Extendibility of Ergodic Actions of Abelian Groups on a Measure Space

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Abstract

Let E be a group extension of an abelian l.c.s.c. group A by an amenable l.c.s.c. group G. We say that an ergodic action V of A is extendible to an action W of E if $V(A)$ is isomorphic to $W(A)$. It turns out that the extendibility property can be described in terms of cocycles over a skew product taking values in A. For topologically trivial group extensions $E(G, A)$, we prove that the extendibility property is not generic. We give an example of $\mathbb{R}$-action that is not extendible to an action of $\mathbb{R}_+^* \ltimes \mathbb{R}$. We answer the question of when two isomorphic actions of A can be extended to isomorphic actions of $E(G, A)$.

Introduction. Let A be an abelian locally compact second countable (l.c.s.c.) group and let G be an amenable l.c.s.c. group acting on A by group automorphisms. Denote by E the group extension of A by G. Then A can be identified with a normal subgroup of E. The group extension concept becomes more transparent in case of topologically trivial group extensions $E_f(G, A)$ with $f : G \times G \to A$ being a 2-cocycle. An action V of A on a measure space is called extendible to an action W of E if $V(A)$ is isomorphic

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to $W(A)$. In [B], the question when an action V of A can be extended to an action W of E_f was answered. It turns out that the extendibility property can be reformulated in terms of properties of cocycles with values in A. In the present paper, we study a circle of problems that is concentrated around actions of group extensions. It is worthy to note that we are mainly interested in topologically trivial group extensions E_f because in this case one can prove deeper results. On the other hand, we believe that the theorems may be generalized to arbitrary group extensions as was done in [Dan] where some of the results of [B] were extended. We first prove that the extendibility property is not generic in some sense. In particular, it proves the existence of non-extendible actions. We also give an explicit example of $\mathbb{R}$-action that cannot be extended to an action of the semi-direct product $\mathbb{R}_+^* \ltimes \mathbb{R}$. Assuming that G is countable, we also answer the question of when two isomorphic actions of A can be extended to isomorphic actions of E_f.

Our study is based on two (important for us) results about actions of amenable groups proved in [BG1, GS1, GS2]. The first result says that any ergodic nonsingular action of an amenable l.c.s.c. group is isomorphic to the Mackey action of this group defined by an ergodic countable approximately finite (a.f.) group Γ of measure preserving automorphisms and a recurrent cocycle over Γ. Moreover since all such automorphism groups are orbit equivalent, we can fix some Γ, then the variety of Mackey actions is determined, up to isomorphism, by classes of weakly equivalent cocycles. The other result states that, roughly speaking, two Mackey actions are isomorphic if and only if the corresponding cocycles are weakly equivalent (see Section 1 for exact definitions and references). Then one can determine the size of the set of cocycles that generate extendible Mackey actions. It turns out that this set is nowhere dense.

The outline of the paper is as follows. In Section 1, we collect all necessary definitions and facts that are used in the article. Section 2 contains the basic results about extendible and non-extendible actions. In the next section, we study an example of a Mackey action of $\mathbb{R}$ that is not extendible to an action of $ax + b$-group. The last section is devoted to the solution of the following problem: find necessary and sufficient conditions under which two isomorphic (extendible) actions of A can be extended to isomorphic actions of E_f.

We will use freely the notions of the full group and its normalizer, approximative finiteness, cocycles, Mackey actions. The necessary definitions can be found, for example, in [HO, Sch]. All equalities below hold a.e. on the appropriate measure space.

1 Preliminaries

We establish the following notations which we will use throughout the paper.

Notations:

- A is an abelian l.c.s.c. group that will be written additively;
- G is an amenable l.c.s.c. group with the identity e;
- $(g, a) \xrightarrow{R} g \cdot a \ : \ G \times A \xrightarrow{R} A$ denotes a Borel action of G on A by group automorphisms. R is jointly continuous by a theorem from [M];
- Γ is a countable ergodic group of automorphisms of a measure space $(X, \mathcal{B}, \mu)$ (as a rule, Γ is measure preserving);
- $Z^1(X \times \Gamma,\ G)$ stands for the set of G-valued cocycles over Γ, (we say $c \in Z^1(X \times \Gamma,\ G)$ if $c(x, \gamma_2\gamma_1) = c(\gamma_1 x, \gamma_2)c(x, \gamma_1)$ for any $\gamma_1, \gamma_2 \in \Gamma$ and a.e. $x \in X$).

Let

$$1 \longrightarrow B \xrightarrow{i} E \xrightarrow{j} G \longrightarrow 1 \tag{1.1}$$

be a topological group extension of a l.c.s.c. group B by G. This means that (i) (1.1) is a short exact sequence where i is a homeomorphism from B onto a normal closed subgroup $i(B) \subset E$, (ii) j is a homomorphism of E onto G which induces a homeomorphism of $E/i(B)$ and G such that a natural action of G by conjugation on $i(B) \simeq B$ coincides with the given action of G on B. Throughout the paper we will identify B and $i(B)$ and refer to E as a *group extension*. Let q be a normalized Borel section from G into E, i.e. $j \circ q = \mathrm{id}$ and $q(e) = e$. Then every $k \in E$ can be uniquely represented as $k = q(g)b$ where $b \in B$. If q can be chosen as a group homomorphism from G into E, then we say that E *splits*. Given (1.1) and a Borel section q, we can define a map $f : G \times G \to B$, called a 2-cocycle, by:

$$f(g, h) = q(gh)^{-1}q(g)q(h).$$

The above definitions become simplier in the case of topologically trivial group extensions of an abelian group A by G. In this settings, we introduce the set $Z^2(G, A)$ of continuous 2-cocycles: $f \in Z^2(G, A)$ if $f(g, e) = f(e, g) = 0$ and

$$g_3^{-1} \cdot f(g_1, g_2) + f(g_1 g_2,\ g_3) = f(g_2, g_3) + f(g_1,\ g_2 g_3), \tag{1.2}$$

where $g, g_1, g_2, g_3 \in G$. We may equip $E = G \times A$ with the product topology and for each $f \in Z^2(G, A)$ we define a group structure on E as follows:

$$(g_1, a_1)(g_2, a_2) = (g_1 g_2,\ f(g_1, g_2) + g_2^{-1} \cdot a_1 + a_2), \tag{1.3}$$

$$(g, a)^{-1} = (g^{-1},\ -f(g, g^{-1}) - g \cdot a). \tag{1.4}$$

The set E, equipped with the group structure (1.3) and (1.4), is called a *topologically trivial group extension* of A by means of G and denoted by $E_f(G, A)$ (or simply E_f). For a continuous map $p : G \to A,\ p(e) = 0$, define the 2-cocycle $f_p \in Z^2(G, A)$ by

$$f_p(g_1, g_2) = \ -g_2^{-1} \cdot p(g_1) + p(g_1 g_2) - p(g_2). \tag{1.5}$$

Then f_p is called a 2-coboundary. The set of all 2-coboundaries is denoted by $B^2(G, A)$. The quotient $H_c^2(G, A) = Z^2(G, A)/B^2(G, A)$ is the group of continuous 2-cohomologies. It is well known that $H_c^2(G, A)$ is isomorphic to $Ext_t(G, A)$, the group of equivalence classes of topologically trivial group extensions. Note that $E_f(G, A)$ is isomorphic to $E_{f'}(G, A)$ if and only if $f - f'$ is a 2-coboundary. The case when $f = 0$ (or f is a 2-coboundary) is of a crucial importance. The group extension $E_0(G, A)$ is called a *semi-direct product* of G and A. The notation $G \ltimes A$ is also used for $E_0(G, A)$.

Later we will use the following statement. Its proof is a slight modification of an argument given by Banach [Ba].

Lemma 1.1. *Let p be a normalized Borel map from G into A and let f_p be defined by* (1.5)*. If $f_p : G \times G \to A$ is separately continuous, then p is also continuous.*

Proof. Let M be a meager set such that $p(g)$ is continuous for all $g \in G - M$. Take some $g_0 \in G$ and let $g_n \to g_0$. We will show that $p(g_n) \to p(g_0)$. The set $M' = \cup_n M g_n^{-1}$ is also meager. Since G is of the second category, $G - M'$ is not empty and there is some $g' \in G - M'$. Then $g' g_n \in G - M$ for all n. It follows from (1.5) that

$$p(g_n) = \ -f_p(g', g_n) - g_n^{-1} \cdot p(g') + p(g' g_n).$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} p(g_n) = \ -f_p(g', g_0) - g_0^{-1} \cdot p(g') + p(g' g_0) = p(g_0).$$

□

We will use the notions of cocycles and H-cocycles over an automorphism group Γ of $(X, \mathcal{B}, \mu)$. H-cocycles appeared first in [U] and then studied in [B, DaD, Da1, Da2]).

Definition 1.2. *Let $f \in Z^2(G, A)$ and $c \in Z^1(X \times \Gamma,\ G)$. A measurable map $\alpha : X \times \Gamma \to A$ is called an H-cocycles if it satisfies the following conditions for $\gamma_1, \gamma_2 \in \Gamma$ and a.e. $x \in X$:*

$$\alpha(x, \mathbb{I}) = 0,$$

$$\alpha(x, \gamma_2\gamma_1) = f(c(\gamma_1 x, \gamma_2), c(x, \gamma_1)) + c(x, \gamma_1)^{-1} \cdot \alpha(\gamma_1 x, \gamma_2) + \alpha(x, \gamma_1) \quad (1.6)$$

where $\mathbb{I}$ is the identity map. The set of all H-cocycles is denoted by $Z^1_{f,c}(X \times \Gamma, A)$ (or $Z^1_{f,c}(A)$). If for an H-cocycle $\delta(x, \gamma)$ there exist a normalized Borel map $p : G \to A$ and a measurable map $a : X \to A$ such that

$$\delta(x, \gamma) = p(c(x, \gamma)) + c(x, \gamma)^{-1} \cdot a(\gamma x) - a(x)$$

then δ is called an H-coboundary.

H-cocycles arise naturally in the following way. Let π be a cocycle over $X \times \Gamma$ with values in E_f. Then $\pi = (c, \alpha)$ where c and α are the projections of π onto G and A respectively. It is easily seen that $c \in Z^1(X \times \Gamma, G)$ and $\alpha \in Z^1_{f,c}(X \times \Gamma, A)$. The converse is also true [B, Da2].

Let K be a l.c.s.c. group with the Haar measure m_K. Let $\Gamma \subset Aut(X, \mathcal{B}, \mu)$, and $c \in Z^1(X \times \Gamma, K)$. Define the group of automorphisms $\Gamma(c) \subset Aut(X \times K, \mu \times m_K)$ whose elements act by the formula:

$$\gamma(c)(x, k) = (\gamma x, c(x, \gamma)k), \quad (x, k) \in X \times K,\ \gamma \in \Gamma. \quad (1.7)$$

The group $\Gamma(c)$ is called the *skew product.* If $\Gamma(c)$ is ergodic on $(X \times K, \mu \times m_K)$, then the cocycle c is said to be of *dense range* in K [Sch].

Let us consider the action V of K on $(X \times K, \mu \times m_K)$:

$$V(h)(x, k) = (x, kh^{-1}), \quad h \in K.$$

Denote by ξ the measurable partition of $X \times K$ into $\Gamma(c)$-ergodic components. The groups $\Gamma(c)$ and $V(K)$ pairwise commute. Therefore, V generates on $((X \times K)/\xi,\ (\mu \times m_K)/\xi)$ a new action $W_{(\Gamma,c)}$ of K which is called the *Mackey action* (or the *action associated to the pair* (Γ, c)). Note that $W_{(\Gamma,c)}(K)$ is ergodic if and only if Γ is ergodic.

Remark 1.3. Recall some results from [BG1, GS1, GS2] about Mackey actions that will be used later on.

