SLIDECONT: AN AUTO97 DRIVER FOR SLIDING BIFURCATION ANALYSIS

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ABSTRACT

SLIDECONT, an AUTO97 driver for sliding bifurcation analysis of discontinuous piecewise-smooth autonomous systems, known as Filippov systems, is described in detail. Sliding bifurcations are those in which some sliding on the discontinuity boundary is critically involved. The software allows for detection and continuation of codimension-1 sliding bifurcations as well as detection of some codimension-2 singularities, with special attention to planar systems ($n = 2$). Some bifurcations are considered also for n-dimensional systems. SLIDECONT contains source code, detailed documentation, and tutorial examples, one of which is described at the end of this paper.

Key words & Phrases: Sliding bifurcations, discontinuous piecewise smooth systems, Filippov systems, continuation techniques, AUTO97

1 Introduction

SlideCont (version 1.0) is a suite of routines accompanying Auto97 (Doedel & Kernévez, 1986; Doedel *et al.*, 1997) which allow one to perform bifurcation analysis of generic discontinuous piecewise smooth autonomous systems (Filippov, 1964, 1988), here called *Filippov systems*, with special attention to planar systems.

Bifurcation analysis of Filippov systems is important in many applications in various fields of science and engineering. Unfortunately, the complete catalogue of sliding bifurcations in n-dimensional systems is not yet available. There is a growing number of interesting results on bifurcations of periodic solutions in specific 3-dimensional and in general n-dimensional Filippov systems (see, for example Feigin (1994); Bernardo di *et al.* (1999, 1998a,b, 2001), and, in particular, Bernardo di *et al.* (2002)). Much less is known about local bifurcations in n-dimensional systems.

However, for the case of planar systems ($n = 2$), codimension-1 sliding bifurcations have been recently completely analyzed (Kuznetsov *et al.*, 2002) and suitable *defining systems* have been proposed for the numerical computation of bifurcation curves with standard continuation techniques. In this paper we revise such defining systems and discuss their implementation in SlideCont, indicating explicitly when they are also applicable to general n-dimensional systems.

The paper is organized as follows. In the next section we recall the definition of Filippov systems and some of their properties (for details and references, see Kuznetsov *et al.* (2002)). Then, we focus on SlideCont, assuming that the reader is acquainted with Auto97. In particular, we give an overview of the capabilities and limitations of SlideCont (Section 3) and we describe its structure (Section 4), as well as the problems it can solve (Section 5). Finally, we present a brief user guide (Section 6), including the information on availability of the software. Then we give a tutorial example from ecological modelling (Section 7). Some comments on possible extensions of SlideCont close the paper.

2 Preliminaries

We consider a generic Filippov system (Filippov, 1964)

$$\dot{x} = \begin{cases} f^{(1)}(x), & x \in S_1, \\ f^{(2)}(x), & x \in S_2, \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n$,

$$S_1 = \{x \in \mathbf{R}^n : H(x) < 0\}, \;\; S_2 = \{x \in \mathbf{R}^n : H(x) > 0\},$$

H is a smooth scalar function with non-vanishing gradient $H_x(x)$ on the discontinuity boundary

$$\Sigma = \{x \in \mathbf{R}^n : H(x) = 0\},$$

and $f^{(i)} : \mathbf{R}^n \to \mathbf{R}^n$ are smooth functions. For $n = 2$ we denote

$$H_x^{\perp} = \begin{pmatrix} H_{x_2} \\ -H_{x_1} \end{pmatrix}.$$

Solutions of (1) can be constructed by concatenating *standard solutions* in $S_{1,2}$ and *sliding solutions* on Σ obtained with the Filippov convex method (Filippov, 1964, 1988; Kuznetsov *et al.*, 2002) described below. Let

$$\sigma(x) = \langle H_x(x), f^{(1)}(x)\rangle\langle H_x(x), f^{(2)}(x)\rangle, \tag{2}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbf{R}^n$. The *crossing set* $\Sigma_c \subset \Sigma$ is defined by

$$\Sigma_c = \{x \in \Sigma : \sigma(x) > 0\}.$$

By definition, at points in Σ_c the orbit of (1) crosses Σ. The *sliding set* Σ_s is the complement to Σ_c in Σ, i.e.

$$\Sigma_s = \{x \in \Sigma : \sigma(x) \leq 0\}.$$

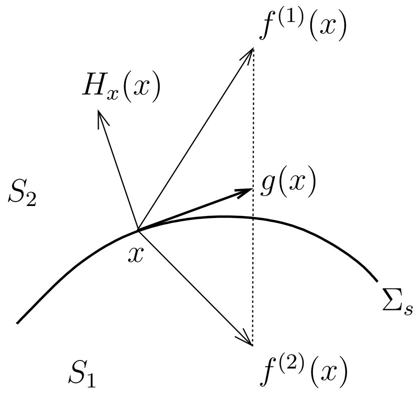

Figure 1: Filippov construction.

Points $x \in \Sigma_s$, where

$$\langle H_x(x), f^{(2)}(x) - f^{(1)}(x) \rangle = 0$$

are called *singular sliding points*. At such points, either both vectors $f^{(1)}(x)$ and $f^{(2)}(x)$ are tangent to Σ, or one of them vanishes while the other is tangent to Σ, or they both vanish. The Filippov method (see Fig. 1) associates the following convex combination $g(x)$ of the two vectors $f^{(i)}(x)$ to each nonsingular sliding point $x \in \Sigma_s$:

$$g(x) = \lambda f^{(1)}(x) + (1-\lambda) f^{(2)}(x), \;\; \lambda = \frac{\langle H_x(x), f^{(2)}(x) \rangle}{\langle H_x(x), f^{(2)}(x) - f^{(1)}(x) \rangle}. \tag{3}$$

Excluding infinitely-degenerate cases, $g(x)$ and its derivatives can be defined by continuity at all singular sliding points, which are not isolated sliding points. Set $g(x) = 0$ at isolated singular sliding points. Thus,

$$\dot{x} = g(x), \;\; x \in \Sigma_s, \tag{4}$$

defines a system of differential equations on Σ_s, which is smooth on codimension-1 sliding domains of Σ_s. Solutions of this system are called *sliding solutions*.

Equilibria of (4), where the vectors $f^{(i)}(x)$ are transversal to Σ_s and anti-collinear, are called *pseudo-equilibria* of (1). An equilibrium X of (4), where one of the vectors $f^{(i)}(X)$ vanishes, is called a *boundary equilibrium*. In this setting, all isolated singular sliding points are equilibria of (4). The boundary of a sliding domain is composed of *tangent points*, T, where the vectors $f^{(i)}(T)$ are nonzero but one of them is tangent to Σ, i.e.

$$\langle H_x(T), f^{(i)}(T)\rangle = 0,$$

and boundary equilibria. Tangent points are called *visible* (*invisible*) if the orbits of $\dot{x} = f^{(i)}(x)$ starting from them at time $t = 0$ belong to S_i (S_j, $j \neq i$) for all sufficiently small $|t| \neq 0$.

Orbits of (1) can overlap when sliding. Three types of periodic orbits can occur in (1): *standard*, *crossing* (i.e. passing through both domains S_i but with no points in Σ_s), and *sliding* (i.e. with at least a segment of the orbit in Σ_s).

Two Filippov systems of the form (1) are *topologically equivalent* if there is a homeomorphism $h : \mathbf{R}^n \to \mathbf{R}^n$ that maps the state portrait of one system onto the state portrait of the other, preserving orientation of the orbits. It is also required that h maps the discontinuity boundary Σ of one system onto the discontinuity boundary of the other system. Now, consider a Filippov system depending on parameters

$$\dot{x} = \begin{cases} f^{(1)}(x,\alpha), & x \in S_1(\alpha), \\ f^{(2)}(x,\alpha), & x \in S_2(\alpha), \end{cases} \tag{5}$$

where $x \in \mathbf{R}^n, \alpha \in \mathbf{R}^m$, and $f^{(i)}, i = 1, 2$, are smooth functions of (x, α), while

$$S_1(\alpha) = \{x \in \mathbf{R}^n : H(x,\alpha) < 0\}, \; S_2(\alpha) = \{x \in \mathbf{R}^n : H(x,\alpha) > 0\},$$

for some smooth function $H(x, \alpha)$ with $H_x(x, \alpha) \neq 0$ for all (x, α) such that $H(x, \alpha) = 0$. System (5) exhibits a *bifurcation* at $\alpha = \alpha_0$ if by an arbitrarily small parameter perturbation we get a topologically nonequivalent system. All bifurcations of (5) are classified as *local* or *global*. A local bifurcation can be detected by looking at an arbitrarily small neighborhood of a point in the state space. All other bifurcations are called global.

