any $N \in I\!N$, there exists a matrix on distance of order N^{-s} having at most N non-zeros in each row and column.

The main theorem from [CDD00] now says that if (1.2) is valid for some s, and $\mathbf{M}$ is s^*-compressible with $s^* > s$, then the number of arithmetic operations and storage locations used by the adaptive wavelet algorithm for computing an approximation for $\mathbf{u}$ within tolerance ϵ is of the order $\epsilon^{-1/s}$. Since in view of (1.2) the same order of operations is already needed to approximate $\mathbf{u}$ within this tolerance assuming that all its entries would be explicitly known, this result shows that this solution method has *optimal computational complexity.*

It remains to determine the value of s^*. First of all, note that even for a differential operator, $\mathbf{M}$ itself is not sparse. Indeed, any two wavelets ψ_λ, $\psi_{\lambda'}$ from the infinite collection with $\mathrm{vol}(\mathrm{supp}\,\psi_\lambda \cap \mathrm{supp}\,\psi_\lambda) > 0$ give rise to a generally non-zero entry. Furthermore, in contrast to the non-adaptive setting, here we do not have the possibility for the matrix-vector multiplication to switch to a single-scale representation, which for differential operators would be sparse. On the other hand, it can be shown that for wavelets that both have vanishing moments and have some global smoothness, the modulus of an entry decreases with increasing distance in scale of the involved wavelets. Assuming that for some $\sigma > 0$, L and its adjoint L' are bounded from $H^{t+\sigma} \to H^{-t+\sigma}$, by substituting the estimates [Dah97, (9.4.5), (9.4.8)] into [CDD01, Proposition 6.6.2] we infer that $\mathbf{M}$ is s^*-compressible with

$$s^* = \frac{\min\{t + \tilde{d}, \sigma, \gamma - t\}}{n} - \frac{1}{2}, \tag{1.3}$$

where $\tilde{d}$ is the order of the dual wavelets, i.e., the number of vanishing moments, or more generally, the order of cancellation properties of the primal wavelets, and $\gamma = \sup_s\{\Psi \subset$

$H^s\}$ (here we used that the condition $\sigma < t + \tilde{\gamma}$ imposed for [Dah97, (9.4.8)] can actually be relaxed to $\sigma \leq t + \tilde{d}$). This result holds true for differential operators as well as for singular integral operators. Note that in contrast to the non-adaptive setting discussed at the beginning of this introduction, global smoothness of the wavelets is required.

The result (1.3) however is not satisfactory. Indeed, since $\gamma < d$ and so $s^* \leq \frac{\gamma - t}{n} - \frac{1}{2} < \frac{d-t}{n}$, on basis of (1.3) optimal computational complexity of the adaptive wavelet method can be concluded only for solutions u that have limited Besov regularity. Indeed, when $u \in B^{sn+t}_\tau(L_\tau)$ with $\tau = (\frac{1}{2} + s)^{-1}$ and $s > s^*$, then the best N-term approximations converge at a rate higher than can be shown for the approximations yielded by the adaptive wavelet method.

For the special case of L being the Laplace operator and spline wavelets, in [BBC+01, DDU01] it has been proved that $s^* = \frac{\gamma - t}{n}$, which however is still less than $\frac{d-t}{n}$.

The goal of this paper is to prove that for both differential as singular integral operators, for wavelets that are sufficiently smooth and have cancellation properties of sufficiently high order, $\mathbf{M} = \langle \Psi, L\Psi \rangle$ is actually s^*-compressible with $s^* > \frac{d-t}{n}$, showing that the adaptive wavelet method has optimal computational complexity independent of the regularity of u. The key to obtain this improved result is that on essential places we estimate directly norms of blocks of the matrix $\mathbf{M}$, instead of deriving such estimates in terms of the sizes of the individual entries via the Schur lemma.

This paper is organized as follows: In §2 we prove s^*-compressibility with $s^* > \frac{d-t}{n}$ for differential operators on a domain.

In §3 we prove this result for a class of singular integral operators on a manifold, which includes operators resulting from applying the boundary integral method. Since the regularity of the manifold imposes a limit to the smoothness of the wavelets, depending on the other parameters it may restrict the compressibility. In §3.3, we give a general proof of a decay estimate for entries corresponding to wavelets with disjoint supports, which so far was only shown in specific situations. In §3.4, we prove a new decay estimate for entries corresponding to wavelets, or more generally, corresponding to a linear combination of wavelets and another wavelet, that may have overlapping supports, but for which the support of the linear combination has empty intersection with the singular support of the other wavelet. In contrast to available estimates from [Sch98, DHS02], this estimate benefits from global smoothness of the wavelets. Apart from its use in the analysis of adaptive schemes, this estimate also may result in quantitatively better compression rates when applied in the analysis of non-adaptive schemes.

At the end of this introduction, we fix some notations. We always think of the space L_2 of all measurable square integrable functions on a domain Ω or manifold Γ as being equipped with the *standard* scalar product $\langle\,,\,\rangle$ and corresponding norm $\|\,\|$, defined by $\langle u, v \rangle = \int_\Omega u(x)v(x)dx$ or $\langle u, v \rangle = \int_\Gamma u(x)v(x)d\mu(x)$, with $d\mu$ being the induced Lebesgue measure.

For H being a Hilbert space embedded in L_2 and any $u \in L_2$, the mapping $v \mapsto \langle v, u \rangle$ is continuous on H. This procedure defines an embedding of L_2 into H', or equivalently, it fixes an interpretation of a function in L_2 as a functional in H'. Different scalar products on L_2, defining equivalent norms on L_2, give rise to different embeddings of L_2 into H', that may lead to non-equivalent H'-norms of L_2-functions (cf. [NS01, §4]). This observation, together with the fact that on a few places in the wavelet literature non-standard L_2-scalar are applied is the reason here to emphasize our choice of the L_2-scalar product.

If H is dense in L_2, then the above embedding of L_2 into H' is dense, meaning that the L_2-scalar product restricted to $H \times L_2$ has an unique extension to the duality product on $H \times H'$. For such an H, without risk of confusion, we may use $\langle\,,\,\rangle$ to denote either product.

For any countable index set Λ, the notations $\langle\,,\,\rangle$ and $\|\;\|$ will also be used to denote the standard scalar product and norm, as well as the resulting operator norm, on the space $\ell_2(\Lambda)$ of square summable scalar sequences.

Finally, in order to avoid the repeated use of generic but unspecified constants, by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2. Compressibility of differential operators

For some domain $\Omega \subset \mathbb{R}^n$, $t \in \mathbb{N}_0$ and $\Gamma_D \subset \partial\Omega$, possibly with $\Gamma_D = \emptyset$, let

$$H^t_{0,\Gamma^D}(\Omega) = \operatorname{clos}_{H^t(\Omega)}\{u \in H^t(\Omega) \cap C^\infty(\Omega) : \operatorname{supp} u \cap \Gamma^D = \emptyset\},$$

and let $L : H^t_{0,\Gamma^D}(\Omega) \to (H^t_{0,\Gamma^D}(\Omega))'$ be defined by

$$\langle u, Lv\rangle = \sum_{|\alpha|,|\beta| \leq t} \langle \partial^\alpha u, a_{\alpha\beta}\partial^\beta v\rangle,$$

where $a_{\alpha\beta} \in L_\infty(\Omega)$ so that L is bounded. Obviously L as an extension, that we will also denote by L, as a bounded operator from $H^t(\Omega) \to H^{-t}(\Omega)$. For completeness, $H^s(\Omega)$ for $s < 0$ denotes the dual of $H^{-s}(\Omega)$.

We assume that there exists a $\sigma > 0$, such that

$$L, L' : H^{t+\sigma}(\Omega) \to H^{t-\sigma}(\Omega) \qquad \text{are bounded.}$$

Sufficient is that for arbitrary $\varepsilon > 0$, and all α, β with $\min\{|\alpha|, |\beta|\} > m - \sigma$, it holds that $a_{\alpha\beta} \in \begin{cases} W^{\sigma-m+\min\{|\alpha|,|\beta|\}}_\infty(\Omega) & \text{when } \sigma \in \mathbb{N}, \\ C^{\sigma-m+\min\{|\alpha|,|\beta|\}+\varepsilon}(\Omega) & \text{when } \sigma \notin \mathbb{N}. \end{cases}$

Let

$$\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$$

be a *Riesz basis for* $H^t_{0,\Gamma^D}(\Omega)$ of wavelet type. The index λ encodes both the level, denoted by $|\lambda| \in \mathbb{N}_0$, and the location of the wavelet ψ_λ.

We assume that the wavelets are *local*, in the sense that

$$\operatorname{diam}(\operatorname{supp}\psi_\lambda) \lesssim 2^{-|\lambda|} \quad \text{and} \quad \sup_{x\in\Omega,\ell\in\mathbb{N}_0} \#\{|\lambda| = \ell : x \in \operatorname{supp}\psi_\lambda\} < \infty,$$

and that they are *piecewise smooth*, with which we mean that $\operatorname{supp}\psi_\lambda \backslash \operatorname{sing supp}\psi_\lambda$ is the disjoint union of m open "uniformly Lipschitz" domains $\Xi_{\lambda,1},\ldots,\Xi_{\lambda,m}$, with $\cup_{i=1}^m \overline{\Xi_{\lambda,i}} = \operatorname{supp}\psi_\lambda$, and that $\psi_\lambda|_{\Xi_{\lambda,i}}$ is smooth with

$$\sup_{x\in\Xi_{\lambda,i}} |\partial^\beta \psi_\lambda(x)| \lesssim 2^{(|\beta|+\frac{n}{2}-t)|\lambda|}, \quad (\beta \in I\!N_0^n). \tag{2.1}$$

Remark 2.1. Precisely, we will call a collection domains $\{A_\nu\} \subset I\!R^n$ to be uniformly Lipschitz domains, when there exist affine mappings B_ν with $|DB_\nu| \lesssim \operatorname{vol}(A_\nu)^{-1}$ and $|(DB_\nu)^{-1}| \lesssim \operatorname{vol}(A_\nu)$, such that the sets $B_\nu(A_\nu)$ satisfy the condition of "minimal smoothness" ([Ste70, Ch.VI, §3]) with uniform parameters "ε", "N" and "M". Examples are given by, the interiors of, "non-degenerate" polygons.

For minimally smooth domains it is known ([Ste70]) that there exist universal extension operators, with Sobolev norms only dependent on the aforementioned parameters. So in particular, by first transforming $\psi_\lambda|_{\Xi_{\lambda,i}}$ using such an affine mapping to a function on a minimally smooth domain with volume one, then making the extension, and finally applying the inverse transformation, by (2.1) we conclude that there exists a smooth function $\varphi_{\lambda,i}$ on $I\!R^n$, equal to ψ_λ on $\Xi_{\lambda,i}$, with for any $s \geq 0$, $p \in [1,\infty]$,

$$\|\varphi_{\lambda,i}\|_{W_p^s(I\!R^n)} \lesssim 2^{(s-t+\frac{n}{2}-\frac{n}{p})|\lambda|}.$$

Furthermore, we assume that there exist $\gamma > t$, $\tilde{d} > -t$ such that for $r \in [-\tilde{d},\gamma)$, $s < \gamma$,

$$\|\cdot\|_{H^r(\Omega)} \lesssim 2^{\ell(r-s)}\|\cdot\|_{H^s(\Omega)}, \qquad \text{on } W_\ell := \operatorname{span}\{\psi_\lambda : |\lambda| = \ell\}. \tag{2.2}$$

Remark 2.2. It is known that the above wavelet assumptions are satisfied by biorthogonal wavelets when the primal and dual spaces have regularity indices $\gamma > \max\{0,t\}$, $\tilde{\gamma} > \max\{0,-t\}$ and orders $d > \gamma$, $\tilde{d} > \tilde{\gamma}$ respectively (cf. [Dah96, DS99c]), the primal spaces consist of "piecewise" smooth functions, and finally, no boundary conditions are imposed on the dual spaces ("complementary boundary conditions", see [DS98]). In particular, (2.2) for $r \in [-\tilde{d},-\tilde{\gamma}]$ can be deduced from the lines following (A.2) in [DS99c]. In Remark 2.5 we will comment on the case when the dual spaces satisfy the same boundary conditions as the primal ones.

