Asymptotic radial speed of the support of supercritical branching and super-Brownian motion in $\mathbb{R}^d$

A.E. Kyprianou*

September 24, 2002

Abstract

It has long been known that the left-most or right-most particle in a one dimensional dyadic branching Brownian with constant branching rate $\beta > 0$ has almost sure asymptotic speed $\sqrt{2\beta}$, (cf. McKean (1975)). Recently similar results for higher dimensional branching Brownian motions and super-Brownian motion have also been established the weaker sense of convergence in probability; see Pinsky (1995) and Engländer and den Hollander (2002). In this short note we confirm the 'folklore' for higher dimensions and establish an asymptotic radial speed of the support of the latter two processes in the almost sure sense. The proofs rely on the local extinction dichotomy proved in Pinsky (1996) and Engländer and Kyprianou (2002) together with simple geometrical considerations.

AMS 1991 Subject classification: 60J80

Keywords and phrases: Spatial branching processes, super-Brownian motion, branching Brownian motion, local extinction.

1 Introduction

In this short note we shall consider X as either a $(\triangle/2, \beta; \mathbb{R}^d)$ binary branching Brownian motion with $\beta > 0$ or a $(\triangle/2, \beta, \alpha; \mathbb{R}^d)$ super-Brownian motion with $\beta, \alpha > 0$ together with probabilities $\{P_\mu : \mu \in \mathcal{M}_c(\mathbb{R}^d)\}$. Here $\mathcal{M}_c(\mathbb{R}^d)$ denotes the space of Borel measures on $\mathbb{R}^d$ which are finite and compactly supported in $\mathbb{R}^d$.

The branching Brownian motion $(\triangle/2, \beta; \mathbb{R}^d)$ under measure P_μ with $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ is constructed as follows. Start with an initial (finite) configuration of points represented by μ in $\mathbb{R}^d$. From each point an independent Brownian motion is initiated. Each particle diffuses until an independent exponentially distributed time with mean $1/\beta$ at which point it undergoes fission producing

*Department of Mathematics, University of Utrecht, P.O.box 80010, NL-3508 TA Utrecht, The Netherlands. Email: kyprianou@math.uu.nl

two particles. The two particles move and reproduce independently starting from their point of creation in a way that is stochastically identical to their parent and so on. A simple generalization of this process in one dimension that we shall also work with is the $(\triangle/2-\gamma d/dx, \beta; \mathbb{R})$ branching process for $\gamma \in \mathbb{R}$ and $\beta > 0$. This process has virtually the same definition but for the fact that particles diffuse as a standard Brownian motion with drift $-\gamma t$. For any of the aforementioned particle processes, since each particle has exactly two offspring, they are supercritical (in the traditional sense of Galton-Watson processes) and survive with probability one.

The super-Brownian motion $(\triangle/2, \beta, \alpha; \mathbb{R}^d)$ under P_μ with $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ arises as the weak limit of an appropriately rescaled, branching Brownian motion in which particles have a random number of offspring with finite variance and mean which is greater than 1. Pinsky (1996) gives specific details of the limiting procedure. It suffices to note for our purposes that the resulting process (X, P_μ) is valued in $\mathcal{M}_c(\mathbb{R}^d)$ with $X_0 = \mu$ and has the property that for each g belonging to the cone of non-negative, continuous bounded functions, $E_\mu(\exp\langle -g, X_t\rangle) = \exp\langle -u(\cdot, t), \mu\rangle$ where $\langle \cdot, \cdot \rangle$ is the usual innerproduct and u is the unique solution to the PDE

$$\begin{aligned} \partial u/\partial t &= \frac{1}{2}\triangle u + \beta u - \alpha u^2 \text{ for } (x,t) \in \mathbb{R}^d \times \mathbb{R}^+ \\ u(x,0) &= g(x) \text{ for } x \in \mathbb{R}^d. \end{aligned} \tag{1}$$

In one dimension we shall also talk of a $(\triangle/2-\gamma d/dx, \beta, \alpha; \mathbb{R}^d)$ superprocess for $\beta > 0$ and $\gamma \in \mathbb{R}$. This process satisfies a similar dynamic to (1) with the exception that $\triangle/2$ is replaced by $\triangle/2 - \gamma d/dx$. In any of the aforementioned superprocesses, the positivity of β qualifies this process as supercritical as, contrary to the case $\beta \leq 0$, the process may survive. Unlike its particle counterpart however, supercritical super-Brownian motion survives only with strictly positive probability rather than with probability one. Given $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, the probability of survival is given by $1 - \exp\{\beta||\mu||/\alpha\}$ where $||\mu|| = \langle 1, \mu\rangle$ is the total mass of μ; see Engländer and Pinsky (1999) for details.