(1) It was proved that if $U(K)$ is an amenable ergodic nonsingular action of K on a measure space, then there exists a pair (Γ, d), where Γ is a countable ergodic approximately finite (a.f.) group of measure preserving automorphisms and d is a recurrent cocycle from $Z^1(X \times \Gamma, K)$, such that $U(K)$ and $W_{(\Gamma,d)}(K)$ are isomorphic. In particular, Γ may be taken to be of the form $\Gamma(c)$ where c is a cocycle with dense range in some amenable l.c.s.c. group G.

(2) Let Γ_i be an ergodic a.f. measure preserving group of automorphisms of $(X_i, \mathcal{B}_i, \mu_i)$ and let $d_i \in Z^1(X \times \Gamma_i, K)$ be a recurrent cocycle, $i = 1, 2$. Then, the Mackey actions $W_{(\Gamma_1,d_1)}(K)$ and $W_{(\Gamma_2,d_2)}(K)$ are isomorphic if and only if there is an isomorphism $R : X_1 \to X_2$ such that $R[\Gamma_1]R^{-1} = [\Gamma_2]$ and

cocycles $d_1(x,\gamma_1)$ and $d_2\circ R(x,\gamma_1) := d_2(Rx, R\gamma_1 R^{-1}),\ (x,\gamma_1)\in X_1\times\Gamma_1$, are cohomologous, i.e. there exists a measurable map $\varphi : X_1 \to K$ such that $d_2\circ R(x,\gamma_1) = \varphi(\gamma_1 x)d_1(x,\gamma_1)\varphi(x)^{-1}$. Such cocycles (or, more generally, the pairs (Γ_1, d_1) and (Γ_2, d_2)) are called *weakly equivalent*. We will use the fact that if c and c_1 are cocycles with dense range over Γ, then they are weakly equivalent.

(3) Let $\{U(K)\}$ be the class of K-actions isomorphic to an action U of K. Let $\Gamma(c)$ be an ergodic countable a.f. measure preserving group where c is a cocycle with dense range in G. It follows from the above facts that $\{W(K)\}$ contains the Mackey action $W_{(\Gamma(c),d)}(K)$ where d is a recurrent cocycle over $\Gamma(c)$ with values in K. Conversely, if $\Gamma(c)$ is fixed, then every cocycle d over $\Gamma(c)$ determines a class of isomorphic K-actions. Furthermore, two cocycles d and d_1 over $\Gamma(c)$ determine the same class if and only if they are weakly equivalent.

We will also need the following statement.

Lemma 1.4. *Let (X,μ) be a Lebesgue space and let K and H be l.c.s.c. groups with the Haar measures m_K and m_H respectively. Suppose that F is a measurable map from $(X\times K, \mu\times m_K)$ into (H, m_H) such that $F(x,k) = h_0$ for a.e. $(x,k)\in X\times K$ where $h_0\in H$. Assume that F is continuous in k for μ-a.e. $x\in X$. Then there exists a measurable set $D\subset X,\ \mu(X-D)=0$, such that $F(x,k)=h_0$ for all $(x,k)\in D\times K$.*

Proof. Let $N = \{(x,k)\in X\times K\ :\ F(x,k)\neq h_0\}$, then $(\mu\times m_K)(N) = 0$. Denote $N(x) = \{k\in K\ :\ (x,k)\in N\}$ and define $D = \{x\in X\ :\ m_K(N(x)) = 0\}\cap\{x\in X\ :\ k\mapsto F(x,k)$ is continuous in $k\}$. Clearly, $\mu(X-D)=0$. Let $x_0\in D$ and $k_0\in N(x)$. Since $m_K(N(x_0))=0$, there exists a sequence $\{k_n\}\subset K$ such that $k_n\to k_0$ and $k_n\in G-N(x_0),\ n\in\mathbb{N}$. (If there were no such a sequence, then $N(x_0)$ would contain a neighborhood of k_0, i.e. $m_K(N(x_0))$ would be greater than 0). Therefore for $(x_0,k_0)\in D\times K$ we have $F(x_0,k_0) = \lim_n F(x_0,k_n) = h_0$. □

2 Extendible and non-extendible actions

In this section, we will answer the question whether the extendibilty property is typical. We first give a precise definition of extendibility. As we mentioned above, our prime interest is focused on topologically trivial group extensions.

Definition 2.1. *Let V be an ergodic action of an abelian l.c.s.c. group A by nonsingular automorphisms on a measure space $(X,\mathcal{B},\mu)$. Let $0\longrightarrow A\stackrel{i}{\longrightarrow}$*

$E \xrightarrow{j} G \longrightarrow 1$ be a group extension of A by an amenable l.c.s.c. group G. We say that V is extendible to an action W of E if there exists an action of E on $(X, \mathcal{B}, \mu)$ such that $V(A)$ is isomorphic to $W(i(A))$. In particular, E can be taken as a topologically trivial group extension $E_f(G, A)$ defined by a continuous 2-cocycle $f : G \times G \to A$.

Remark. Let W be an ergodic action of E (or $E_f(G, A)$) on a measure space and let θ be a group automorphism of E that acts identically on A. We denote the group of all such automorphisms by $Aut(E; A)$. In Appendix 2, elements from $Aut(E_f; A)$ are described explicitly. Given $\theta \in Aut(E; A)$, one can define a new action $\theta^*(W)$ of E by setting up $\theta^*(W)(k) = W(\theta(k)),\ k \in E$. It is clear that W and $\theta^*(W)$ have the same action of A. Moreover, if an action V of A is extendible to an action W of E, then V is also extendible to the action $\theta^*(W)$ of E for any group automorphism $\theta \in Aut(E; A)$. Therefore, a given extendible action V of A corresponds to a family of actions of E such that each of them extends V. From this point of view, we may identify W and $\theta^*(W)$ for every group automorphism $\theta \in Aut(E; A)$.

Suppose that we are given a group extension $0 \longrightarrow A \xrightarrow{i} E \xrightarrow{j} G \longrightarrow 1$. It is natural to ask whether one can extend the translation of A onto itself to an action of E? Clearly, such an action must be transitive on A.

We note that since A is considered as a subgroup of E, the group operation in E (and hence in A) is written multiplicatively.

Theorem 2.2. *Let T be the translation on $A : T(a)(b) = ab,\ a, b \in A$. Then: (1) T is extendible to an action of E if and only if E splits. (2) If $E = E_f(G, A)$, then T is extendible to an action of E_f if and only if f is a 2-coboundary.*

Proof. (1) Suppose that $q : G \to E$ is a group homomorphism such that $j \circ q = \mathrm{id}$. We can easily find an action of E that extends T. Given $k \in E$ find $g \in G,\ a \in A$, such that $k = q(g)a$. Define $g \cdot a = q(g)aq(g)^{-1}$. Then set up

$$W(k)(b) = W(q(g)a)(b) = g \cdot (ab), \qquad k \in E.$$

Clearly, W extends T. Next, if $k_1 = q(g_1)a_1,\ k_2 = q(g_2)a_2$, then

$$\begin{aligned} W(k_1k_2)(b) &= W(q(g_1)a_1q(g_2)a_2))(b) \\ &= W(q(g_1g_2)(g_2^{-1} \cdot a_1)a_2))(b) \\ &= (g_1g_2) \cdot [g_2^{-1} \cdot a_1)a_2b)] \\ &= (g_1 \cdot a_1)(g_1g_2) \cdot (a_2b) \end{aligned}$$

$$
\begin{aligned}
&= g_1 \cdot [a_1 g_2 \cdot (a_2 b)] \\
&= W(q(g_1)a_1)W(q(g_2)a_2)(b) \\
&= W(k_1)W(k_2)(b).
\end{aligned}
$$

Note that if $k = q(g)a$, then $k^{-1} = q(g^{-1})(g \cdot a^{-1})$. Therefore

$$
\begin{aligned}
W(k^{-1})(b) &= g^{-1} \cdot [(g \cdot a^{-1})b] \\
&= a^{-1}(g^{-1} \cdot b) \\
&= W(k)^{-1}(b)
\end{aligned}
$$

since the map $b \mapsto a^{-1}(g^{-1} \cdot b)$ is inverse to $b \mapsto g \cdot (ab)$. In this proof we have not used that A is abelian.

To prove the converse we have to assume that A is abelian (the group operation in E and A will be again written multiplicatively). Let T be extendible to an action W of E. Let $q : G \to E$ be a Borel normalized section. Denote $W(q(g)) = \tau(g)$, then $(g, b) \mapsto \tau(g)(b)$ is a Borel map from $G \times A$ into A that leaves the Haar measure m_A quasi-invariant for every $g \in G$. Then $g \in G$ defines a group homomorphism $a \mapsto g \cdot a$ where, by definition, $g \cdot a = q(g)aq(g)^{-1}$. Let $f : G \times G \to A$ be a 2-cocycle such that $q(g_1g_2)f(g_1, g_2) = q(g_1)q(g_2)$. Then we get the following relations:

$$\tau(g)T(a) = T(g \cdot a)\tau(g). \tag{2.1}$$

$$\tau(g_1)\tau(g_2) = \tau(g_1g_2)T(f(g_1, g_2)). \tag{2.2}$$

(2.1) implies that $\tau(g)(ab) = (g \cdot a)\tau(g)(b)$ for all $b \in A$. Then, for $b = 1$, we have

$$\tau(g)(a) = (g \cdot a)s_g \tag{2.3}$$

where $s_g = \tau(g)(1)$ is a Borel map from G into A. In such a way, the W-"action" of G (i.e. the maps $\tau(g)$) can be found by (2.3). Note that (2.1) holds automatically if (2.3) is true. Furthermore, it follows from (2.2) and (2.3) that there is a connection between f and s_g. We have

$$\tau(g_1)\tau(g_2)(b) = [(g_1g_2) \cdot b](g_1 \cdot s_{g_2})s_{g_1}$$

$$\tau(g_1g_2)T(f(g_1, g_2))(b) = [(g_1g_2) \cdot b][(g_1g_2) \cdot f(g_1, g_2)]s_{g_1g_2}.$$

Thus, we get the following relation on A:

$$(g_1g_2) \cdot f(g_1, g_2) = (g_1 \cdot s_{g_2})(s_{g_1g_2})^{-1}s_{g_1}$$

or

$$f(g_1, g_2) = (g_2^{-1} \cdot s_{g_2})[(g_1g_2)^{-1}) \cdot s_{g_1g_2}]^{-1}[(g_2^{-1}g_1^{-1}) \cdot s_{g_1}].$$

Denote $p(g) = (g^{-1} \cdot s_g)^{-1}$. Then

$$f(g_1, g_2) = [g_2^{-1} \cdot p(g_1)]^{-1} p(g_1 g_2)(p(g_2))^{-1}.$$

Clearly, $p(e) = 0$. Therefore, E splits since f is a 2-coboundary.

(2) In the case $E = E_f(G, A)$ the proof is the same. We should only note that because f is continuous and p is Borel, then, by Lemma 1.1, we get that p is a continuous map, and therefore, f is a 2-coboundary. □

Remark. The fact that $\tau(g)$ (in the proof of Theorem 2.2) must be actually continuous we can get also from (2.3). Indeed, since $\tau(g)(a)(g \cdot a)^{-1}$ does not depend on a, we see that $\tau(g)(a)(\tau(g)(b))^{-1} = g \cdot (ab^{-1})$. The latter and continuity of G-action on A imply that $\tau(g)$ is continuous for every $g \in G$.

We recall some notations and results from [B] that will be used later on.

Let $\pi : X \times \Gamma \to E_f(G, A)$ be a cocycle over a countable ergodic measure preserving group of automorphisms Γ acting on $(X, \mathcal{B}, \mu)$. Then $\pi = (c, \alpha)$ where $c \in Z^1(X \times \Gamma, G)$ and $\alpha \in Z^1_{f,c}(X \times \Gamma, A)$.