Theorem 2.3. *Let* $\mathbf{M} = \langle\Psi, L\Psi\rangle$. *For* $j \in I\!N$, *and with* $k(j,n) := \begin{cases} \frac{j}{n-1} & \textit{when } n > 1, \\ 2^j & \textit{when } n = 1, \end{cases}$ *we define the infinite matrix* $\mathbf{M}_j$ *by replacing all entries* $\mathbf{M}_{\lambda,\lambda'} = \langle\psi_\lambda, L\psi_{\lambda'}\rangle$ *by zeros when*

$$\big||\lambda| - |\lambda'|\big| > k(j,n), \quad \textit{or} \tag{2.3}$$

$$\big||\lambda| - |\lambda'|\big| > \tfrac{j}{n} \quad \textit{and for some } 1 \leq i \leq m, \begin{cases} \operatorname{supp}\psi_\lambda \subset \overline{\Xi_{\lambda',i}} & \textit{when } |\lambda| > |\lambda'|, \\ \operatorname{supp}\psi_{\lambda'} \subset \overline{\Xi_{\lambda,i}} & \textit{when } |\lambda| < |\lambda'|. \end{cases} \tag{2.4}$$

Then the number of non-zero entries in each row and column of $\mathbf{M}_j$ *is of order* 2^j, *and for any*

$$s \leq \min\{\tfrac{t+\tilde{d}}{n}, \tfrac{\sigma}{n}\}, \textit{ with } s < \tfrac{\gamma-t}{n-1} \textit{ when } n > 1,$$

it holds that $\|\mathbf{M} - \mathbf{M}_j\| \lesssim 2^{-sj}$.

Remark 2.4. In view of Definition 1.1, by taking $\tilde{\mathbf{M}}_j = \mathbf{M}_{\lceil j+\log(\alpha_j)\rceil}$, with for example $\alpha_j = j^{-(1+\varepsilon)}$ for some $\varepsilon > 0$, we infer that $\mathbf{M}$ is s^*-compressible with

$$s^* = \min\{\tfrac{t+\tilde{d}}{n}, \tfrac{\sigma}{n}, \tfrac{\gamma-t}{n-1}\}$$

($s^* = \min\{t+\tilde{d}, \sigma\}$ when $n = 1$). So, with d being the order of the wavelets, if $\tilde{d} > d - 2t$, $\sigma > d - t$ and when $n > 1$, $\frac{\gamma-t}{n-1} > \frac{d-t}{n}$, then indeed $s^* > \frac{d-t}{n}$. The condition involving γ when $n > 1$ is satisfied for instance when $\frac{d-t}{n} > \frac{1}{2}$ and $\gamma = d - \frac{1}{2}$ (spline wavelets).

Proof of Theorem 2.3. Let λ be some given index. By the locality of the wavelets, the number of indices λ' with fixed $|\lambda'|$ for which $\operatorname{vol}(\operatorname{supp}\psi_{\lambda'} \cap \operatorname{supp}\psi_\lambda) > 0$ is of order $\max\{1, 2^{(|\lambda'|-|\lambda|)n}\}$. By using in addition the piecewise smoothness of the wavelets, the number of indices λ' with fixed $|\lambda'| > |\lambda|$ for which $\operatorname{vol}(\operatorname{supp}\psi_{\lambda'} \cap \operatorname{supp}\psi_\lambda) > 0$ and $\operatorname{supp}\psi_{\lambda'}$ is not contained in some $\overline{\Xi_{\lambda,i}}$ is of order $2^{(|\lambda'|-|\lambda|)(n-1)}$. We conclude that the number of non-zero entries in the λth row and column of $\mathbf{M}_j$ is of order

$$\sum_{||\lambda'|-|\lambda|| \leq \frac{j}{n}} \max\{1, 2^{(|\lambda'|-|\lambda|)n}\} + \sum_{\frac{j}{n} < ||\lambda'|-|\lambda|| \leq k(j,n)} \max\{1, 2^{(|\lambda'|-|\lambda|)(n-1)}\} \eqsim 2^j.$$

Let $\hat{\mathbf{M}}_j$ be defined by $(\mathbf{M} - \hat{\mathbf{M}}_j)_{\lambda,\lambda'} = \begin{cases} \mathbf{M}_{\lambda,\lambda'} & \text{when } \big||\lambda| - |\lambda'|\big| > k(j,n), \\ 0 & \text{otherwise.} \end{cases}$ The continuity assumptions on L, L', together with (2.2) show that for

$$r \in (0, t+\tilde{d}] \cap (0,\sigma] \cap (0, \gamma - t),$$

and $w_\ell \in W_\ell$, $w_{\ell'} \in W_{\ell'}$,

$$\begin{aligned} |\langle w_\ell, Lw_{\ell'}\rangle| &\lesssim \|w_\ell\|_{H^{t-r}(\Omega)} \|Lw_{\ell'}\|_{H^{-t+r}(\Omega)} \\ &\lesssim \|w_\ell\|_{H^{t-r}(\Omega)} \|w_{\ell'}\|_{H^{t+r}(\Omega)} \lesssim 2^{r(\ell'-\ell)} \|w_\ell\|_{H^t(\Omega)} \|w_{\ell'}\|_{H^t(\Omega)}, \end{aligned} \tag{2.5}$$

and analogously, $|\langle w_\ell, Lw_{\ell'}\rangle| = |\langle L'w_\ell, w_{\ell'}\rangle| \lesssim 2^{r(\ell-\ell')} \|w_\ell\|_{H^t(\Omega)} \|w_{\ell'}\|_{H^t(\Omega)}$. So for arbitrary $\mathbf{c}, \mathbf{d} \in \ell_2(\Lambda)$, we have

$$\begin{aligned} |\langle \mathbf{c}, (\mathbf{M} - \hat{\mathbf{M}}_j)\mathbf{d}\rangle| &= \Big| \sum_{|\ell-\ell'|>k(j,n)} \langle \sum_{|\lambda|=\ell} \mathbf{c}_\lambda \psi_\lambda, L \sum_{|\lambda'|=\ell'} \mathbf{d}_{\lambda'} \psi_{\lambda'} \rangle \Big| \\ &\lesssim \sum_{|\ell-\ell'|>k(j,n)} 2^{-r|\ell-\ell'|} \Big\| \sum_{|\lambda|=\ell} \mathbf{c}_\lambda \psi_\lambda \Big\|_{H^t(\Omega)} \Big\| \sum_{|\lambda'|=\ell'} \mathbf{d}_{\lambda'} \psi_{\lambda'} \Big\|_{H^t(\Omega)} \\ &\lesssim 2^{-k(j,n)r} \sqrt{\sum_\ell \Big\| \sum_{|\lambda|=\ell} \mathbf{c}_\lambda \psi_\lambda \Big\|^2_{H^t(\Omega)}} \sqrt{\sum_{\ell'} \Big\| \sum_{|\lambda'|=\ell'} \mathbf{d}_{\lambda'} \psi_{\lambda'} \Big\|^2_{H^t(\Omega)}} \\ &\eqsim 2^{-k(j,n)r} \|\mathbf{c}\| \|\mathbf{d}\|, \end{aligned}$$

or $\|\mathbf{M} - \hat{\mathbf{M}}_j\| \lesssim 2^{-k(j,n)r}$.

Finally, we analyze the error as a consequence of dropping entries with indices that satisfy criterion (2.4). For $\lambda \in \Lambda$, $1 \leq i \leq m$ and $\ell > |\lambda'|$, let

$$A_{\ell,\lambda',i} := \{|\lambda| = \ell : \operatorname{supp}\psi_\lambda \subset \overline{\Xi_{\lambda',i}}\}.$$

For $w_{\ell,\lambda',i} \in \operatorname{span}\{\psi_\lambda : \lambda \in A_{\ell,\lambda',i}\} \subset W_\ell$ and

$$q \in (0, t+\tilde{d}] \cap (0,\sigma],$$

we will prove that

$$|\langle w_{\ell,\lambda',i}, L\psi_{\lambda'}\rangle| \lesssim 2^{q(|\lambda'|-\ell)} \|w_{\ell,\lambda',i}\|_{H^t(\Omega)}. \tag{2.6}$$

Because of (2.5), it is sufficient to consider $q \geq \gamma - t \geq -t$. From Remark 2.1, recall that $\varphi_{\lambda',i}$ is the extension of $\psi_{\lambda'}|_{\Xi_{\lambda',i}}$ to a smooth function on $I\!R^n$ with for $s \geq 0$, $\|\varphi_{\lambda',i}\|_{H^s(I\!R^n)} \lesssim 2^{(s-t)|\lambda'|}$. From the locality and continuity of L, we conclude that

$$\begin{aligned}|\langle w_{\ell,\lambda',i}, L\psi_{\lambda'}\rangle| &= |\langle w_{\ell,\lambda',i}, L\varphi_{\lambda',i}\rangle| \lesssim \|w_{\ell,\lambda',i}\|_{H^{t-q}(\Omega)} \|L\varphi_{\lambda',i}\|_{H^{-t+q}(\Omega)}\\ &\lesssim \|w_{\ell,\lambda',i}\|_{H^{t-q}(\Omega)} \|\varphi_{\lambda',i}\|_{H^{t+q}(\Omega)} \lesssim 2^{-q(\ell-|\lambda'|)} \|w_{\ell,\lambda',i}\|_{H^t(\Omega)}.\end{aligned}$$

For any $\mathbf{c}, \mathbf{d} \in \ell_2(\Lambda)$ and $q \in (0, t+\tilde{d}] \cap (0,\sigma]$, from (2.6) we have

$$\begin{aligned}&|\sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)} \sum_{|\lambda'|=\ell'} \mathbf{d}_{\lambda'} \langle \sum_{i=1}^m \sum_{\lambda\in A_{\ell,\lambda',i}} \mathbf{c}_\lambda \psi_\lambda, L\psi_{\lambda'}\rangle|\\ &\lesssim \sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)} \sum_{|\lambda'|=\ell'} |\mathbf{d}_{\lambda'}| 2^{-q(\ell-\ell')} \sum_{i=1}^m \|\sum_{\lambda\in A_{\ell,\lambda',i}} \mathbf{c}_\lambda\psi_\lambda\|_{H^t(\Omega)}\\ &\lesssim \sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)} 2^{-q(\ell-\ell')} \sqrt{\sum_{|\lambda'|=\ell'} |\mathbf{d}_{\lambda'}|^2} \sqrt{\sum_{|\lambda'|=\ell'} \left(\sum_{i=1}^m \sqrt{\sum_{\lambda\in A_{\ell,\lambda',i}} |\mathbf{c}_\lambda|^2}\right)^2}\\ &\lesssim \sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)} 2^{-q(\ell-\ell')} \sqrt{\sum_{|\lambda'|=\ell'} |\mathbf{d}_{\lambda'}|^2} \sqrt{\sum_{|\lambda|=\ell} |\mathbf{c}_\lambda|^2} \lesssim 2^{-\frac{j}{n}q} \|\mathbf{d}\|\|\mathbf{c}\|,\end{aligned}$$

where for the last line we have used that for fixed $|\lambda'|$, each λ is contained in at most a uniformly bounded number of sets $A_{|\lambda|,\lambda',i}$.