3 Overview

SLIDECONT can be used to perform a partial bifurcation analysis of n-dimensional Filippov systems (5) and a much more complete bifurcation analysis in the planar case ($n = 2$). No more than two *control parameters* are allowed ($m \leq 2$). Specifically, using the terminology introduced in Kuznetsov *et al.*

(2002), SLIDECONT can:

- compute the boundary Σ in planar systems ($n = 2$) for fixed parameter values;
- compute a curve of tangent points in three-dimensional systems ($n = 3$) for fixed parameter values;
- continue a tangent point in planar systems ($n = 2$) in one control parameter;
- continue a standard equilibrium in one control parameter;
- continue a pseudo-equilibrium in one control parameter;
- continue a standard periodic solution in one control parameter;
- continue a standard orbit, possibly crossing the boundary Σ, connecting a tangent point with the boundary Σ in planar systems ($n = 2$) (e.g. the standard part of a sliding cycle) in one control parameter;
- continue a standard orbit, possibly crossing the boundary Σ, connecting a pseudo-equilibrium with the boundary Σ in one control parameter;
- continue a crossing periodic solution in one control parameter;
- continue a boundary equilibrium in two control parameters;
- continue a pseudo-saddle-node bifurcation in two control parameters;
- continue a double tangency bifurcation in planar systems ($n = 2$) in two control parameters;
- continue coinciding tangent points in planar systems ($n = 2$) in two control parameters;
- continue a grazing periodic solution (touching bifurcation) in two control parameters;
- continue a standard orbit, possibly crossing the boundary Σ, connecting two tangent points of the same vector field in planar systems ($n = 2$) (e.g. a crossing-crossing bifurcation) in two control parameters;
- continue a standard orbit, possibly crossing the boundary Σ, connecting two tangent points of different vector fields in planar systems ($n = 2$) (e.g. a buckling or sliding-crossing bifurcation) in two control parameters;

- continue a standard orbit, possibly crossing the boundary Σ, connecting a tangent point with a pseudo-equilibrium (e.g. the standard part of a sliding homoclinic orbit to a pseudo-saddle in planar systems ($n = 2$)) in two control parameters;

- continue a standard orbit, possibly crossing the boundary Σ, connecting a tangent point with a standard saddle in planar systems ($n = 2$) (e.g. the standard part of a sliding homoclinic orbit to a saddle) in two control parameters;

- continue a standard orbit, possibly crossing the boundary Σ, connecting a pseudo-equilibrium with a tangent point in two control parameters;

- continue a standard orbit, possibly crossing the boundary Σ, connecting two pseudo-equilibria in planar systems ($n = 2$) in two control parameters;

- continue a standard orbit, possibly crossing the boundary Σ, connecting a pseudo-equilibrium with a standard saddle with one-dimensional unstable manifold in two control parameters;

Accurate detection of additional local degeneracies along these computations is supported together with switching possibilities between different types of problems (see Section 5 for details).

4 STRUCTURE OF SLIDECONT

In this section the structure of SLIDECONT is presented, together with some comments on its implementation (which is further described in the next two sections). SLIDECONT solves, through numerical continuation techniques, several problems, a list of which has been summarized in the previous section. The general idea is that SLIDECONT sets up the proper defining equations of the user-selected problem in AUTO97 format, so that the computation can be performed by means of standard AUTO97 routines. This is why SLIDECONT is an AUTO97 driver. The driving process and the overall structure of SLIDECONT are illustrated in Figure 2.

As in AUTO97, the user must provide three files. An equations file (`<name>.f`, where `<name>` is a user-selected name), a constants file (`sc.<name>`), and, possibly, a data file (`<name>.dat`) (see Fig. 2). The equations file contains a set of Fortran subroutines specifying model (5), namely the two

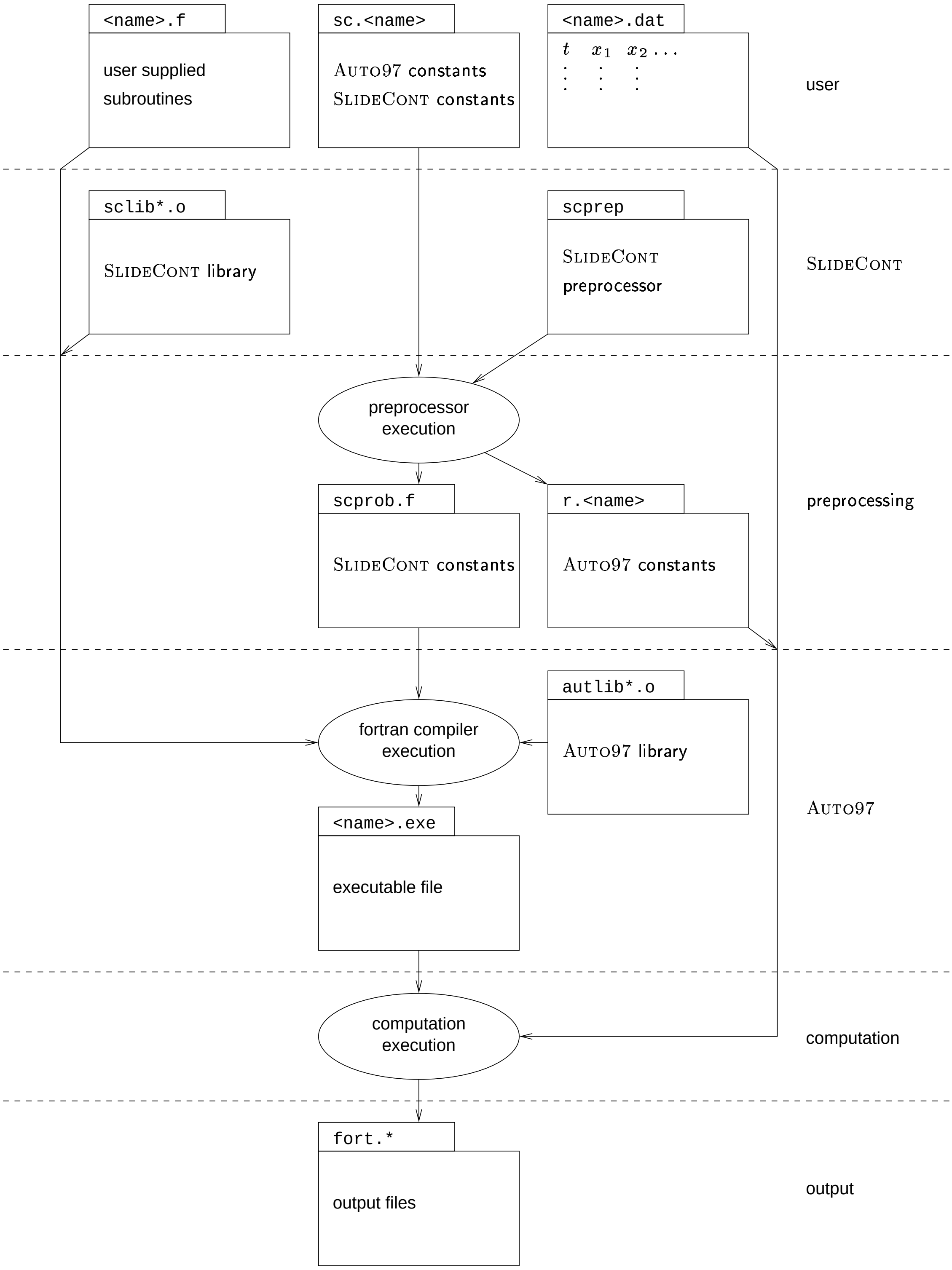

Figure 2: SLIDECONT implementation structure.

vector fields, $f^{(1)}$ and $f^{(2)}$, and the scalar function H, the starting solution, either analytically or numerically, and possible state and parameter user functions to be monitored during continuation. Analytical derivatives of $f^{(1)}$, $f^{(2)}$, and H are required by some problems (see Section 5). The constants file specifies all parameters qualifying the AUTO97 continuation algorithms plus some SLIDECONT specific constants, including, of course, the problem type, namely a constant indicating the problem to be solved. As in AUTO97, problem types are coded by means of integer numbers. This coding is done in such a way that SLIDECONT problem types do not overlap with those of AUTO97, so that the SLIDECONT user can access all AUTO97 facilities. Finally, the data file is required to numerically specify the starting solution of boundary-value problems.

As shown in Figure 2, SLIDECONT is composed of two parts: the SLIDECONT preprocessor (`scprep`) and the SLIDECONT library (`sclib*.o`). Preprocessing takes the user constants file and produces the corresponding AUTO97 constants file (`r.<name>`) and a problem specific Fortran file (`scprob.f`) containing the definition of all SLIDECONT constants, as global variables, and an initialization subroutine which sets these variables at the values specified by the user in the constants file. The initialization subroutine is called by the subroutine of the SLIDECONT library first called by AUTO97, so that during the computation SLIDECONT constants are well defined. The library contains the standard AUTO97 user subroutines for each problem and additional support routines.

The compilation of the user and problem specific Fortran files and the linking with the SLIDECONT and AUTO97 libraries produce the executable file (`<name>.exe`), whose execution finally gives the standard AUTO97 output files (`fort.*`).