It can be easily verified that every $\alpha \in Z^1_{f,c}(X \times \Gamma, A)$ generates a cocycle b_α from $Z^1(X \times G \times \Gamma(c), A)$ where

$$b_\alpha(x, h, \gamma(c)) = h^{-1} \cdot \alpha(x, \gamma) + f(c(x, \gamma), h), \quad \gamma \in \Gamma. \tag{2.4}$$

We obtain two simple consequences of this fact. Firstly, $\alpha \mapsto b_\alpha$ defines a map S from $Z^1_{f,c}(X \times \Gamma, A)$ into $Z^1(X \times G \times \Gamma(c), A)$ where $f \in Z^2(G, A)$, $c \in Z^1(X \times \Gamma, G)$. Denote by $I(f, c)$ the image of $Z^1_{f,c}(X \times \Gamma, A)$ under the map S. Then $I(f, c) \subset Z^1(X \times G \times \Gamma(c), A)$ and we will see below that cocycles from $I(f, c)$ produce extendible actions of A via the Mackey construction. One can show (see [B]) that a cocycle $d : X \times G \times \Gamma(c) \to A$ belongs to $I(f, c)$ if and only if

$$d(x, h, \gamma(c)) = h^{-1} \cdot d(x, e, \gamma(c)) + f(c(x, \gamma), h) \tag{2.5}$$

for a.e. $(x, h) \in X \times G$. Indeed, since the right hand side in (2.5) is continuous in h, (2.5) holds for a.e. $x \in X$ and all $h \in G$, due to Lemma 1.4. Define $\alpha(x, \gamma) = d(x, e, \gamma(c))$, then $d = b_\alpha$. Secondly, we can consider the Mackey action $W_{(\Gamma(c), b_\alpha)}$ of A associated with $(\Gamma(c), b_\alpha)$ as well as the Mackey action $W_{(\Gamma, \pi)}$ of (e, A) associated with (Γ, π). It turns out that these two actions are isomorphic.

Theorem 2.3. [B] *Given a cocycle $\pi = (c, \alpha) : X \times \Gamma \to E_f(G, A)$, let b_α be defined by* (2.4)*. Then, $W_{(\Gamma(c), b_\alpha)}(A)$ is isomorphic to $W_{(\Gamma, \pi)}(e, A)$.*

We will study only ergodic actions of E_f and (e, A). Theorem 2.3 shows that, in this case, cocycles $c \in Z^1(X \times \Gamma,\ G)$ must necessarily be of dense range.

The next theorem answers the question when an ergodic action of A can be extended to an action of $E_f(G,A)$. The key point here is that, without loss of generality, we may deal only with Mackey actions of A and E_f and therefore use the results mentioned in Remark 1.3.

Theorem 2.4. [B] *Let V be an ergodic nonsingular action of A on a measure space (Ω, m) and let f be a 2-cocycle from $Z^2(G,A)$. Then the following statements are equivalent:*

(i) V is extendible to an action of $E_f(G,A)$;

(ii) for some cocycle $c \in Z^1(X\times\Gamma, G)$ with dense range, there exists a cocycle $d \in I(f,c)$ such that $V(A)$ is isomorphic to the Mackey action $W_{(\Gamma(c),d)}(A)$;

(iii) for every cocycle $c \in Z^1(X\times\Gamma, G)$ with dense range, there exists a cocycle $d \in I(f,c)$ such that $V(A)$ is isomorphic to the Mackey action $W_{(\Gamma(c),d)}(A)$.

The last statement of Theorem 2.4 asserts that extendibility of $V(A)$ does not depend on a choice of c. In other words, this property does not depend on a realization of $V(A)$ as an associated action.

To clarify Theorem 2.4, we note that if $d \in I(f,c)$, then $W_{(\Gamma(c),d)}(A)$ is extendible to an action of $E_f(G,A)$. Indeed, since $d = b_\alpha$, we see by Theorem 2.3 that

$$W_{(\Gamma(c),d)}(A) \simeq W_{(\Gamma(c),b_\alpha)}(A) \simeq W_{(\Gamma,\pi)}(e,A).$$

The last Mackey action is obviously extendible to the action $W_{(\Gamma,\pi)}(E_f)$. To see that (ii) and (iii) are equivalent, we can use the following statement proved in [B]: If c and c_1 are two cocycles from $Z^1(X\times\Gamma, G)$ with dense ranges in G, then for any cocycle $d \in I(f,c) \subset Z^1(X\times G\times\Gamma(c), A)$ there exists a cocycle $d_1 \in I(f,c_1) \subset Z^1(X\times G\times\Gamma(c_1), A)$ such that the Mackey actions $W_{(\Gamma(c),d)}(A)$ and $W_{(\Gamma(c_1),d_1)}(A)$ are isomorphic.

The next proposition gives another approach to the extendibility problem. We will work here with a group extension E of an abelian group A by G as in Theorem 2.2. Let V be an ergodic action of the group A on a measure space $(X,\mathcal{B},\mu)$, and suppose that $\tau : G \to Aut(X,\mathcal{B},\mu)$ is a map satisfying the conditions: $\tau(g_1)\tau(g_2) = \tau(g_1g_2)V(f(g_1,g_2))$ and $\tau(g^{-1}) = \tau(g)^{-1}V(f(g,g^{-1}))$ (the latter is equivalent to $\tau(e) = \mathbb{I}$) where f is a 2-cocycle defined by a normalized section $g : G \to E$ as in Section 1 and $g, g_1, g_2 \in G$. We call such a τ an f-action of G with respect to V.

Proposition 2.5. *An ergodic action V of A is extendible to an action W of E if and only if there exists a f-action τ of G such that*

$$\tau(g)V(a)\tau(g)^{-1} = V(g\cdot a),\ g\in G,\ a\in A \tag{2.6}$$

where $g \cdot a = q(g)aq(g)^{-1}$.

Proof. Here we use the same notations as in Theorem 2.2. Assume that V is extendible to an action W of E. Then it is straightforward to check that $\tau(g) = W(q(g))$, $g \in G$, is the required f-action of G. Conversely, suppose that an f-action τ of G satisfying (2.6) is given. Every $k \in E$ can be written as $k = q(g)a$. Set up $W(q(g)a) = \tau(g)V(a)$. Let us check that W is the action of E which extends V. Obviously, $W(a) = V(a)$, $a \in A$. Next, given $k_1 = q(g_1)a_1$, $k_2 = q(g_2)a_2$, we have

$$\begin{aligned}
W(q(g_1)a_1)W(q(g_2)a_2) &= V(g_1 \cdot a_1)\tau(g_1)\tau(g_2)V(a_2) \\
&= V(g_1 \cdot a_1)\tau(g_1 g_2)V(f(g_1, g_2)a_2) \\
&= \tau(g_1 g_2)V(g_2^{-1} \cdot a_1)V(f(g_1, g_2)a_2) \\
&= W(q(g_1 g_2)f(g_1, g_2)(g_2^{-1} \cdot a_1)a_2) \\
&= W(q(g_1)q(g_2)(g_2^{-1} \cdot a_1)a_2) \\
&= W(q(g_1)a_1 q(g_2)a_2)
\end{aligned}$$

and

$$\begin{aligned}
W((q(g)a)^{-1}) &= W(q(g)^{-1}(g \cdot a^{-1})) \\
&= W(q(g^{-1})[f(g, g^{-1})]^{-1}(g \cdot a^{-1})) \\
&= \tau(g^{-1})V(f(g, g^{-1})^{-1})V(g \cdot a^{-1}) \\
&= \tau(g)^{-1}V(g \cdot a)^{-1} \\
&= (\tau(g)V(a))^{-1} \\
&= W(q(g)a)^{-1}.
\end{aligned}$$

□

Based on Theorem 2.2, we can deduce some results about extendibility of Mackey actions associated to $\Gamma(c)$-coboundaries. We recall that cocycles $c \in Z^1(X \times \Gamma,\ G)$ are assumed to be of dense range because we study ergodic actions. From now, we work with topologically trivial group extensions.

Lemma 2.6. *Let* $c \in Z^1(X \times \Gamma,\ G)$ *be a cocycle with dense range and let* $d \in Z^1(X \times G \times \Gamma(c), A)$ *be a* $\Gamma(c)$*-coboundary. Then* $d \in I(0, c)$ *if and only if for a.e.* $x \in X$ *and all* $h \in G$ *there exists a measurable map* $\xi : X \to A$ *such that*

$$d(x, h, \gamma(c)) = (h^{-1}c(x, \gamma)^{-1}) \cdot \xi(\gamma x) - h^{-1} \cdot \xi(x). \tag{2.7}$$

Proof. It is straightforward to check that if d satisfies (2.7), then $d \in I(0,c)$. Conversely, let d be a $\Gamma(c)$-coboundary, i.e. we assume that there exists a measurable map $s : X \times G \to A$ such that $d(x,h,\gamma(c)) = s(\gamma(c)(x,h)) - s(x,h)$ a.e. Since $d \in I(0,c)$, we have from (2.5) that

$$s(\gamma(c)(x,h)) - s(x,h) = h^{-1} \cdot s(\gamma(c)(x,e)) - h^{-1} \cdot s(x,e)$$

or

$$s(\gamma(c)(x,h)) - h^{-1} \cdot s(\gamma(c)(x,e)) = s(x,h) - h^{-1} \cdot s(x,e).$$

In view of ergodicity of $\Gamma(c)$, we see that there exists $a \in A$ such that $s(x,h) = h^{-1} \cdot s(x,e) + a$ for a.e. $x \in X$ and all $h \in G$ (Lemma 1.4). Therefore, $d(x,h,\gamma(c))$ has the desired form. □

Theorem 2.7. *Let $d \in Z^1(X \times G \times \Gamma(c), A)$ and suppose that d is a $\Gamma(c)$-coboundary. Then $W_{(\Gamma(c),d)}(A)$ is extendible to an action $E_f(G,A)$ if and only if f is a 2-coboundary.*

Proof. The assertion is a direct consequence of Theorem 2.2 because $W_{(\Gamma(c),d)}(A)$ is isomorphic to the translation T on A. □

Corollary 2.8. *Let $B^1(X \times G \times \Gamma(c), A) \subset Z^1(X \times G \times \Gamma(c), A)$ be the subgroup of all $\Gamma(c)$-coboundaries. Then (i) $B^1(X \times G \times \Gamma(c), A) \cap I(0,c) \neq \emptyset$ and (ii) $B^1(X \times G \times \Gamma(c), A) \cap I(f,c) = \emptyset$ if f is not a 2-coboundary.*

Proof. The statements follow immediately from Theorem 2.7. Here we will give a direct proof of the corollary where we assume for simplicity that G is countable.

Let $d \in I(f,c)$ be a $\Gamma(c)$-coboundary. This means that there exists a measurable function $\xi : X \times \Gamma \to A$ such that

$$d(x,h,\gamma(c)) = \xi(\gamma(c)(x,h)) - \xi(x,h)$$

and d satisfies (2.5). Then

$$\xi(\gamma(c)(x,h)) - \xi(x,h) = h^{-1} \cdot \xi(\gamma(c)(x,e)) - h^{-1} \cdot \xi(x,e) + f(c(x,\gamma),h). \quad (2.8)$$

Let $\Gamma_0 = \{\gamma \in [\Gamma] : c(x,\gamma) = e \text{ a.e.}\}$, then Γ_0 is ergodic (recall that G is countable and $\Gamma(c)$ is ergodic). We get from (2.8) that for $\gamma_0 \in \Gamma_0$,

$$\xi(\gamma_0 x, h) - \xi(x,h) = h^{-1} \cdot \xi(\gamma x, e) - h^{-1} \cdot \xi(x,e).$$

It follows from ergodicity of Γ_0 that

$$\xi(x,h) = h^{-1} \cdot \zeta(x) + r(h) \quad (2.9)$$

where $\zeta(x) = \xi(x, e)$, and r is a normalized Borel map from G into A. If we substitute (2.9) into (2.8), then we get

$$f(c(x,\gamma), h) = -h^{-1} \cdot r(c(x,\gamma)) + r(c(x,\gamma)h) - r(h) \tag{2.10}$$

Let $G = \{g_i : i \in \mathbb{N}\}$ and let $\gamma_i \in [\Gamma]$ be such that $c(x, \gamma_i) = g_i$ a.e. Taking $\gamma = \gamma_i$ in (2.10), we get that f is a 2-coboundary. $\square$

We consider on $Z^1(X \times G \times \Gamma(c), A)$ the topology of convergence in measure. It is well known that $Z^1(X \times G \times \Gamma(c), A)$ is a Polish space in this topology and the set of all $\Gamma(c)$-coboundaries is dense in $Z^1(X \times G \times \Gamma(c), A)$ when Γ is approximately finite [Sch].

