THE MARKOV PROPERTY OF RANDOM β-EXPANSIONS

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ABSTRACT. Let $\beta > 1$, we show that if 1 has a finite greedy expansion in base β, $1=b_1/\beta + b_2/\beta^2 + \dots b_n/\beta^n$ with $b_1 \geq b_i \geq 1$ for $i = 1, 2, \dots, n$, then given an infinite coin toss, one can associate a new β-expansion of a point x in $[0, \frac{\lfloor\beta\rfloor}{\beta-1}]$. We show that all such expansions can be seen as realizations of an appropriate Markov Chain. We also discuss the uniqueness of the β-expansion of 1.

1. Introduction

Let $\beta > 1$ be a non-integer. There are two well-known expansions of numbers x in $[0, \frac{\lfloor\beta\rfloor}{\beta-1}]$ of the form,

$$x = \sum_{i=1}^{\infty} b_i/\beta^i,$$

with $b_i \in \{0, 1, \dots, \lfloor\beta\rfloor\}$. The largest in lexicographical order is the *greedy expansion*; [P], [R1], [R2], and the smallest is the *lazy expansion*; [JS], [EJK]. The greedy expansion is obtained by iterating the transformation T_β defined on $[0, \lfloor\beta\rfloor/(\beta-1)]$ by

$$T_\beta(x) = \begin{cases} \beta x \pmod 1, & 0 \leq x < 1, \\ \beta x - \lfloor\beta\rfloor, & 1 \leq x \leq \lfloor\beta\rfloor/(\beta-1). \end{cases}$$

The lazy expansion is obtained by iterating the map S_β on $[0, \lfloor\beta\rfloor/(\beta-1)] \to [0, \lfloor\beta\rfloor/(\beta-1)]$, and defined by

$$S_\beta(x) = \beta x - d \quad \text{for } x \in \Delta(d),$$

where

$$\Delta(0) = \left[0, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}\right],$$

and

$$\begin{aligned}\Delta(d) &= \left(\frac{\lfloor\beta\rfloor}{\beta-1} - \frac{(\lfloor\beta\rfloor - d + 1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1} - \frac{\lfloor\beta\rfloor - d}{\beta}\right] \\ &= \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{d-1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{d}{\beta}\right], \quad d \in \{1, 2, \dots, \lfloor\beta\rfloor\}.\end{aligned}$$

1991 *Mathematics Subject Classification.* 28D05.
Key words and phrases. greedy expansions, lazy expansions, Markov Chains.

We denote by μ, the *Parry* measure (see [P],[G]) on $[0, \lfloor\beta\rfloor/(\beta-1)]$ which is absolutely continous with respect to Lebesque measure λ, and with density

$$h_\beta(x) = \begin{cases} \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \frac{1}{\beta^n} 1_{[0,T^n(1))}(x) & 0 \le x < 1, \\ 0 & 1 \le x \le \lfloor\beta\rfloor/(\beta-1), \end{cases}$$

where $F(\beta) = \int_0^1 (\sum_{x<T^n(1)} \frac{1}{\beta^n})dx$ is a normalizing constant.

Define $\psi : [0, \lfloor\beta\rfloor/(\beta-1)) \to (0, \lfloor\beta\rfloor/(\beta-1)]$ by

$$\psi(x) = \frac{\lfloor\beta\rfloor}{\beta-1} - x,$$

and consider the measure ρ defined on $[0, \lfloor\beta\rfloor/(\beta-1)]$ by $\rho(A) = \mu(\psi(A))$ for every measurable set A. Then it is easy to see ([DK1]) that ψ is a measurable isomorphism between $([0, \lfloor\beta\rfloor/(\beta-1)], \mu, T_\beta)$ and $([0, \lfloor\beta\rfloor/(\beta-1)], \rho, S_\beta)$.