This paper concerns the how fast the support of these processes grows in time. In order to describe how this will happen, we need the following definitions.

Definition 1 *Let B_r be the closed ball of radius $r > 0$ centered at the origin. We define the radius of the support $M_t = \inf\{r > 0 : X_t(B_r^c) = 0\}$.*

Definition 2 *For one dimensional processes, we shall also talk about the right most extreme of the support,*

$$R_t = \inf\{y \in \mathbb{R} : X_t(y, \infty) = 0\}.$$

It is well known that when $d = 1$ and X is a branching Brownian motion, there exists an asymptotic speed for extreme particles almost surely (cf. McKean (1975), Bramson (1978, 1983)). This is captured in the following proposition.

Proposition 3 *If X is a supercritical branching Brownian motion in one dimension and $\mu \in \mathcal{M}_c(\mathbb{R})$ then*

$$\frac{R_t}{t} \to \sqrt{2\beta}$$

P_μ-almost surely.

A generalized version of this result for both types of process in any Euclidian dimension reads as follows.

Theorem 4 *Suppose that X is either the $(\triangle/2, \beta; \mathbb{R}^d)$ branching Brownian motion or the $(\triangle/2, \beta, \alpha; \mathbb{R}^d)$ super-Brownian motion. Assume that $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, then*

$$\frac{M_t}{t} \to \sqrt{2\beta}$$

P_μ-almost surely on the survival set of X. (Note that when X is a particle process it survives with probability 1 and hence the last qualifier is superfluous).

Although Theorem 4 would seem to be an intuitively obvious and natural extension to Proposition 3, the existence in the literature of this *strong law of large numbers* for the radius of the support would seem to be for the most part folklore. Recently weaker versions of this result have appeared in related problems. Specifically in Engländer and Hollander (2002) for branching Brownian motion where the convergence occurs in probability and in Pinsky (1995) for super-Brownian motion where again convergence in probability is proved. In related work, Engländer (2002) also offers large deviation asymptotics concerning mass in supercritical branching Brownian motion and supercritical super-Brownian motion close to the asymptotic speed $\sqrt{2\beta}$.

The aim of this note is, in a certain sense, to round off these existing results by proving Theorem 4. For the most part, the proof is based on standard Euclidian geometry and the following result which is a specialization of the combined conclusions of Theorem 6 in Pinsky (1996) and Theorem 3 in Engländer and Kyprianou (2002).

Theorem 5 *Suppose that $\mu \in \mathcal{M}_c(\mathbb{R})$. Let $\gamma \in \mathbb{R}$ and take X^γ as either the $(\triangle/2-\gamma d/dx, \beta, \mathbb{R})$ branching process or the $(\triangle/2-\gamma d/dx, \beta, \alpha, \mathbb{R})$ superprocess. Denote the generalized principle eigenvalue*

$$\begin{aligned} \lambda_c &= \lambda_c(\triangle/2 - \gamma d/dx + \beta) \\ &= \inf\{\lambda \in \mathbb{R} : (\triangle/2 - \gamma d/dx + \beta - \lambda)h = 0 \textit{ for some } h > 0 \textit{ in } C^2(\mathbb{R})\}. \end{aligned}$$

Then for all bounded intervals $I \subset \mathbb{R}$ either

$\lambda_c > 0$ *: in which case $P_\mu(X_t^\gamma(I) > 0$ for arbitrarily large $t) > 0$.*

$\lambda_c \leq 0$: *in which case there exists a P_μ-almost surely finite T such that*

$$P_\mu\left(X_t^\gamma\left(I\right) = 0 \textit{ for all } t > T\right) = 1.$$

Remark 6 Since all positive eigenfunctions of the operator $(\triangle/2 - \gamma d/dx + \beta)$ are exponential it follows from a straightforward calculation that

$$\lambda_c(\triangle/2 - \gamma d/dx + \beta) = \beta - \frac{\gamma^2}{2}.$$

2 Proofs

In order to prove the Theorem 4 we shall first prove a slightly stronger version of Proposition 3 for one dimensional branching Brownan motion or super-Brownian motion. Indeed the following result contains the statement of Proposition 3.