As we mentioned in Remark 1.3, every ergodic nonsingular A-action is isomorphic to the associated action $W_{(\Gamma(c),d)}(A)$ where an ergodic group $\Gamma(c)$ may be chosen a priori and d is a cocycle from $Z^1(X \times G \times \Gamma(c), A)$. If d is of the form (2.5), then this action is extendible to an action of E_f. Our goal now is to answer the question: How typical is such a cocycle d? In other words, we want to find out if the extendibility property is typical or not. Theorem 2.9 (below) gives the answer: The extendibility property is "nowhere dense".

Let $[d]$ denote the set of cocycles from $Z^1(X \times G \times \Gamma(c), A)$ weakly equivalent to d. Then $[I(f, c)]$ is formed by all classes $[d]$ where $d \in I(f, c)$. Let

$$I(c) = \bigcup_{f \in Z^2(G,A)} I(f, c), \qquad [I(c)] = \bigcup_{f \in Z^2(G,A)} [I(f, c)].$$

Since $I(f, c) + I(f_1, c) = I(f + f_1, c)$, we get that $I(c)$ is a subgroup in $Z^1(X \times G \times \Gamma(c), A)$. Note that $I(f, c) \cap I(f_1, c) = \emptyset$ when $f \neq f_1$. In [B], we showed that $I(c)$ does not depend on c up to isomorphism. If $d \in I(c)$, then d defines (explicitly, see (2.4) and Theorem 2.3) an extendible action for some E_f. If $d \in [I(c)]$, then there exists $f \in Z^2(G, A)$ such that the associated action defined by d is isomorphic to an A-action that can be explicitely extended to an action of E_f.

Theorem 2.9. *Let Γ be an ergodic a.f. group of measure preserving automorphisms and let $c \in Z^1(X \times \Gamma,\ G)$ be a cocycle with dense range. Then $[I(c)]$ is nowhere dense in $Z^1(X \times G \times \Gamma(c), A)$ endowed with the topology of convergence in measure. In other words, a typical action of A is not extendible to an action of E_f.*

Proof. We first note that $I(0, c)$ is a closed subgroup in $Z^1(X \times G \times \Gamma(c), A)$. For this, take $d_k \in I(0, c)$ such that $d_k \to d$ (in measure), $k \in \mathbb{N}$. We see from Lemma 1.4 that the relation $d_k(x, h, \gamma(c)) = h^{-1} \cdot d_k(x, e, \gamma(c))$ holds for all $k \in \mathbb{N}$ and all $(x, h) \in D \times G$ where $\mu(X - D) = 0$. Taking the pointwise limit in the above relation, we obtain that $d \in I(0, c)$.

Then, the formula

$$I(f,c) = d_f + I(0,c) \tag{2.10}$$

showes that $I(f,c)$ is also closed where d_f is some fixed cocycle from $I(f,c)$. Note that the map $f \mapsto d_f : Z^2(G,A) \to Z^1(X \times G \times \Gamma(c), A)$ can be chosen continuous. Indeed, since Γ is an a.f. group of automorphisms, we can take $\Gamma = \{T^n : n \in \mathbb{N}\}$ [CFW]. Let $u : X \to A$ be a measurable function. For given $f \in Z^2(G,A)$, we set

$$\alpha^f(x,T) = u(x),$$

$$\alpha^f(x,T^2) = f(c(Tx,T), c(x,T)) + c(x,T)^{-1} \cdot u(Tx) + u(x)$$

and so on. In such a way, we define an H-cocycle α^f (see (1.6)). Denote

$$d_f(x,h,T^n(c)) = h^{-1} \cdot \alpha^f(x,T^n) + f(c(x,T^n),h).$$

Clearly, $d_f \in I(f,c)$ and the map $f \mapsto d_f$ is continuous. Therefore $I(c)$ is a closed subgroup in $Z^1(X \times G \times \Gamma(c), A)$. Since Γ (hence $\Gamma(c)$) is a.f., the set of all $\Gamma(c)$-coboundaries, $B^1(X \times G \times \Gamma(c), A)$, is dense in $Z^1(X \times G \times \Gamma(c), A)$. It follows from Corollary 2.8 that $I(f,c)$ is nowhere dense when f is not a 2-coboundary. Although the sets $I(0,c)$ and $I(f_p,c)$ can contain some $\Gamma(c)$-coboundaries (Lemma 2.6), relation (2.10) proves that they are also nowhere dense. Note that $[d]$ contains a $\Gamma(c)$-coboundary if and only if d is a coboundary. Therefore the same argument works for $[I(f,c)]$ proving that this set is also nowhere dense. Next, it follows from the proved facts that $Z^1(X \times G \times \Gamma(c), A) - I(c)$ and $Z^1(X \times G \times \Gamma(c), A) - [I(c)]$ are dense. This remark implies that $[I(c)]$ is nowhere dense. $\square$

Let us consider the case of a *countable* group G. The next theorem gives a sufficient condition for a cocycle $d \in Z^1(X \times G \times \Gamma(c), A)$ to belong to $I(c)$. In other words, for d satisfying the condition of the theorem, there exists $f \in Z^2(G,A)$ such that $W_{(\Gamma(c),d)}(A)$ may be extended to an action of E_f.

Theorem 2.10. *Let Γ be an ergodic a.f. countable group of automorphisms of $(X, \mathcal{B}, \mu)$ and let c be a cocycle over Γ with dense range in a countable amenable group G. Assume that for given $d \in Z^1(X \times G \times \Gamma(c),\ A)$ there exists a subset $N \subset X,\ \mu(N) = 0$, such that $d(x,h,\gamma(c)) - h^{-1} \cdot d(x,e,\gamma(c))$ does not depend on $x \in X - N$. Then $d \in I(c)$.*

Proof. It follows from [BG2] that Γ is generated by ergodic subgroup $\Gamma_0 = \{\gamma \in [\Gamma] \ : \ c(x,\gamma) = e$ for a.e. $x \in X\}$ and the automorphisms $\gamma_g \in [\Gamma] \cap N[\Gamma_0]$ such that $c(x,\gamma_g) = g$ for a.e. $x \in X$ and $g \in G$. Set $\gamma_e = \mathbb{I}$. Then, for any $\gamma \in \Gamma$ and a.e. $x \in X$, there are $g = g(x) \in G$ and $\gamma_0 \in \Gamma_0$ such that $\gamma x = \gamma_g \gamma_0 x$. We also note that, without loss of generality, Γ may be taken as a free a.f. group automorphisms. It allows one to extend d to the full group $[\Gamma]$. By the assumption of the theorem, we may define

$$F(\gamma, h) = d(x, h, \gamma(c)) - h^{-1} \cdot d(x, e, \gamma(c)), \tag{2.11}$$

where $\gamma \in \Gamma$ and $x \in X - N$, $\mu(N) = 0$. We note that when h is fixed and γ runs over the full subgroup $\Gamma_0 \subset [\Gamma]$, the function $F(\gamma, h)$ defines a group homomorphism F_h from Γ_0 into A. Furthermore, F_h is continuous with respect to the uniform topology. Thus, the kernel of F_h, $ker(F_h)$, must be a normal closed subgroup in the ergodic group Γ_0. It follows from [Dye] that $ker(F_h)$ is either $\{\mathbb{I}\}$ or Γ_0. But if for $\gamma_1, \gamma_2 \in \Gamma_0$, $\mu(\{x \in X : \gamma_1 x = \gamma_2 x\}) > 0$, then $F_h(\gamma_1) = F_h(\gamma_2)$. This shows that $ker(F_h) = \Gamma_0$. In other words, we proved that for all $\gamma_0 \in \Gamma_0$

$$d(x, h, \gamma_0(c)) - h^{-1} \cdot d(x, e, \gamma_0(c)) = 0. \tag{2.12}$$

Define

$$f(g, h) = F(\gamma_g, h), \qquad g, h \in G. \tag{2.13}$$

Note that $f(e, g) = f(g, e) = 0$, $g \in G$. Next, take $\gamma \in G$, then $\gamma x = \gamma_g \gamma_0 x$ where g depends on x and $\gamma_0 \in \Gamma_0$. We get for a.e. $x \in X$ that

$$\begin{aligned}
& d(x, h, \gamma(c)) - h^{-1} \cdot d(x, e, \gamma(c)) \\
= \; & d(x, h, \gamma_g(c)\gamma_0(c)) - h^{-1} \cdot d(x, e, \gamma_g(c)\gamma_0(c)) \\
= \; & d(\gamma_0 x, h, \gamma_g(c)) - h^{-1} \cdot d(\gamma_0 x, e, \gamma_g(c)) \\
& + d(x, h, \gamma_0(c)) - h^{-1} d(x, e, \gamma_0(c)) \\
= \; & f(g, h) \qquad \text{(in view of (2.12), (2.13))} \\
= \; & f(c(\gamma_0 x, \gamma_g) c(x, \gamma_0), h) \\
= \; & f(c(x, \gamma), h).
\end{aligned} \tag{2.14}$$

If we proves that f is a 2-cocycle, then (2.14) would imply that $d \in I(f, c) \subset I(c)$. Thus, it remains to show that $f \in Z^2(G, A)$. We have for $g_1, g_2, g_3 \in G$ that

$$g_3^{-1} \cdot f(g_1, g_2) = g_3^{-1} \cdot d(x, g_2, \gamma_{g_1}(c)) - g_3^{-1} g_2^{-1} \cdot d(x, e, \gamma_{g_1}(c)),$$

$$f(g_1 g_2, g_3) = d(x, g_3, \gamma_{g_1 g_2}(c)) - g_3^{-1} \cdot d(x, e, \gamma_{g_1 g_2}(c)),$$

$$f(g_2, g_3) = d(x, g_3, \gamma_{g_2}(c)) - g_3^{-1} \cdot d(x, e, \gamma_{g_2}(c)),$$

$$f(g_1, g_2 g_3) = d(x, g_2 g_3, \gamma_{g_1}(c)) - g_3^{-1} g_2^{-1} \cdot d(x, e, \gamma_{g_1}(c)).$$

To see that the 2-cocycle identity (1.2) holds, we have to show that

$$\begin{aligned} & g_3^{-1} \cdot d(x, g_2, \gamma_{g_1}(c)) + d(x, g_3, \gamma_{g_1 g_2}(c)) - g_3^{-1} \cdot d(x, e, \gamma_{g_1 g_2}(c)) \\ & = d(x, g_3, \gamma_{g_2}(c)) - g_3^{-1} \cdot d(x, e, \gamma_{g_2}(c) + d(x, g_2 g_3, \gamma_{g_1}(c)). \end{aligned} \tag{2.15}$$

is true. It is easily seen that $\gamma_{g_1 g_2}(c) = \gamma_{g_1}(c)\gamma_{g_2}(c)\gamma_0(c)$ for some γ_0 from Γ_0. Therefore, we can get from (2.12) and the assumption of the theorem that

$$\begin{aligned} & d(x, g_3, \gamma_{g_1}(c)\gamma_{g_2}(c)\gamma_0(c)) - g_3^{-1} \cdot d(x, e, \gamma_{g_1}(c)\gamma_{g_2}(c)\gamma_0(c)) \\ = \; & d(\gamma_0 x, g_3, \gamma_{g_1}(c)\gamma_{g_2}(c)) + d(x, g_3, \gamma_0(c)) \\ & - g_3^{-1} \cdot d(\gamma_0 x, e, \gamma_{g_1}(c)\gamma_{g_2}(c)) - g_3^{-1} \cdot d(x, e, \gamma_0(c)) \\ = \; & d(\gamma_0 x, g_3, \gamma_{g_1}(c)\gamma_{g_2}(c)) - g_3^{-1} \cdot d(\gamma_0 x, e, \gamma_{g_1}(c)\gamma_{g_2}(c)) \\ = \; & d(\gamma_{g_2}\gamma_0 x, g_2 g_3, \gamma_{g_1}(c)) + d(\gamma_0 x, g_3, \gamma_{g_2}(c)) \\ & - g_3^{-1} \cdot d(\gamma_{g_2}\gamma_0 x, g_2, \gamma_{g_1}(c)) - g_3^{-1} \cdot d(\gamma_0 x, e, \gamma_{g_2}(c)) \\ = \; & d(\gamma_{g_2}\gamma_0 x, g_2 g_3, \gamma_{g_1}(c)) - g_3^{-1} \cdot d(\gamma_{g_2}\gamma_0 x, g_2, \gamma_{g_1}(c)) \\ & + d(x, g_3, \gamma_{g_2}(c)) - g_3^{-1} \cdot d(x, e, \gamma_{g_2}(c)). \end{aligned} \tag{2.16}$$

