APPLICATIONS OF CANONICAL RELATIONS IN GENERIC DIFFERENTIAL GEOMETRY

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ABSTRACT. Conflict sets are loci of intersecting wavefronts emanating from l different surfaces. We show that generically conflict sets are Legendrian: locally they admit the structure of wavefronts. Simple stable singularities for this problem in $\mathbb{R}^n$ occur when $0 \leq n - l \leq 4$. Other related sets, such as kite curves and centre sets are also defined and discussed. Throughout canonical relations are used as an essential tool to carry out several of these geometrical constructions.

INTRODUCTION

The symmetry set of a manifold $M \subset \mathbb{R}^n$ is defined as the closure of the set in $u \in \mathbb{R}^n$ where the distance function

$$s \overset{F}{\mapsto} \|u - \gamma(s)\|$$

has a double extremum. Here $\gamma \colon M \to \mathbb{R}^n$ is an embedding of the surface. The symmetry set is the closure of the A_1A_1 stratum in the parameter space $\mathbb{R}^n$ of the family of functions $F(x, s)$. The symmetry set was studied in [JB85]. There also a number of other related sets that measure

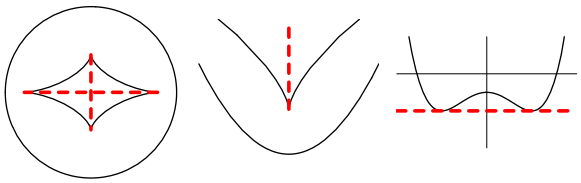

FIGURE 1. Two symmetry sets and a double extremum

symmetry are discussed: medial axis, cut-locus. For references to the many variants that exist we refer to this paper and other work of Giblin.

In this paper we do not take one manifold, but l manifolds of codimension 1 in $\mathbb{R}^n$. Hence, we have l distance functions. We are interested in the points where all the the distance functions have an extremum at the same time. So we attempt to find those $x \in \mathbb{R}^n$ for which there are s_i on the

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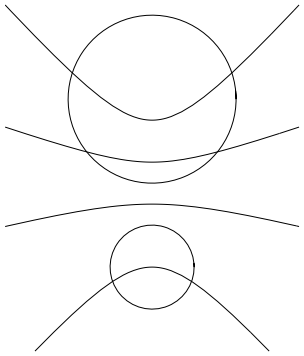

FIGURE 2. A conflict set of two circles

M_i such that the following equations hold true:

$$\begin{aligned} F_i(x,s) =& \|x-\gamma_i(s_i)\| \qquad F_i = F_j \qquad 1 \leq i,j \leq l \\ & \frac{\partial F_i}{\partial s_i} = 0 \in \mathbb{R}^{n-1} \qquad 1 \leq i \leq l \end{aligned} \tag{0.1}$$

Here $\gamma_i \colon M_i \mapsto \mathbb{R}^n$ are again embeddings for the hypersurfaces M_i.

Definition 0.1. The x_0 for which such $\{s_i\}_{1\leq i\leq l}$ exist make up the conflict set.

We measure not so much symmetry but more that what is in the middle. The conflict set of two lines is another pair of lines: nl. their bisectors. For a line and a circle the conflict set is already a more complicated object, while the conflict set of two circles generally will consist of four conic sections. This is due to the fact that the distance function from a fixed point to a circle mostly has two extrema. With other curves such phenomena happen too.
To be able to single out a certain component we define an oriented conflict set. Let the M_i be oriented manifolds. Then the orientation defines a direction for a flow induced by the unit normal vectorfield $n_i \in NM_i$.

Definition 0.2. The union of the intersection of the wavefronts at all times $t \in \mathbb{R}$ is called the oriented conflict set.

The main result of this paper comes in two parts.

- The conflict set of $\{M_i\}_{1\leq i\leq l}$ is for generic embeddings of the surfaces M_i a Legendrian manifold, that it is the projection of a smooth set in $PT^*\mathbb{R}^n$.
- The conflict set generically has the singularities that occur in $n-l$ parameter families of wavefront in $\mathbb{R}^{n-l+2}$. If $n-l+2 \leq 6$ then the conflict set generically only has singularities that are combinations of the well-known ADE singularities.

Our approach is such that it will entail several other results. Among these are results concerning the center symmetry set, and others concerning the Gauss map. Also when $n = l$ we will define a sort of dual to the conflict set, akin to the dual of a curve in projective space.

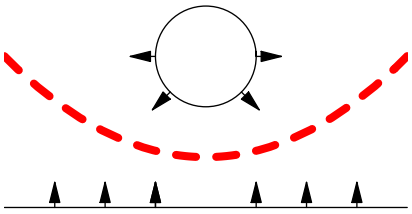

FIGURE 3. An oriented conflict set of a line and a circle. The arrows indicate the orientations.

The paper has the following set up. In the first section we redefine the conflict set as the intersection of so called big fronts. In the second section we use canonical relations to carry out a number geometrical constructions. In the third section we use the Thom transversality theorem to proof the first part of out main result. In the fourth section we recall some results on Lagrangian and Legendrian singularities. These mainly concern so-called phase functions. We use these to proof the second part of our main result.
The main results of this paper have been the subject of talks held by the author at several conferences in the years 2000-2002.

1. STATEMENT OF THE MAIN RESULT

Denote by $T^*\mathbb{R}^n \setminus 0$ the slit cotangent bundle, that is the cotangent bundle without the zero section. Coordinates for the slit cotangent bundle are (x, ξ). The ξ are coordinates in the fiber. Let $H_i \colon T^*\mathbb{R}^n \setminus 0 \to \mathbb{R}_{\geq 0}$ be $\mathcal{C}^\infty$ functions positively homogeneous of degree 1, and independent of x. If also the matrices

$$\frac{\partial^2 H_i^2(\xi)}{\partial \xi^2}$$

are positive definite these functions define a Finsler metric and as Hamiltonians they define trajectories. These trajectories are particularly simple. They are straight lines. More precisely we have that the time that it takes to travel from p_0 to p_1 in $\mathbb{R}^n$ is a function whose squared value is a $\mathcal{C}^\infty$ function on $\mathbb{R}^n \times \mathbb{R}^n$ minus the diagonal and whose first derivative is never zero.
To a certain extent we could drop the condition that the H_i are translation invariant or the condition that the H_i^2 only take on positive values but that would lead us too far afield.
Now let M_i be smoothly embedded manifolds of codimension 1 in $\mathbb{R}^n$. We could take smoothly embedded submanifolds of any codimension, but we consider wavefronts emanating from these submanifolds and after some time they become submanifolds of codimension 1.
Denote coordinates on

$$T^*\mathbb{R}^{n+1} \setminus 0$$

by (x, ξ, t, τ).
For each of the M_i we can define the (oriented) big wavefront, by means of a map

$$\begin{aligned} \Psi \colon T^*(\mathbb{R}^{n+1}) \setminus 0 &\rightarrow : T^*(\mathbb{R}^{n+1}) \setminus 0 \\ \Psi \colon (x, \xi, t, \tau) &\rightarrow (\exp(tX_H), t, \tau - H) \end{aligned}$$

Definition 1.1. The image of

$$\mathbb{R} \times \{0\} \times (N^*M_i \cap \{H_i = 1\}) \subset T^*\mathbb{R} \times T^*\mathbb{R}^n \setminus 0$$

by the map Ψ or rather its image under multiplication in the fibers by $\lambda \in \mathbb{R}_{>0}$, is called the big wavefront. It is denoted by $N^*M_i^h$.