In [DK2], a random mixture of the greedy and the lazy expansions was introduced, and it was shown that for special values of β, the underlying random β-transformation is isomorphic to a mixing Markov Chain. The result was shown for β satisfying $\beta^2 = n\beta + k$ (with $1 \le k \le n$) and $\beta^n = \beta^{n-1} + \cdots + \beta + 1$. In this paper we show that the Markov property holds for all values of β satisfying $\beta^n = b_1\beta^{n-1} + b_2\beta^{n-2} + \ldots + b_{n-1}\beta + b_n$, with $b_1 \ge b_i \ge 1$, $i = 1, \ldots n$. We first outline the random procedure just mentioned.

If we super impose the greedy map and the corresponding lazy map on $[0, \lfloor\beta\rfloor/(\beta-1)]$, we get $\lfloor\beta\rfloor$ overlapping regions of the form

$$S_\ell = \left[\frac{\ell}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{\ell-1}{\beta}\right], \quad \ell = 1, \ldots, \lfloor\beta\rfloor,$$

which we will refer to as *switch regions*. On S_ℓ, the greedy map assigns the digit ℓ, while the lazy map assigns the digit $\ell - 1$. Outside these switch regions both maps are identical, and hence they assign the same digits. We will now define a new random expansion in base β by randomizing the choice of the map used in the switch regions. So, whenever x belongs to a switch region we flip a coin to decide which map will be applied to x, and hence which digit will be assigned. To be more precise, we partition the interval $[0, \lfloor\beta\rfloor/(\beta-1)]$ into switch regions S_ℓ and *equality regions* E_ℓ, where

$$E_\ell = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{\ell-1}{\beta}, \frac{\ell+1}{\beta}\right), \quad \ell = 1, \ldots, \lfloor\beta\rfloor - 1,$$

$$E_0 = \left[0, \frac{1}{\beta}\right) \quad \text{and} \quad E_{\lfloor\beta\rfloor} = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{\lfloor\beta\rfloor-1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}\right].$$

Let

$$S = \cup_{\ell=1}^{\lfloor\beta\rfloor} S_\ell, \quad \text{and} \quad E = \cup_{\ell=0}^{\lfloor\beta\rfloor} E_\ell,$$

and consider $\Omega = \{0, 1\}^{\mathbb{N}}$ with product σ-algebra. Let $\sigma : \Omega \to \Omega$ be the left shift, and define $K : \Omega \times [0, \lfloor\beta\rfloor/(\beta-1)] \to \Omega \times [0, \lfloor\beta\rfloor/(\beta-1)]$ by

(1)
$$K(\omega, x) = \begin{cases} (\omega, \beta x - \ell) & x \in E_\ell,\ \ell = 0, 1, \dots, \lfloor\beta\rfloor, \\ \\ (\sigma(\omega), \beta x - \ell) & x \in S_\ell \text{ and } \omega_1 = 1,\ \ell = 1, \dots, \lfloor\beta\rfloor, \\ \\ (\sigma(\omega), \beta x - \ell + 1) & x \in S_\ell \text{ and } \omega_1 = 0,\ \ell = 1, \dots, \lfloor\beta\rfloor. \end{cases}$$

The elements of Ω represent the coin tosses ('heads'=1 and 'tails'=0) used every time the orbit hits a switch region. We assume that the probability that the coin lands heads is p with $0 \le p \le 1$. Let

$$d_1 = d_1(\omega, x) = \begin{cases} \ell & \text{if } x \in E_\ell,\ \ell = 0, 1, \dots, \lfloor\beta\rfloor, \\ & \text{or } (\omega, x) \in \{\omega_1 = 1\} \times S_\ell,\ \ell = 1, 2, \dots, \lfloor\beta\rfloor, \\ \\ \ell - 1 & \text{if } (\omega, x) \in \{\omega_1 = 0\} \times S_\ell,\ \ell = 1, 2, \dots, \lfloor\beta\rfloor, \end{cases}$$

then

$$K(\omega, x) = \begin{cases} (\omega, \beta x - d_1) & \text{if } x \in E, \\ \\ (\sigma(\omega), \beta x - d_1) & \text{if } x \in S. \end{cases}$$

In Section 2, we use this dynamical view to give necessary and sufficient conditions for 1 to have a unique β-expansion. These necessary and sufficient conditions were also obtained independently, and by a different method in [KL, Theorem 3.1]. In Section 3, we extend the proof in [DK2] showing the Markov property for all bases β satisfying $\beta^n = b_1\beta^{n-1} + b_2\beta^{n-2} + \dots + b_{n-1}\beta + b_n$, with $b_1 \ge b_i \ge 1$, $i = 1, \dots n$.