It follows from (2.16) that (2.15) may be transformed into the following relation:

$$\begin{aligned} & d(\gamma_{g_2}\gamma_0 x, g_2 g_3, \gamma_{g_1}(c)) - g_3^{-1} \cdot d(\gamma_{g_2}\gamma_0 x, g_2, \gamma_{g_1}(c)) \\ & \quad = d(x, g_2 g_3, \gamma_{g_1}(c)) - g_3^{-1} \cdot d(x, g_2, \gamma_{g_1}(c)). \end{aligned} \tag{2.17}$$

To see that (2.17) is true, let us add and subtract $g_3^{-1} g_2^{-1} \cdot d(\gamma_2\gamma_0 x, e, \gamma_{g_1}(c))$ from the left-hand side of (2.17), and $g_3^{-1} g_2^{-1} \cdot d(x, e, \gamma_{g_1}(c))$ from the right-hand side. Then

$$\begin{aligned} & d(\gamma_{g_2}\gamma_0 x, g_2 g_3, \gamma_{g_1}(c)) - g_3^{-1} g_2^{-1} \cdot d(\gamma_2\gamma_0 x, e, \gamma_{g_1}(c)) \\ & - g_3^{-1} \cdot \left[d(\gamma_2\gamma_0 x, g_2, \gamma_{g_1}(c)) - g_2^{-1} \cdot d(\gamma_2\gamma_0 x, e, \gamma_{g_1}(c)) \right] \\ = \; & d(x, g_2 g_3, \gamma_{g_1}(c)) - g_3^{-1} g_2^{-1} \cdot d(x, e, \gamma_{g_1}(c)) \\ & - g_3^{-1} \cdot \left[d(x, g_2, \gamma_{g_1}(c)) - g_2^{-1} \cdot d(x, e, \gamma_{g_1}(c)) \right]. \end{aligned}$$

Clearly, the last relation (and therefore (2.15)) holds. □

3 Example

In this section we consider the case when $A = \mathbb{R}$, $G = \mathbb{R}_+^*$, and $E_0 = \mathbb{R}_+^* \ltimes \mathbb{R}$. Our aim is to find an explicit example of $\mathbb{R}$-action that cannot be extended to an action of $\mathbb{R}_+^* \ltimes \mathbb{R}$.

Let Γ be a group of automorphism of $(X, \mathcal{B}, \mu)$, $\mu(X) = \infty$, that is generated by three pairwise commuting automorphisms S, T_1, and T_2. We will assume that S is ergodic and measure preserving. Let λ_1 and λ_2 be positive real numbers such that $\lambda_1 > \lambda_2 > 1$ and $\log \lambda_1$, $\log \lambda_2$ are rationally independent, i.e. $\{\lambda_1^n, \lambda_2^m \mid n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}_+^*$. We assume that T_i is an automorphism of X such that $\mu \circ T_i = \lambda_i^{-1}\mu$, $i = 1, 2$. Define a cocycle $c : X \times \Gamma \to \mathbb{R}_+^*$ by its values on generators S, T_1, T_2:

$$c(x, S) = 1, \quad c(x, T_1) = \lambda_1, \quad c(x, T_2) = \lambda_2. \tag{3.1}$$

Then the skew product $\Gamma(c)$ acts on $(X \times \mathbb{R}_+^*,\ \mu \times du)$ (here du is the Lebesgue measure on $\mathbb{R}_+^*$). The generators of $\Gamma(c)$ are measure preserving automorphisms defined by formulae:

$$S(c)(x, p) = (Sx, p), \qquad T_i(c)(x, p) = (T_i x, \lambda_i p) \tag{3.2}$$

where $(x, p) \in X \times \mathbb{R}_+^*$ and $i = 1, 2$. Obviously, $\Gamma(c)$ is ergodic since c has a dense range in $\mathbb{R}_+^*$.

Define now a cocycle d over $X \times \mathbb{R}_+^* \times \Gamma(c)$ with values in $\mathbb{R}$. We set for $i = 1, 2$

$$d(x, p, S(c)) = 0, \quad d(x, p, T_i(c)) = 1. \tag{3.3}$$

Then, d can be extended to a cocycle over $\Gamma(c)$. In fact, d is a homomorphism of the orbit equivalence relation generated by $\Gamma(c)$ into $\mathbb{Z}^2$. Let us write down the generators of $\Gamma(c)(d)$. This group acts on $(X \times \mathbb{R}_+^* \times \mathbb{R},\ \mu \times \nu_+ \times \nu)$ and the generators can be written down as

$$S(c)(d)(x, p, u) = (Sx, p, u)$$

$$T_i(c)(d)(x, p, u) = (T_i x,\ \lambda_i p,\ u + 1), \quad i = 1, 2.$$

We need to find the partition $\mathcal{P}$ into ergodic components of $\Gamma(c)(d)$ and the quotient space $(X \times \mathbb{R}_+^* \times \mathbb{R})/\mathcal{P}$.

Lemma 3.1. *$(X \times \mathbb{R}_+^* \times \mathbb{R})/\mathcal{P}$ is isomorphic to $Y = [1, \alpha) \times [0, 1)$ where $\alpha = \lambda_1 \lambda_2^{-1}$.*

Proof. It follows from ergodicity of S that a $\Gamma(c)(d)$-ergodic component that intersects $X \times \{1\} \times \{0\}$ must contain $X \times \{1\} \times \{0\}$. Then $(X \times \mathbb{R}_+^* \times \mathbb{R})/\mathcal{P}$ is isomorphic to $(\mathbb{R}_+^* \times \mathbb{R})/\mathcal{E}$ where $\mathcal{E}$ is equivalence relation defined by $\tilde{T}_i(p, u) = (\lambda_i p,\ u + 1)$, $i = 1, 2$. Obviously, $\mathcal{E}$ is of type I_∞ since $\tilde{T}_1$ and $\tilde{T}_2$ commute. Let us show that Y is a fundamental set for $\mathcal{E}$. Take some point $(p, u) \in Y$. Then $\tilde{T}_i^n(\mathbb{R}_+^* \times [0, 1)) = \mathbb{R}_+^* \times [n, n + 1)$, $\forall n \in \mathbb{Z}$, $i = 1, 2$. This means that $\tilde{T}_i^n$ $(\neq \mathbb{I})$ cannot send (p, u) to Y. On the other hand,

$\tilde{T}_1\tilde{T}_2^{-1}(\mathbb{R}_+^* \times \{p\}) = \mathbb{R}_+^* \times \{p\}$ for all p. Thus, $\tilde{T}_1\tilde{T}_2^{-1}$ acts on $\mathbb{R}_+^*$ by multiplication by $\alpha = \lambda_1\lambda_2^{-1}$. It follows that $\tilde{T}_1\tilde{T}_2^{-1}$ is again of type I_∞ and $[1,\alpha) \times \mathbb{R}$ is a fundamental set for $\tilde{T}_1\tilde{T}_2^{-1}$. To complete the proof, we note that every $\Gamma(c)(d)$-orbit meets the set Y only once. $\square$

We have shown that the Mackey action of $\mathbb{R}$ associated to the pair $(\Gamma(c), d)$ is defined on Y. Let us compute $W_{(\Gamma(c),d)}(\mathbb{R})$. The translation $V(a) : \ (x,p,u) \mapsto (x,p,u+a)$ determines the transformation $\tilde{V}(a)$ of $\mathbb{R}_+^* \times \mathbb{R} : \ \tilde{V}(a)(p,n) = (p, u+a)$. Obviously, $\tilde{V}(\cdot)$ commutes with $\tilde{T}_i$. This means that $\tilde{V}$ generates the Mackey action of $\mathbb{R}$ on Y. Given $(p,u) \in Y$, find the unique point $(p_1,u_1) \in Y$ such that $(p, u+a)$ and (p_1,u_1) are in the same $\Gamma(c)(d)$-orbit. Let $[x]$ ($\{x\}$) denote the integer (resp. fractional) part of $x \in \mathbb{R}$. Take $n = [u+a]$, then $\tilde{T}_1^{-n}(p, u+a) = (\lambda_1^{-n}p, \ u+a-[u+a]) = (\lambda_1^{-n}p, \{u+a\})$ belongs to $\mathbb{R}_+^* \times [0,1)$. To return the found point into Y, we determine $m \in \mathbb{Z}$ such that $\lambda_1^{-n}p_1 \in [\alpha^{-m}, \alpha^{-m+1})$. Then $\alpha^m\lambda_1^{-n} \in [1,\alpha)$. Therefore for $(p,u) \in Y$ and $a \in \mathbb{R}$, we get

$$W_{(\Gamma(c),d)}(a)(p,u) = (\alpha^m\lambda_1^{-n}p, \ \{u+a\}) = (\lambda_1^{m-n}\lambda_2^{-m}p, \ \{u+a\}) \tag{3.4}$$

Thus, we have proved the following statement.

Proposition 3.2. *The Mackey action $W_{(\Gamma(c),\ d)}(\mathbb{R})$ associated to $(\Gamma(c), \ d)$ with the pair $(\Gamma(c), d)$ defined by* (3.1) $-$ (3.3), *is isomorphic to the special flow $W_{(Q,\varphi)}(\mathbb{R})$ built under the constant ceiling function $\varphi = 1$ and with base automorphism $Q : \ [1,\alpha) \to [1,\alpha)$ such that $Qp = \alpha^m\lambda_1^{-1}p, \ \ m = m(p)$ where m is defined by* (3.4). (See Appendix 1 for an explicit description of Q). $\square$

Theorem 3.3. *$W_{(\Gamma(c),d)}(\mathbb{R})$ is not extendible to an action of $\mathbb{R}_+^* \ltimes \mathbb{R}$.*

Proof. To prove that $W_{(\Gamma(c),d)}(\mathbb{R}) \simeq W_{(Q,1)}(\mathbb{R})$ is not extendible to an action of $\mathbb{R}_+^* \ltimes \mathbb{R}$, we will show that Proposition 2.5 fails. Let us assume that there exists an action τ of $\mathbb{R}_+^*$ on $Y = [1,\alpha) \times [0,1)$ such that for $(x,u) \in Y, \ p \in \mathbb{R}_+^*, \ t \in \mathbb{R}$, one has

$$\tau(p)W(t)(x,u) = W(pt)\tau(p)(x,u) \tag{3.5}$$

where $W(t) = W_{(Q,1)}(t)$. Note that $W(1)(x,u) = (Qx, u), \ (x,u) \in Y$. If $t = 1$, then we get from in (3.5) that

$$\tau(p)(Qx, 0) = W(p)\tau(p)(x,0). \tag{3.6}$$

Denote $\tau(p)(x,0) = (T_p(x), v_p(x))$ where $x \mapsto T_p(x)$ and $x \mapsto v_p(x)$ are measurable.