5 PROBLEM DESCRIPTION

Among other things, AUTO97 can compute curves of solutions to “algebraic problems”:

$$F(U,\mu) = 0, \quad U, F \in \mathbf{R}^{n_d},\ \mu \in \mathbf{R}^1,$$

which we rewrite as

$$F(x,\alpha,\beta) = 0, \qquad x \in \mathbf{R}^n,\ F \in \mathbf{R}^{n_d},\ \alpha \in \mathbf{R}^m,\ \beta \in \mathbf{R}^{m_d}, \tag{6}$$

as well as paths of solutions to boundary-value problems with non-separated boundary conditions:

$$\dot{U}(\tau) - F(U(\tau), \alpha, \beta) = 0, \qquad U, F \in \mathbf{R}^{n_d},\ \alpha \in \mathbf{R}^m,\ \beta \in \mathbf{R}^{m_d},\ \tau \in [0,1] \tag{7}$$

$$b(U(0), U(1), \alpha, \beta) = 0, \qquad b \in \mathbf{R}^{n_b}. \tag{8}$$

In both cases, m control parameters α_i are allowed to vary and the following conditions on dimensions are imposed: $n + m + m_d = n_d + 1$ for equation (6) and $m + m_d = n_b - n_d + 1$ for equations (7)–(8). Moreover, AUTO97 can accurately locate zeros along the solution branch of several test functions (see AUTO97 documentation).

For each SLIDECONT problem, we present in a separated subsection the corresponding defining system (6) or (7)–(8) and some details on its implementation. As for the defining system, we report its analytical formulation and specify the following informations: the state space dimension n for which the defining system is valid; the number m of control parameters; the list of other active parameters β of the defining system (i.e., different from $\alpha_1, \dots, \alpha_m$) and their total number m_d; for algebraic defining systems, the dimension n_d of the defining system; for boundary-value problems, the number n_d of differential conditions and the number n_b of boundary conditions.

As for the implementation, we specify, in accordance with AUTO97 notation, the following informations: the AUTO97 problem type `IPS` used to perform the computation (for example: `IPS=0` for (6) and `IPS=4` for (7)–(8)); the order `SCIDIFF` up to which analytical derivatives of $f^{(1)}$, $f^{(2)}$, and H are required; the problem dimension `NDIM` (except for Subsections 5.1 and 5.2 where an extra state variable is used, `NDIM`=n_d); the composition of the state vector `U(1)`, ..., `U(NDIM)`; the right-hand side vector `F(1)`, ..., `F(NDIM)`; the total number of active parameters `NICP` ($m + m_d$) and the list of active parameter indexes `ICP(1)`, ..., `ICP(NICP)`, denoting by I_i the index of the user parameter α_i ($i = 1, \dots, m$) and reporting the other m_d active parameter symbols in parenthesis after the corresponding indexes; the list of test functions for detecting additional local degeneracies with switching possibilities (a switch to a problem is denoted by a reference to the problem subsection, indicating the vector field(s) at which the problem is applied, and adding a star ($*$) if the switch is not automatic, namely the user must set up the starting solution manually). Notice that boundary conditions `FB(1)`, ..., `FB(NBC)` (`NBC`=n_b) are not reported, since their definition is always clear.

It is worth to remark that for boundary-value problems there can be more switching possibilities than those listed, which, however, are not possible with certainty when the corresponding test function vanishes. For example, continuing, for $n = 2$, an orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with the boundary Σ (see Subsection 5.7), a zero of test function 3 detects a tangent point of $f^{(i)}$ at the right boundary-value. If this tangent point is the same tangent point present at the left boundary-value, a condition that does not imply a codimension-2 bifurcation, then left and right boundary-values coincide and one can switch to the continuation of a standard cycle (Subsection 5.6) or of a touching bifurcation (Subsection 5.16). Thus, when a test function vanishes during the continuation of a boundary-value problem, one should check if the critical solution satisfies the defining system of some problems not listed among the switching possibilities.

A common practice in AUTO97 is 'parameter overspecification', namely the number of active parameters `NICP` is allowed to be greater than the number required by the specified problem. In such cases, the extra activated parameters located at the end of the `ICP` list are not true continuation parameters, but their values appear in the output. Overspecified parameters are denoted in the parameter index list `ICP` by indicating in parenthesis the value at which they are set (see Subsections 5.12 and 5.15). Parameter overspecification is also used by SLIDECONT for the implementation of test functions and by the user for defining user test functions (see Section 6), so that the actual number of active parameters `NICP` can be greater than the value specified here.

Notice that the defining systems for standard equilibrium and cycle continuation (see Subsections 5.4 and 5.6) are not reported, since they correspond to AUTO97 built-in problems. Similarly, the defining systems for the continuation of pseudo-saddle-node and double tangency bifurcations (see Subsections 5.13 and 5.14) are not reported, since they are implemented indirectly by using the defining systems for pseudo-equilibrium and tangent point continuation (Subsections 5.5 and 5.3) and enabling AUTO97 limit point continuation (`ISW=2`, see AUTO97 documentation).

All the informations described above are organized in a two-columns tabular format where the left column reports symbols or names of the informations, while the right column reports their values or expressions. Recall that SLIDECONT does not support the automatic switches marked by a star ($*$) in the right column. Finally, refer to Figures 5-13 for a graphical representation of boundary-value problems in the case of planar systems ($n = 2$).

5.1 *Computation of the discontinuity boundary*

$$H(x,\alpha) = 0. \tag{9}$$

n, m	2, 0
m_d	0
n_d	1
`IPS`, `SCIDIFF`, `NDIM`	0, 0, 2
`U(1)`, ..., `U(NDIM)`	x_1, x_2
`F(1)`	$H(x,\alpha)$
`F(2)`	$x_1 - \mathtt{PAR}(13)$
`NICP`, (`ICP(1)`, ..., `ICP(NICP)`)	1, (13)
t. f. 1: tangent point of $f^{(1)}$ switch to	$\langle H_x(x,\alpha), f^{(1)}(x,\alpha)\rangle$ problem 5.3 for $f^{(1)}$
t. f. 2: tangent point of $f^{(2)}$ switch to	$\langle H_x(x,\alpha), f^{(2)}(x,\alpha)\rangle$ problem 5.3 for $f^{(2)}$
t. f. 3: pseudo-equilibrium switch to	$\langle H_x^{\perp}(x,\alpha), g(x,\alpha)\rangle$ problem 5.5

5.2 *Computation of a curve of tangent points of vector field $f^{(i)}$ in three-dimensional systems*

$$\left\{ \begin{array}{rcl} H(x,\alpha) & = & 0, \\ \langle H_x(x,\alpha), f^{(i)}(x,\alpha)\rangle & = & 0. \end{array} \right. \tag{10}$$

n, m	3, 0
m_d	0
n_d	2
`IPS`, `SCIDIFF`, `NDIM`	0, 1, 3
`U(1)`, ..., `U(NDIM)`	x_1, x_2, x_3
`F(1)`	$H(x,\alpha)$
`F(2)`	$\langle H_x(x,\alpha), f^{(i)}(x,\alpha)\rangle$
`F(3)`	$x_1 - \mathtt{PAR}(13)$
`NICP`, (`ICP(1)`, ..., `ICP(NICP)`)	1, (13)
t. f. 1: tangent point of $f^{(j)}$, $j \neq i$ switch to	$\langle H_x(x,\alpha), f^{(j)}(x,\alpha)\rangle$ problem 5.2 for $f^{(j)}$

5.3 Continuation of a tangent point of vector field $f^{(i)}$

$$
\begin{cases} H(x,\alpha) = 0, \\ \langle H_x(x,\alpha), f^{(i)}(x,\alpha)\rangle = 0. \end{cases} \tag{11}
$$

n, m	2, 1
m_d	0
n_d	2
`IPS`, `SCIDIFF`, `NDIM`	0, 1, 2
`U(1)`, ..., `U(NDIM)`	x_1, x_2
`F(1)`	$H(x,\alpha)$
`F(2)`	$\langle H_x(x,\alpha), f^{(i)}(x,\alpha)\rangle$
`NICP`, (`ICP(1)`, ..., `ICP(NICP)`)	1, (I_1)
t. f. 1: boundary equilibrium of $f^{(i)}$ switch to	$\langle H_x^{\perp}(x,\alpha), f^{(i)}(x,\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$, $j \neq i$ switch to	$\langle H_x(x,\alpha), f^{(j)}(x,\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.15
t. f.: double tangency switch to	AUTO97 limit point problem 5.14 for $f^{(i)}$