If the M_i are orientable we can consider a section n_i of N^*M_i contained in $\{H_i = 1\}$.

Definition 1.2. The image of

$$\mathbb{R} \times \{0\} \times \mathrm{Image}(n_i) \subset T^*\mathbb{R} \times T^*\mathbb{R}^n \setminus 0$$

by the map Ψ or rather its image under multiplication in the fibers by $\lambda \in \mathbb{R}_{>0}$, is called the oriented big wavefront. It is denoted by $N^*M_i^b$.

The (oriented) big wavefront are conic Lagrangian submanifolds of

$$T^*(\mathbb{R} \times \mathbb{R}^n) \setminus \{0\}$$

We can now define a more general conflict set in a Hamiltonian context.

Definition 1.3. The oriented conflict set of (M_i, n_i, H_i), $1 \leq i \leq l$ is

$$C\left((M_i, n_i, H_i)_{i=1,\cdots,l}\right) = \pi_n\left(\bigcap_{i=1}^{l} \pi_{n+1}\left(N^*M_i^b\right)\right)$$

where $\pi_n(x, t) = x$ and $\pi_{n+1}(t, \tau, x, \xi) = (t, x)$.

Remark 1.4. The conflict set of $\{(M_i, H_i)\}_{1 \leq i \leq l}$ is defined similarly. Also the sets $\pi_{n+1}(N^*M_i^h)$ and $\pi_{n+1}(N^*M_i^b)$ are sometimes called "the graph of the time function", see [Arn90]. The "time function" is of course not really a function. It is multi-valued.

Examples of these concepts are readily provided. For instance we can consider the Hamiltonians

$$H_1 = |\xi| \; H_2 = \eta|\xi| \; \eta \in \mathbb{R}_{>0}$$

for two circles. If we draw the conflict set for different values of η, we get a plethora of curves , see figure 4.
Denote by $\mathrm{Emb}(M_i, \mathbb{R}^n)$ the space of embeddings of the manifold M_i in $\mathbb{R}^n$. This space is an open subset of $\mathcal{C}^\infty(M_i, \mathbb{R}^n)$. To take into account the l embeddings we deal with we introduce the space

$$\bigoplus_{i=1}^{l} \mathrm{Emb}(M_i, \mathbb{R}^n)$$

The main result of our paper reads as follows.

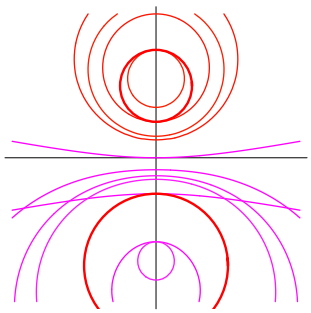

FIGURE 4. Conflict sets for different values of η

Theorem 1.5. *For a residual subset of $\bigoplus_{i=1}^{l} \mathrm{Emb}(M_i, \mathbb{R}^n)$ the (oriented) conflict set can be realized as the projection of a conic Lagrange submanifold in $T^*\mathbb{R}^n$. In addition if the M_i are compact the generic singularities of the (oriented) conflict set are the generic singularities of $n-l$ parameter families of fronts in $\mathbb{R}^{n-l+2}$ embedded in $\mathbb{R}^n$. In particular if $2 \leq n-l+2 \leq 6$ the (oriented) conflict set generically only has combinations of simple singularities of ADE type.*

Remark 1.6. The dimension of the conflict set in $\mathbb{R}^n$ is $n-l+1$.

We point out one particularly nice consequence of this theorem. If we have n surfaces in $\mathbb{R}^n$ the conflict set is one dimensional and generically only has self-intersections and cusps as singularities.

2. CANONICAL RELATIONS AND ASSOCIATED GEOMETRICAL CONSTRUCTS

We will need the notion of a canonical relation between two symplectic manifolds.

Definition 2.1. A canonical relation between two symplectic manifolds $\{(S_i, \omega_i)\}_{i=1,2}$ is a Lagrangian submanifold of

$$S_1 \times S_2, \pi_1^* \omega_1 - \pi_2^* \omega_2$$

The π_i denote projections $S_1 \times S_2 \to S_i$.

Graphs of symplectomorphisms $\chi \colon S_1 \to S_2$ are important examples of canonical relations. Symplectomorphisms can be composed, canonical relation can be composed as well. The composition of symplectomorphisms is a special case of a theorem of Hörmander, see [Hör85], chapter 21.

Theorem 2.2. *Let $S_i, i = 1 \cdots 3$ be three symplectic manifolds. Let G_1 be a canonical relation between S_1 and S_2 and G_2 one between S_2 and S_3. If $G_1 \times G_2$ intersects $S_1 \times \Delta(S_2) \times S_3$ transversally then the image G_3 under the projection $S_1 \times S_2 \times S_2 \times S_3 \mapsto S_1 \times S_3$ is a canonical relation between S_1 and S_3. We call it the composition $G_1 \circ G_2$ of G_1 and G_2.*

This theorem allows one to create new Lagrangian manifolds from old. Canonical relations also formalize remarks of Arnol'd who has regularly stated that in symplectic geometry one "decreases dimensions" by "sectioning and projection", see for instance [Arn90].
Often we will be interested in a particular case where S_3 is a point. So we use the next proposition,

that rephrases the demand in the theorem of Hörmander. Note that a canonical relation between S_3 and a point is just a Lagrange manifold in S_3.

Proposition 2.3. *Let G_1 be a canonical relation between S_1 and S_2 whose projection to S_2 is an immersion and let G_2 be a canonical relation between S_2 and a point. Then the composition $G_1 \circ G_2$ is a canonical relation , and thus a Lagrangian manifold in S_1, if $\pi_2(G_1) \pitchfork G_2$.*

Proof. We need that

$$G_1 \times G_2 \pitchfork S_1 \times \Delta(S_2) \tag{2.1}$$

This intersection is contained in the graph of the projection $\pi_2 \colon G_1 \to S_2$. We have that (2.1) holds iff.

$$\mathrm{gr}(\pi_2) \pitchfork G_1 \times G_2$$

this in turn is true iff.

$$\pi_2(G_1) \pitchfork G_2$$

□

Now the theorem of Hörmander has a more familiar interpretation: through this manifold $\pi_2(G_1)$ we can pull back Lagrangian manifolds from S_2 to S_1.