Since, analogous to (2.6), $|\langle \psi_\lambda, Lw_{\ell',\lambda,i}\rangle| = |\langle L'\psi_\lambda, w_{\ell',\lambda,i}\rangle| \lesssim 2^{-q(\ell'-|\lambda|)} \|w_{\ell',\lambda,i}\|_{H^t(\Omega)}$ when $w_{\ell',\lambda,i} \in \operatorname{span}\{\psi_{\lambda'} : \lambda' \in A_{\ell',\lambda,i}\}$, and so

$$|\sum_{\frac{j}{n}<\ell'-\ell\leq k(j,n)} \sum_{|\lambda|=\ell} \mathbf{c}_\lambda \langle \psi_\lambda, L \sum_{i=1}^m \sum_{\lambda'\in A_{\ell',\lambda,i}} \mathbf{d}_{\lambda'}\psi_{\lambda'}\rangle| \lesssim 2^{-\frac{j}{n}q} \|\mathbf{c}\|\|\mathbf{d}\|,$$

we conclude that $\|\hat{\mathbf{M}}_j - \mathbf{M}_j\| \lesssim 2^{-\frac{j}{n}q}$.

A combination of the estimates for $\mathbf{M}-\hat{\mathbf{M}}_j$ and $\hat{\mathbf{M}}_j-\mathbf{M}_j$ shows that for $s \leq \min\{\frac{t+\tilde{d}}{n}, \frac{\sigma}{n}\}$, with $s < \frac{\gamma-t}{n-1}$ when $n > 1$, it holds that $\|\mathbf{M}-\mathbf{M}_j\| \lesssim 2^{-sj}$. □

Remark 2.5. Let us consider the situation that $t \in I\!N$, $\Gamma_D \neq \emptyset$, and that Ψ is a biorthogonal basis for $H^t_{0,\Gamma_D}(\Omega)$, where now also the dual spaces satisfy homogeneous Dirichlet boundary conditions on Γ_D. Then since for $s \geq 0$, $H^s(\Omega) \cap H^t_{0,\Gamma_D}(\Omega)$ is only dense in $H^s(\Omega)$ when

$s < \frac{1}{2}$, we can expect (2.2) only for $r \in [-\tilde{d}, \gamma)$, $s < \gamma$ with $r > -\frac{1}{2}$. As a consequence, in the proof of Theorem 2.3, the range of r for which (2.5) holds is restricted to $r \in (0, t + \tilde{d}] \cap (0, \sigma] \cap (0, \gamma - t) \cap (0, t + \frac{1}{2})$.

The same problems are encountered for proving (2.6). However, instead of restricting the range of q, here another solution is possible. The homogeneous Dirichlet boundary conditions on the dual spaces only affect wavelets with supports near Γ_D. More precisely, one can expect that there exists a constant $\theta > 0$, such that for any $r \in [-\tilde{d}, \gamma)$, $s < \gamma$,

$$\|\cdot\|_{H^r(\Omega)} \lesssim 2^{\ell(r-s)} \|\cdot\|_{H^s(\Omega)} \quad \text{on } \operatorname{span}\{\psi_\lambda : |\lambda| = \ell, \operatorname{dist}(\operatorname{supp}\psi_\lambda, \Gamma_D) \geq \theta 2^{-|\lambda|}\}. \tag{2.7}$$

Let us now add to the dropping criterium (2.4) the condition that $\operatorname{dist}(\operatorname{supp}\psi_\lambda, \Gamma_D) \geq \theta 2^{-|\lambda|}$ when $|\lambda| > |\lambda'|$, or $\operatorname{dist}(\operatorname{supp}\psi_{\lambda'}, \Gamma_D) \geq \theta 2^{-|\lambda'|}$ when $|\lambda| < |\lambda'|$. Then it is easily verified that the resulting $\mathbf{M}_j$, although a little bit less sparse, still has at most order 2^j non-zero entries in each row and column. On the other hand, changing the definition of $A_{\ell,\lambda',i}$ into

$$A_{\ell,\lambda',i} := \{|\lambda| = \ell : \operatorname{supp}\psi_\lambda \subset \overline{\Xi_{\lambda',i}}, \operatorname{dist}(\operatorname{supp}\psi_\lambda, \Gamma_D) \geq \theta 2^{-|\lambda|}\},$$

for $w_\ell \in \operatorname{span}\{\psi_\lambda : \lambda \in A_{\ell,\lambda',i}\}$ and $q \in (0, t + \tilde{d}] \cap (0, \sigma)$, using (2.7) we can prove that $|\langle w_{\ell,\lambda',i}, L\psi_{\lambda'}\rangle| \lesssim 2^{-q(\ell - |\lambda'|)} \|w_{\ell,\lambda',i}\|_{H^t(\Omega)}$. By copying the remainder of the proof, we conclude that for

$$s \leq \min\{\tfrac{t+\tilde{d}}{n}, \tfrac{\sigma}{n}\}, \text{ with } s < \min\{\tfrac{\gamma - t}{n-1}, \tfrac{t+\frac{1}{2}}{n-1}\} \text{ when } n > 1,$$

it holds that $\|\mathbf{M} - \mathbf{M}_j\| \lesssim 2^{-sj}$.

Remark 2.6. As follows from Remark 2.4, to show s^*-compressibility with $s^* > \frac{d-t}{n}$, it will be necessary that both $\tilde{d}$ and γ increase linearly as function of d. To benefit from an often much higher regularity of the solution in Besov than in Sobolev scale, we are interested to apply the adaptive method with a relatively large value of $d - t$. Indeed, for small $d - t$, the adaptive method can give at most a small improvement in the order of convergence compared to non-adaptive methods, which in practice might not compensate for the overhead it requires.

Unfortunately, on general, non tensor product domains, smooth wavelets, i.e., with large values of γ, are hard to construct. The approach from [DS99b] based on a non-overlapping domain decomposition yields wavelet bases that in principal for any d satisfy all requirements concerning smoothness and cancellation properties to obtain $s^* > \frac{d-t}{n}$. Yet, since suitable extension operators from one subdomain into neighboring subdomains enter the construction, it seems not easy to implement. Other approaches based on non-overlapping domain decomposition ([DS99a, CTU99, CM00]) yield wavelets which over the interfaces between subdomains are only continuous. Note that although for non-adaptive wavelet methods the fact that wavelets along some lower-dimensional interface are less smooth or have reduced cancellation properties might not influence the overall complexity-accuracy balance, for adaptive methods it generally does. Indeed, without assuming more than membership of a Besov space on the "critical line", i.e., a space that is only just embedded in $H^t_{0,\Gamma_D}(\Omega)$, it might happen that the solution is smooth everywhere except exactly along that interface, meaning that the adaptive method mainly produces coefficients corresponding to wavelets with degenerated properties. Also finite element wavelets as constructed in [DS99c, CES00, Ste00] are only continuous. For example for $t = 1$ and $n = 2$, with continuous wavelets only for orders $d \leq 2$, $s^* \geq \frac{d-t}{n}$ can be shown.

Still thinking of a non-overlapping decomposition of the domain into a number of subdomains or patches $\Omega_1, \ldots, \Omega_M$, as an alternative it seems not too difficult, in any case if one refrains from having local dual wavelets which are not needed here anyway, to construct wavelets of some given order d, which restricted to each patch are again wavelets characterized by parameters $\gamma \leq d - \frac{1}{2}$ and $\tilde{d}$ ('*patchwise*' smoothness and cancellation properties) that can be chosen at ones convenience. If in addition, these wavelets have a sufficient global smoothness such that they generate a Riesz basis for $H^t_{0,\Gamma_D}(\Omega)$, then $\langle \Psi, L\Psi \rangle = \sum_q \langle \Psi|_{\Omega_q}, L\Psi|_{\Omega_q} \rangle$. Theorem 2.3 now directly applies to the matrices in the sum with conditions in terms of the 'local' γ and $\tilde{d}$, and so when these are sufficiently large, $s^* > \frac{d-t}{n}$ follows.

Finally, in [Ste02] we generalized the adaptive wavelet method from [CDD00] to the case that Ψ is a *frame* for $H^t_{0,\Gamma_D}(\Omega)$ instead of a Riesz basis. Writing the domain as an *overlapping* union of subdomains Ω_q, a suitable frame Ψ is given by $\cup_{q=1}^M \omega_q \Psi_q$, where Ψ_q is a Riesz basis of wavelet type order d for a Sobolev space of order t on Ω_q, and ω_q a smooth weight function that vanishes at the internal boundary of Ω_q. As in the Riesz basis case, the optimal computational complexity of this adaptive solution method was proven when (1.2) is valid for some s, necessarily with $s \leq \frac{d-t}{n}$, and $\mathbf{M} := \langle \Psi, L\Psi \rangle = \sum_{q,q'} \langle \omega_q \Psi_q, L\omega_{q'} \Psi_{q'} \rangle$, or equivalently, any of these matrices in the sum are s^*-compressible for some $s^* > s$. Since both the presence of smooth weight functions and the fact that for $q \neq q'$, Ψ_q and $\Psi_{q'}$ are different are harmless, Theorem 2.3 directly applies to all these matrices. Indeed, note that we have not used that the wavelets are piecewise smooth with respect to partitions that are nested as function of the level. The advantage of this frame approach is that smoothness requirements on the wavelets Ψ_q are easily satisfied, since this construction requires no linking of functions from different subdomains over interfaces, and so $s^* > \frac{d-t}{n}$ can easily be realized.

3. Compressibility of boundary integral operators

3.1. **Definitions and main result.** For some $\mu \in I\!N$, let Γ be a patchwise smooth, compact n-dimensional, globally $C^{\mu-1,1}$ manifold in $I\!R^{n+1}$. Following [DS99b], we assume that $\Gamma = \cup_{q=1}^M \overline{\Gamma_q}$, with $\Gamma_q \cap \Gamma_{q'} = \emptyset$ when $q \neq q'$, and that for each $1 \leq q \leq M$, there exists

- a domain $\Omega_q \subset I\!R^n$ and a C^∞-parametrization $\kappa_q : I\!R^n \to I\!R^{n+1}$ with $\mathrm{Im}(\kappa_q|_{\Omega_q}) = \Gamma_q$,
- a domain $I\!R^n \supset \hat{\Omega}_q \supset\supset \Omega_q$, and an extension of $\kappa_q|_{\Omega_q}$ to a $C^{\mu-1,1}$ parametrization $\hat{\kappa}_q : \hat{\Omega}_q \to \mathrm{Im}(\hat{\kappa}_q) \subset \Gamma$.