5.4 Continuation of a standard equilibrium of vector field $f^{(i)}$

n, m	n, 1
`IPS`, `SCIDIFF`, `NDIM`	1, 0, n
`U(1)`, ..., `U(NDIM)`	$x_1, \ldots, x_n$
`F(`k`)`, $k = 1, \ldots, n$	$f_k^{(i)}(x,\alpha)$
`NICP`, (`ICP(1)`, ..., `ICP(NICP)`)	1, (I_1)
t. f. 1: boundary equilibrium of $f^{(i)}$ switch to	$H(x,\alpha)$ - problem 5.1 ($n = 2$) - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.5 - problem 5.12 for $f^{(i)}$
t. f.: branch switch to	AUTO97 branch problem 5.4 for $f^{(i)}$ (branch switching)
t. f.: limit point switch to	AUTO97 limit point continuation of a limit point bifurcation
t. f.: Hopf	AUTO97 Hopf

switch to	continuation of a Hopf bifurcation

5.5 *Continuation of a pseudo-equilibrium*

$$\begin{cases} H(x,\alpha) &= 0, \\ \lambda_1 f^{(1)}(x,\alpha) + \lambda_2 f^{(2)}(x,\alpha) &= 0, \\ \lambda_1 + \lambda_2 - 1 &= 0. \end{cases} \tag{12}$$

n, m	n, 1
m_d, $(\beta_1, \ldots, \beta_{m_d})$	2, (λ_1, λ_2)
n_d	$n+2$
IPS, SCIDIFF, NDIM	0, 0, $n+2$
U(1), $\ldots$, U(NDIM)	$x_1, \ldots, x_n, \lambda_1, \lambda_2$
F(k), $k = 1, \ldots, n$	$\lambda_1 f_k^{(1)}(x,\alpha) + \lambda_2 f_k^{(2)}(x,\alpha)$
F($n+1$)	$H(x,\alpha)$
F($n+2$)	$\lambda_1 + \lambda_2 - 1$
NICP, (ICP(1), $\ldots$, ICP(NICP))	1, (I_1)
t. f. 1: boundary equilibrium of $f^{(1)}$ switch to	λ_2 - problem 5.2 for $f^{(1)}$ ($n = 3$) - problem 5.3 for $f^{(1)}$ ($n = 2$) - problem 5.4 for $f^{(1)}$ - problem 5.12 for $f^{(1)}$
t. f. 2: boundary equilibrium of $f^{(2)}$ switch to	λ_1 - problem 5.2 for $f^{(2)}$ ($n = 3$) - problem 5.3 for $f^{(2)}$ ($n = 2$) - problem 5.4 for $f^{(2)}$ - problem 5.12 for $f^{(2)}$
t. f. 3: singular sliding point switch to	$\langle H_x(x,\alpha), f^{(1)}(x,\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n = 2$) - problem 5.15 ($n = 2$)
t. f.: pseudo-saddle-node bifurcation switch to	AUTO97 limit point problem 5.13

5.6 *Continuation of a standard cycle of vector field $f^{(i)}$*

n, m	n, 1
IPS, SCIDIFF, NDIM	2, 0, n
U(1), $\ldots$, U(NDIM)	$x_1, \ldots, x_n$

`F(k)`, $k = 1, \ldots, n$	$f_k^{(i)}(x, \alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	2, (I_1,11)
t. f. 1: touching bifurcation switch to	$\min_{\{k=1,\ldots,NTST*NCOL\}}\{(-1)^i H(x_k,\alpha)\}$ - problem 5.1 ($n = 2$) - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.7 for $f^{(i)}$ ($n = 2$) * - problem 5.16 for $f^{(i)}$ *
t. f.: branch switch to	AUTO97 branch problem 5.6 for $f^{(i)}$ (branch switching)
t. f.: limit point switch to	AUTO97 limit point continuation of a limit point bifurcation
t. f.: period doubling switch to	AUTO97 period doubling continuation of a period doubling bifurcation
t. f.: torus switch to	AUTO97 torus continuation of a torus bifurcation

5.7 *Continuation of an orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with the boundary Σ*

$$\left\{\begin{array}{rcl} \dot{u} - Tf^{(i)}(u,\alpha) &=& 0, \\ H(u(0),\alpha) &=& 0, \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle &=& 0, \\ H(u(1),\alpha) &=& 0. \end{array}\right. \quad (13)$$

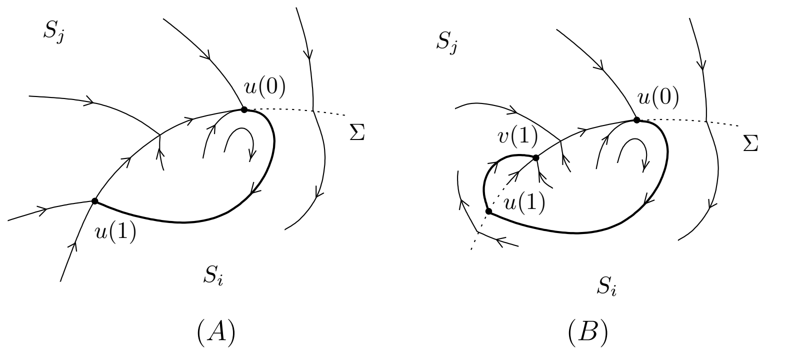

Figure 3: Boundary-value problems corresponding to (A): subsection 5.7; (B): subsection 5.8.

n, m	2, 1
m_d, ($\beta_1, \ldots, \beta_{m_d}$)	1, (T)
n_d, n_b	2, 3
`IPS`, `SCIDIFF`, `NDIM`	4, 1, 2

`U(1)`, $\ldots$, `U(NDIM)`	u_1, u_2
`F(`k`)`, $k = 1, 2$	$Tf_k^{(i)}(u, \alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	2, (I_1, 11(T))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^{\perp}(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$, $j \neq i$, at $u(0)$ switch to	$\langle H_x(u(0), \alpha), f^{(j)}(u(0), \alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ - problem 5.15
t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.17 for $f^{(i)}$
t. f. 4: tangent point of $f^{(j)}$, $j \neq i$, at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(j)}(u(1), \alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.8 for $f^{(i)}$, $f^{(j)}$ * - problem 5.19 for $f^{(i)}$, $f^{(j)}$
t. f. 5: pseudo-equilibrium at $u(1)$ switch to	$\langle H_x^{\perp}(u(1), \alpha), g(u(1), \alpha)\rangle$ - problem 5.5 - problem 5.21 for $f^{(i)}$

5.8 Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a tangent point of $f^{(i)}$ with the boundary Σ

$$\left\{\begin{array}{rcl} \dot{u} - T_i f^{(i)}(u, \alpha) & = & 0, \\ \dot{v} - T_j f^{(j)}(v, \alpha) & = & 0, \\ H(u(0), \alpha) & = & 0, \\ \langle H_x(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle & = & 0, \\ H(u(1), \alpha) & = & 0, \\ u(1) - v(0) & = & 0, \\ H(v(1), \alpha) & = & 0. \end{array}\right. \tag{14}$$

n, m	2, 1
m_d, ($\beta_1, \ldots, \beta_{m_d}$)	2, (T_i, T_j)
n_d, n_b	4, 6
`IPS`, `SCIDIFF`, `NDIM`	4, 1, 4
`U(1)`, $\ldots$, `U(NDIM)`	u_1, u_2, v_1, v_2
`F(`k`)`, $k = 1, 2$	$T_i f_k^{(i)}(u, \alpha)$
`F(`k`)`, $k = 3, 4$	$T_j f_k^{(j)}(v, \alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	3, (I_1, 11(T_i), 12(T_j))

t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^{\perp}(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$ at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(j)}(u(0),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ * - problem 5.10 for $f^{(i)}$, $f^{(j)}$ - problem 5.15
t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.17 for $f^{(i)}$ *
t. f. 4: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.7 for $f^{(j)}$ * - problem 5.19 for $f^{(i)}$, $f^{(j)}$ *
t. f. 5: tangent point of $f^{(i)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.18 for $f^{(i)}$, $f^{(j)}$
t. f. 6: tangent point of $f^{(j)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.20 for $f^{(i)}$, $f^{(j)}$
t. f. 7: pseudo-equilibrium at $v(1)$ switch to	$\langle H_x^{\perp}(v(1),\alpha), g(v(1),\alpha)\rangle$ - problem 5.5 - problem 5.22 for $f^{(i)}$, $f^{(j)}$

5.9 *Continuation of an orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium with the boundary Σ*

$$\left\{\begin{array}{rcll} \dot u - Tf^{(i)}(u,\alpha) & = & 0, & \\ H(u(0),\alpha) & = & 0, & \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) & = & 0, & j \neq i, \\ \lambda_i + \lambda_j - 1 & = & 0, & \\ H(u(1),\alpha) & = & 0. & \end{array}\right. \tag{15}$$

n, m	n, 1
m_d, $(\beta_1,\dots,\beta_{m_d})$	3, (T,λ_i,λ_j)
n_d, n_b	n, $n+3$
`IPS`, `SCIDIFF`, `NDIM`	4, 0, n
`U(1)`, $\dots$, `U(NDIM)`	$u_1,\dots,u_n$
`F(`k`)`, $k=1,\dots,n$	$Tf_k^{(i)}(u,\alpha)$