2.1. **Conflict sets.** Our foremost example of this procedure will be where

$$S_1 = T^*\mathbb{R}^{n+1} \setminus 0$$

As coordinates for S_1 we will take

$$\left(\bar{x}, \bar{\xi}\right) \ \bar{x} = (x_0, x) \in \mathbb{R}^{1+n} \ \bar{\xi} = (\xi_0, \xi) \in \mathbb{R}^{1+n}$$

or $(\bar{y}, \bar{\eta})$. In $T^*\mathbb{R}^n$ we will write (x, ξ) or (y, η).
If we put $S_2 = (S_1)^l$ then conflict sets can be constructed by means of a canonical relation as follows. Set $G_1 \subset S_1 \times S_2$

$$\left\{ \bar{y}, \bar{\eta}, \bar{x}_1, \bar{\xi}_1, \cdots, \bar{x}_l, \bar{\xi}_l \mid \bar{\eta} = \sum_{i=1}^{l} \bar{\xi}_i, \ \bar{y} = \bar{x}_i, \ 1 \leq i \leq l \right\}$$

The manifold G_1 is clearly conic Lagrange and also it projects as an embedding to S_2. For G_2 we take the product of the big wavefronts

$$G_2 = \times_{i=1}^{l} N^* M_i^h \text{ or } G_2 = \times_{i=1}^{l} N^* M_i^b$$

The manifold $\pi_2(G_1)$ is the diagonal in $\left(\mathbb{R}^{n+1}\right)^l$ together with all cotangent vectors

$$\pi_2(G_1) = T^*_{\Delta}(S_1^l)$$

The lifted conflict set L^h (or L^b in the oriented case) is the pull-back to S_1 by $T^*_{\Delta}(\mathbb{R}^{(n+1)l})$. The projection of L_h to the base $\mathbb{R}^{1+n}$ is the graph of the time function on the conflict set.

Theorem 2.4. *L^h is conic Lagrange if*

$$\times_{i=1}^{l} N^* M_i^h \pitchfork T^*_{\Delta}(\mathbb{R}^{(n+1)l}) \tag{2.2}$$

Proof. Clear from the above remarks. □

Remark 2.5. The criterion in equation (2.2) is quite computable in practice. Suppose that the time functions for each of the big wave fronts take on the form

$$t = F_i(x, s_i),\ \frac{\partial F_i}{\partial s_i} = 0,\ 1 \leq i \leq l$$

then the intersection in equation (2.2) is transversal iff.

$$F(x, \lambda, s_1, \cdots, s_l) = \sum_{i=1}^{l-1} \lambda_i (F_i(x, s_i) - F_{i+1}(x, s_{i+1}))$$

is a phase function, i.e. the matrix

$$\mathrm{d}_{s,\lambda,x}(F, \mathrm{d}_{s,\lambda} F)$$

has maximal rank there where

$$(F, \mathrm{d}_{s,\lambda} F) = 0$$

In case the Hamiltonians are just Euclidean metrics this leads to a lot of computable examples, in the spirit of [Por94]. For instance, it is verified that with 3 surfaces in $\mathbb{R}^3$ only one of the wavefronts can have a cuspidal edge, if this maximal rank criterion is to hold. All more degenerate cases lead to, albeit interesting, examples of non-Legendrian behavior.

We now have a conflict set L^h in S_1 and this object is the most important one. First, the projection of L^h to $\mathbb{R}^{n+1}$ has the same Legendrian singularities as the projection of the corresponding object in $T^*\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}^n$, and second, there is associated to the conflict set a "kite curve" which can be constructed directly from L^h. The kite curve will be treated below in paragraph 2.2.
To pull L^h into $T^*\mathbb{R}^n \setminus \{0\}$ we need to apply the "sectioning and projection". The section is $\eta_0 = 0$ and the projection is along the y_0 axis. The y_0 is axis is the time axis and as we saw that $\pi_{n+1}(L^h)$ is the graph of the time function on the conflict set it is not surprising that this projection is immersive and that it induces no extra singularities. The conic canonical relation we use is

$$G_1 = \{(x, \xi, \bar{y}, \bar{\eta}) \mid x = y,\ \xi = \eta, \eta_0 = 0\} \subset T^*\mathbb{R}^n \setminus \{0\} \times S_1 \text{ and } G_2 = L^h \tag{2.3}$$

Using proposition 2.3 we obtain that if

$$L^h \pitchfork W \text{ where } W = \{(\bar{y}, \bar{\eta}) \mid \eta_0 = 0\} \tag{2.4}$$

we can pull back L^h to $T^*\mathbb{R}^n \setminus \{0\}$.

Lemma 2.6. $\times_{i=1}^{l} N^* M_i^h \pitchfork T^*_{\Delta}\left(S_1^l\right) \Rightarrow W \pitchfork L^h$

Proof. Suppose that we did not have $W \pitchfork L^h$. Because W is a hypersurface that would mean that at some point p in L^h the tangent space TL^h would be contained in TW. So it would hold

$$\langle (0,0,0,\delta\eta_0), \vec{w} \rangle = 0, \ \forall \vec{w} \in TL^h$$

and consequently

$$\omega((0,0,\delta y_0,0), \vec{w}) = 0, \ \forall \vec{w} \in T_pL^h$$

with ω being the canonical symplectic structure. But L^h is Lagrangian, so we'd have that this vector $(0,0,\delta y_0,0) \in TL^h$. But that is clearly impossible. □

Corollary 2.7. *The conflict set is conic Lagrange if (2.2) holds.*

Our main theorem says that not only generically the conflict set is a Legendrian manifold, but also that its singularities are singularities of fronts in $\mathbb{R}^{n-l+2}$. So we want to pull back not just to $\mathbb{R}^n$ but to an $n-l+2$ dimensional subspace of $\mathbb{R}^n$. This can be done in more or less the same way as in the above where we projected along along the "time" axis. front in an $n-l+2$ dimensional space.
We can project in a Legendrian way along some direction v if

$$W(v) = \{(x,\xi) \in T^*\mathbb{R}^n \mid \langle v, \xi \rangle = 0 \qquad \|\xi\| = 1\}$$

intersects transversely with L^h. This was done in the above along the time axis.
More generally we can section and project along a subspace V spanned by $\{v_1, \cdots, v_k\}$, if

$$W(V) = \{(x,\xi) \in T^*\mathbb{R}^n \mid \langle v_i, \xi \rangle = 0 \quad i = 1, \cdots, l \quad \|\xi\| = 1\}$$

intersects L^h transversely. The assertion is proven by using a canonical relation as the previous G_1. In $\mathbb{R}^n$ the conflict set is $n-l+1$ dimensional. If it is Legendrian "the fiber" has dimension $l-1$. A maximum of $l-2$ directions can thus additionally be sectioned away. We end up in an $n-l+2$ dimensional space, as stated in the main theorem.