For $|s| \leq \mu$, the Sobolev spaces $H^s(\Gamma)$ are well-defined, where for $s < 0$, $H^s(\Gamma)$ is the dual of $H^{-s}(\Gamma)$. For some $|t| \leq \mu$, let L be a bounded operator from $H^t(\Gamma) \to H^{-t}(\Gamma)$, where in this section we have in mind a singular integral operator of order $2t$. Let

$$\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$$

be a *Riesz basis for* $H^t(\Gamma)$ of wavelet type.

We assume that the wavelets are *local*, in the sense that

$$\operatorname{diam}(\operatorname{supp}\psi_\lambda) \lesssim 2^{-|\lambda|} \quad \text{and} \quad \sup_{x\in\Gamma,\ell\in\mathbb{N}_0} \#\{|\lambda| = \ell : x \in \operatorname{supp}\psi_\lambda\} < \infty,$$

and that they are *piecewise smooth*, with which we mean that $\operatorname{supp}\psi_\lambda \backslash \operatorname{sing\,supp}\psi_\lambda$ is the disjoint union of sets $\Xi_{\lambda,1}, \ldots, \Xi_{\lambda,m}$, with $\cup_{i=1}^m \overline{\Xi_{\lambda,i}} = \operatorname{supp}\psi_\lambda$, such that each $\Xi_{\lambda,i}$ is contained in some Γ_q, $\kappa_q^{-1}(\Xi_{\lambda,i})$ are uniformly Lipschitz domains, and $(\psi_\lambda \circ \kappa_q)|_{\kappa_q^{-1}(\Xi_{\lambda,i})}$ is smooth with

$$\sup_{\xi\in\kappa_q^{-1}(\Xi_{\lambda,i})} |\partial^\beta(\psi_\lambda \circ \kappa_q)(\xi)| \lesssim 2^{(|\beta|+\frac{n}{2}-t)|\lambda|}, \quad (\beta \in N_0^n). \tag{3.1}$$

We assume that the wavelets on levels $\ell > 0$ have the *cancellation property of order* $\tilde{d} \in \mathbb{N}$. That is, there exists an $\eta > 0$, and for all $\ell \in \mathbb{N}$ a linear mapping P_ℓ, such that for all sufficiently smooth functions v on Γ and any $x \in \Gamma$,

$$\left\{ \begin{array}{rcl} \langle v, \psi_\lambda \rangle & = & \langle (I - P_{|\lambda|})v, \psi_\lambda \rangle, \\ |(I - P_{|\lambda|}v)(x)| & \lesssim & 2^{-\tilde{d}|\lambda|} \sup\limits_{|\beta|=\tilde{d},\, 1\leq q\leq M,\, \xi\in\kappa_q^{-1}(B(x;2^{-|\lambda|}\eta)\cap\Gamma_q)} |\partial^\beta(v \circ \kappa_q)(\xi)|, \end{array} \right. \tag{3.2}$$

where for $A \subset \mathbb{R}^{n+1}$ and $\varepsilon \geq 0$, $B(A;\varepsilon) := \{y \in \mathbb{R}^{n+1} : \operatorname{dist}(A, y) \leq \varepsilon\}$.

Remark 3.1. In the biorthogonal setting, one may think of $P_{|\lambda|}$ as being an interpolator on the dual space (cf. [DS99c, Prop. 4.7], [NS01, Prop. 3.4]). Noting that $\langle v, \psi_\lambda \rangle$ only depends on $v|_{\operatorname{supp}\psi_\lambda}$, for the combination of both formulas from (3.2), v outside $\operatorname{supp}\psi_\lambda$ can be any suitable extension of $v|_{\operatorname{supp}\psi_\lambda}$. In case $B(\operatorname{supp}\psi_\lambda; 2^{-|\lambda|}\eta)$ is contained in one patch $\overline{\Gamma_q}$, and assuming that $\{\xi : (\psi_\lambda \circ \kappa_q)(\xi) \neq 0\}$ is a uniformly Lipschitz domain, a smooth extension can be chosen that yields the bound $|\langle v, \psi_\lambda \rangle| \lesssim 2^{-\tilde{d}|\lambda|} \sup_{|\beta|=\tilde{d},\, \xi\in\kappa_q^{-1}(\operatorname{supp}\psi_\lambda)} |\partial^\beta(v \circ \kappa_q)(\xi)|$. Yet, for the general case it is not clear whether $|\langle v, \psi_\lambda \rangle|$ can be bounded in terms of derivatives of order $\tilde{d}$ of v on $\operatorname{supp}\psi_\lambda$ only, and we will have to deal with the technical complication of a bound involving derivatives in a small neighborhood of $\operatorname{supp} v$.

Furthermore, for some $k \in \mathbb{N}_0 \cup \{-1\}$, with $k < \mu$ and

$$\gamma := k + \tfrac{3}{2} > t, \tag{3.3}$$

we assume that all $\psi_\lambda \in C^k(\Gamma)$, where $k = -1$ means no global continuity condition, and that for all $r \in [-\tilde{d}, \gamma)$, $s < \gamma$, with $|s|, |r| \leq \mu$,

$$\|\cdot\|_{H^r(\Gamma)} \lesssim 2^{\ell(r-s)} \|\cdot\|_{H^s(\Gamma)} \qquad \text{on } W_\ell := \operatorname{span}\{\psi_\lambda : |\lambda| = \ell\}. \tag{3.4}$$

Finally, we assume that for all $1 \leq q \leq M$, and $r \in [-\tilde{d}, \gamma)$, $s < \gamma$,

$$\|\cdot\|_{H^r(\Gamma_q)} \lesssim 2^{\ell(r-s)} \|\cdot\|_{H^s(\Gamma_q)} \quad \text{on } \operatorname{span}\{\psi_\lambda : |\lambda| = \ell, B(\operatorname{supp}\psi_\lambda; 2^{-\ell}\eta) \subset \overline{\Gamma_q}\}. \tag{3.5}$$

Remark 3.2. For the case that each parametrization κ_q has a constant Jacobian, in [DS99c] a simple construction is given of continuous finite element wavelets, i.e., $k = 0$ and so $\gamma = \frac{3}{2}$, that in principal for any $\tilde{d}$ and order d satisfies the above assumptions. This restriction on the Jacobians was removed in [NS01], however here wavelets with supports that extend to more than one patch have the cancellation property of only order 1. In a forthcoming paper, we will remove this inconvenience for the application in adaptive methods, and construct wavelets that all have the cancellation property of order $\tilde{d}$.

As in the domain case, wavelets that satisfy the assumptions for in principal any d, $\tilde{d}$ and smoothness permitted by both d and the regularity of the manifold were constructed in [DS99b]. As will appear from Theorem 3.3, via several parameters this regularity however seems to impose a principal barrier to the compressibility. In case of differential operators, we could reduce conditions concerning smoothness and cancellation properties to corresponding patchwise conditions. Yet, the arguments used for that do not carry over to the case of non-local, integral operators.

With the constructions from [DS99a, CTU99, CM00], biorthogonality was realized with respect to a modified $L_2(\Gamma)$-scalar product. As a consequence, with the interpretation of functions as functionals via the Riesz mapping with respect to the standard $L_2(\Gamma)$ scalar product, for negative t the wavelets only generate a Riesz basis for $H^t(\Gamma)$ when $t > -\frac{1}{2}$ (cf. [NS01, §4]), and likewise wavelets with supports that extend to more than one patch generally have no cancellation properties in the sense of (3.2).

The frame approach discussed in the previous section seems to be even more attractive in the compact manifold case. Because of the absence of a boundary, all wavelet bases on the overlapping patches may satisfy periodic boundary conditions. One may verify that Theorem 3.3 given below formulated for the Riesz basis case, as well as the verification of the estimates (3.6) and (3.7), extend to any of the matrices $\langle \omega_q \Psi_q, L\omega_{q'}\Psi_{q'}\rangle$. It is easy to construct collections Ψ_q with any smoothness permitted by the regularity of the manifold.

In the following theorem, we assume two decay estimates (3.6) and (3.7) that in §3.3, §3.4 will be verified for an important class of singular integral operators. For (3.7), we will need the additional condition that $\tilde{d} > \gamma - 2t$, which will be needed anyway in Remark 3.4 to conclude s^*-compressibility with $s^* > \frac{d-t}{n}$.

Theorem 3.3. *Let $L : H^t(\Gamma) \to H^{-t}(\Gamma)$ be bounded, such that for some $\sigma \in (0, \mu - t]$, both L and its adjoint L' are bounded from $H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$. For Ψ being a Riesz basis for $H^t(\Gamma)$ as described above with $\tilde{d} + t > 0$, let $\mathbf{M} = \langle \Psi, L\Psi\rangle$.*

For any $\lambda, \lambda' \in \Lambda$, let

$$|\langle \psi_\lambda, L\psi_{\lambda'}\rangle| \lesssim \left(\frac{2^{-||\lambda|-|\lambda'||/2}}{\delta(\lambda,\lambda')}\right)^{n+2\tilde{d}+2t} \quad \textit{when } \delta(\lambda,\lambda') \geq 3\eta, \tag{3.6}$$

where

$$\delta(\lambda, \lambda') := 2^{\min\{|\lambda|,|\lambda'|\}} \operatorname{dist}(\operatorname{supp}\psi_\lambda, \operatorname{supp}\psi_{\lambda'}),$$

and η is from (3.2).

For some $\tau \geq \sigma$, and with

$$\tilde{\delta}(\lambda,\lambda') := 2^{\min\{|\lambda|,|\lambda'|\}} \times \begin{cases} \mathrm{dist}(\mathrm{supp}\,\psi_\lambda, \mathrm{sing\,supp}\,\psi_{\lambda'}) & \text{when} |\lambda| > |\lambda'|, \\ \mathrm{dist}(\mathrm{sing\,supp}\,\psi_\lambda, \mathrm{supp}\,\psi_{\lambda'}) & \text{when} |\lambda| < |\lambda'|, \end{cases}$$

for any $\ell > |\lambda'|$, $\varepsilon > 0$ and

$$w_{\ell,\lambda',\varepsilon} \in \mathrm{span}\{\psi_\lambda; |\lambda| = \ell,\ 1 \gtrsim \tilde{\delta}(\lambda,\lambda') \geq \max\{\varepsilon, 2\eta 2^{|\lambda'|-\ell}\}\},$$

let

$$\left.\begin{array}{l} |\langle w_{\ell,\lambda',\varepsilon}, L\psi_{\lambda'}\rangle| \\ |\langle L'\psi_{\lambda'}, w_{\ell,\lambda',\varepsilon}\rangle| \end{array}\right\} \lesssim \|w_{\ell,\lambda',\varepsilon}\|_{H^t(\Gamma)} \max\left\{\frac{2^{(|\lambda'|-\ell)(\tilde{d}+t)}}{\varepsilon^{2t+\tilde{d}-\gamma}}, 2^{(|\lambda'|-\ell)\tau}\right\}. \tag{3.7}$$

Let $\alpha \in (\frac{1}{2}, 1)$ and $b_i := (1+i)^{-1-\varepsilon}$ for some $\varepsilon > 0$. Then for $j \in \mathbb{N}$, and with $k(j,n) := \begin{cases} \frac{j}{n-1} & \text{when } n > 1, \\ 2^j & \text{when } n = 1, \end{cases}$ we define the infinite matrix $\mathbf{M}_j$ by replacing all entries $\mathbf{M}_{\lambda,\lambda'} = \langle \psi_\lambda, L\psi_{\lambda'}\rangle$ by zeros when

$$\big||\lambda| - |\lambda'|\big| > k(j,n), \quad \text{or} \tag{3.8}$$

$$\big||\lambda| - |\lambda'|\big| \leq \tfrac{j}{n} \quad \text{and} \quad \delta(\lambda,\lambda') \geq \max\{3\eta, 2^{\alpha(\frac{j}{n} - ||\lambda|-|\lambda'||)}\}, \quad \text{or} \tag{3.9}$$

$$\big||\lambda| - |\lambda'|\big| > \tfrac{j}{n} \quad \text{and} \quad \tilde{\delta}(\lambda,\lambda') \geq \max\{2^{n(\frac{j}{n} - ||\lambda|-|\lambda'||)} b_{||\lambda|-|\lambda'||-\frac{j}{n}}, 2\eta 2^{-||\lambda|-|\lambda'||}\}. \tag{3.10}$$

Then the number of non-zero entries in each row and column of $\mathbf{M}_j$ is of order 2^j, and for any

$$s \leq \min\left\{\tfrac{t+\tilde{d}}{n}, \tfrac{\tau}{n}\right\}, \ \text{with } s < \tfrac{\gamma-t}{n-1},\ s \leq \tfrac{\sigma}{n-1} \text{ and } s \leq \tfrac{t+\mu}{n-1} \text{ when } n > 1,$$

it holds that $\|\mathbf{M} - \mathbf{M}_j\| \lesssim 2^{-sj}$.