Assume for definiteness that $p < 1$. Let us find the properties of T_p. We see that the $\tau(p)$-image of $\{x\} \times [0,1)$ is either a subinterval in $\{T_p(x)\} \times [0,1)$ of length p (when $v_p(x) + p < 1$) or the union of two subintervals from $(\{T_p(x)\} \times [0,1)) \cup (\{QT_p(x)\} \times [0,1))$ of total length p (when $v_p(x) + p \geq 1$). It is clear that, for every $p \in \mathbb{R}_+^*$, $\tau(p)$ is a nonsingular automorphism defined everywhere on Y that preserves the partition into W-orbits. Then T_p is a measurable map of X that sends Q-orbits onto Q-orbits. It is easily seen that T_p is not one-to-one and $\nu(T_pD) > 0$ if and only if $\nu(D) > 0$ where ν is the Lebesgue measure on $Z = [1, \alpha)$.

We claim that $T_p : Z \to Z$ is onto. Indeed, let $y = T_p(x)$. Then we observe that every part of W-orbit of $(y, 0)$ that lies between $Z \times \{0\}$ and $Z \times \{1\}$ contains at least one point from $\tau(p)(Z \times \{0\})$. Since every W-orbit is dense in Y as well as every Q-orbit is dense in Z (see Appendix 1), we obtain that $T_p(Z) = Z$.

Next, let $K_y(p) = \{x \in Z : T_p(x) = y\}$, $y \in Z$. Take $p = 1/n$ where n is an integer greater than 1. We note that for every $y \in Z$, the set $K_y(1/n)$ has exactly n points that belong to the same Q-orbit. This fact follows from (3.6) and from the observation that every subinterval $\{y\} \times [(i-1)/n,\ i/n),\ i = 1, ..., n$, contains exactly one point from $\tau(1/n)(Z \times \{0\})$.

Let $Y_i = Z \times [(i-1)/n,\ i/n)$, $i = 1, ..., n$. Denote by

$$Z_i = \{x \in Z \mid \frac{i-1}{n} \leq v_{1/n}(x) < \frac{i}{n}\}.$$

One can easily seen that $T_{1/n}(Z_i) = Z,\ i = 1, ..., n$. In such a way, the above remarks allows one to describe the image of $Z \times \{0\}$ under the map $\tau(p)$.

We state that for $p = 1/n$, the set Z is partitioned into $Z_i,\ i = 1, ..., n$, and

$$Q(Z_i) = Z_{i+1}, \qquad Q(Z_n) = Z_1, \tag{3.7}$$

i.e. $(Z_i : i = 1, ..., n)$ form a Q-tower. In fact, it follows from the following remark: If $\tau(1/n)(x, 0) \in Y_i$, then $\tau(1/n)(Qx, 0) \in Y_{i+1}$ for $i = 1, ..., n-1$ (the case $i = n$ is considered analogously).

We get from relation (3.7) that $Q^n(Z_1) = Z_1$. This contradicts the ergodicity of Q^n (see again Appendix 1). □

Remark 3.4. (1) Theorem 3.3 may be reformulated in terms of cocycles. In other words, it has been proved that the cocycle d, defined by (3.3), cannot be weakly equivalent to any cocycle d_1 from $I(0, c)$, i.e. for all $R \in N[\Gamma]$, $d \circ R(x, p, \gamma(c))$ is not $\Gamma(c)$-cohomologous to $d_1(x, p, \gamma(c)) = p^{-1} d_1(x, 1, \gamma(c))$ (see (2.5) for $f = 0$).

(2) Relation (3.5) (or, more generally, (2.6)) shows that, for every $p \in \mathbb{R}_+^*$ and $t \in \mathbb{R}$, the automorphisms $W(t)$ and $W(pt)$ must be isomorphic. This simple observation allows us to produce a family of $\mathbb{R}$-actions that cannot be

extended to actions of $\mathbb{R}_+^* \ltimes \mathbb{R}$. Let $W_{(Q,\varphi)}(\mathbb{R})$ be an ergodic special flow of measure preserving automorphisms such that the entropy $h(Q) > 0$. Then $W_{(Q,\varphi)}(\mathbb{R})$ is not extendible to an action $\mathbb{R}_+^* \ltimes \mathbb{R}$ since $W(t)$ and $W(pt)$ have different entropies. On the other hand, suppose we are given by an ergodic measure preserving action U of $\mathbb{R}_+^* \ltimes \mathbb{R}$. Then the automorphism group $U(\{1\} \times \mathbb{R})$ has the property: the entropy of every $U(\{1\} \times t)$, $t \in \mathbb{R}$, is either 0 or ∞.

(3) Let $U(t)$ be the horocycle flow and let $\psi(s)$ be the geodesic flow in the Poincaré half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, $s, t \in \mathbb{R}$. It is well known that they are related in the following way:

$$\psi(s)U(t)\psi(-s) = U(te^s). \tag{3.8}$$

Define $\tau(p) = \psi(\log p)$. Then we see that (3.8) is transformed into (2.6). By Proposition 2.5, the horocycle flow can be extended to the action of $\mathbb{R}_+^* \ltimes \mathbb{R}$ defined by τ and U.

4 Isomorphic actions of group extensions

This section is devoted to the solution of the following problem: Suppose we are given by two isomorphic actions of A, V_1 and V_2. Assume they are extendible to actions U_1 and U_2 of $E_f(G, A)$. The question is: Under what conditions U_1 and U_2 are isomorphic?

It was remarked in Section 2 that we can represent any ergodic nonsingular action of E_f as the Mackey action $W_{(\Gamma,\pi)}(E_f)$ where Γ is an ergodic measure preserving a.f. countable group of automorphisms of $(X, \mathcal{B}, \mu)$ and $\pi : X \times \Gamma \to E_f$ is a cocycle. The following proposition is a direct corollary of the results from [GS1, GS2] discussed in Section 1.

Proposition 4.1. *Let U_1 and U_2 be two isomorphic ergodic actions of E_f. Then there exist pairs (Γ, π_1) and (Γ, π_2) such that U_i is isomorphic to $W_{(\Gamma,\pi_i)}, i = 1, 2$, and cocycles π_1, π_2 are weakly equivalent, i.e. there is $R \in N[\Gamma]$ such that $\pi_1 \circ R$ is cohomologous to π_2 where Γ is an ergodic countable a.f. group of measure preserving automorphisms.*

We start with a theorem that solves the problem which is converse to the question formulated above.

Theorem 4.2. *Let $W_{(\Gamma,\pi)}$ and $W_{(\Gamma,\pi_1)}$ be isomorphic ergodic actions of E_f where $\pi = (c, \alpha)$ and $\pi_1 = (c_1, \alpha_1)$ are cocycles with values in $E_f(G, A)$. Then there exists an automorphism $R'(x, h) = (Rx,\ k(x)h)$ of $(X \times G, \mu \times m_G)$, $R \in N[\Gamma]$, such that cocycles $b_\alpha \circ R'$ and b_{α_1} are $\Gamma(c_1)$-cohomologous.*

Proof. It follows from Proposition 4.1 that there exist a measurable function $\Phi(x) = (k(x), u(x)) : X \to E_f$ and an automorphism $R \in N[\Gamma]$ such that for any $\gamma \in \Gamma$ and a.e. $x \in X$,

$$\pi \circ R(x,\gamma) = \Phi(\gamma x)\pi_1(x,\gamma)\Phi(x)^{-1}.$$

In view of (1.4), $\Phi(x)^{-1} = (k(x)^{-1},\ -f(k(x), k(x)^{-1}) - k(x) \cdot u(x))$. Then

$$\pi \circ R(x,\gamma) = (c \circ R(x,\gamma), \alpha \circ R(x,\gamma))$$

$$= (k(\gamma x),\ u(\gamma x))(c_1(x,\gamma),\ \alpha_1(x,\gamma))(k(x)^{-1},\ -f(k(x), k(x)^{-1}) - k(x) \cdot u(x)),$$

and therefore

$$c \circ R(x,\gamma) = k(\gamma x)c_1(x,\gamma)k(x)^{-1}, \tag{4.1}$$

$$\alpha \circ R(x,\gamma) = f(k(\gamma x)c_1(x,\gamma), k(x)^{-1}) + k(x) \cdot f(k(\gamma x), c_1(x,\gamma))$$

$$+k(x)c_1(x,\gamma)^{-1} \cdot u(\gamma x) + k(x) \cdot \alpha_1(x.\gamma) - f(k(x), k(x)^{-1}) - k(x) \cdot u(x). \tag{4.2}$$

Applying (1.2), we get

$$k(x) \cdot f(k(\gamma x), c_1(x,\gamma)) + f(k(\gamma x)c_1(x,\gamma), k(x)^{-1})$$

$$= f(k(\gamma x), c_1(x,\gamma)k(x)^{-1}) + f(c_1(x,\gamma), k(x)^{-1})$$

and (4.2) is transformed into

$$\alpha \circ R(x,\gamma) = f(k(\gamma x), c_1(x,\gamma)k(x)^{-1}) + f(c_1(x,\gamma), k(x)^{-1}) - f(k(x), k(x)^{-1})$$

$$+k(x) \cdot \alpha_1(x,\gamma) + k(x)c_1(x,\gamma)^{-1} \cdot u(\gamma x) - k(x) \cdot u(x). \tag{4.3}$$

It is easy to check that if $\alpha \in Z^1_{f,c}(A)$, then $\alpha \circ R \in Z^1_{f,c\circ R}(A)$.

Define the automorphism $R' \in Aut(X \times G,\ \mu \times m_G)$ by setting up $R'(x,h) = (Rx, k(x)h)$. Let us show that

$$R'\gamma(c_1)(R')^{-1} = R\gamma R^{-1}(c). \tag{4.4}$$

Indeed, we obtain from (4.1) that

$$\begin{aligned}
& R'\gamma(c_1)(R')^{-1}(x,h) \\
= \;& (R\gamma R^{-1}, k(\gamma R^{-1}x)c_1(R^{-1}x,\gamma)k(R^{-1}x)^{-1}h) \\
= \;& (R\gamma R^{-1}x, c \circ R(R^{-1}x,\gamma)h) \\
= \;& (R\gamma R^{-1}x, c(x, R\gamma R^{-1})h) \\
= \;& R\gamma R^{-1}(c)(x,h).
\end{aligned}$$

To compute $b_\alpha \circ R'$ and b_{α_1} we use (2.4), (4.3), and (4.4):

$$b_{\alpha_1}(x,h,\gamma(c_1)) = h^{-1}\cdot \alpha_1(x,\gamma) + f(c_1(x,\gamma),h)$$

$$\begin{array}{rl} & b_\alpha \circ R'(x,h,\gamma(c_1)) \\ = & b_\alpha(Rx, k(x)h,\ R'\gamma(c_1)(R')^{-1}) \\ = & h^{-1}k(x)^{-1}\cdot \alpha(Rx, R\gamma R^{-1}) + f(c(Rx,R\gamma R^{-1}), k(x)h) \\ = & h^{-1}k(x)^{-1}\cdot f(k(\gamma x), c_1(x,\gamma)k(x)^{-1}) + h^{-1}k(x)^{-1}\cdot f(c_1(x,\gamma),k(x)^{-1}) \\ & -h^{-1}k(x)^{-1}\cdot f(k(x),k(x)^{-1}) + h^{-1}\cdot\alpha_1(x,\gamma) + h^{-1}c_1(x,\gamma)^{-1}\cdot u(\gamma x) \\ & -h^{-1}\cdot u(x) + f(c\circ R(x,\gamma),k(x)h). \end{array} \tag{4.5}$$