2.2. **The kite curve.** Associated to the conflict set is the kite curve. It is some sort of dual to the conflict set. This occurs when $l = n$. Suppose we are at a point of the conflict set that corresponds to l Morse extrema of the time function and such that the normals to the front are affinely in general position. The conflict set is smooth there and at the l basepoints on the M_i we have tangent planes,

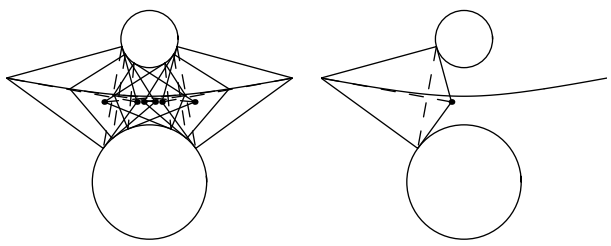

FIGURE 5. Oriented conflict sets and some kites

that are naturally thought of as affine hyperplanes in $\mathbb{R}^n$. They usually intersect in a point. This point traces out a curve in $\mathbb{R}^n$.
The kite curve can most conveniently be thought of the intersection of the tangent developable of $\pi_{n+1}(L^h)$ with a plane $t = \text{constant}$, see figure 6. This explains immediately that the kite curve is a line whenever the M_i are all spheres (in sufficiently general position).
To form a good idea of the kite curve one could consult [Sie99]. There the case with 2 curves in $\mathbb{R}^2$ is thoroughly examined. The kite curve is exhibited there as the locus of points where two tangent lines have equal length. As said for two circles in the plane the kite curve is a straight line, see the pictures in 5.
If one wants to, the kite curve can also be constructed at singular points of the conflict set. First of all we are interested in the intersection of the big wave fronts, so we need:

$$\bar{x}_i = \bar{x}_j,\ 1 \leq i, j \leq l \tag{2.5}$$

Our y variable should be such that it is in the intersection of the tangent planes to the big fronts. So the vector that runs from the point $(y, 0)$ to $\bar{x}_i$ should lie in all the tangent planes. So the line from $(y, 0)$ to $\bar{x}_i$ should be orthogonal to each $\bar{\xi}_i$. That is:

$$\langle x - y, \xi_i \rangle + x_0 \xi_{i,0} = 0,\ 1 \leq i \leq l = n \tag{2.6}$$

Equations (2.5) and (2.6) define a set in $\mathbb{R}^n$. Possibly this is a Legendrian curve, but this is not so clear. Further on we will consider a more natural candidate for a "kite curve".

2.3. **Gauss maps and parallel tangent planes.** The Gauss map can also be used to define sets that measure symmetry. The center symmetry set is defined as the locus of the midpoints of chords connecting two distinct points with parallel tangent space on a surface $M_1 \subset \mathbb{R}^n$. Apparently this was introduced by Giblin, see [GH99]. The center symmetry set of a conic section is a point.
Clearly we can repeat this with two distinct manifolds, looking for pairs of parallel tangent planes. As set we can consider

1. the midpoints of the chords connecting the tangent planes
2. the normals themselves as a subset of the family of oriented lines in $\mathbb{R}^n$, T^*S^{n-1}

Definition 2.8. The midpoints of the chords form the **center set** and the chords themselves form the **normal chord set**.

These constructions can both be carried out with canonical relations. The first curve will turn out to be Legendrian iff. the second is Lagrangian.
We do a recap of the construction of the space of oriented lines in $\mathbb{R}^n$. A directed line in $\mathbb{R}^n$ has a direction, that is a unit vector v in S^{n-1}. At $v \in S^{n-1}$ there is a tangent plane. This tangent plane can be identified with a plane in $T_0\mathbb{R}^n$. The intersection point of this tangent plane with the directed line leaves us with a vector in $T_v S^{n-1}$. On the tangent space we have the Legendre transform that maps the vector in $T_v S^{n-1}$ to $T_v^* S^{n-1}$.
On a hypersurface M in $\mathbb{R}^n$ we have the Gauss map. We can view it as a map to T^*S^{n-1}: assign to $p \in M$ the normal as a directed line in $\mathbb{R}^n$. It is a standard theorem that the image of the Gauss

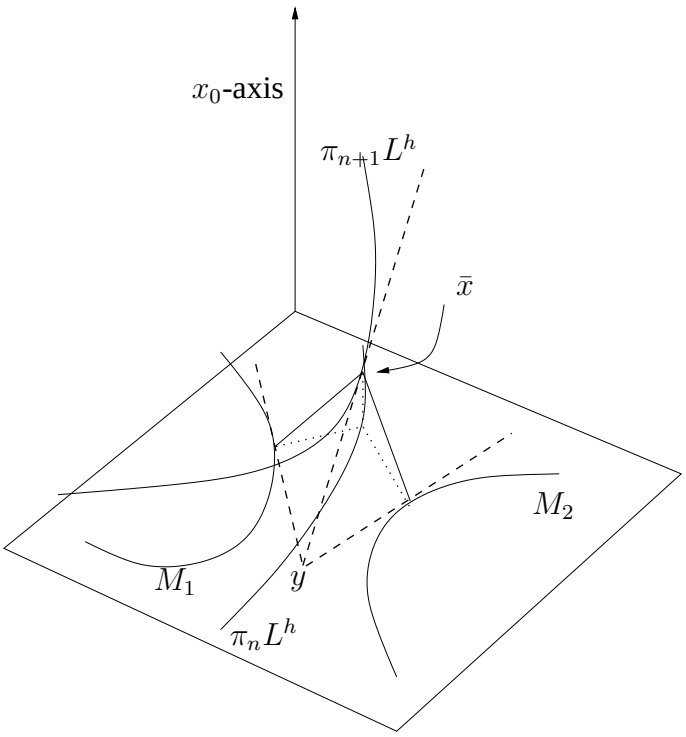

FIGURE 6. The construction of the kite curve.

map is Lagrangian, see again [Arn90].
One can proof this theorem by proving that the set

$$v, \mu, x, \xi \in T^*S^{n-1} \times T^*\mathbb{R}^n \text{ such that}$$
$$v = \frac{\xi}{\|\xi\|},\ \mu = x - \frac{\langle x, \xi \rangle \xi}{\|\xi\|^2},\ \|\xi\| = 1 \tag{2.7}$$

is a canonical relation between T^*S^{n-1} and $T^*\mathbb{R}^n$. The pull-back to T^*S^{n-1} of the conormal bundle to a manifold is the image of the Gauss map. We could try to pull back any Lagrangian manifold to T^*S^{n-1} by computing in this way "the image of the Gauss map". Using proposition 2.3 we see that this only makes sense if the Lagrange manifold is $T^*\mathbb{R}^n$ lies transverse to all the level sets of $\|\xi\|$, that is, "the image of the Gauss map" is defined for all conic Lagrange manifolds. This is a rather important remark, we formulate it in a theorem - for which we claim no originality whatsoever:

Theorem 2.9. *For each Legendre manifold in $\mathbb{R}^n$ we can define an "image of the Gauss map". If the Legendre manifold is the conormal bundle of a smooth submanifold of codimension 1 in $\mathbb{R}^n$ this coincides with the usual image of the usual Gauss map.*

2.4. **The center set.** Let us now look at the center set. We introduce a canonical relation between

$$S_1 = T^*\mathbb{R}^n \setminus 0$$

and

$$S_2 = (S_1)^2$$

We will as in the above use coordinates

$$(y, \eta, x_1, \xi_1, x_2, \xi_2)$$

Define G_1 by

$$y = \frac{x_1 + x_2}{2},\ \eta = 2\xi_1, \xi_1 = \xi_2 \tag{2.8}$$

The manifold G_1 is clearly a conic canonical relation and G_1 projects immersively into S_2. For G_2 we take the product of the conormal bundles of M_1 and M_2. The proposition 2.3 can be applied to yield that

Theorem 2.10. *The center set is conic Lagrange if*

$$N^*M_1 \times N^*M_2 \pitchfork \{(x_1, \xi, x_2, \xi\} \tag{2.9}$$

Remark 2.11. Note the reciprocity between (2.9) and (2.2). The criterion for the conflict set deals with a diagonal in the base and the criterion for the center set deals with a diagonal in the fiber

Remark 2.12. Clearly, lots of other interesting and less interesting sets can be constructed in this way. We could take l manifolds M_l and consider the relation

$$y = \sum_{i=1}^{l} a_i x_i \qquad \eta a_i = \xi_i \tag{2.10}$$

where the a_i are a set of nonzero numbers. This again is a canonical relation. If the product of the conormal bundles of the M_i is transverse to the "diagonal in the fiber" as in (2.9) then the resulting set is a Legendre manifold. For instance with three surfaces we could take the centroid of a triangle.

2.5. **The normal chord set.** Next comes the normal chord set. Denote ν_i the Gauss map from N^*M_i to T^*S^{n-1}. We have an image of the product of the two Gauss maps, the chords we use are on the diagonal. After the many examples above the following theorem is obvious:

Theorem 2.13. *If*

$$\nu_1(N^*M_1) \times \nu_2(N^*M_2) \pitchfork \{(v, \mu_1, v, \mu_2)\} \tag{2.11}$$

the normal chord set is Lagrangian.

Remark 2.14. For the normal chord set we can write down maximal rank criteria as we did for the conflict set in remark 2.5. A phase function for the image of the Gauss map is

$$\begin{aligned} F_i \colon S^{n-1} \times M_i &\to \mathbb{R} \\ (v, s) &\mapsto \langle v, \gamma(s) \rangle \end{aligned} \tag{2.12}$$

The image of the Gauss map is described by

$$\frac{\partial F_i}{\partial s} = 0 \tag{2.13}$$

To get a maximal rank criterion under which the normal chord set is Lagrangian we use as in the remark 2.5 a special phase function:

$$F_1(v, s_1) + F_2(v, s_2)$$

And the maximal rank criterion that is equivalent to the transversality in (2.11) is that the matrix

$$\mathrm{d}_{v,s_1,s_2} \left(\mathrm{d}_{s_1,s_2} F \right)$$

has maximal rank there where

$$\mathrm{d}_{s_1,s_2} F = 0$$

If so, the normal chord set is Lagrangian. If one chooses local coordinates on S^{n-1}, as is done in [BGM82], this is a nicely computable criterion.

2.6. **The kite curve revisited.** As a definition of the kite curve it seems more useful to consider the image in T^*S^n of the lifted conflict set L^h: the image of its Gauss map. This is also defined when $l < n$. The kite curve can be reconstructed from it, when $l = n$.
If we define the kite curve in this way we can summarize our reasoning in a tentative diagram, that illustrates the dualities mentioned.

	Diagonal in the base	Diagonal in the fibre	
Legendrian lift is smooth	Conflict set	Centre set	in $T^*\mathbb{R}^{n+1}$ (resp. $T^*\mathbb{R}^n$)
	↓	↓	
Lagrangian lift is smooth	Kite curve	Chord set	in T^*S^n (resp. T^*S^{n-1})

3. GENERICITY OF TRANSVERSALITY CONDITIONS

We proof that for a residual set of embeddings in $\bigoplus_{i=1}^{l} \mathrm{Emb}(M_i, \mathbb{R}^n)$ the transversality conditions that ensure that the conflict set and the center set (resp. the kite curve and the normal chord set) are Legendrian (resp. Lagrangian), are satisfied.

3.1. **That the conflict set is generically Legendre.** To proof the genericity of the criterion (2.2) we will make use of the map that defines the big wavefront, described in definition 1.1. This is a map

$$\times_{i=1}^{l}(M_i \times (\mathbb{R} \setminus 0) \times \mathbb{R}) \to T^*(\mathbb{R}^{n+1})^l \tag{3.1}$$

More precisely, we associate such a map to each $(\gamma_1, \cdots, \gamma_l)$.

$$\bigoplus_{i=1}^{l} \mathrm{Emb}(M_i, \mathbb{R}^n) \to \mathcal{C}^\infty(\times_{i=1}^{l}(M_i \times (\mathbb{R} \setminus 0) \times \mathbb{R}), T^*(\mathbb{R}^{n+1})^l) \tag{3.2}$$

To simplify matters we look at each embedding individually.

$$\mathrm{Emb}(M_i, \mathbb{R}^n) \to \mathcal{C}^\infty(M_i \times (\mathbb{R} \setminus 0) \times \mathbb{R}, T^*(\mathbb{R}^{n+1})) \tag{3.3}$$

These can be put in a family. Namely just translate the embeddings by a (small) vector e_i. For simplicity drop the index i.

$$\mathbb{R}^n \times \mathrm{Emb}(M, \mathbb{R}^n) \to \mathcal{C}^\infty(M \times (\mathbb{R} \setminus 0) \times \mathbb{R}, T^*(\mathbb{R}^{n+1})^l)$$

$$(e, \gamma\colon M \to \mathbb{R}^n) \longmapsto e, s, \lambda, x_0 \xrightarrow{\Psi} \begin{pmatrix} \pi_x(\exp(x_0 X_H)(\gamma(s) + e)) \\ \lambda \pi_\xi(\exp(x_0 X_H)(\gamma(s) + e)) \\ x_0 \\ \lambda \end{pmatrix} \tag{3.4}$$