`NICP`, (`ICP(1)`, ..., `ICP(NICP)`)	4, (I_1, 11(T), 13(λ_i), 14(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.4 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.4 for $f^{(j)}$ - problem 5.12 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n = 2$) - problem 5.7 for $f^{(i)}$ ($n = 2$) - problem 5.15 ($n = 2$)
t. f. 4: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.25 for $f^{(i)}$
t. f. 5: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(j)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.10 for $f^{(i)}$, $f^{(j)}$ * - problem 5.27 for $f^{(i)}$, $f^{(j)}$
t. f. 6: pseudo-equilibrium at $u(1)$ ($n = 2$) switch to	$\langle H_x^{\perp}(u(1), \alpha), g(u(1), \alpha)\rangle$ - problem 5.5 - problem 5.29 for $f^{(i)}$

5.10 Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium with the boundary Σ

$$\left\{ \begin{array}{rcl} \dot{u} - T_i f^{(i)}(u, \alpha) & = & 0, \\ \dot{v} - T_j f^{(j)}(v, \alpha) & = & 0, \\ H(u(0), \alpha) & = & 0, \\ \lambda_i f^{(i)}(u(0), \alpha) + \lambda_j f^{(j)}(u(0), \alpha) & = & 0, \\ \lambda_i + \lambda_j - 1 & = & 0, \\ H(u(1), \alpha) & = & 0, \\ u(1) - v(0) & = & 0, \\ H(v(1), \alpha) & = & 0. \end{array} \right. \tag{16}$$

n, m	n, 1

m_d, $(\beta_1,\ldots,\beta_{m_d})$	4, $(T_i, T_j, \lambda_i, \lambda_j)$
n_d, n_b	$2n$, $2n+4$
`IPS`, `SCIDIFF`, `NDIM`	4, 0, $2n$
`U(1)`$,\ldots,$`U(NDIM)`	$u_1,\ldots,u_n, v_1,\ldots,v_n$
`F(`k`)`, $k=1,\ldots,n$	$T_i f_k^{(i)}(u,\alpha)$
`F(`k`)`, $k=n+1,\ldots,2n$	$T_j f_k^{(j)}(v,\alpha)$
`NICP`, (`ICP(1)`$,\ldots,$`ICP(NICP)`)	5, (I_1, 11(T_i), 12(T_j), 13(λ_i), 14(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.4 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.4 for $f^{(j)}$ - problem 5.12 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n=2$) - problem 5.7 for $f^{(i)}$ ($n=2$) * - problem 5.8 for $f^{(i)}$, $f^{(j)}$ ($n=2$) - problem 5.15 ($n=2$)
t. f. 4: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.25 for $f^{(i)}$ *
t. f. 5: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.7 for $f^{(j)}$ ($n=2$) * - problem 5.27 for $f^{(i)}$, $f^{(j)}$ *
t. f. 6: tangent point of $f^{(i)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.26 for $f^{(i)}$, $f^{(j)}$
t. f. 7: tangent point of $f^{(j)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.28 for $f^{(i)}$, $f^{(j)}$
t. f. 8: pseudo-equilibrium at $v(1)$ ($n=2$) switch to	$\langle H_x^{\perp}(v(1),\alpha), g(v(1),\alpha)\rangle$ - problem 5.5 - problem 5.30 for $f^{(i)}$, $f^{(j)}$

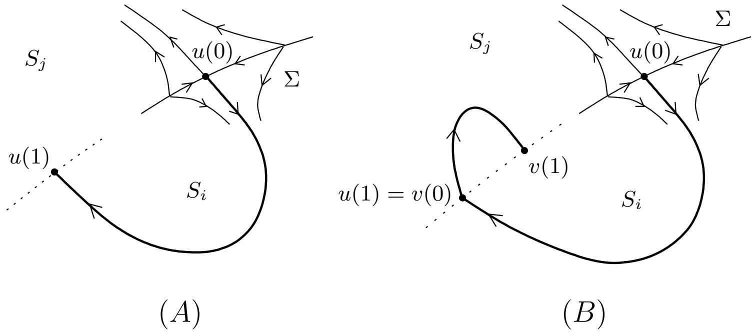

Figure 4: Boundary-value problems corresponding to (A): subsection 5.9; (B): subsection 5.10.

5.11 Continuation of a crossing cycle

$$\begin{cases} \dot{u} - T_1 f^{(1)}(u, \alpha) &= 0, \\ \dot{v} - T_2 f^{(2)}(v, \alpha) &= 0, \\ H(u(0), \alpha) &= 0, \\ H(u(1), \alpha) &= 0, \\ u(1) - v(0) &= 0, \\ v(1) - u(0) &= 0. \end{cases} \tag{17}$$

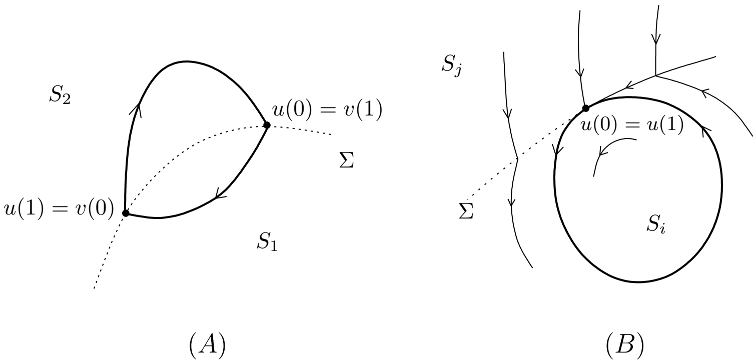

Figure 5: Boundary-value problems corresponding to (A): subsection 5.11; (B): subsection 5.16.

n, m	n, 1
m_d, $(\beta_1, \ldots, \beta_{m_d})$	2, (T_1, T_2)
n_d, n_b	$2n$, $2n + 2$
`IPS`, `SCIDIFF`, `NDIM`	4, 0, $2n$
`U(1)`, $\ldots$, `U(NDIM)`	$u_1, \ldots, u_n$, $v_1, \ldots, v_n$
`F(`k`)`, $k = 1, \ldots, n$	$T_1 f_k^{(1)}(u, \alpha)$
`F(`k`)`, $k = n + 1, \ldots, 2n$	$T_2 f_k^{(2)}(v, \alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	3, (I_1, 11(T_1), 12(T_2))

t. f. 1: tangent point of $f^{(1)}$ at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(1)}(u(0),\alpha)\rangle$ - problem 5.2 for $f^{(1)}$ ($n=3$) - problem 5.3 for $f^{(1)}$ ($n=2$) - problem 5.7 for $f^{(1)}$ ($n=2$) * - problem 5.8 for $f^{(1)}$, $f^{(2)}$ ($n=2$) - problem 5.18 for $f^{(1)}$, $f^{(2)}$ ($n=2$)
t. f. 2: tangent point of $f^{(2)}$ at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(2)}(u(0),\alpha)\rangle$ - problem 5.2 for $f^{(2)}$ ($n=3$) - problem 5.3 for $f^{(2)}$ ($n=2$)
t. f. 3: tangent point of $f^{(1)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(1)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(1)}$ ($n=3$) - problem 5.3 for $f^{(1)}$ ($n=2$)
t. f. 4: tangent point of $f^{(2)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(2)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(2)}$ ($n=3$) - problem 5.3 for $f^{(2)}$ ($n=2$) - problem 5.7 for $f^{(2)}$ ($n=2$) * - problem 5.8 for $f^{(2)}$, $f^{(1)}$ ($n=2$) * - problem 5.18 for $f^{(2)}$, $f^{(1)}$ ($n=2$) *

5.12 Continuation of a boundary equilibrium of vector field $f^{(i)}$

$$\begin{cases} f^{(i)}(x,\alpha) &= 0, \\ H(x,\alpha) &= 0. \end{cases} \tag{18}$$

n, m	n, 2
m_d	0
n_d	$n+1$
`IPS`, `SCIDIFF`, `NDIM`	0, 0, $n+1$
`U(1)`, $\ldots$, `U(NDIM)`	$x_1,\ldots,x_n$, α_2
`F(`k`)`, $k=1,\ldots,n$	$f_k^{(i)}(x,\alpha)$
`F(`$n+1$`)`	$H(x,\alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	2, (I_1, I_2(`PAR(`I_2`)`=`U(`$n+1$`)`))
t. f. 1: Hopf switch to	$\mathrm{Re}\lambda_p \,:\, p=\arg\min_q\{\lvert\mathrm{Re}\lambda_q\rvert \,:\, \mathrm{Im}\lambda_q\neq 0\}$ continuation of a Hopf bifurcation *
t. f. 2: branch/limit point switch to	$\mathrm{Re}\lambda_p \,:\, p=\arg\min_q\{\lvert\mathrm{Re}\lambda_q\rvert \,:\, \mathrm{Im}\lambda_q = 0\}$ problem 5.4 for $f^{(i)}$ (branch switching) */ continuation of a limit point bifurcation *
t. f. 3: tangent point of $f^{(j)}$, $j\neq i$ switch to	$\langle H_x(x,\alpha), f^{(j)}(x,\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.15 ($n=2$)