Remark 3.4. As in Remark 2.4, we infer that $\mathbf{M}$ is s^*-compressible with

$$s^* = \min\{\tfrac{t+\tilde{d}}{n}, \tfrac{\tau}{n}, \tfrac{\gamma-t}{n-1}, \tfrac{\sigma}{n-1}, \tfrac{t+\mu}{n-1}\}$$

($s^* = \min\{t+\tilde{d}, \tau\}$ when $n = 1$). So, if $\tilde{d} > d - 2t$, $\tau > d - t$, and when $n > 1$, $\frac{\min\{\gamma-t,\sigma,t+\mu\}}{n-1} > \frac{d-t}{n}$, then $s^* > \frac{d-t}{n}$.

Proof. Let λ be some given index. Since Γ is a Lipschitz manifold, by the locality of the wavelets, the number of indices λ' with fixed $|\lambda'| \geq |\lambda|$ and $\mathrm{dist}(\mathrm{supp}\,\psi_\lambda, \mathrm{supp}\,\psi_{\lambda'}) \leq R$ is of order $(2^{|\lambda'|}(2^{-|\lambda|} + R))^n$. By using in addition the piecewise smoothness of the wavelets, the number of indices λ' with fixed $|\lambda'| > |\lambda|$ and $\mathrm{dist}(\mathrm{sing\,supp}\,\psi_\lambda, \mathrm{supp}\,\psi_{\lambda'}) \leq R$, where $2^{-|\lambda'|} \lesssim R \lesssim 2^{-|\lambda|}$ is of order $2^{(|\lambda'|-|\lambda|)(n-1)} 2^{|\lambda'|} R$. From this, one may infer that the number of non-zero entries in the λth row or column of $\mathbf{M}_j$ is of order

$$\sum_{-k(j,n)\leq|\lambda'|-|\lambda|<0} 1 \quad + \sum_{0\leq|\lambda'|-|\lambda|\leq\frac{j}{n}} (2^{|\lambda'|}(2^{-|\lambda|}+2^{-|\lambda|}\max\{3\eta, 2^{\alpha(\frac{j}{n}-|\lambda'|+|\lambda|}\}))^n \quad +$$
$$\sum_{\frac{j}{n}<|\lambda'|-|\lambda|\leq k(j,n)} 2^{(|\lambda'|-|\lambda|)(n-1)}2^{|\lambda'|}2^{-|\lambda|}\max\{2^{n(\frac{j}{n}-|\lambda'|+|\lambda|)}b_{|\lambda'|-|\lambda|-\frac{j}{n}}, 2\eta 2^{|\lambda|-|\lambda'|}\} \eqsim 2^j,$$

because of $\alpha<1$ and $\sum_i b_i<\infty$.

Let $\hat{\mathbf{M}}_j$ be defined by $(\mathbf{M}-\hat{\mathbf{M}}_j)_{\lambda,\lambda'} = \begin{cases} \mathbf{M}_{\lambda,\lambda'} & \text{when } \big||\lambda|-|\lambda'|\big|>k(j,n), \\ 0 & \text{otherwise.}\end{cases}$ The continuity assumptions on L, L', together with (3.4) show that for

$$r\in(0,t+\tilde{d}]\cap(0,t+\mu]\cap(0,\sigma]\cap(0,\gamma-t),$$

and $w_\ell\in W_\ell$, $w_{\ell'}\in W_{\ell'}$,

$$\begin{aligned}|\langle w_\ell, Lw_{\ell'}\rangle| &\lesssim \|w_\ell\|_{H^{t-r}(\Gamma)}\|Lw_{\ell'}\|_{H^{-t+r}(\Gamma)}\\ &\lesssim \|w_\ell\|_{H^{t-r}(\Gamma)}\|w_{\ell'}\|_{H^{t+r}(\Gamma)} \lesssim 2^{r(\ell'-\ell)}\|w_\ell\|_{H^t(\Gamma)}\|w_{\ell'}\|_{H^t(\Gamma)},\end{aligned}$$

and analogously, $|\langle w_\ell, Lw_{\ell'}\rangle| = |\langle L'w_\ell, w_{\ell'}\rangle| \lesssim 2^{r(\ell-\ell')}\|w_\ell\|_{H^t(\Gamma)}\|w_{\ell'}\|_{H^t(\Gamma)}$. As in the proof of Theorem 2.3, we conclude that $\|\mathbf{M}-\hat{\mathbf{M}}_j\|\lesssim 2^{-k(j,n)r}$.

As a second step, let $\tilde{\mathbf{M}}_j$ be defined by

$$(\hat{\mathbf{M}}_j-\tilde{\mathbf{M}}_j)_{\lambda,\lambda'} =$$
$$\begin{cases} \mathbf{M}_{\lambda,\lambda'} & \text{when } \delta(\lambda,\lambda')\geq\max\{1,3\eta,2^{\alpha(\frac{j}{n}-||\lambda|-|\lambda'||)}\} \text{ and } \big||\lambda|-|\lambda'|\big|\leq k(j,n),\\ 0 & \text{otherwise.}\end{cases}$$

Let us recall the Schur lemma: If, for some positive scalars ω_λ, $\sum_{\lambda'}\frac{\omega_\lambda|\mathbf{b}_{\lambda,\lambda'}|}{\omega_{\lambda'}}\leq c$, and $\sum_\lambda\frac{\omega_{\lambda'}|\mathbf{b}_{\lambda,\lambda'}|}{\omega_\lambda}\leq c$, then $\|(\mathbf{b}_{\lambda,\lambda'})_{\lambda,\lambda'\in\Lambda}\|\leq c$. We apply this lemma onto $\hat{\mathbf{M}}_j-\tilde{\mathbf{M}}_j$ with $\omega_\lambda=2^{|\lambda|\frac{n}{2}}$. By the locality of the wavelets, for each λ, ℓ', $R\gtrsim 1$ and $\beta>n$, one has

$$\sum_{\{\lambda':|\lambda'|=\ell',\,\delta(\lambda,\lambda')>R\}}\delta(\lambda,\lambda')^{-\beta}\lesssim R^{-\beta+n}2^{n\max\{0,\ell'-|\lambda|\}}.$$

Because of $\tilde{d}+t>0$, from the decay estimate (3.6) we obtain that

$$\begin{aligned}&\sum_{\lambda'}\frac{\omega_\lambda|(\hat{\mathbf{M}}_j-\tilde{\mathbf{M}}_j)_{\lambda,\lambda'}|}{\omega_{\lambda'}}\\ &\lesssim\sum_{|\lambda'|}2^{(|\lambda|-|\lambda'|)\frac{n}{2}}2^{-||\lambda|-|\lambda'||(\frac{n}{2}+\tilde{d}+t)}\Big(\max\{1,2^{\alpha(\frac{j}{n}-||\lambda|-|\lambda'||)}\}\Big)^{-(2\tilde{d}+2t)}2^{n\max\{0,|\lambda'|-|\lambda|\}}\\ &\eqsim 2^{-\frac{j}{n}(\tilde{d}+t)},\end{aligned}$$

by $\alpha>\frac{1}{2}$. By the symmetry of the right-hand side of (3.6) in λ, λ', analogously we have $\sum_\lambda\frac{\omega_\lambda|(\hat{\mathbf{M}}_j-\tilde{\mathbf{M}}_j)_{\lambda,\lambda'}|}{\omega_{\lambda'}}\lesssim 2^{-\frac{j}{n}(\tilde{d}+t)}$, and so $\|\hat{\mathbf{M}}_j-\tilde{\mathbf{M}}_j\|\lesssim 2^{-\frac{j}{n}(\tilde{d}+t)}$.

Given λ' and $\ell > |\lambda'|$, let

$$A_{\ell,\lambda'} = \{|\lambda| = \ell : \tilde{\delta}(\lambda,\lambda') \geq \max\{2^{n(\frac{j}{n}-\ell+|\lambda'|)} b_{\ell-|\lambda'|-\frac{j}{n}}, 2\eta 2^{|\lambda'|-\ell}\},\ \delta(\lambda,\lambda') < \max\{1, 3\eta\}\}.$$

Since for $\big||\lambda| - |\lambda'|\big| > \frac{j}{n}$, entries $\mathbf{M}_{\lambda,\lambda'}$ with $\delta(\lambda,\lambda') \geq \max\{1,3\eta\}$ were already been removed from $\hat{\mathbf{M}}_j$, we have

$$(\hat{\mathbf{M}}_j - \mathbf{M}_j)_{\lambda,\lambda'} = \begin{cases} \mathbf{M}_{\lambda,\lambda'} & \begin{cases} \text{when} & \frac{j}{n} < |\lambda| - |\lambda'| \leq k(j,n) \quad \text{and} \quad \lambda \in A_{|\lambda|,\lambda'}, \\ \text{or} & \frac{j}{n} < |\lambda'| - |\lambda| \leq k(j,n) \quad \text{and} \quad \lambda' \in A_{|\lambda'|,\lambda}, \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