Proposition 7 *Suppose that X is either the $(\triangle/2, \beta; \mathbb{R})$ branching Brownian motion or the $(\triangle/2, \beta, \alpha; \mathbb{R})$ super-Brownian motion (hence $d = 1$). Assume that $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and let I be an arbitrary bounded interval in $\mathbb{R}$.*

(i) When $\gamma \geq \sqrt{2\beta}$

$$P_\mu(X_t(I + \gamma t) = 0 \textit{ for all sufficiently large } t) = 1.$$

(ii) When $\gamma \in [0, \sqrt{2\beta})$

$$P_\mu(X_t(I + \gamma t) > 0 \textit{ for arbitrarily large } t | X \textit{ survives}) = 1.$$

(iii) On the survival set of X,

$$\frac{R_t}{t} \to \sqrt{2\beta}$$

P_μ-almost surely.

Remark 8 Note that if X is a branching Brownian motion then survival occurs with probability one.

Proof of Proposition 7. We shall only offer proofs for the case of super-Brownian motion. The case of branching Brownian motion follows by simpler if not identical arguments.

(i) Let X^γ be the superprocess with characteristics, $(\triangle/2 - \gamma d/dx, \beta, \alpha, \mathbb{R})$ where $\gamma \geq 0$. We have

$$\begin{aligned} &P_\mu(X_t(I + \gamma t) = 0 \text{ for all sufficiently large } t) \\ &\quad = P_\mu(X_t^\gamma(I) = 0 \text{ for all sufficiently large } t). \end{aligned} \tag{2}$$

From Remark 6 and Theorem 5 it follows that when $\gamma \geq \sqrt{2\beta}$,

$$P_\mu(X_t(I+\gamma t) = 0 \text{ for all sufficiently large } t) = 1.$$

(ii) Now suppose that $\gamma \in [0, \sqrt{2\beta})$. We have

$$\begin{aligned} & P_\mu(X \text{ becomes extinct}) \\ & \quad \leq \; P_\mu(X_t(I+\gamma t) = 0 \text{ for all sufficiently large } t) \\ & \quad = \; e^{-\langle \rho, \mu \rangle} \end{aligned} \tag{3}$$

where $\rho(x) = -\log P_{\delta_x}(X_t(I+\gamma t) = 0$ for all sufficiently large $t)$ is monotone in x. It is known that the left hand side of (3) is equal to $\exp\{-||\mu||\beta/\alpha\}$ for all $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and hence $0 < \rho \leq \beta/\alpha$; the strict inequality in this last lower bound on ρ follows from Theorem 5 as $\lambda_c > 0$ for this regime of γ. Our aim is thus to show that $\rho = \beta/\alpha$ and then (ii) follows easily.

To this end note that an application of the Markov Property in the equality of (3) shows that

$$E_\mu(e^{-\langle \rho, X_t \rangle}) = e^{-\langle \rho, \mu \rangle}.$$

for all $\mu \in \mathcal{M}_c(\mathbb{R})$ and $t \geq 0$. By taking expectation of $\exp\langle -\rho, X_{t+s}\rangle$ conditional on $\mathcal{F}_t = \sigma(X_u : u \leq t)$, applying the Markov property together with the last equality, one can easily deduce that $\{\exp\langle -\rho, X_t\rangle : t \geq 0\}$ is a martingale. It follows from an application of Theorem II.3.1 in Dynkin (1993) that $\frac{1}{2}\rho'' - \gamma\rho' + \beta\rho - \alpha\rho^2 = 0$. Now write $\rho = f\beta/\alpha$ so that $0 < f \leq 1$. A simple calculation shows that $1-f$ solves the classic Kolmogorov-Petrovski-Piscounov traveling wave equation

$$\frac{1}{2}f'' - \gamma f' + \beta(f^2 - f) = 0$$

for which it is known there are no non-trivial solutions in $[0,1]$ when $\gamma \in [0, \sqrt{2\beta})$ (cf. Kolmogorov et al (1937)). We are forced to conclude that $f = 1$ and hence $\rho = \beta/\alpha$ and the proof of (ii) is complete.