We use (1.2) three times to get

$$\begin{array}{rl} & h^{-1}k(x)^{-1}\cdot f(k(\gamma x),\ c_1(x,\gamma)k(x)^{-1}) \\ = & -f(k(\gamma x)c_1(x,\ \gamma)k(x)^{-1}, k(x)h) + f(c_1(x,\gamma)k(x)^{-1},\ k(x)h) \\ & +\ f(k(\gamma x),\ c_1(x,\gamma)h), \\ & h^{-1}k(x)^{-1}\cdot f(c_1(x,\gamma),\ k(x)^{-1}) \\ = & -f(c_1(x,\gamma)k(x)^{-1},\ k(x)h) + f(c_1(x,\gamma),h) + f(k(x)^{-1},\ k(x)h), \\ & h^{-1}k(x)^{-1}\cdot f(k(x),\ k(x)^{-1}) = -f(k(x)^{-1},\ k(x)h) - f(k(x),h) \end{array}$$

Substitute the three formulae into (3.5). Then

$$b_\alpha\circ R'(x,h,\gamma(c_1)) = f(k(\gamma x),\ c_1(x,\gamma)h) + f(c_1(x,\gamma),h)$$

$$-f(k(x),h) + h^{-1}\cdot\alpha_1(x,\gamma) + h^{-1}c_1(x,\gamma)^{-1}\cdot u(\gamma x) - h^{-1}\cdot u(x).$$

Thus,

$$b_\alpha\circ R'(x,h,\gamma(c_1)) - b_{\alpha_1}(x,h,\gamma(c_1))$$

$$= f(k(\gamma x),\ c_1(x,\gamma)h) - f(k(x),h) + h^{-1}c_1(x,\gamma)^{-1}\cdot u(\gamma x) - h^{-1}\cdot u(x).$$

Denote by $\xi(x,h) = f(k(x),h) + h^{-1}\cdot u(x)$. To complete the proof, we note that

$$b_\alpha\circ R'(x,h,\gamma(c_1)) - b_{\alpha_1}(x,h,\gamma(c_1)) = \xi(\gamma(c_1)(x,h)) - \xi(x,h).$$

□

Remark 4.3. It is not surprising that cocycles b_α and b_{α_1} are weakly equivalent since, as it follows from Theorem 4.2, the Mackey actions $W_{(\Gamma(c),b_\alpha)}(A)$ and $W_{(\Gamma(c_1),b_{\alpha_1})}(A)$ must be isomorphic by Theorem 2.3. The non-trivial part of Theorem 4.2 consists of explicit description of the automorphism R' that implements the isomorphism of these Mackey actions.

Now our goal is to prove the converse statement. To do this, we will have to assume that G is countable.

It is known [GS2] that if c and c_1 are cocycles over an ergodic a.f. automorphism group Γ both with dense ranges in G, then they are weakly equivalent, i.e. there exist an $R \in N[\Gamma]$ and a measurable map $k : X \to G$ such that $c \circ R(x, \gamma) = k(\gamma x)c_1(x, \gamma)k(x)^{-1}$, $(x, \gamma) \in X \times \Gamma$.

Theorem 4.4. *Let G be a countable amenable group and let $c, c_1, R,$ and $k(x)$ be as above. Define $R'(x, h) = (Rx, k(x)h)$, $(x, h) \in X \times G$. For given $\alpha \in Z^1_{f,c}(A)$ and $\alpha_1 \in Z^1_{f,c_1}(A)$, assume that the cocycles $b_\alpha \circ R'$ and b_{α_1} are $\Gamma(c_1)$-cohomologous, that is $W_{(\Gamma(c),\alpha)}(A)$ and $W_{(\Gamma(c_1),\alpha_1)}(A)$ are isomorphic. Then there exists a group automorphism θ of E_f such that the Mackey actions $W_{(\Gamma,\pi_1)}(E_f)$ and $\theta^*(W)_{(\Gamma,\pi)}(E_f))$ are isomorphic where $\pi = (c, \alpha)$ and $\pi_1 = (c_1, \alpha_1)$.*

Proof. It follows from our assumption that there exists a measurable map $q : X \times G \to A$ such that for any $\gamma \in \Gamma$ and a.e. $(x, h) \in X \times G$

$$b_\alpha \circ R'(x, h, \gamma(c_1)) - b_{\alpha_1}(x, h, \gamma(c_1)) = q(\gamma(c_1)(x, h)) - q(x, h).$$

By (2.4), we have

$$h^{-1}k(x)^{-1} \cdot \alpha \circ R(x, \gamma) + f(c \circ R(x, \gamma),\ k(x)h) - h^{-1} \cdot \alpha_1(x, \gamma) - f(c_1(x, \gamma), h)$$
$$= q(\gamma(c_1)(x, h)) - q(x, h)$$

or

$$k(x)^{-1} \cdot \alpha \circ R(x, \gamma) - \alpha_1(x, \gamma)$$
$$= h\cdot f(c_1(x, \gamma), h) - h\cdot f(c\circ R(x, \gamma),\ k(x)h) + h\cdot q(\gamma(c_1)(x, h)) - h\cdot q(x, h). \quad (4.6)$$

The right-hand side in (4.6) does not depend on $h \in G$. Therefore, we can set $h = e$ in (4.6). We then have

$$k(x)^{-1} \cdot \alpha \circ R(x, \gamma) - \alpha_1(x, \gamma)$$
$$= -f(k(\gamma x)c_1(x, \gamma)k(x)^{-1},\ k(x)) + q(\gamma(c_1)(x, e)) - q(x, e)$$

or

$$h \cdot f(c_1(x, \gamma), h) - h \cdot f(c \circ R(x, \gamma),\ k(x)h) + h \cdot q(\gamma(c_1)(x, h)) - h \cdot q(x, h)$$

$$= -f(k(\gamma x)c_1(x,\gamma)k(x)^{-1},\ k(x)) + q(\gamma(c_1)(x,e)) - q(x,e). \tag{4.7}$$

Since

$$h \cdot f(c_1(x,\gamma),h) = -f(c_1(x,\gamma)h,\ h^{-1}) + f(h,h^{-1}),$$

and

$$-h \cdot f(c \circ R(x,\gamma),\ k(x)h)$$
$$= f(k(\gamma x)c_1(x,\gamma)h,\ h^{-1}) - f(k(x)h,\ h^{-1}) - f(k(\gamma x)c_1(x,\gamma)k(x)^{-1},\ k(x)),$$

relation (4.7) can be written in the following form:

$$\begin{array}{rl} & q(\gamma(c_1)(x,e)) - q(x,e) \\ = & -f(c_1(x,\gamma)h,h^{-1}) + f(h,h^{-1}) + f(k(\gamma x)c_1(x,\gamma)h,h^{-1}) \\ & -f(k(x)h,h^{-1}) + h\cdot q(\gamma(c_1)(x,h)) - h\cdot q(x,h). \end{array} \tag{4.8}$$

We use (1.2):

$$\begin{array}{l} f(k(\gamma x)c_1(x,\gamma)h,h^{-1}) - f(c_1(x,\gamma)h,h^{-1}) \\ = -h\cdot f(k(\gamma x),\ c_1(x,\gamma)h) + f(k(\gamma x),\ c_1(x,\gamma)), \end{array}$$

$$f(k(x)h,h^{-1}) - f(h,h^{-1}) = -h \cdot f(k(x),h).$$

Then it follows from (4.8) that

$$-h\cdot f(k(\gamma x),\ c_1(x,\gamma)h) + f(k(\gamma x),\ c_1(x,\gamma)) + h\cdot q(\gamma(c_1)(x,h)) - q(\gamma(c_1)(x,e))$$
$$= -h\cdot f(k(x),h) + h\cdot q(x,h) - q(x,e). \tag{4.9}$$

Consider the measurable function $F : X \times E_f \to A$

$$F(x,h) = -h\cdot f(k(x),h) + h\cdot q(x,h) - q(x,e). \tag{4.10}$$

Note that for every fixed $h \in G$, F is constant a.e. on X. To see this, define the ergodic subgroup $\Gamma_0 = \{\gamma \in [\Gamma] : c_1(x,\gamma) = 0 \text{ for a.e. } x \in X\}$. For $h \in G,\ \gamma \in \Gamma_0$, we get from (4.9) that $F(\gamma x,h) = F(x,h)$, i.e. $F(x,h) = \varphi(h)$ a.e.

We show that $\varphi(h)$ satisfies the relation

$$\varphi(gh) = \varphi(g) + g\cdot\varphi(h). \tag{4.11}$$

Indeed, let $\gamma_g \in [\Gamma]$ be chosen such that $c_1(x,\gamma_g) = g,\ g \in G$ where $x \in X - N,\ \mu(N) = 0$. Denote $y = \gamma_g x$. Then we get from (4.9) and (4.10) that

$$\begin{aligned} \varphi(h) &= -h\cdot f(k(y),gh) + f(k(y),g) + h\cdot q(y,gh) - q(y,g) \\ &= g^{-1}[-(gh)\cdot f(k(y),gh) + g\cdot f(k(y),g) + gh\cdot q(y,gh) - g\cdot q(y,g)] \\ &= g^{-1}\cdot(\varphi(gh) - \varphi(g)). \end{aligned}$$

It follows from (4.11) that $\varphi(h)$ generates a group automorphism $\theta \in Aut(E_f; A)$ defined by $\theta(h,a) = (h, a - \varphi(h^{-1}))$ (see Appendix 2).

Finally, let us define the measure space isomorphism $\Phi : X \times E_f \to X \times E_f$ as follows:

$$\Phi(x,h,a) = (Rx,\ k(x)h,\ a + q(x,h)).$$

Note that, due to (4.10), Φ can be written in the following form

$$\Phi(x,h,a) = (Rx,\ k(x)h,\ a + h^{-1} \cdot q(x,e) - \varphi(h^{-1}) + f(k(x),\ h))$$

Recall that the translation on $X \times E_f$ by the group E_f is defined by

$$\begin{aligned} T_{(h,a)}(x,g,b) &= (x, (g,b)(h,a)^{-1}) \\ &= (x,\ gh^{-1},\ h \cdot b + f(g,h^{-1}) - f(h,h^{-1}) - h \cdot a) \end{aligned}$$

Claim 1. $\Phi \cdot T_{(h,a)} = T_{\theta(h,a)} \cdot \Phi$

We compute

$$\begin{aligned} & \Phi \cdot T_{(h,a)}(x,g,b) \\ = \ & (Rx,\ k(x)gh^{-1},\ \ f(g,h^{-1}) + h \cdot b - f(h,h^{-1}) - h \cdot a \\ & \qquad + hg^{-1} \cdot q(x,e) - \varphi(hg^{-1}) + f(k(x),\ gh^{-1})) \end{aligned}$$

and

$$\begin{aligned} & T_{\theta(h,a)} \cdot \Phi(x,g,b) \\ = \ & (Rx,\ k(x)gh^{-1},\ \ h \cdot b + hg^{-1} \cdot q(x,e) - h \cdot \varphi(g^{-1}) + h \cdot f(k(x),g) \\ & \qquad + f(k(x)g, h^{-1}) - f(h,h^{-1}) - h \cdot a + h \cdot \varphi(h^{-1})) \end{aligned}$$

Note that $f(k(x), gh^{-1}) = -f(g,h^{-1}) + h \cdot f(k(x),g) + f(k(x)g, h^{-1})$. To compare the third coordinates in $\varphi \cdot T_{(h,a)}(x,g,b)$ and $T_{\theta(h,a)} \cdot \Phi(x,g,b)$, we notice that their difference is equal to $-\varphi(hg^{-1}) + h \cdot \varphi(g^{-1}) - h \cdot \varphi(h^{-1}) = 0$. The claim is proved.

Claim 2. Let $x \in X$ and let γ and $\gamma' \in \Gamma$, be such that $\gamma' Rx = R\gamma x$. Then $\Phi(\gamma(\pi_1)(x,h,a)) = \gamma'(\pi)\Phi(x,h,a)$.