So from the decay estimate (3.7), for any $\mathbf{c}, \mathbf{d} \in \ell_2(\Lambda)$ we have

$$\begin{aligned}
&|\sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)} \sum_{|\lambda'|=\ell'} \mathbf{d}_{\lambda'} \langle \sum_{\lambda\in A_{\ell,\lambda'}} \mathbf{c}_\lambda \psi_\lambda, L\psi_{\lambda'}\rangle| \\
&\lesssim \sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)} \sum_{|\lambda'|=\ell'} |\mathbf{d}_{\lambda'}| \max\{\frac{2^{(\ell'-\ell)(\tilde{d}+t)}}{(2^{n(\frac{j}{n}-\ell+\ell')} b_{\ell-\ell'-\frac{j}{n}})^{2t+\tilde{d}-\gamma}}, 2^{(\ell'-\ell)\tau}\} \|\sum_{\lambda\in A_{\ell,\lambda'}} \mathbf{c}_\lambda \psi_\lambda\|_{H^t(\Gamma)} \\
&\lesssim \sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)} \max\{\frac{2^{(\ell'-\ell)(\tilde{d}+t)}}{(2^{n(\frac{j}{n}-\ell+\ell')} b_{\ell-\ell'-\frac{j}{n}})^{2t+\tilde{d}-\gamma}}, 2^{(\ell'-\ell)\tau}\} \sqrt{\sum_{|\lambda'|=\ell'} |\mathbf{d}_{\lambda'}|^2} \sqrt{\sum_{|\lambda'|=\ell'} \sum_{\lambda\in A_{\ell,\lambda'}} |\mathbf{c}_\lambda|^2} \\
&\lesssim \sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)} \max\{\frac{2^{(\ell'-\ell)(\tilde{d}+t)}}{(2^{n(\frac{j}{n}-\ell+\ell')} b_{\ell-\ell'-\frac{j}{n}})^{2t+\tilde{d}-\gamma}}, 2^{(\ell'-\ell)\tau}\} \sqrt{\sum_{|\lambda'|=\ell'} |\mathbf{d}_{\lambda'}|^2} \sqrt{\sum_{|\lambda|=\ell} |\mathbf{c}_\lambda|^2},
\end{aligned}$$

where for the last line we have used that for fixed $|\lambda'|$, each λ is contained in at most a uniformly bounded number of sets $A_{|\lambda|,\lambda'}$. Since the analogous estimate is valid with interchanged roles of ℓ and ℓ', and for $s \leq \min\{\frac{t+\tilde{d}}{n}, \frac{\tau}{n}\}$, with $s < \frac{\gamma-t}{n-1}$ when $n > 1$,

$$\sum_{m=1}^{k(j,n)-\frac{j}{n}} \max\{2^{-(m+\frac{j}{n})(\tilde{d}+t)} \left(2^{-mn} b_m\right)^{\gamma-2t-\tilde{d}}, 2^{-(m+\frac{j}{n})\tau}\} \lesssim 2^{-sj},$$

we conclude that for such s, $\|\tilde{\mathbf{M}}_j - \mathbf{M}_j\| \lesssim 2^{-sj}$.

A combination of the estimates for $\mathbf{M} - \hat{\mathbf{M}}_j$, $\hat{\mathbf{M}}_j - \tilde{\mathbf{M}}_j$ and $\tilde{\mathbf{M}}_j - \mathbf{M}_j$ shows that for $s \leq \min\{\frac{t+\tilde{d}}{n}, \frac{\tau}{n}\}$, with for $n > 1$, $s < \frac{\gamma-t}{n-1}$, $s \leq \frac{\sigma}{n-1}$ and $s \leq \frac{t+\mu}{n-1}$, it holds that $\|\mathbf{M} - \mathbf{M}_j\| \lesssim 2^{-sj}$. □

3.2. **Singular integral operators.** In §3.3, §3.4, we verify the decay estimates (3.6) and (3.7) for operators

$$Lu(x) = \int_\Gamma K(x,y)u(y)d\Gamma_y, \qquad (x \in \Gamma),$$

with kernels that satisfy for all $1 \leq q, q' \leq M$, $\xi \in \Omega_q$, $\eta \in \Omega_{q'}$,

$$|\partial_\xi^\alpha \partial_\eta^\beta K(\kappa_q(\xi), \kappa_{q'}(\eta))| \lesssim \operatorname{dist}(\kappa_q(\xi), \kappa_{q'}(\eta)))^{-(n+2t+|\alpha|+|\beta|)}, \quad (n+2t+|\alpha|+|\beta| > 0), \tag{3.11}$$

(cf. [DHS02, Def. 2.1]). Following [DHS02], we emphasize that (3.11) requires patchwise smoothness but no global smoothness of Γ. Only assuming global Lipschitz continuity of Γ, the kernel of a boundary integral operator of order $2t$ can be shown to satisfy (3.11).

If Γ is a C^∞-manifold, then these boundary integral operators are known to be pseudo-differential operators, meaning that for any $\sigma \in \mathbb{R}$ they define bounded mappings from $H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$. In this case we may conclude that $\mathbf{M}$ is s^*-compressibility with $s^* = \min\{\frac{t+\tilde{d}}{n}, \frac{\gamma-t}{n-1}\}$. For Γ being only Lipschitz continuous, for the classical boundary integral equations the question of continuity in Sobolev spaces has been answered in [Cos88]. With increasing smoothness of Γ one may expect larger ranges of σ for which boundedness from $H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$ is valid. Little results in this direction seem yet available.

3.3. **Decay estimate** (3.6)**.** This estimate for singular integral operators with wavelets that satisfy the cancellation property (3.2) of order $\tilde{d}$ has first been proved in [Sch98] for C^∞-manifolds. In [DS99c], it has been shown for Lipschitz manifolds for a specific wavelet construction. For convenience, in this subsection we recall the arguments used there, and show that they also apply to the general setting discussed in this paper.

With η from (3.2), let $\lambda, \lambda' \in \Lambda$ with $\delta(\lambda, \lambda') \geq 3\eta$ and $\delta(\lambda, \lambda') > 0$. Then with $\Gamma_{\lambda,\eta} := B(\operatorname{supp}\psi_\lambda; 2^{-|\lambda|}\eta)$, it holds that

$$2^{\min\{|\lambda|,|\lambda'|\}}\operatorname{dist}(\Gamma_{\lambda,\eta}, \Gamma_{\lambda',\eta}) \geq \tfrac{1}{3}\delta(\lambda, \lambda') > 0.$$

Because of

$$n + 2t + 2\tilde{d} > 0,$$

from (3.2), (3.11) and $\int_\Gamma |\psi_\lambda| d\Gamma \lesssim 2^{-|\lambda|\frac{n}{2}} 2^{-|\lambda|t}$ (by the locality and (3.4)), we infer that

$$\begin{aligned}
|\langle \psi_\lambda, L\psi_{\lambda'}\rangle| &= |\int_\Gamma \psi_\lambda(x) \int_\Gamma K(x,y)\psi_{\lambda'}(y) d\Gamma_y d\Gamma_x| \\
&= |\int_\Gamma \psi_\lambda(x) \int_\Gamma [(I - P_{|\lambda'|})K(x,\cdot)](y)\psi_{\lambda'}(y) d\Gamma_y d\Gamma_x| \\
&= |\int_\Gamma \psi_\lambda(x)(I - P_{|\lambda|})[x \mapsto \int_\Gamma [(I - P_{|\lambda'|})K(x,\cdot)](y)\psi_{\lambda'}(y) d\Gamma_y] d\Gamma_x| \\
&\lesssim 2^{-(\tilde{d}+\frac{n}{2}+t)|\lambda|} \sup_{\substack{|\alpha|=\tilde{d}, 1\leq q\leq M, \\ \xi \in \kappa_q^{-1}(\Gamma_{\lambda,\eta}\cap\Gamma_q)}} |\partial_\xi^\alpha \int_\Gamma [(I - P_{|\lambda'|})K(\kappa_q(\xi),\cdot)](y)\psi_{\lambda'}(y) d\Gamma_y|
\end{aligned}$$

$$
\begin{aligned}
&= 2^{-(\tilde{d}+\frac{n}{2}+t)|\lambda|} \sup_{\substack{|\alpha|=\tilde{d}, 1\leq q\leq M,\\ \xi\in\kappa_q^{-1}(\Gamma_{\lambda,\eta}\cap\Gamma_q)}} \Big|\int_\Gamma [(I-P_{|\lambda'|})\partial_\xi^\alpha K(\kappa_q(\xi),\cdot)](y)\psi_{\lambda'}(y)d\Gamma_y\Big| \\
&\lesssim 2^{-(\tilde{d}+\frac{n}{2}+t)(|\lambda|+|\lambda'|)} \sup_{\substack{|\alpha|=\tilde{d}, 1\leq q\leq M,\\ \xi\in\kappa_q^{-1}(\Gamma_{\lambda,\eta}\cap\Gamma_q)}} \sup_{\substack{|\beta|=\tilde{d}, 1\leq q'\leq M,\\ \zeta\in\kappa_{q'}^{-1}(\Gamma_{\lambda',\eta}\cap\Gamma_{q'})}} |\partial_\xi^\alpha\partial_\zeta^\beta K(\kappa_q(\xi),\kappa_{q'}(\zeta))| \\
&\lesssim 2^{-(\tilde{d}+\frac{n}{2}+t)(|\lambda|+|\lambda'|)}(2^{-\min\{|\lambda|,|\lambda'|\}}\delta(\lambda,\lambda'))^{-(n+2t+2\tilde{d})} \\
&= \left(\frac{2^{-||\lambda|-|\lambda'||/2}}{\delta(\lambda,\lambda')}\right)^{n+2\tilde{d}+2t}.
\end{aligned}
$$

3.4. **Decay estimate** (3.7)**.** Let $\Gamma = \cup_{q=1}^M \overline{\Gamma_q}$ be a compact n-dimensional, globally $C^{\mu-1,1}$-manifold in $\mathbb{R}^{n+1}$, where Γ_q are C^∞-manifolds as described in §3.1. For some $|t| \leq \mu$, let L be singular integral operator of order $2t$ as described in §3.2, which is bounded from $H^t(\Gamma) \to H^{-t}(\Gamma)$, and for which there exists a $\sigma > 0$ such that $L, L' : H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$ are bounded. Let Ψ be a Riesz basis for $H^t(\Gamma)$ as described in §3.1, consisting of local and piecewise smooth $C^k(\Gamma)$ wavelets, that have cancellation properties of order $\tilde{d}$, where $k \in \mathbb{N}_0 \cup \{-1\}$, $k < \mu$ and $\gamma := k + \frac{3}{2} > t$.

In addition, in this subsection we assume that

$$\tilde{d} > \gamma - 2t. \tag{3.12}$$

Furthermore, with $\tilde{H}^s(\Gamma_q) := \begin{cases} H^s(\Gamma_q) & \text{when } s \geq 0, \\ (H_0^{-s}(\Gamma_q))' & \text{when } s < 0, \end{cases}$ we assume that there exists a $\tau \in (0, \mu - t]$ such that for all $1 \leq q \leq M$,

$$L : H^{t+\tau}(\Gamma) \to \tilde{H}^{-t+\tau}(\Gamma_q) \quad \text{is bounded.} \tag{3.13}$$

Remark 3.5. Since for any $|s| \leq \mu$, the restriction of functions on Γ to Γ_q is a bounded mapping from $H^s(\Gamma)$ to $\tilde{H}^s(\Gamma_q)$, from the boundedness of $L : H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$, it follows that in any case (3.13) is valid for $\tau = \sigma$. So for example for Γ being a C^∞-manifold, (3.13) is valid for any $\tau \in \mathbb{R}$. Yet, in particular when $t < 0$, for Γ being less smooth it might happen that (3.13) is valid for a τ that is strictly larger that any σ for which $L : H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$ is bounded.