5.13 Continuation of a pseudo-saddle-node bifurcation

n, m	n, 2
IPS, SCIDIFF, NDIM	0, 0, $n+2$
U(1), $\ldots$, U(NDIM)	$x_1, \ldots, x_n, \lambda_1, \lambda_2$
F(k), $k = 1, \ldots, n$	$\lambda_1 f_k^{(1)}(x, \alpha) + \lambda_2 f_k^{(2)}(x, \alpha)$
F($n+1$)	$H(x, \alpha)$
F($n+2$)	$\lambda_1 + \lambda_2 - 1$
NICP, (ICP(1), $\ldots$, ICP(NICP))	2, (I_1, I_2)
t. f. 1: boundary equilibrium of $f^{(1)}$ switch to	λ_2 - problem 5.2 for $f^{(1)}$ ($n = 3$) - problem 5.3 for $f^{(1)}$ ($n = 2$) - problem 5.4 for $f^{(1)}$ - problem 5.12 for $f^{(1)}$
t. f. 2: boundary equilibrium of $f^{(2)}$ switch to	λ_1 - problem 5.2 for $f^{(2)}$ ($n = 3$) - problem 5.3 for $f^{(2)}$ ($n = 2$) - problem 5.4 for $f^{(2)}$ - problem 5.12 for $f^{(2)}$
t. f. 3: singular sliding point switch to	$\langle H_x(x, \alpha), f^{(1)}(x, \alpha) \rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n = 2$) - problem 5.15 ($n = 2$)
note	ISW$= 2$

5.14 Continuation of a double tangency bifurcation of vector field $f^{(i)}$

n, m	2, 2
IPS, SCIDIFF, NDIM	0, 1, 2
U(1), $\ldots$, U(NDIM)	x_1, x_2
F(1)	$H(x, \alpha)$
F(2)	$\langle H_x(x, \alpha), f^{(i)}(x, \alpha) \rangle$
NICP, (ICP(1), $\ldots$, ICP(NICP))	1, (I_1, I_2)
t. f. 1: boundary equilibrium of $f^{(i)}$ switch to	$\langle H_x^{\perp}(x, \alpha), f^{(i)}(x, \alpha) \rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$, $j \neq i$ switch to	$\langle H_x(x, \alpha), f^{(j)}(x, \alpha) \rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.15

note	`ISW`$=2$

5.15 Continuation of coinciding tangent points

$$\left\{\begin{array}{rcl} H(x,\alpha) & = & 0, \\ \langle H_x(x,\alpha), f^{(1)}(x,\alpha)\rangle & = & 0, \\ \langle H_x(x,\alpha), f^{(2)}(x,\alpha)\rangle & = & 0. \end{array}\right. \tag{19}$$

n, m	2, 2
m_d	0
n_d	3
`IPS`, `SCIDIFF`, `NDIM`	0, 1, 3
`U(1)`$,\ldots,$`U(NDIM)`	x_1, x_2, α_2
`F(1)`	$H(x,\alpha)$
`F(2)`	$\langle H_x(x,\alpha), f^{(1)}(x,\alpha)\rangle$
`F(3)`	$\langle H_x(x,\alpha), f^{(2)}(x,\alpha)\rangle$
`NICP`, (`ICP(1)`$,\ldots,$`ICP(NICP)`)	2, (I_1, I_2(`PAR(`I_2`)`=`U(3)`))
t. f. 1: boundary equilibrium of $f^{(1)}$ switch to	$\langle H_x^{\perp}(x,\alpha), f^{(1)}(x,\alpha)\rangle$ - problem 5.4 for $f^{(1)}$ - problem 5.12 for $f^{(1)}$
t. f. 2: boundary equilibrium of $f^{(2)}$ switch to	$\langle H_x^{\perp}(x,\alpha), f^{(2)}(x,\alpha)\rangle$ - problem 5.4 for $f^{(2)}$ - problem 5.12 for $f^{(2)}$
t. f.: double tangency switch to	AUTO97 limit point problem 5.14 for $f^{(1)}$ or for $f^{(2)}$ *

5.16 Continuation of a touching bifurcation of vector field $f^{(i)}$

$$\left\{\begin{array}{rcl} \dot{u} - Tf^{(i)}(u,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle & = & 0, \\ u(0) - u(1) & = & 0. \end{array}\right. \tag{20}$$

n, m	n, 2
m_d, ($\beta_1,\ldots,\beta_{m_d}$)	1, (T)
n_d, n_b	n, $n+2$
`IPS`, `SCIDIFF`, `NDIM`	4, 1, n
`U(1)`$,\ldots,$`U(NDIM)`	$u_1,\ldots,u_n$

`F(k)`, $k=1,\ldots,n$	$Tf_k^{(i)}(u,\alpha)$
`NICP`, (`ICP(1)`, ..., `ICP(NICP)`)	3, (I_1, I_2, 11(T))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ ($n=2$) switch to	$\langle H_x^{\perp}(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$, $j \neq i$, at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(j)}(u(0),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.5 ($n=2$) - problem 5.8 for $f^{(i)}$, $f^{(j)}$ ($n=2$) * - problem 5.9 for $f^{(i)}$ ($n=2$) - problem 5.10 for $f^{(i)}$, $f^{(j)}$ ($n=2$) * - problem 5.15 ($n=2$) - problem 5.19 for $f^{(i)}$, $f^{(j)}$ ($n=2$) - problem 5.21 for $f^{(i)}$ ($n=2$) - problem 5.25 for $f^{(i)}$ ($n=2$) - problem 5.27 for $f^{(i)}$, $f^{(j)}$ ($n=2$)

5.17 Continuation of an orbit of vector field $f^{(i)}$ connecting two tangent points of $f^{(i)}$

$$\left\{\begin{array}{rcl} \dot{u} - Tf^{(i)}(u,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle & = & 0, \\ H(u(1),\alpha) & = & 0, \\ \langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle & = & 0. \end{array}\right. \tag{21}$$

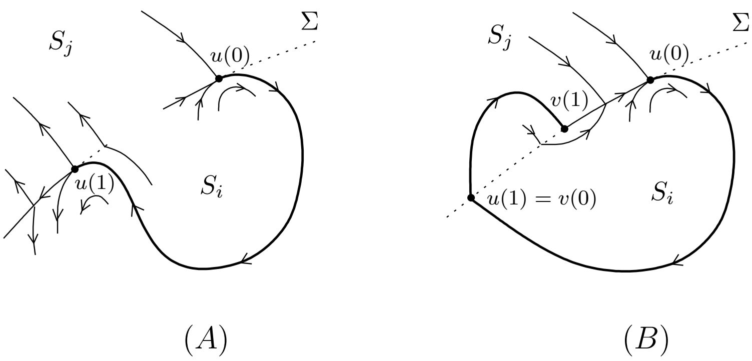

Figure 6: Boundary-value problems corresponding to (A): subsection 5.17; (B): subsection 5.18.

n, m	2, 2
m_d, $(\beta_1, \ldots, \beta_{m_d})$	1, (T)
n_d, n_b	2, 4
`IPS`, `SCIDIFF`, `NDIM`	4, 1, 2
`U(1)`$, \ldots,$ `U(NDIM)`	u_1, u_2
`F(`k`)`, $k = 1, 2$	$Tf_k^{(i)}(u, \alpha)$
`NICP`, (`ICP(1)`$, \ldots,$ `ICP(NICP)`)	3, $(I_1, I_2, 11(T))$
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^{\perp}(u(0), \alpha), f^{(i)}(u(0), \alpha) \rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$, $j \neq i$, at $u(0)$ switch to	$\langle H_x(u(0), \alpha), f^{(j)}(u(0), \alpha) \rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ - problem 5.15 - problem 5.25 for $f^{(i)}$
t. f. 3: boundary equilibrium of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x^{\perp}(u(1), \alpha), f^{(i)}(u(1), \alpha) \rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$ - problem 5.21 for $f^{(i)}$
t. f. 4: tangent point of $f^{(j)}$, $j \neq i$, at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(j)}(u(1), \alpha) \rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.8 for $f^{(i)}$, $f^{(j)}$ * - problem 5.15 - problem 5.19 for $f^{(i)}$, $f^{(j)}$ - problem 5.21 for $f^{(i)}$