(iii) In view of the afore mentioned weaker versions of Theorem 4 we know that mass propagates no faster than linearly in time. Hence part (i) implies that

$$\limsup_{t\uparrow\infty} \frac{R_t}{t} \leq \sqrt{2\beta}$$

P_μ-almost surely and (ii) implies that

$$\liminf_{t\uparrow\infty} \frac{R_t}{t} \geq \sqrt{2\beta}$$

P_μ-almost surely on the survival set of X. ■

Proof of Theorem 4. As usual X denotes either a supercritical branching Brownian motion or a supercritical super-Brownian motion. Let $\mathbf{e}$ be any vector on the sphere of unit radius $S_1 = \{\mathbf{e} \in \mathbb{R}^d : |\mathbf{e}| = 1\}$ and for each $\mathbf{e} \in S_1$ define the 'slab'

$$H(\mathbf{e}) := \{\mathbf{r} \in \mathbb{R}^d : \mathbf{r} \cdot \mathbf{e} \in [x, x + x')\}$$

where $x' > 0$ is some arbitrary constant (the thickness of the slab) and $x > 0$ is the perpendicular distance of the slab from the origin. We claim now that for all $\mu \in \mathcal{M}_c(\mathbb{R}^d)$

$$P_\mu(X_t(H(\mathbf{e}) + t\gamma\mathbf{e}) = 0 \text{ for all sufficiently large } t) = 1$$

when $\gamma \geq \sqrt{2\beta}$ and

$$P_\mu(X_t(H(\mathbf{e}) + t\gamma\mathbf{e}) > 0 \text{ for arbitrarily large } t | X \text{ survives}) = 1$$

when $\gamma \in [0, \sqrt{2\beta})$. To see why these two claims are true, note that we can always choose our axes so that $\mathbf{e}$ lies along, say, the first axis. By considering the process X projected onto this first axis we have simply a one dimensional supercritical branching Brownian motion or supercritical super-Brownian motion with the same parameters. Proposition 7 then justifies the claims. [Indeed when $d = 1$ these claims are nothing more than part (i) and (ii) of Proposition 7]. It is also immediate from this projection that

$$\liminf_{t\uparrow\infty} \frac{M_t}{t} \geq \sqrt{2\beta}$$

on the survival set of X.

Suppose now that $d \geq 2$. We shall need standard geometrical results which trace back to Euclid's works, *Elements* (cf. Heath (1956)). For any constant $c > 0$ define S_c the sphere with radius $c > 0$. Let x be any positive number. Then for each $\epsilon > 0$ there exists a regular polytope $\mathcal{P}$ symmetrically centred at the origin which contains S_x and is contained in $S_{x+\epsilon}$. Let $\mathbf{e}_1, ..., \mathbf{e}_n$ be the outward normal unit vectors to each surface of $\mathcal{P}$. For each $\mathbf{e}_i : i = 1, ..., n$ let $H(\mathbf{e}_i)$ be a slab of arbitrary positive thickness whose surface closest to the origin contains the side of $\mathcal{P}$ which has normal vector $\mathbf{e}_i$. Write $T^{\mathbf{e}_i}$ for the last time that the space-time slab $\{H(\mathbf{e}_i) + t\sqrt{2\beta}\mathbf{e}_i : t \geq 0\}$ is charged. From the previous paragraph we know that $T^{\mathbf{e}_i} < \infty$ P_μ-almost surely for each $i = 1, ..., n$.