We need that l copies of maps Ψ ,each for a different M_i map transversal to $T^*_\Delta(\mathbb{R}^{1+n})^l$. We see that it suffices to proof that l copies of

$$e, s, x_0, \lambda \to \pi_x(\exp(x_0 X_H)(\gamma(s) + e)), x_0 \tag{3.5}$$

map transversal to the diagonal $\Delta \subset (\mathbb{R}^{1+n})^l$. Because

$$\pi_x(\exp(x_0 X_H)(\gamma(s) + e)) = e + \pi_x(\exp(x_0 X_H)(\gamma(s)))$$

this is clear: the derivatives for the e vectors and the time variable x_0 already cause the maximal rank to be attained. So the product of l maps (3.5) is submersive onto $(\mathbb{R}^{n+1})^l$. We have thus shown that

$$(\mathbb{R}^n)^l \times \bigoplus_{i=1}^{l} \mathrm{Emb}(M_i, \mathbb{R}^n) \to \mathcal{C}^\infty(\times_{i=1}^{l}(M_i \times (\mathbb{R} \setminus 0) \times \mathbb{R}), T^*(\mathbb{R}^{n+1})^l) \tag{3.6}$$

gives for all

$$(\gamma_1, \cdots, \gamma_l)$$

a family of mappings , parametrized by $(\mathbb{R}^n)^l$, whose image is transverse to the closed manifold $T^*_\Delta(\mathbb{R}^{1+n})^l$, or any closed submanifold of $T^*(\mathbb{R}^{1+n})^l$ for that matter. Thus most members of this family are transversal to $T^*_\Delta(\mathbb{R}^{1+n})^l$. It now follows from results of Abraham, see [Wal77], that for a residual set of embeddings the transversality condition is satisfied, which is exactly what we needed. For clarity we cite the theorem:

Theorem 3.1 (Thom Transversality theorem)**.** *Let A be a manifold of mappings , let X, Y be manifolds. Let*

$$\alpha\colon A \to \mathcal{C}^\infty(X, Y)$$

be a map such that

$$\mathrm{ev}(\alpha)\colon A \times X \times Y \tag{3.7}$$

is a smooth submersion at every $(a, x) \in A \times X$. Then for every closed submanifold $W \subset Y$ we have that

$$\{a \in A \mid \ \alpha(a) \pitchfork W\}$$

is a residual subset of A.

In particular $\oplus_{i=1}^l \mathrm{Emb}(M_i, \mathbb{R}^n)$ is a manifold of mappings, the map α we use is in (3.3). For any fixed $(\gamma_1, \cdots, \gamma_l)$ l copies of the family in (3.5) are submersive. These are all embeddings, so locally the map from (3.7) is a smooth submersion.

Proposition 3.2. *The conflict set is generically Legendre and the kite curve in T^*S^n is generically Lagrange.*

Remark 3.3. Roughly speaking the family of translations (3.5) produces all first order perturbations. This family will be of much use further on.

3.2. **That the center set is generically Legendre.** The aforementioned family seemingly can not be used to proof that generically the center set is conic Lagrange. We need a covering $\{U_\alpha\}$ of $M_1 \times M_2$ and in each $\{U_\alpha\}$ perturb the tangent space a little, as indicated in figure 7. It is enough to proof that , if $\vec{n}_i$ is the map that assigns the normal to M_i, that the map $(\vec{n}_1, \vec{n}_2)$ is transverse to the diagonal. We first show that locally families exist that are indeed transverse to the diagonal. Denote by

$$\phi_{r,A,p},\ r \in \mathbb{R}, A \in SO(n, \mathbb{R}), p \in \mathbb{R}^n$$

a diffeomorphism, which is the identity on $\mathbb{R}^n$ where we're outside the sphere of radius $2r$ round p and equal to $q \to A(x - q)$, inside a circle of radius r round q. Now compose an embedding $\gamma\colon M \to \mathbb{R}^n$ with the map $\phi_{r,A,p(\alpha)}$ and we get a map that in some environment U'_α of of $p(\alpha) \in M$ is submersive. Looking at a product $\phi_{r,A,q(\alpha)} \circ \gamma_1, \phi_{r',A',p(\alpha)} \circ \gamma_2$ we see that in a neighborhood U_α the transversality condition is satisfied. Indeed, at $p(\alpha)$ the normal looks like $A\vec{n}$.
One can pick a countable number of points $p(\alpha)$ such that the U_α cover $M_1 \times M_2$. We have proven:

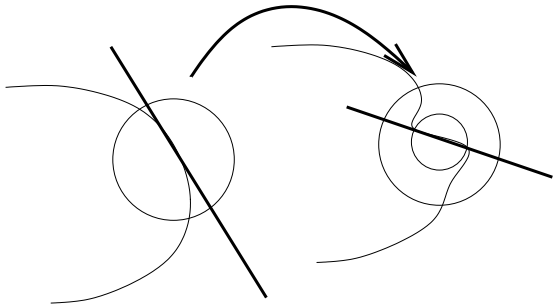

FIGURE 7. The map $\phi_{r,A,p=\gamma(s')} \circ \gamma(s)$

Theorem 3.4. *For a countable intersection of open and dense subsets of*

$$\bigoplus_{i=1}^{2} \mathrm{Emb}(M_i, \mathbb{R}^n)$$

the center set is Legendrian (and the normal chord set therefore Lagrangian)

4. PROOF OF MAIN THEOREM

The arguments presented in this section are standard and so we do not provide all the details. More details can be found in the excellent surveys [Dui74] and [Wal77]. See also [AGZV85].
The general idea of the proof of the theorem is that we stratify for each M_i some part B_i of the graph of the time function $\pi_{n+1}(N^*M_i^h)$. The strata of B_i correspond to singularity types of individual momental fronts. Then the intersections of these graphs are generically such that they miss the non-stratified part. So in the intersection we will only meet singularities that are well-known singularity types of individual fronts.
Our proof will consist of purely local considerations, they can be patched together as the transversality theorem.

4.1. **Stratification by codimension of the big wavefront.** We want to define equisingularity manifolds. They will be defined using the notion of codimension for germs. We use codimension wrt. to contact equivalence or V-equivalence. This is motivated by our Legendrian point of view. Two germs of unfoldings are V-isomorphic iff. the germs of Legendrian immersions they determine are equivalent, see [AGZV85], §20.
We consider locally $x_0 = F(x, s)$ the time function belonging to the embedding $M \hookrightarrow \mathbb{R}^n$. That is we consider it near $\bar{x}', s' \in \mathbb{R}^{n+1}$. Then we put $G(\bar{x}, s) = x_0 - F(x, s)$. The equations

$$\frac{\partial G}{\partial s} = 0,\ G = 0 \tag{4.1}$$

define a (germ of a) surface $Z_1 \subset \mathbb{R}^{1+n+n-1}$, near $\bar{x}', s'$. We want to consider closed parts of the surface Z_1, namely those where the codimension of the germ

$$\jmath G \in \mathcal{C}^\infty(s'),\ s \xrightarrow{\jmath G} G(\bar{x}', s) = x_0 \tag{4.2}$$

has a suitable value:

$$1 \leq \operatorname{codim} \jmath G \leq N$$

We define the codimension by the dimension of the real vector space

$$\dim_{\mathbb{R}} \frac{\mathcal{C}^\infty(s')}{(G, \mathrm{d}_s G)}$$

As an example consider

$$x_0 = s^3 + x_1 s + x_2,\ \bar{x}' = 0,\ s' = 0$$

The codimension to calculate is

$$\dim_{\mathbb{R}} \frac{\mathcal{C}^\infty(s')}{(s^3, s^2)} = 2$$

There are other points nearby that have codimension 2. These are

$$\{x_0 = a,\ x_1 = 0,\ x_2 = a,\ s = 0\}$$

The projection of this manifold to the $\bar{x}$-space is a affine space. This affine space has codimension 2 in $\mathbb{R}^{n+1}$. (In the example $n = 2$)
We would like this to hold in general. The manifold Z_1 should be stratified.
Put $N = \min(n - l + 2, 6)$. Then in the space of $N + 2$-jets of germs at s' in M, we have a part $\mathcal{C}$ that is stratified according to the codimensions 1 to N. The part of $J^{N+2}(s')$ with codimension $> N$ is an algebraic variety that can be stratified in some canonical way.
The surface Z_1 is to be divided into a part $\mathcal{B}$ and its complement $\complement\mathcal{B}$. The complement should have codimension $> N$ and the part $\mathcal{B}$ should have a Whitney stratification such that on each stratum the codimension is constant.
To stratify Z_1 we consider the map

$$\bar{x}, s \overset{\jmath^{N+2}G}{\to} \mathbb{R}^n \times J^{N+2}(s)$$

If $\jmath^{N+2}G$ is transverse to the stratification of $\mathcal{C}$ and its complement then this induces the division of Z_1 in $\mathcal{B}$ and $\complement\mathcal{B}$. Each of the strata of $\mathcal{B}$ corresponds to finitely many types of singularities and they project to $\mathbb{R}^{1+n}$ immersively.
We next pass to intersections.
Above one $\bar{x}$ there might be several pairs of $(\bar{x}, s^{(i)}),\ i = 1 \cdots r$. They are only finitely many because M is compact. For a residual subset of G and thus for a residual subset of the embeddings γ the projection $\pi^r \colon Z_1^{(r)} \to (\mathbb{R}^{1+n})^r$ is transverse to the diagonal stratification of $(\mathbb{R}^{n+1})^r$. We can have $\pi^r(\mathcal{B}) \pitchfork \mathcal{D}^{(r)}$ generically for $r = n$, which will be enough.
For this reason generically the stratification of $\mathcal{B}$ has regular intersections relative $\pi \colon Z_1 \to \mathbb{R}^{1+n}$.

Lemma 4.1. *For a residual set of embeddings $M \to \mathbb{R}^n$ the "graph of the time function" $\pi_{n+1}(N^*M^h)$ has a subset $\mathcal{B}$ of codimension $\leq N$ that is Whitney stratified and whose strata correspond to singularity types of individual momental fronts.*

4.2. **Intersecting l big fronts.** We want the intersection of the l bigfronts be such that in the intersection we only find elements of $\bigcap_{i=1}^{l} \mathcal{B}_i$ and no points of one the complements $\complement\mathcal{B}_i$. With the family (3.6) we see that the intersection of the strata of the bigfronts is transversal.
In fact the maximal codimension of a stratum of say $\pi_{n+1}(N^*M_1^h)$ that can appear in the intersection of the big fronts, appears when the other $l-1$ big fronts are smooth, so this maximal codimension is $\leq n+1-(l-1)$. Thus if $N \leq n-l+2$ we have only strata of the $\mathcal{B}_i$ in the intersection. This is where the condition $n-l+2 \leq 6$ in the theorem comes in.
The last thing we need to know to hold generically is alike what we needed to know for the projection $Z_1 \to \mathbb{R}^{1+n}$, namely that $\pi\colon \cap_{i=1}^{l} \pi_{n+1}(L^h) \to \mathbb{R}^n$ has regular intersections relative $\pi_n\colon L^h \to \mathbb{R}^n$. This is again achieved with the family (3.6).
If $n-l+2=7$ then it will be possible to find a stratum of the complement of say $\mathcal{B}_2$ in the intersection of the big fronts. Such a stratum can represent a modulus. If we move $\pi_{n+1}(N^*M_2^h)$ a little the stratum will still be there, but the singularity type will have changed. We conclude that $n-l+2 \leq 6$ are the nice dimensions for conflict sets.

4.3. **Geometrical description of different cases.** Once we know that the stratified big wavefronts intersect transversally to determine what sort of singularities can occur will follow from a codimension count.
For the description of these singularities the main distinction is the difference $n-l$.
Indeed if $(\mu_1, \mu_2, \cdots, \mu_l)$ is the list of codimensions then we seek μ_i with $1 \leq \mu_i$ and $\sum_{i=1}^{l} \mu_i \leq n+1$. Those μ_i that are 1 correspond to smooth hypersurfaces. They are not very interesting because they present just a reduction of n and l by 1, because if say $\mu_l = 1$ then $N^*M_l^h$ is smooth, and locally smoothly equivalent to $\mathbb{R}^{n-1} \times \mathbb{R}$. Thus the singularity type reduces to what happens in $N^*M_1^h$, and thus it reduces to a problem with $l-1$ surfaces in $\mathbb{R}^{n-1}$.
If $n-l$ is fixed then for arbitrary n a certain number of parts in the partition have to be 1. Let k be the number of strata that have codimension >1. It follows that $2k+(l-k) \leq n+1$ so that a maximum of $n-l+1$ codimensions is >1. The others are 1.

$\boxed{n-l=0}$ If $n=l$ then at most 1 of the μ_i is >1. So the only case to consider is $l=2$. We can have only two cases: (1), (2).

$\boxed{n-l=1}$ At most 2 of the codimensions are >1. So it is enough to consider $n=3$, $l=2$. In addition to the above combinations we will have: $(2,2)$ and (3).

$\boxed{n-l=2}$ The relevant dimensions are: $n=5$, $l=3$. The new cases are: (4), $(3,2)$ and $(2,2,2)$.

$\boxed{n-l=3}$ Dimensions: $n=7$, $l=4$. New cases: (5), $(4,2)$, $(3,3)$, $(3,2,2)$, $(2,2,2,2)$

$\boxed{n-l=4}$ Dimensions: $n=9$, $l=5$. New cases: (6), $(5,2)$, $(4,3)$, $(4,2,2)$, $(3,3,2)$, $(3,2,2,2)$ and $(2,2,2,2,2)$.

For each of the strata there are only a limited number of singularities, from the ADE list. The conflict set has dimension $n-l+1$. The codimension of a singularity on a generic front of dimension $n-l+1$ is maximally $n-l+2$. If we look at the above list we see that on the conflict set the codimension can add up to $2(n-l+1)$. Thus the singularities we encounter are the ones that we also expect to find in $n-l$ parameter families of $n-l+1$ dimensional fronts, though this last list will typically contain more singularities.