5.18 Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ $(j \neq i)$ connecting two tangent points of $f^{(i)}$

$$\left\{ \begin{array}{rcl} \dot{u} - T_i f^{(i)}(u, \alpha) & = & 0, \\ \dot{v} - T_j f^{(j)}(v, \alpha) & = & 0, \\ H(u(0), \alpha) & = & 0, \\ \langle H_x(u(0), \alpha), f^{(i)}(u(0), \alpha) \rangle & = & 0, \\ H(u(1), \alpha) & = & 0, \\ u(1) - v(0) & = & 0, \\ H(v(1), \alpha) & = & 0, \\ \langle H_x(v(1), \alpha), f^{(i)}(v(1), \alpha) \rangle & = & 0. \end{array} \right. \tag{22}$$

n, m	2, 2

m_d, $(\beta_1, \dots, \beta_{m_d})$	2, (T_i, T_j)
n_d, n_b	4, 7
`IPS`, `SCIDIFF`, `NDIM`	4, 1, 4
`U(1)`, $\dots$, `U(NDIM)`	u_1, u_2, v_1, v_2
`F(`k`)`, $k = 1, 2$	$T_i f_k^{(i)}(u, \alpha)$
`F(`k`)`, $k = 3, 4$	$T_j f_k^{(j)}(v, \alpha)$
`NICP`, (`ICP(1)`, $\dots$, `ICP(NICP)`)	4, (I_1, I_2, 11(T_i), 12(T_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^{\perp}(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$ at $u(0)$ switch to	$\langle H_x(u(0), \alpha), f^{(j)}(u(0), \alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ * - problem 5.10 for $f^{(i)}$, $f^{(j)}$ - problem 5.15 - problem 5.26 for $f^{(i)}$, $f^{(j)}$
t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.17 for $f^{(i)}$ *
t. f. 4: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(j)}(u(1), \alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.7 for $f^{(j)}$ * - problem 5.19 for $f^{(i)}$, $f^{(j)}$ * - problem 5.19 for $f^{(j)}$, $f^{(i)}$ *
t. f. 5: boundary equilibrium of $f^{(i)}$ at $v(1)$ switch to	$\langle H_x^{\perp}(v(1), \alpha), f^{(i)}(v(1), \alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$
t. f. 6: tangent point of $f^{(j)}$ at $v(1)$ switch to	$\langle H_x(v(1), \alpha), f^{(j)}(v(1), \alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.15 - problem 5.20 for $f^{(i)}$, $f^{(j)}$ - problem 5.22 for $f^{(i)}$, $f^{(j)}$

5.19 Continuation of an orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with a tangent point of $f^{(j)}$ ($j \neq i$)

$$
\begin{cases}
\dot{u} - Tf^{(i)}(u,\alpha) &= 0,\\
H(u(0),\alpha) &= 0,\\
\langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle &= 0,\\
H(u(1),\alpha) &= 0,\\
\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle &= 0.
\end{cases}
\tag{23}
$$

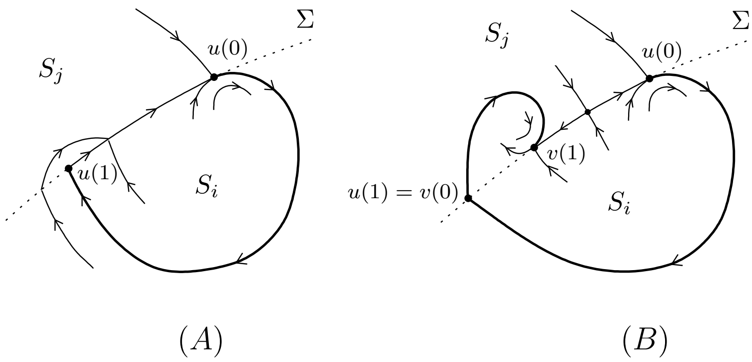

Figure 7: Boundary-value problems corresponding to (A): subsection 5.19; (B): subsection 5.20.

n, m	2, 2
m_d, $(\beta_1,\ldots,\beta_{m_d})$	1, (T)
n_d, n_b	2, 4
`IPS`, `SCIDIFF`, `NDIM`	4, 1, 2
`U(1)`$,\ldots,$ `U(NDIM)`	u_1, u_2
`F(`k`)`, $k = 1,2$	$Tf_k^{(i)}(u,\alpha)$
`NICP`, (`ICP(1)`$,\ldots,$ `ICP(NICP)`)	3, (I_1, I_2, 11(T))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^{\perp}(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$ at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(j)}(u(0),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ - problem 5.10 for $f^{(i)}$, $f^{(j)}$ * - problem 5.15 - problem 5.27 for $f^{(i)}$, $f^{(j)}$
t. f. 3: boundary equilibrium of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x^{\perp}(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.4 for $f^{(j)}$ - problem 5.5 - problem 5.12 for $f^{(j)}$ - problem 5.21 for $f^{(i)}$
t. f. 4: tangent point of $f^{(i)}$ at $u(1)$	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$

switch to	- problem 5.3 for $f^{(i)}$ - problem 5.5 - problem 5.15 - problem 5.17 for $f^{(i)}$ - problem 5.21 for $f^{(i)}$

5.20 Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a tangent point of $f^{(i)}$ with a tangent point of $f^{(j)}$

$$\left\{ \begin{array}{rcl} \dot{u} - T_i f^{(i)}(u,\alpha) & = & 0, \\ \dot{v} - T_j f^{(j)}(v,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle & = & 0, \\ H(u(1),\alpha) & = & 0, \\ u(1) - v(0) & = & 0, \\ H(v(1),\alpha) & = & 0, \\ \langle H_x(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle & = & 0. \end{array} \right. \tag{24}$$

n, m	2, 2
m_d, $(\beta_1,\ldots,\beta_{m_d})$	2, (T_i, T_j)
n_d, n_b	4, 7
IPS, SCIDIFF, NDIM	4, 1, 4
U(1), ..., U(NDIM)	u_1, u_2, v_1, v_2
F(k), $k = 1, 2$	$T_i f_k^{(i)}(u,\alpha)$
F(k), $k = 3, 4$	$T_j f_k^{(j)}(v,\alpha)$
NICP, (ICP(1), ..., ICP(NICP))	4, (I_1, I_2, 11(T_i), 12(T_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^{\perp}(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$ at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(j)}(u(0),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ * - problem 5.10 for $f^{(i)}$, $f^{(j)}$ - problem 5.15 - problem 5.28 for $f^{(i)}$, $f^{(j)}$
t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.17 for $f^{(i)}$ *
t. f. 4: tangent point of $f^{(j)}$ at $u(1)$	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$

switch to	- problem 5.3 for $f^{(j)}$ - problem 5.7 for $f^{(j)}$ * - problem 5.17 for $f^{(j)}$ * - problem 5.19 for $f^{(i)}$, $f^{(j)}$ *
t. f. 5: boundary equilibrium of $f^{(j)}$ at $v(1)$ switch to	$\langle H_x^\perp(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle$ - problem 5.4 for $f^{(j)}$ - problem 5.5 - problem 5.12 for $f^{(j)}$
t. f. 6: tangent point of $f^{(i)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.5 - problem 5.15 - problem 5.18 for $f^{(i)}$, $f^{(j)}$

5.21 Continuation of an orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with a pseudo-equilibrium

$$\left\{\begin{array}{rcll} \dot{u} - Tf^{(i)}(u,\alpha) &=& 0, & \\ H(u(0),\alpha) &=& 0, & \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle &=& 0, & \\ H(u(1),\alpha) &=& 0, & \\ \lambda_i f^{(i)}(u(1),\alpha) + \lambda_j f^{(j)}(u(1),\alpha) &=& 0, & j \neq i, \\ \lambda_i + \lambda_j - 1 &=& 0. & \end{array}\right. \tag{25}$$

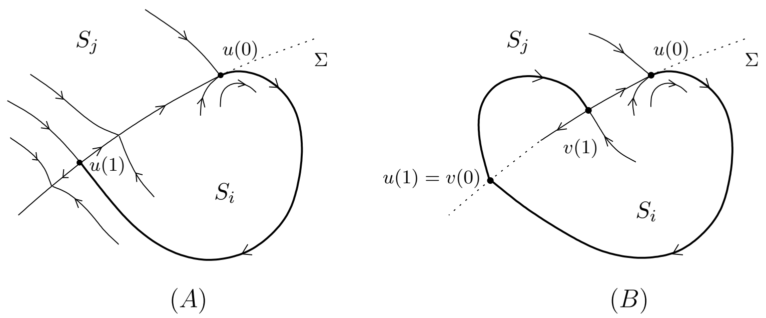

Figure 8: Boundary-value problems corresponding to (A): subsection 5.21; (B): subsection 5.22.

n, m	n, 2
m_d, $(\beta_1, \ldots, \beta_{m_d})$	3, $(T, \lambda_i, \lambda_j)$
n_d, n_b	n, $n+4$
IPS, SCIDIFF, NDIM	4, 1, n
U(1), ..., U(NDIM)	$u_1, \ldots, u_n$