To show this, set $y = \gamma x$. Then

$$\begin{aligned} & \Phi(\gamma(\pi_1)(x,h,a)) \\ = \ & \Phi(y,\ c_1(x,\gamma)h,\ b_{\alpha_1}(x,h,\gamma(c_1)) + a) \\ = \ & (Ry,\ k(y)c_1(x,\gamma)h,\ b_{\alpha_1}(x,h,\gamma(c_1)) + a + q(y, c_1(x,\gamma)h)) \end{aligned}$$

and

$$
\begin{aligned}
& \gamma'(\pi)\Phi(x,h,a) \\
= \;& \gamma'(\pi)(Rx,\ k(x)h,\ a+q(x,h)) \\
= \;& (\gamma' Rx,\ c(Rx,\gamma')k(x)h,\ b_\alpha(Rx,k(x)h,\gamma'(c))+a+q(x,h))
\end{aligned}
$$

Since $\gamma'(c) = R'\gamma(c_1)(R')^{-1}$, we have that

$$
b_\alpha(Rx,k(x)h,\gamma'(c)) - b_{\alpha_1}(x,h,\gamma(c_1)) = q(\gamma x, c_1(x,\gamma)h) - q(x,h).
$$

Thus the proof of the claim is complete.

Now let us return to the theorem proof. Claims 1 and 2 imply that the map Φ sends every $\Gamma(\pi_1)$-ergodic component to a $\Gamma(\pi)$-ergodic component. We denote by $\tilde{\Phi}$ the map, induced by Φ, from the quotient space $(X\times E_f)/\xi(\Gamma(\pi))$ onto $(X\times E_f)/\xi(\Gamma(\pi_1))$ where $\xi(\Gamma(\pi))$ and $\xi(\Gamma(\pi_1))$ are partitions into ergodic components of $\Gamma(pi)$ and $\Gamma(\pi_1)$ respectively. Then the map $\tilde{\Phi}$ gives a conjugacy between the Mackey actions, that is,

$$
\tilde{\Phi} W_{(\Gamma,\pi_1)}(h,a) = \theta^*(W)_{(\Gamma,\pi)}(\tilde{\Phi}(h,a)), \quad \text{for all } (h,a)\in E_f.
$$

□

The following statement is an immediate consequence of Theorems 4.2 and 4.4. Recall that any ergodic nonsingular action V of A can be represented as the Mackey action $W_{(\Gamma(c),d)}(A)$ where $\Gamma(c)$ is an ergodic a.f. measure preserving group of automorphisms, $c : X\times\Gamma\to G$ is a cocycle with dense range and $d\in Z^1(X\times G\times\Gamma(c), A)$. We consider the case of a countable group G.

Theorem 4.5. *Assume that two ergodic nonsingular actions V and V_1 of A are isomorphic, and they are extendible to actions of $E_f(G,A)$. This means that for the corresponding Mackey actions $W_{(\Gamma(c),d)}(A)$ and $W_{(\Gamma(c_1),d_1)}(A)$, there exists an automorphism R' on $X\times G$ such that $R'[\Gamma(c_1)](R')^{-1} = [\Gamma(c)]$ and $d\circ R'$ is $\Gamma(c_1)$-cohomologous to d_1. Then the actions of E_f extended from $W_{(\Gamma(c),d)}(A)$ and $W_{(\Gamma(c_1),d_1)}(A)$ are isomorphic if and only if R' is of the skew product form, i.e. $R'(x,h) = (Rx,\ k(x)h)$.*

Proof. It follows from Theorem 2.4 that the Mackey actions $W_{(\Gamma(c),d)}(A)$ and $W_{(\Gamma(c_1),d_1)}(A)$ can be chosen such that $d\in I(f,c)$ and $d_1\in I(f,c_1)$, i.e. $d = b_\alpha$ and $d_1 = b_{\alpha_1}$. Therefore, the automorphism R' satisfies the conditions of Theorem 4.4, and we get that the theorem holds. □

Remark 4.6. It is not difficult to point out nonisomorphic actions of $E_f(G, A)$ that have isomorphic actions of A. Assume, for simplicity, that G and A are countable groups. For every $f \in Z^2(G, A)$ consider a Bernoulli action U of E_f with infinite entropy. Then the subgroup $U(e, A)$ also has infinite entropy. Therefore if f_1 and f_2 non-cohomologous (i.e. E_{f_1} and E_{f_2} are nonisomorphic) the corresponding Bernoulli actions $U_1(e, A)$ and $U_2(e, A)$ of A are isomorphic.

Appendix 1

To complete our study of the associated action $W_{(\Gamma(c),d)}(\mathbb{R})$ described in Section 3, let us find the explicit form of Q. It follows immediately from ergodicity of $\Gamma(c)$ that Q is ergodic. Denote $I_s = [\alpha^{-s}, \alpha^{-s+1})$, then $I_0 = [1, \alpha)$. Note that if $\lambda_1^{-1} \in I_s$, then $\lambda_2^{-1} \in I_{s-1}$ because $\alpha I_s = I_{s-1}$ and $\alpha\lambda_1^{-1} = \lambda_2^{-1}$. Find a formula for s, in terms of $\lambda_1\lambda_2$, such that $\lambda_1^{-1} \in I_s$.

Lemma A1.1.
$$s = 1 + \left[\frac{\log \lambda_1}{\log \alpha}\right].$$

Proof. We have that $\alpha^{-s} \leq \lambda_1^{-1} < \alpha^{-s+1}$, then
$$\lambda_2^s \leq \lambda_1^{s-1} < \lambda_2^{s-1}\lambda_1 \tag{A1.1}$$
¿From the left inequality in (A1.1), $s(\log \lambda_1 - \log \lambda_2) \geq \log \lambda_1$ and
$$s \geq \left[\frac{\log \lambda_1}{\log \alpha}\right].$$
¿From the right inequality in (A1.1),
$$s(\log \lambda_1 - \log \lambda_2) \leq 2 \log \lambda_1 - \log \lambda_2,$$
$$s \leq 1 + \frac{\log \lambda_1}{\log \alpha}.$$
Hence
$$s = 1 + \left[\frac{\log \lambda_1}{\log \alpha}\right].$$
□

To clarify the definition of Q, we note that the map $l_1 : p \mapsto \lambda_1^{-1}p$ sends $I_0 = [1, \alpha)$ onto $[\lambda_1^{-1}, \lambda_2^{-1})$ and this interval has a non-empty intersection with I_s and I_{s-1}. Let $r \in I_0$ be such that $r\lambda_1^{-1} = \alpha^{-s+1}$. Then $[1, r) \xrightarrow{l_1} [\lambda_1^{-1}, \alpha^{-s+1})$ and $[r, \alpha) \xrightarrow{l_1} [\alpha^{-s+1}, \alpha^{-s+2})$. By definition of Q, we get
$$Qp = \begin{cases} qp, & 1 \leq p < r \\ \alpha^{-1}qp, & r \leq p < \alpha \end{cases} \tag{A1.2}$$

where $q = \alpha^s \lambda_1^{-1}$. Then Q is the automorphism of $I_0 = [1, \alpha)$ such that $Q : [1, r) \to [q, \alpha)$ and $Q : [r, \alpha) \to [1, q)$.

Note that our assumption on λ_1, λ_2 implies that $r \neq q$ and $\log q$, $\log \alpha$ are rationally independent.

In fact, the map Q depends on two parameters (q, α) (or (q, r)) which in turn can be written in terms of λ_1 and λ_2. If we identify $[1, \alpha)$ and the circle $\mathbb{S}$, of the circumference α, then Q can be represented as a homeomorphism of $\mathbb{S}$. The dynamics of Q is completely determined by the following theorem proved by M. Misiurewicz.

Theorem A1.2. *The rotation number of Q on $\mathbb{S}$ is $\rho = (\log \alpha)^{-1} \log q$.*

It follows from the theorem that Q is a minimal homeomorphism of $\mathbb{S}$ because ρ is irrational. In particular, Q^n is ergodic for every $n \in N$.

Appendix 2: on automorphisms of group extensions

Let $0 \longrightarrow A \xrightarrow{i} E \xrightarrow{j} G \longrightarrow 1$ be a group extension of an abelian group A by G. If $q : G \to E$ is a normalized section of E, $j \circ q = \mathrm{id}$, $q(e) = e$, then we denote $g \cdot a = q(g)aq(g)^{-1}$. Now we are going to describe the group $Aut(E; A)$ of all Borel (continuous) group automorphisms of E that leave A fixed. Let $Z^1(G, A)$ denote the group of algebraic 1-cocycles, i.e. $p \in Z^1(G, A)$ if and only if $p(gh) = h^{-1} \cdot p(g)p(h)$ and $p(e) = e$.

Theorem A2.1. *Assume that G "acts" freely on A, i.e. if $g \cdot a = a$ for some $a \neq 0$, then $g = e$. Then $\theta \in Aut(E; A)$ if and only if there exists $p \in Z^1(G, A)$ such that*

$$\theta(g, a) = (g,\ a + p(g)). \tag{A2.1}$$

Proof. Let θ be as in (A2.1). Then, it is straightforward to check that $\theta \in Aut(E; A)$:

$$\begin{aligned}
\theta[q(g_1)a_1q(g_2)a_2] &= \theta[q(g_1)q(g_2)(g_2^{-1} \cdot a_1)a_2] \\
&= \theta[q(g_1g_2)f(g_1, g_2)(g_2^{-1} \cdot a_1)a_2] \\
&= q(g_1g_2)f(g_1, g_2)(g_2^{-1} \cdot a_1)a_2p(g_1g_2) \\
&= q(g_1)q(g_2)(g_2^{-1} \cdot a_1p(g_1))a_2p(g_2) \\
&= [q(g_1)a_1p(g_1)][q(g_2)a_2p(g_2)] \\
&= \theta[q(g_1)a_1]\theta[q(g_2)a_2]
\end{aligned}$$

and

$$\theta[(q(g)a)^{-1}] = \theta[q(g)^{-1}g \cdot a^{-1}]$$

$$
\begin{aligned}
&= \theta[q(g^{-1})f(g,g^{-1})^{-1}g\cdot a^{-1}] \\
&= q(g^{-1})f(g,g^{-1})^{-1}(g\cdot a^{-1})\cdot p(g^{-1}) \\
&= q(g)^{-1}(g\cdot a^{-1})p(g^{-1})) \\
&= (ap(g))^{-1}q(g)^{-1} \\
&= \theta[q(g)ap(g)]^{-1}.
\end{aligned}
$$

Conversely, let $\theta \in Aut(E;A)$. Denote $\theta[q(g)a] = q(g')a'$ where $g' = g'(g,a),\ a' = a'(g,a)$. It is easily seen that for any $q(g)a \in E$ and $b \in A$

$$(q(g)a)b(q(g)a)^{-1} = g\cdot b.$$

Thus

$$\theta[(q(g)a)b(q(g)a)^{-1}] = \theta(g\cdot b) = g\cdot b.$$

On the other hand,

$$
\begin{aligned}
theta[(q(g)a)b(q(g)a)^{-1}] &= (q(g')a')b(q(g')a')^{-1} \\
&= g\cdot b.
\end{aligned}
$$

Therefore $g\cdot b = g'\cdot b$ which implies $g' = g$. Now we can denote $\theta[q(g)] = q(g)p(g)$ where $p(g) \in A$. Then

$$\theta[q(g)a] = \theta[q(g)]\theta[a] = q(g)ap(g).$$

It remains to check that $p \in Z^1(G,A)$. The fact that $p(e) = 0$ is obvious. Take $g_1, g_2 \in G$, then

$$\theta[q(g_1)q(g_2)] = q(g_1g_2)f(g_1,g_2)p(g_1g_2) \tag{A2.2}$$

and

$$
\begin{aligned}
\theta[q(g_1)]\theta[q(g_2)] &= [q(g_1)p(g_1)][q(g_2)p(g_2)] \\
&= q(g_1g_2)f(g_1,g_2)(g_2^{-1}\cdot p(g_1))p(g_2).
\end{aligned}
$$

Then (A2.2) and the last relation show that $p(g) \in Z^1(G,A)$. □

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