2. Uniqueness of the β-expansion of 1

Using the same notation as in the previous section, we first observe that 1 has a unique β-expansion if and only if $T_\beta^n 1 \in E_{b_{n+1}}$ for all $n \ge 0$. Further, $1 \in S_{\lfloor\beta\rfloor} \cup E_{\lfloor\beta\rfloor}$, and $1 \in E_{\lfloor\beta\rfloor}$ if and only if $\frac{b_1}{\beta-1} - 1 \in E_0$. The following proposition gives a characterization of each case in terms of the second digit of the greedy expansion of 1.

Proposition 1. *Suppose* 1 *has a finite or infinite greedy expansion of the form* $1 = b_1/\beta + b_2/\beta^2 + \dots$.

(i) *If* $b_i = 0$ *for all* $i \ge 3$, *then* $1 \in E_{b_1}$ *if and only if* $b_2 \ge 2$. *Moreover, if* $b_2 = 1$, *then* $1 = \frac{\lfloor\beta\rfloor}{\beta-1} - \frac{1}{\beta}$.

(ii) *If* $b_i \ge 1$ *for some* $i \ge 3$, *then* $1 \in E_{b_1}$ *if and only if* $b_2 \ge 1$.

Proof. First observe that $\lfloor\beta\rfloor = b_1$, and that $1 = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \frac{1}{\beta^2} T_\beta^2 1$. The latter implies that $\beta^2 - b_1\beta = b_2 + T_\beta^2 1$. Now, by definition $1 \in E_{b_1}$ if and only if $1 > \frac{b_1}{\beta-1} - \frac{1}{\beta}$, or equivalently $\beta^2 - b_1\beta > 1$.

In case (i), we have $T_\beta^2 1 = 0$ which implies that $\beta^2 - b_1\beta = b_2$. Hence, $1 \in E_{b_1}$ if and only if $b_2 \ge 2$. If $b_2 = 1$, then $\beta^2 - b_1\beta = 1$; equivalently, $1 = \frac{\lfloor\beta\rfloor}{\beta-1} - \frac{1}{\beta}$.

In case (ii), we have that $0 < T_\beta^2 1 < 1$. Hence, $\beta^2 - b_1\beta = b_2 + T_\beta^2 1 > 1$ if and only if $b_2 \ge 1$. $\square$

Before we proceed to the characterization of the uniqueness of the β-expansion of 1, we need the following simple lemma.

Lemma 1. *Suppose* 1 *has a finite or infinite greedy expansion of the form* $1 = b_1/\beta + b_2/\beta^2 + \dots$ *. If* $b_{n+1} \geq 1$*, then* $T_\beta^n 1 \in E_{b_{n+1}}$ *if and only if* $T_\beta^{n+1} 1 > \frac{b_1}{\beta-1} - 1$.

Proof. Notice that $T_\beta^n 1 = \frac{b_{n+1}}{\beta} + \frac{1}{\beta} T_\beta^{n+1} 1$. Since $T_\beta^{n+1} 1 < 1$, we have that $T_\beta^n 1 < \frac{b_{n+1}+1}{\beta}$. Thus, $T_\beta^n 1 \in E_{b_{n+1}}$ if and only if $T_\beta^n 1 > \frac{b_1}{\beta(\beta-1)} + \frac{b_{n+1}-1}{\beta}$. Rewriting one gets that $T_\beta^n 1 \in E_{b_{n+1}}$ if and only if $T_\beta^{n+1} 1 > \frac{b_1}{\beta-1} - 1$. □

Note that if $b_{n+1} = 0$, then $T_\beta^n 1 \in E_0$.