Proposition 3.6. *In the above setting, for any $\ell > |\lambda'|$, $\varepsilon > 0$, and*

$$w_{\ell,\lambda',\varepsilon} \in \operatorname{span}\{\psi_\lambda : |\lambda| = \ell, \tilde{\delta}(\lambda,\lambda') \geq \max\{\varepsilon, 2\eta 2^{|\lambda'|-\ell}\}\},$$

with $\delta := 2^{|\lambda'|}\operatorname{diam}(\operatorname{supp} w_{\ell,\lambda',\varepsilon})$ it holds that

$$\left.\begin{array}{l} |\langle w_{\ell,\lambda',\varepsilon}, L\psi_{\lambda'}\rangle| \\ |\langle L'\psi_{\lambda'}, w_{\ell,\lambda',\varepsilon}\rangle| \end{array}\right\} \lesssim$$

$$\|w_{\ell,\lambda',\varepsilon}\|_{H^t(\Gamma)} \max\left\{2^{(|\lambda'|-\ell)(\tilde{d}+t)}\varepsilon^{-2t-\tilde{d}+k+1}\delta^{\frac{n-1}{2}}\min\{\varepsilon,\delta\}^{\frac{1}{2}}, 2^{(|\lambda'|-\ell)\min\{\tau,t+\tilde{d}\}}\right\}.$$

By substituting $\delta \lesssim 1$ and by using that $k+\frac{3}{2}=\gamma$, the decay estimate (3.7) *is obtained.*

Remark 3.7. If for some $|\lambda|>|\lambda'|$, $\operatorname{dist}(\operatorname{supp}\psi_\lambda, \operatorname{sing\,supp}\psi_{\lambda'}) \geq 2\eta 2^{-|\lambda|}$, then by substituting $w_{\ell,\lambda',\varepsilon}=\psi_\lambda$ in Proposition 3.6, and so $\delta = 2^{|\lambda'|-|\lambda|}$, we obtain that

(3.14)
$$\left.\begin{array}{l}|\langle \psi_\lambda, L\psi_{\lambda'}\rangle|\\ |\langle L'\psi_{\lambda'}, \psi_\lambda\rangle|\end{array}\right\} \lesssim \max\left\{\frac{2^{(|\lambda'|-|\lambda|)(\tilde{d}+t+\frac{n}{2})}}{(2^{|\lambda'|}\operatorname{dist}(\operatorname{supp}\psi_\lambda, \operatorname{sing\,supp}\psi_{\lambda'}))^{2t+\tilde{d}-(k+1)}}, 2^{(|\lambda'|-|\lambda|)\min\{\tau, t+\tilde{d}\}}\right\}.$$

In [Sch98], for the case of a C^∞-manifold, and in [DHS02], for the case of a piecewise smooth, globally Lipschitz manifold, it was proved that

$$\left.\begin{array}{l}|\langle \psi_\lambda, L\psi_{\lambda'}\rangle|\\ |\langle L'\psi_{\lambda'}, \psi_\lambda\rangle|\end{array}\right\} \lesssim \frac{2^{(|\lambda'|-|\lambda|)(\tilde{d}+t+\frac{n}{2})}}{(2^{|\lambda'|}\operatorname{dist}(\operatorname{supp}\psi_\lambda, \operatorname{sing\,supp}\psi_{\lambda'}))^{2t+\tilde{d}}}.$$

An adaptation of these proofs using Lemma 3.8 show that in the general situation of Proposition 3.6, it holds that

$$\left.\begin{array}{l}|\langle w_{\ell,\lambda',\varepsilon}, L\psi_{\lambda'}\rangle|\\ |\langle L'\psi_{\lambda'}, w_{\ell,\lambda',\varepsilon}\rangle|\end{array}\right\} \lesssim \|w_{\ell,\lambda',\varepsilon}\|_{H^t(\Gamma)} 2^{(|\lambda'|-\ell)(\tilde{d}+t)}\varepsilon^{-2t-\tilde{d}}\delta^{\frac{n-1}{2}}\min\{\varepsilon,\delta\}^{\frac{1}{2}}.$$

A comparison of these results shows that for $k>-1$, the bounds derived in this paper are sharper when ε or $\operatorname{dist}(\operatorname{supp}\psi_\lambda, \operatorname{sing\,supp}\psi_{\lambda'})$ are sufficiently small. Moreover they improve with increasing k which is essential to be able to prove s^*-compressibility with $s^* > \frac{d-t}{n}$ for relative large $d-t$. The estimates from [Sch98, DHS02] do not exploit global smoothness of the wavelets, but on the other hand, the continuity assumption (3.13) is avoided. Recall that in Theorem 3.3 we make no use of (3.14), but instead apply Proposition 3.6 for $w_{\ell,\lambda',\varepsilon}$ with $\delta = 2^{|\lambda'|}\operatorname{diam}(\operatorname{supp} w_{\ell,\lambda',\varepsilon}) \eqsim 1$, with which we estimate directly norms of rows of $\hat{\mathbf{M}}_j - \mathbf{M}_j$, i.e., not via estimates of the individual entries using the Schur lemma.

To prove Proposition 3.6 we start with a lemma.

Lemma 3.8. *For any $\ell \in I\!N$, $\lambda' \in \Lambda$, $J \subset \{1,\ldots,m\}$, let either $E=\cup_{i\in J}\overline{\Xi_{\lambda',i}}$ or $E=\overline{\Gamma\backslash \cup_{i\in J}\overline{\Xi_{\lambda',i}}}$. Then for any $v\in L_\infty(\Gamma)$, $w\in L_2(\Gamma)$ with $\operatorname{supp} w \subset E$, $\operatorname{supp} v \subset \overline{\Gamma\backslash E}$, $\varepsilon := 2^{|\lambda'|}\operatorname{dist}(\operatorname{supp} w, \partial E) \in (0,\infty)\cap 2^{|\lambda'|-\ell}[2\eta,\infty)$, and*

$$|v(y)| \lesssim 2^{(k+1+\frac{n}{2}-t)|\lambda'|}\operatorname{dist}(y,E)^{k+1},$$

with $\delta := 2^{|\lambda'|}\operatorname{diam}(\operatorname{supp} w)$ it holds that

$$\left|\int_\Gamma w(x)(I-P_\ell)[x\mapsto \int_\Gamma K(x,y)v(y)d\Gamma_y]d\Gamma_x\right| \lesssim \|w\| 2^{-\ell\tilde{d}}2^{|\lambda'|(\tilde{d}+t)}\varepsilon^{-2t-\tilde{d}+k+1}\delta^{\frac{n-1}{2}}\min\{\varepsilon,\delta\}^{\frac{1}{2}}.$$

Proof. By applying (3.2) and (3.11), interchanging the order of differentiation and integration, and by using that $\operatorname{dist}(B(x;2^{-\ell}\eta),y) \geq \frac{1}{2}|x-y|$ for any $x\in\operatorname{supp} w$, $y \in \operatorname{supp} v$, it follows that the expression in the statement of the lemma can be bounded by a multiple of

$$2^{-\ell\tilde{d}}2^{(k+1+\frac{n}{2}-t)|\lambda'|}\int_{\operatorname{supp} w}|w(x)|\int_{\operatorname{supp} v}|x-y|^{-(n+2t+\tilde{d})}|x-y|^{k+1}d\Gamma_y d\Gamma_x. \tag{3.15}$$

Because of $-2t-\tilde{d}+k+1<0$ by (3.12), for all $x \notin \operatorname{supp} v$,

$$\int_{\operatorname{supp} v} |x-y|^{k+1-(n+2t+\tilde{d})} d\Gamma_y \lesssim \int_{z\in \mathbb{R}^n,\, |z|\geq \operatorname{dist}(x,\operatorname{supp} v)} |z|^{k+1-(n+2t+\tilde{d})} dz$$
$$\eqsim \operatorname{dist}(x,\operatorname{supp} v)^{-2t-\tilde{d}+k+1}.$$

By applying the Cauchy-Schwarz inequality, (3.15) can now be bounded by a multiple of

$$2^{-\ell\tilde{d}} 2^{(k+1+\frac{n}{2}-t)|\lambda'|}\|w\| \sqrt{\int_{\operatorname{supp} w} \operatorname{dist}(x,\operatorname{supp} v)^{-4t-2\tilde{d}+2k+2} d\Gamma_x}. \tag{3.16}$$

Since $-4t-2\tilde{d}+2k+2<-1$ by (3.12), because of the geometry of E we may estimate

$$\int_{\operatorname{supp} w} \operatorname{dist}(x,\operatorname{supp} v)^{-4t-2\tilde{d}+2k+2} d\Gamma_x$$
$$\lesssim \operatorname{diam}(\operatorname{supp} w)^{n-1} \int_{\operatorname{dist}(\operatorname{supp} w,\partial E)}^{\operatorname{dist}(\operatorname{supp} w,\partial E)+\operatorname{diam}(\operatorname{supp} w)} z^{-4t-2\tilde{d}+2k+2} dz$$
$$\eqsim (2^{-|\lambda'|}\delta)^{n-1}[(2^{-|\lambda'|}\varepsilon)^{-4t-2\tilde{d}+2k+3} - (2^{-|\lambda'|}(\varepsilon+\delta))^{-4t-2\tilde{d}+2k+3}]$$
$$\eqsim 2^{|\lambda'|(4t+2\tilde{d}-2k-2-n)}\delta^{n-1}\varepsilon^{-4t-2\tilde{d}+2k+2}\min\{\varepsilon,\delta\}.$$

By substituting this result into (3.16) the proof is completed. □

Proof of Proposition 3.6. **(I)**. Let $\lambda' \in \Lambda$, $\ell > |\lambda'|$ and $\varepsilon > 0$, and let

$$w_{\ell,\lambda',\varepsilon} \in \operatorname{span}\{\psi_\lambda : |\lambda| = \ell, \tilde{\delta}(\lambda,\lambda') \geq \max\{\varepsilon, 2\eta 2^{|\lambda'|-\ell}\}\},$$

and put $\delta := 2^{|\lambda'|}\operatorname{diam}(\operatorname{supp} w_{\ell,\lambda',\varepsilon})$. It suffices to prove the bound for $\langle w_{\ell,\lambda',\varepsilon}, L\psi_{\lambda'}\rangle$, since the proof for $\langle L'\psi_{\lambda'}, w_{\ell,\lambda',\varepsilon}\rangle$ is similar.

For $1 \leq i \leq m$, we define $w^{(i)}_{\ell,\lambda',\varepsilon}$ by

$$w^{(i)}_{\ell,\lambda',\varepsilon}(x) = \begin{cases} w_{\ell,\lambda',\varepsilon}(x) & \text{when } x \in \Xi_{\lambda',i}, \\ 0 & \text{elsewhere,} \end{cases}$$

and put $w^{(0)}_{\ell,\lambda',\varepsilon} = w_{\ell,\lambda',\varepsilon} - \sum_{i=1}^m w^{(i)}_{\ell,\lambda',\varepsilon}$, meaning that $\operatorname{supp} w^{(0)}_{\ell,\lambda',\varepsilon} \cap \operatorname{supp}\psi_{\lambda'} = \emptyset$.