One might ask whether the above multi-singularities do not present any moduli. This is not the case. Even though our singularities are not, if $n - l > 0$, singularities of generic fronts we are still allowed to produce local models - as is remarked in [JB85] for the case $n = 3$, $l = 2$ - because they present R^+-versal unfoldings of multigerms.
So if $n - l = 0$ the singularities of the conflict set are the generic singularities of 2-dimensional fronts.
If $n - l = 1$ the codimension can add up to 4. The cases to consider are A_2A_2, $A_1^2A_2$, $A_1^2A_1^2$. All other singularities are just those of generic 2-dimensional fronts. Pictures are again in [JB85], but we take some time to discuss a nice example.
The A_2A_2 singularity is a generic projection of two transversely intersecting cuspidal edges in $\mathbb{R}^4$. To obtain a picture of this we take two copies of our previous example

$$G_1 \colon x_0 = s_1^3 + x_1 s_1 + x_2 \qquad G_2 \colon x_0 = s_2^3 + x_3 s_2 - x_2$$

At zero these two intersect transversally. Next we project the intersection along the time axis x_0 to $\mathbb{R}^3$. The surface we get is the following picture. This is also known as D_4^+ if we view it as a

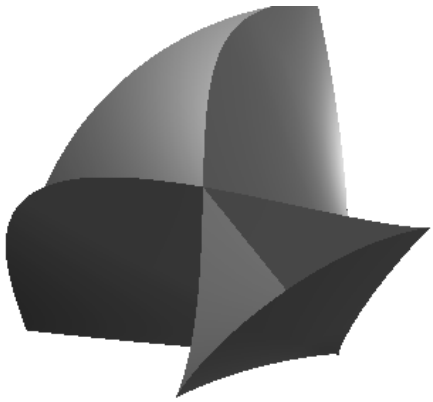

FIGURE 8. The A_2A_2 surface

metamorphosis of a wavefront in $\mathbb{R}^3$. Recall that a metamorphosis is a one dimensional family of fronts, see[Arn90]. The name D_4^+ is chosen because the surface is also obtained with an unfolding

$$G_1 - G_2 = s_1^3 - s_2^3 + x_1 s_1 - x_3 s_2 + 2x_2$$

This is not a versal unfolding. If we want to unfold the D_4^+ germ $s_1^3 - s_2^3$ with V-versal unfolding we need 4 parameters.
If $n - l = 2$ we need at least $n = 4$ and $l = 2$ to obtain an interesting new local model. Indeed the case (4) has A_4 and $D_4^\pm$ and suspensions of the cases that occur with $n - l = 1$. So the first really new case is $(3, 2)$. On this stratum we have amongst others A_3A_2. This is a metamorphosis of a 3-dimensional front. Some sections of this surface are in figure 9. In one them we see a swallowtail meeting a cuspidal edge.

Remark 4.2. All pictures here were obtained with the help of the software [GPS01] and the program "`surf`", written by Stephan Endrass.

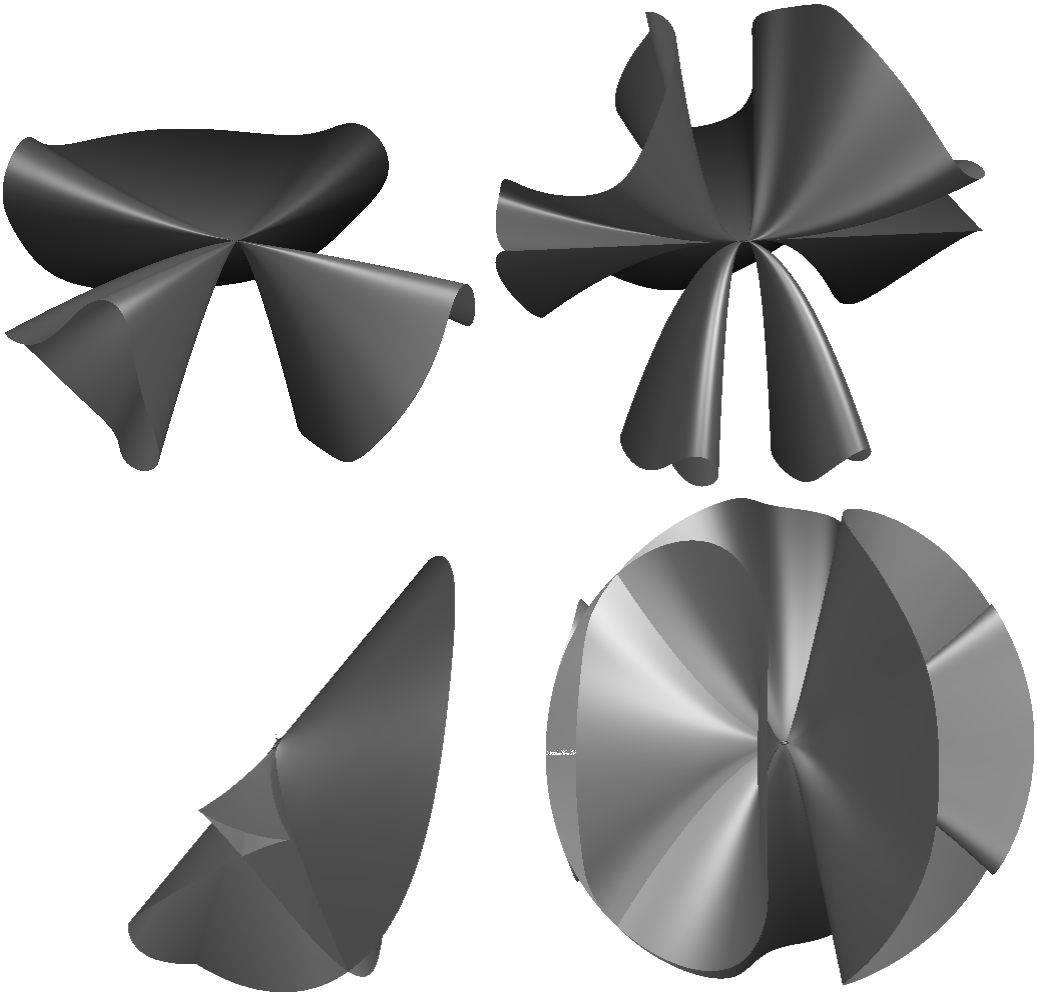

FIGURE 9. Sections of A_3A_2

5. CONCLUDING REMARK

Similar results should hold for the center set. We expect that if $n \leq 6$ the center set only has the multi-singularities of wavefronts in that dimension. We also expect that smooth boundaries of strictly convex compact domains in $\mathbb{R}^n$ the center symmetry set should generically have only ADE-sings if $n \leq 6$. That is because on such a boundary points with parallel tangent planes always stay at a distance from each other. This distance is uniformly bounded for such a surface, because of the compactness. So they stay away from the diagonal and the local situation can be treated as if the patches round the two points with parallel tangent planes originated from two distinct surfaces. Again this is all conjectural, though Giblin and Holtom have proved some assertions in this direction, see [GH99].

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