The following theorem is an immediate consequence of the above lemma. We remark that this theorem was obtained independently, and via other methods in [KL, Theorem 3.1].

Theorem 1. *Let* $\beta > 1$*, and suppose that* 1 *has an infinite greedy expansion of the form* $1 = b_1/\beta + b_2/\beta^2 + \dots$ *. Then* 1 *has a unique expansion in base* β *if and only if for all* $n \geq 1$ *with* $b_{n+1} \geq 1$*, we have* $T_\beta^{n+1} 1 > \frac{b_1}{\beta-1} - 1$.

Note that if 1 has a finite greedy expansion, then 1 has infinitely many expansions in base β.

Corollary 1. *If* 1 *has an infinite greedy expansion of the form* $1 = b_1/\beta + b_2/\beta^2 + \dots$ *, with* $b_i \geq 1$ *for all* $i \geq 1$*. Then,* 1 *has a unique* β*-expansion.*

Proof. Since $b_2 \geq 1$, it follows from Proposition 1 that $1 \in E_{b_1}$, and hence $\psi(1) = \frac{b_1}{\beta-1} - 1 \in E_0$. Now, $b_i \geq 1$ for all $i \geq 1$, implies that $T_\beta^n 1 > \frac{b_{n+1}}{\beta} > \frac{1}{\beta}$. Hence, $T_\beta^n 1 > \frac{b_1}{\beta-1} - 1$, for all $n \geq 1$. The result follows from Theorem 1. □

Corollary 2. *If* 1 *has a unique* β*-expansion, then there exists* $k \geq 0$ *such that in the greedy expansion of* 1*, every block of consecutive zeros consists of at most* k *terms.*

Proof. Let $1 = b_1/\beta + b_2/\beta^2 + \dots$ be the greedy expansion, then by Theorem 1 we have $\frac{b_1}{\beta-1} - 1 < \frac{1}{\beta}$. Hence, there exists a k such that $\frac{1}{\beta^{k+1}} < \frac{b_1}{\beta-1} - 1 < \frac{1}{\beta^k}$. If $b_{i-1}b_i \dots b_j$ is a block with $b_{i-1} \geq 1$, $b_i = \dots = b_j = 0$ and $j - i + 1 \geq k + 1$, then $T_\beta^{i-1} 1 < \frac{1}{\beta^{k+1}} < \frac{b_1}{\beta-1} - 1$, contradicting Theorem 1. □

Another immediate Corollary of Theorem 1, and Proposition 1 is the following.

Corollary 3. *Suppose* 1 *has an infinite greedy expansion of the form* $1 = b_1/\beta + b_2/\beta^2 + \dots$ *, with* $b_2 \geq 1$*. Let* $k \geq 1$ *be the unique integer such that* $\frac{1}{\beta^{k+1}} < \frac{b_1}{\beta-1} - 1 < \frac{1}{\beta^k}$*. If in the greedy expansion of* 1*, every block of consecutive zeros contains at most* $k-1$ *terms, then* 1 *has a unique* β*-expansion.*

3. Finite greedy expansion of 1 with positive coefficients, and the Markov property of the random β-expansions

In this section we assume that the greedy expansion of 1 in base β satisfies $1 = b_1/\beta + b_2/\beta^2 + \dots + b_n/\beta^n$ with $b_1 \geq b_i \geq 1$, and $n \geq 3$. We begin by a proposition that is an immediate consequence of Lemma 1, and which plays a crucial role in finding the Markov partition describing the dynamics of the map K, as defined in section 1.