We assume that $\tilde{\delta}(\lambda,\lambda') \geq \max\{\varepsilon, 2\eta 2^{|\lambda'|-|\lambda|}\}$ implies that either $\operatorname{supp}\psi_\lambda \subset \overline{\Xi_{\lambda',i}}$ for some $1 \leq i \leq m$, or $\operatorname{supp}\psi_\lambda \cap \operatorname{supp}\psi_{\lambda'} = \emptyset$. In the very unlikely situation that this does not hold "automatically", we can always increase the parameter η such that this is true, since $\operatorname{diam}(\operatorname{supp}\psi_\lambda) \lesssim 2^{-|\lambda|}$. Under this assumption, for all i we have that $w^{(i)}_{\ell,\lambda',\varepsilon} \in W_\ell$, and so $\|w^{(i)}_{\ell,\lambda',\varepsilon}\| \lesssim 2^{-\ell t}\|w^{(i)}_{\ell,\lambda',\varepsilon}\|_{H^t(\Gamma)} \lesssim \|w_{\ell,\lambda',\varepsilon}\|_{H^t(\Gamma)}$

(II). We consider $\langle w^{(0)}_{\ell,\lambda',\varepsilon}, L\psi_{\lambda'}\rangle$. Let $E = \overline{\Gamma\backslash\operatorname{supp}\psi_{\lambda'}}$. If i is such that $\overline{\Xi_{\lambda',i}} \cap E \neq \emptyset$, then because of $\psi_{\lambda'} \in C^k(\Gamma)$ and (3.1) it follows that

$$|\psi_{\lambda'}(y)| \lesssim 2^{(k+1+\frac{n}{2}-t)|\lambda'|}\operatorname{dist}(y,E)^{k+1}, \qquad (y \in \Xi_{\lambda',i}). \tag{3.17}$$

If $\overline{\Xi_{\lambda',i}} \cap E = \emptyset$, then by the "shape regularity" of all sets $\Xi_{\lambda',i'}$ for $i' \neq i$, we have $\operatorname{dist}(\Xi_{\lambda',i}, E) \gtrsim 2^{-|\lambda'|}$, and so (3.17) follows from $|\psi_{\lambda'}(y)| \lesssim 2^{(\frac{n}{2}-t)|\lambda'|}$. From an application of Lemma 3.8 with $w = w^{(0)}_{\ell,\lambda',\varepsilon}$ and $v = \psi_{\lambda'}$, we conclude that

$$|\langle w^{(0)}_{\ell,\lambda',\varepsilon}, L\psi_{\lambda'}\rangle| = |\int_\Gamma w^{(0)}_{\ell,\lambda',\varepsilon}(x)(I-P_\ell)[x \mapsto \int_\Gamma K(x,y)\psi_{\lambda'}(y)d\Gamma_y]d\Gamma_x| \lesssim$$
$$\|w_{\ell,\lambda',\varepsilon}\|_{H^t(\Gamma)} 2^{(|\lambda'|-\ell)(\tilde{d}+t)} \varepsilon^{-2t-\tilde{d}+k+1} \delta^{\frac{n-1}{2}} \min\{\varepsilon,\delta\}^{\frac{1}{2}}. \tag{3.18}$$

(III). Let $1 \leq i \leq m$, and let $1 \leq q \leq M$ such that $\Xi_{\lambda',i} \subset \Gamma_q$. From (3.5), for $r \in [-\tilde{d}, \gamma)$, $s < \gamma$, we have $\|w^{(i)}_{\ell,\lambda',\varepsilon}\|_{H^r(\Gamma_q)} \lesssim 2^{\ell(r-s)} \|w^{(i)}_{\ell,\lambda',\varepsilon}\|_{H^s(\Gamma_q)}$.

(a). If $\tau < \gamma - t$, and so $\tau \leq t + \tilde{d}$ by (3.12), then by $\tau \leq t - \mu$, the continuity of L as stated in (3.13), and when $t - \tau > 0$ in addition by $w^{(i)}_{\ell,\lambda',\varepsilon} \in H_0^{t-\tau}(\Gamma_q)$, we have

$$\begin{aligned} |\langle w^{(i)}_{\ell,\lambda',\varepsilon}, L\psi_{\lambda'}\rangle| &\lesssim \|w^{(i)}_{\ell,\lambda',\varepsilon}\|_{H^{t-\tau}(\Gamma_q)} \|L\psi_{\lambda'}\|_{\tilde{H}^{\tau-t}(\Gamma_q)} \\ &\lesssim 2^{(t-\tau)\ell} \|w^{(i)}_{\ell,\lambda',\varepsilon}\| \|\psi_{\lambda'}\|_{H^{\tau+t}(\Gamma)} \lesssim 2^{(|\lambda'|-\ell)\tau} \|w_{\ell,\lambda',\varepsilon}\|_{H^t(\Gamma)}, \end{aligned}$$

which completes the proof in this case.

(b). Let now $\tau + t \geq \gamma \geq 0$. By assumption, $\psi_{\lambda'} \circ \kappa_q$ is smooth on $\kappa_q^{-1}(\Xi_{\lambda',i})$, which is a uniformly Lipschitz domain. From (3.1) and Remark 2.1, we learn that $\psi_{\lambda'} \circ \kappa_q|_{\kappa_q^{-1}(\Xi_{\lambda',i})}$ has an extension to a smooth function $\varphi_{\lambda',i}$, with for $s \geq 0$ and $p \in [1,\infty]$, $\|\varphi_{\lambda',i}\|_{W^s_p(\mathbb{R}^n)} \lesssim 2^{(s-t+\frac{n}{2}-\frac{n}{p})|\lambda'|}$. By multiplying $\varphi_{\lambda',i}$ by a smooth function that is one on Ω_q, and has support inside the "extended" domain $\hat{\Omega}_q$, we may assume that $\operatorname{supp} \varphi_{\lambda',i} \subset \hat{\Omega}_q$, so that $\varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1} \in H^\mu(\Gamma)$ with for $s \in [0,\mu]$, $\|\varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1}\|_{H^s(\Gamma)} \lesssim 2^{(s-t)|\lambda'|}$. With $\max\{0,-t\} \leq s := \min\{\tau, t+\tilde{d}\} \leq \mu - t$, the same arguments as applied in **(a)** show that

$$\begin{aligned} |\langle w^{(i)}_{\ell,\lambda',\varepsilon}, L(\varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1})\rangle| &\lesssim \|w^{(i)}_{\ell,\lambda',\varepsilon}\|_{H^{t-s}(\Gamma_q)} \|L(\varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1})\|_{\tilde{H}^{s-t}(\Gamma_q)} \\ &\lesssim 2^{(t-s)\ell} \|w^{(i)}_{\ell,\lambda',\varepsilon}\| \|\varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1}\|_{H^{s+t}(\Gamma)} \lesssim 2^{(|\lambda'|-\ell)s} \|w_{\ell,\lambda',\varepsilon}\|_{H^t(\Gamma)}. \end{aligned} \tag{3.19}$$

There remains to estimate $|\langle w^{(i)}_{\ell,\lambda',\varepsilon}, L(\psi_{\lambda'} - \varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1})\rangle|$ which we will do by applying Lemma 3.8. Recall that $\operatorname{supp} w^{(i)}_{\ell,\lambda',\varepsilon} \subset \Xi_{\lambda',i}$, whereas $\psi_{\lambda'} - \varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1}$ vanishes on $\Xi_{\lambda',i}$. The global smoothness of $\psi_{\lambda'}$ will ensure that directly outside $\Xi_{\lambda',i}$, $\psi_{\lambda'}$ is sufficiently close to the smooth extension $\varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1}$ of $\psi_{\lambda'}|_{\Xi_{\lambda',i}}$. We have to distinguish between a number of cases.

Suppose $i' \neq i$ with $\overline{\Xi_{\lambda',i'}} \cap \overline{\Xi_{\lambda',i}} \neq \emptyset$, and let q' such that $\Xi_{\lambda',i'} \subset \Gamma_{q'}$. Then from

(a) $\sup_{\xi \in \kappa_{q'}^{-1}(\Xi_{\lambda',i'})} |\partial^\beta(\psi_{\lambda'} \circ \kappa_{q'})(\xi)| \lesssim 2^{(|\beta|+\frac{n}{2}-t)|\lambda'|}$, $(\beta \in \mathbb{N}_0)$ ((3.1)),
(b) $\kappa_{q'}^{-1} \circ \hat{\kappa}_q \in C^{\mu-1,1}(\hat{\kappa}_q^{-1}(\Gamma_{q'} \cap \operatorname{Im} \hat{\kappa}_q))$, where $\mu > k$,
(c) $\sup_{\xi \in \hat{\Omega}_q} |\partial^\beta \varphi_{\lambda',i}(\xi)| \lesssim 2^{(|\beta|+\frac{n}{2}-t)|\lambda'|}$, $(\beta \in \mathbb{N}_0)$,
(d) $\psi_{\lambda'} \circ \hat{\kappa}_q - \varphi_{\lambda',i} \in C^k(\hat{\Omega}_q)$, and it vanishes on $\hat{\kappa}_q^{-1}(\Xi_{\lambda',i})$,

one infers that $|(\psi_{\lambda'} \circ \hat{\kappa}_q - \varphi_{\lambda',i})(\xi)| \lesssim 2^{(k+1+\frac{n}{2}-t)|\lambda'|}\mathrm{dist}(\xi, \hat{\kappa}_q^{-1}(\Xi_{\lambda',i}))^{k+1}$ when $\xi \in \hat{\kappa}_q^{-1}(\Xi_{\lambda',i'} \cap \operatorname{Im}\hat{\kappa}_q)$, and so for $y \in \Xi_{\lambda',i'} \cap \operatorname{Im}\hat{\kappa}_q$,

$$|(\psi_{\lambda'} - \varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1})(y)| \lesssim 2^{(k+1+\frac{n}{2}-t)|\lambda'|}\mathrm{dist}(y, \Xi_{\lambda',i})^{k+1}. \tag{3.20}$$

If $\overline{\Gamma\backslash\operatorname{supp}\psi_{\lambda',i}} \cap \overline{\Xi_{\lambda',i}} \neq \emptyset$, then (c), (d) show that (3.20) is also valid for $y \in (\Gamma\backslash\operatorname{supp}\psi_{\lambda',i}) \cap \operatorname{Im}\hat{\kappa}_q$. For the remaining cases that either $y \in \Xi_{\lambda',i'}$ with $\overline{\Xi_{\lambda',i'}} \cap \overline{\Xi_{\lambda',i}} = \emptyset$, or $y \in (\Gamma\backslash\operatorname{supp}\psi_{\lambda',i}) \cap \operatorname{Im}\hat{\kappa}_q$ whereas $\overline{\Gamma\backslash\operatorname{supp}\psi_{\lambda',i}} \cap \overline{\Xi_{\lambda',i}} = \emptyset$, or $y \notin \operatorname{Im}\hat{\kappa}_q$, then the "shape regularity" of all sets $\Xi_{\lambda',\breve{i}}$ show that $\mathrm{dist}(y, \Xi_{\lambda',i}) \gtrsim 2^{-|\lambda'|}$, and so from $|(\psi_{\lambda'} - \varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1})(y)| \lesssim 2^{(\frac{n}{2}-t)|\lambda'|}$, we conclude that (3.20) is valid for *all* $y \in \Gamma\backslash\Xi_{\lambda',i}$. An application of Lemma 3.8 with $E = \overline{\Xi_{\lambda',i}}$, $w = w^{(i)}_{\ell,\lambda',\varepsilon}$ and $v = \psi_{\lambda'} - \varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1}$ now shows that

$$|\langle w^{(i)}_{\ell,\lambda',\varepsilon}, L(\psi_{\lambda'} - \varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1})\rangle| \lesssim \|w_{\ell,\lambda',\varepsilon}\|_{H^t(\Gamma)} 2^{(|\lambda'|-\ell)(\tilde{d}+t)} \varepsilon^{-2t-\tilde{d}+k+1} \delta^{\frac{n-1}{2}} \min\{\varepsilon, \delta\}^{\frac{1}{2}},$$

which together with (3.18) and (3.19) completes the proof for the case that $\tau + t \geq \gamma$. $\square$

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