Now define $\mathcal{P}(t)$ to be the radially scaled version of $\mathcal{P}$ (scaling factor $(\sqrt{2\beta}t + x)/x$) which contains $S_{\sqrt{2\beta}t+x}$ and is contained in $S_{(1+\epsilon)\sqrt{2\beta}t+x}$. Since $\sup_{i=1,...,n} T^{\mathbf{e}_i} < \infty$ P_μ-almost surely, it follows that all mass is eventually contained in the inflating polytope $\{\mathcal{P}(t) : t > 0\}$ P_μ-almost surely. This implies in turn that all mass is eventually contained in the inflating sphere $\{S_{(1+\epsilon)\sqrt{2\beta}t+x} : t > 0\}$ P_μ-almost surely. (Again we take advantage of the existing weaker forms of Theorem 4 as assurance that the support cannot travel faster than linearly in time). That is to say,

$$\limsup_{t\uparrow\infty} \frac{M_t}{t} \leq (1 + \epsilon)\sqrt{2\beta}.$$

Since ϵ can be chosen arbitrarily close to zero we have shown that

$$\limsup_{t\uparrow\infty} \frac{M_t}{t} \leq \sqrt{2\beta}$$

and the proof is complete for dimension $d \geq 2$.

For the one dimensional case, it suffices to note that S_1 consists of two vectors, $\mathbf{e}_1$ and $\mathbf{e}_2$ say and since $T^{\mathbf{e}_1} \vee T^{\mathbf{e}_2} < \infty$ P_μ-almost surely the limsup is easily deduced. ■

Remark 9 The proof of Theorem 4 shows something a little stronger than the statement of the theorem. Namely that the (space-time) exterior of the inflating sphere $\{S_{x+\gamma t} : t > 0\}$, for arbitrary $x > 0$, is charged for arbitrarily large t when X survives P_μ-almost surely when $\gamma \in [0, \sqrt{2\beta})$ and not charged for all sufficiently large t P_μ-almost surely when $\gamma > \sqrt{2\beta}$. As ususal it is understood that $\mu \in \mathcal{M}_c(\mathbb{R}^d)$.

Remark 10 In principle it is possible to extract the result in Theorem 4 for branching Brownian motion from the results in Biggins (1997) which concern general branching walks. However, it is not clear how to proceed to the case of super-Brownian motion from Biggins' results.

References

[1] Biggins, J.D. (1997) How fast does a general branching random walk spread? *In Classical and Modern Branching processes, (K.B. Athreya, P. Jagers, eds.).* IMA Volumes in Mathematics and its Applications 84, 19-40. Springer-Verlag, New York.

[2] Bramson (1978) Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.* **31** 531–581.

[3] Bramson, M.(1983) Convergence of solutions to the Kolmogorov nonlinear diffusion equation to travelling waves. *Mem. Amer. Math. Soc.* **44** 1–190.

[4] Dynkin, E. B. (1993) Superprocesses and partial differential equations. *Ann. Probab.* **21** 1185–1262.

[5] Engländer, J. (2002) Large deviations for the growth rate of the support of supercritical super-Brownian motion. *Preprint.*

[6] Engländer, J. and Hollander den, F. (2002) On branching Brownian motion in a Poisson field of traps. *To appear in Markov Processes and Related Fields.*

[7] Engländer, J. and Pinsky, R. (1999) On the construction and support properties of measure-valued diffusions on $D \subseteq \mathbb{R}^d$ with spatially dependent branching. *Ann. Probab.* **27**, 684–730.

[8] Engländer, J. and Kyprianou, A. E. (2002) Local extinction versus local exponential growth for spatial branching processes. *Submitted.*

[9] Heath, T.L. (1956) *Euclid's Elements. Vol 3 (book XII.2 and XII.17)* Dover Publications, New York.

[10] Itô, K. and McKean, H.P. (1974) *Diffusion processes and their sample paths.* Springer.

[11] Kolmogorov, A., Petrovskii, I. and Piskounov, N. (1937) Étude de l'équation de la diffusion avec croissance de la quantité de la matière at son application a un problèm biologique. *Moscow Univ. Bull. Math.* **1**, 1–25.

[12] McKean. H. P. (1975) Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piscounov. *Comm. Pure Appl. Math.* **128** 323–331.

[13] Pinsky (1995) On large time growth rate of the support of supercritical super-Brownian motion. *Ann. Probab.* **23** 1748–1754.

[14] Pinsky, R. G. (1996) Transience, recurrence and local extinction properties of the support for supercritical finite measure-valued diffusions Ann. Probab. 24, 237-267.