Proposition 2. *Suppose* 1 *has a finite greedy expansion of the form* $1 = b_1/\beta + b_2/\beta^2 + \ldots + b_n/\beta^n$. *If* $b_j \geq 1$ *for* $1 \leq j \leq n$, *then* $T_\beta^i 1 = S_\beta^i 1 \in E_{b_{i+1}}$, $i = 0, 1, \ldots, n-2$

We remark that under the hypothesis of the above proposition, we also have that $T_\beta^{n-1}1 = S_\beta^{n-1}1 = \frac{b_n}{\beta}$, $T_\beta^n 1 = 0$, and $S_\beta^n 1 = 1$. Further, by Proposition 1 and Lemma 1, one has that $T_\beta^i 1 = S_\beta^i 1 > \frac{b_1}{\beta-1} - 1$ for all $i = 1, 2, \ldots, n-1$. Similarly, $T_\beta^i(\frac{b_1}{\beta-1} - 1) = S_\beta^i(\frac{b_1}{\beta-1} - 1) < 1$ for all $i = 1, 2, \ldots, n-1$.

We now extend the procedure descibed in [DK2] for the special cases β satisfying $\beta^2 = n\beta + k$ (with $1 \leq k \leq n$) and $\beta^n = \beta^{n-1} + \cdots + \beta + 1$. The analysis is a minor modification of that used in [DK2]. For this reason we shall describe the Markov Chain underlying the map K, and refer the reader to [DK2] for the details of the proof.

To find the Markov Chain behind the map K, one starts by refining the partition

$$\mathcal{E} = \{E_0, S_1, E_1, \ldots, S_{b_1-1}, E_{b_1-1}, E_{b_1}\}$$

of $[0, \frac{b_1}{\beta-1}]$, using the orbits of 1 and $\frac{b_1}{\beta-1} - 1$ under the transformation T_β. We place the end points of $\mathcal{E}$ togther with $T_\beta^i 1$, $T_\beta^i(\frac{b_1}{\beta-1} - 1)$, $i = 0, 1, \ldots b_1 - 1$, in increasing order. We use these points to form a new partition $\mathcal{C}$ which is a refinement of $\mathcal{E}$. We write $\mathcal{C}$ as

$$\mathcal{C} = \{C_0, C_1, \ldots, C_L\}.$$

Notice that each element of $\mathcal{C}$ is an interval which is either open, half-open or closed. From the above discussion, we choose $\mathcal{C}$ so that $C_0 = [0, \frac{b_1}{\beta-1} - 1) \subset E_0$, $C_L = (1, \frac{b_1}{\beta-1}]$, and $E_i = \cup_{j \in L_i} C_j$ for $i = 0, 1, \ldots, b_1$ with $L_0, L_1, \ldots, L_{b_1}$ finite disjoint subsets of $\{0, 1, \ldots L\}$. Further, for each S_i there corresponds exactly one $j \in \{0, 1, \ldots, L\} \setminus \cup_{i=0}^{b_1} L_i$ such that $S_i = C_j$. This is possible since the T_β orbits of 1 and $\frac{b_1}{\beta-1} - 1$ never hit the interior of $\cup_i^{b_1} S_i$. Finally, in case $C_j \subset E_i$, then $T_\beta(C_j) = S_\beta(C_j)$ is a finite disjoint union of elements of $\mathcal{C}$, and in case $C_j = S_i$, then $T_\beta(C_j) = C_0$ and $S_\beta(C_j) = C_L$.

We now use $\mathcal{C}$, the dynamics of K and Lebesque measure λ, in order to define a Markov Chain that describes the map K.

Consider the Markov Chain with state space the set $\mathcal{I} = \{c_0, c_1, \ldots, c_L\}$. The transition probabilities are defined as follows,

$$p_{ij} = \begin{cases} \lambda(C_i \cap T_\beta^{-1} C_j)/\lambda(C_i) & \text{if } C_i \subset \cup_{k=0}^{b_1} E_k, \\ \\ p & \text{if } C_i \subset \cup_{k=1}^{b_1} S_k \text{ and } j = 0, \\ \\ 1 - p & \text{if } C_i \subset \cup_{k=1}^{b_1} S_k \text{ and } j = L. \end{cases}$$

It is easy to see that the above finite state Markov Chain is irreducible and recurrent, hence positively recurrent. Let Y be the space of all realizations of the Markov Chain, and σ_Y the left shift on Y. Denote by π the stationary distribution, and by P the stationary probability measure on Y determined by π, and the transition probabilities p_{ij}. For ease of notation, we denote by $s_1, s_2, \ldots, s_{b_1}$ the states $c_{j_1}, \cdots, c_{j_{b_1}}$ corresonding to the switch regions $S_1, S_2, \ldots, S_{b_1}$ respectively.

We now define P a.e. a map $\phi : Y \to \Omega \times [0, \frac{b_1}{\beta-1}]$ in the following way.

For $y \in Y$, we associate a sequence

$$b_j = \begin{cases} i & \text{if } y_j = c_l \text{ and } l \in L_i, \\ i & \text{if } y_j = s_i \text{ and } y_{j+1} = c_0, \\ i-1 & \text{if } y_j = s_i \text{ and } y_{j+1} = c_L. \end{cases}$$

Define a point $x \in \left[0, \frac{n}{\beta-1}\right]$ by

$$x = \sum_{j=1}^{\infty} \frac{b_j}{\beta^j}. \tag{2}$$

To define a point $\omega \in \Omega$, we first locate the indices $n_i = n_i(y)$ where the realization y of the Markov Chain is in state s_ℓ for some $\ell \in \{1, \dots, b_1\}$. That is, let $n_1 < n_2 < \cdots$ be the indices such that $y_{n_i} = s_\ell$ for some $\ell = 1, \dots, b_1$. Define

$$\omega_j = \begin{cases} 1 & \text{if } y_{n_j+1} = c_0, \\ 0 & \text{if } y_{n_j+1} = c_L. \end{cases}$$

Now set $\phi(y) = (\omega, x)$, and define a measure ν on $\Omega \times [0, \frac{b_1}{\beta-1}]$ by setting $\nu = P \circ \phi^{-1}$. We will show that ϕ is a measurable isomorphism between (Y, P, σ_Y) and $(\Omega \times [0, \frac{b_1}{\beta-1}], \nu, K)$. The proof is a slight modification of the argument in [DK2]. We will outline the proof in the form of two lemmas, and refer the reader to [DK2] for the proofs. These two Lemmas reflect the fact that the dynamics of K is essentially the same as that of the Markov Chain Y.

Lemma 2. *Let $y \in Y$ be such that $\phi(y) = (\omega, x)$. Then,*

(i) $y_1 = c_k$ *for some* $k \in \cup_{i=1}^{b_1} L_i \Rightarrow x \in C_k$.

(ii) $y_1 = s_i$, $y_2 = c_0 \Rightarrow x \in S_i$ *and* $\omega_1 = 1$ *for* $i = 1, \dots, b_1$.

(iii) $y_1 = s_i$, $y_2 = c_L \Rightarrow x \in S_i$ *and* $\omega_1 = 0$ *for* $i = 1, \dots, b_1$.

Lemma 3. *For P a.e. $y \in Y$, we have*

$$\varphi \circ \sigma_Y(y) = K \circ \varphi(y).$$

Using these two lemmas one can easily show the following theorem

Theorem 2. *The map ϕ is a measurable isomorphism between (Y, P, σ_Y) and $(\Omega \times [0, \frac{b_1}{\beta-1}], \nu, K)$.*

Remark: In this section, we have assumed that the greedy expansion of 1 satisfies $1 = b_1/\beta + b_2/\beta^2 + \ldots + b_n/\beta^n$ with $b_1 \geq b_i \geq 1$, and $n \geq 3$. Using Proposition 1 and Lemma 1, one easily sees that the results of this section also hold in two other situations. Namely,

(i) when in the greedy expansion of $1 = b_1/\beta + b_2/\beta^2 + \ldots + b_n/\beta^n$, some of the coefficients are zero, yet $T_\beta^i 1 \in E_{b_{i+1}}$ for all $i = 0, 1, \dots n-2$;

(ii) 1 has an ultimately periodic infinite greedy expansion, and the orbit of 1 satisfies $T_\beta^i 1 \in E_{b_{i+1}}$ for all $i \geq 0$.

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