`F(`k`)`, $k = 1, \ldots, n$	$Tf_k^{(i)}(u, \alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	5, (I_1, I_2, 11(T), 13(λ_i), 14(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ ($n = 2$) switch to	$\langle H_x^{\perp}(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$ at $u(0)$ switch to	$\langle H_x(u(0), \alpha), f^{(j)}(u(0), \alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.5 ($n = 2$) - problem 5.9 for $f^{(i)}$ ($n = 2$) - problem 5.15 ($n = 2$)
t. f. 3: boundary equilibrium of $f^{(i)}$ at $u(1)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.4 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$
t. f. 4: boundary equilibrium of $f^{(j)}$ at $u(1)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.4 for $f^{(j)}$ - problem 5.8 for $f^{(i)}$, $f^{(j)}$ ($n = 2$) * - problem 5.12 for $f^{(j)}$ - problem 5.19 for $f^{(i)}$, $f^{(j)}$ ($n = 2$)
t. f. 5: singular sliding point at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n = 2$) - problem 5.8 for $f^{(i)}$, $f^{(j)}$ ($n = 2$) * - problem 5.15 ($n = 2$) - problem 5.17 for $f^{(i)}$ ($n = 2$) - problem 5.19 for $f^{(i)}$, $f^{(j)}$ ($n = 2$)

5.22 Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a tangent point of $f^{(i)}$ with a pseudo-equilibrium

$$
\begin{cases}
\dot{u} - T_i f^{(i)}(u,\alpha) &= 0,\\
\dot{v} - T_j f^{(j)}(v,\alpha) &= 0,\\
H(u(0),\alpha) &= 0,\\
\langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle &= 0,\\
H(u(1),\alpha) &= 0,\\
u(1) - v(0) &= 0,\\
H(v(1),\alpha) &= 0,\\
\lambda_i f^{(i)}(v(1),\alpha) + \lambda_j f^{(j)}(v(1),\alpha) &= 0,\\
\lambda_i + \lambda_j - 1 &= 0.
\end{cases} \tag{26}
$$

n, m	n, 2
m_d, $(\beta_1,\ldots,\beta_{m_d})$	4, $(T_i, T_j, \lambda_i, \lambda_j)$
n_d, n_b	$2n$, $2n+5$
IPS, SCIDIFF, NDIM	4, 1, $2n$
U(1), $\ldots$, U(NDIM)	$u_1,\ldots,u_n$, $v_1,\ldots,v_n$
F(k), $k=1,\ldots,n$	$T_i f_k^{(i)}(u,\alpha)$
F(k), $k=n+1,\ldots,2n$	$T_j f_k^{(j)}(v,\alpha)$
NICP, (ICP(1), $\ldots$, ICP(NICP))	6, (I_1, I_2, 11(T_i), 12(T_j) 13(λ_i), 14(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ ($n=2$) switch to	$\langle H_x^{\perp}(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$ at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(j)}(u(0),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.5 ($n=2$) - problem 5.9 for $f^{(i)}$ ($n=2$) * - problem 5.10 for $f^{(i)}$, $f^{(j)}$ ($n=2$) - problem 5.15 ($n=2$)
t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.17 for $f^{(i)}$ ($n=2$) *
t. f. 4: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.7 for $f^{(j)}$ ($n=2$) * - problem 5.19 for $f^{(i)}$, $f^{(j)}$ ($n=2$) * - problem 5.21 for $f^{(j)}$ *
t. f. 5: boundary equilibrium of $f^{(i)}$ at $v(1)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.4 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$ - problem 5.18 for $f^{(i)}$, $f^{(j)}$ ($n=2$)

t. f. 6: boundary equilibrium of $f^{(j)}$ at $v(1)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.4 for $f^{(j)}$ - problem 5.12 for $f^{(j)}$
t. f. 7: singular sliding point at $v(1)$ switch to	$\langle H_x(v(1), \alpha), f^{(i)}(v(1), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n = 2$) - problem 5.15 ($n = 2$) - problem 5.18 for $f^{(i)}$, $f^{(j)}$ ($n = 2$) - problem 5.20 for $f^{(i)}$, $f^{(j)}$ ($n = 2$)

5.23 Continuation of an orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with a saddle

$$\left\{\begin{array}{rcl} \dot{u} - Tf^{(i)}(u, \alpha) & = & 0, \\ H(u(0), \alpha) & = & 0, \\ \langle H_x(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle & = & 0, \\ f^{(i)}(y, \alpha) & = & 0, \\ \left[f_x^{(i)}(y, \alpha)\right]^T w - \nu_k w & = & 0, \quad \nu_k > 0, \\ \langle w, w\rangle - 1 & = & 0, \\ \langle w, y - u(1)\rangle & = & 0. \end{array}\right. \tag{27}$$

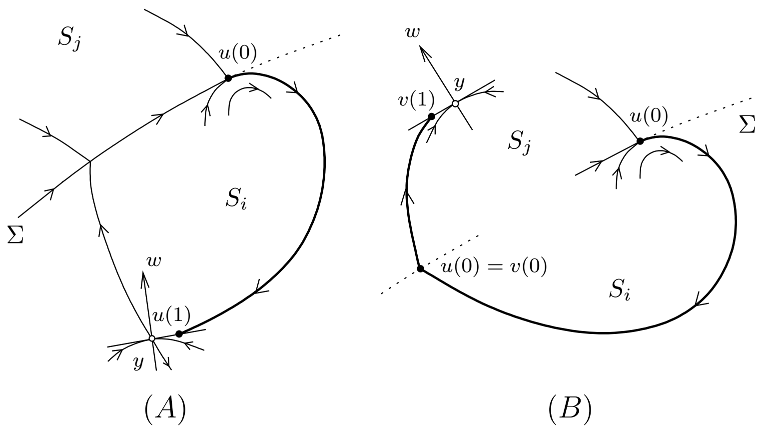

Figure 9: Boundary-value problems corresponding to (A): subsection 5.23; (B): subsection 5.24.

n, m	2, 2
m_d, ($\beta_1, \ldots, \beta_{m_d}$)	5, ($y_1, y_2, w_1, w_2, \nu_k$)
n_d, n_b	2, 8

IPS, SCIDIFF, NDIM	4, 1, 2
U(1), ..., U(NDIM)	u_1, u_2
F(k), $k = 1, 2$	$Tf_k^{(i)}(u, \alpha)$
NICP, (ICP(1), ..., ICP(NICP))	7, (I_1, I_2, 13(y_1), 14(y_2), 15(w_1), 16(w_2), 17(ν_k))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^\perp(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$, $j \neq i$, at $u(0)$ switch to	$\langle H_x(u(0), \alpha), f^{(j)}(u(0), \alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.15 - problem 5.31 for $f^{(i)}$
t. f. 3: boundary equilibrium of $f^{(i)}$ at $u(1)$ switch to	$H(u(1), \alpha)$ - problem 5.1 - problem 5.3 for $f^{(i)}$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.7 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$ - problem 5.17 for $f^{(i)}$ - problem 5.21 for $f^{(i)}$
t. f. 4: branch/limit point switch to	$\mathrm{Re}\nu_p \,:\, p = \arg\min_q\{\lvert\mathrm{Re}\nu_q\rvert\}$ problem 5.4 for $f^{(i)}$ (branch switching) */ continuation of a limit point bifurcation *

5.24 Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ $(j \neq i)$ connecting a tangent point of $f^{(i)}$ with a saddle of $f^{(j)}$

$$\left\{\begin{array}{rcl}
\dot u - T_i f^{(i)}(u, \alpha) & = & 0, \\
\dot v - T_j f^{(j)}(v, \alpha) & = & 0, \\
H(u(0), \alpha) & = & 0, \\
\langle H_x(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle & = & 0, \\
H(u(1), \alpha) & = & 0, \\
u(1) - v(0) & = & 0, \\
f^{(j)}(y, \alpha) & = & 0, \\
\left[f_x^{(j)}(y, \alpha)\right]^T w - \nu_k w & = & 0, \quad \nu_k > 0, \\
\langle w, w\rangle - 1 & = & 0, \\
\langle w, y - v(1)\rangle & = & 0.
\end{array}\right. \qquad (28)$$

n, m	2, 2

m_d, $(\beta_1,\ldots,\beta_{m_d})$	6, $(T_i, y_1, y_2, w_1, w_2, \nu_k)$
n_d, n_b	4, 11
`IPS`, `SCIDIFF`, `NDIM`	4, 1, 4
`U(1)`$,\ldots,$ `U(NDIM)`	u_1, u_2, v_1, v_2
`F(`k`)`, $k=1,2$	$T_i f_k^{(i)}(u,\alpha)$
`F(`k`)`, $k=3,4$	$T_j f_k^{(j)}(v,\alpha)$
`NICP`, (`ICP(1)`$,\ldots,$ `ICP(NICP)`)	8, (I_1, I_2, 11(T_i), 13(y_1), 14(y_2), 15(w_1), 16(w_2), 17(ν_k))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^{\perp}(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$ - problem 5.31 for $f^{(j)}$ *
t. f. 2: tangent point of $f^{(j)}$ at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(j)}(u(0),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ * - problem 5.15 - problem 5.32 for $f^{(i)}$, $f^{(j)}$
t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.17 for $f^{(i)}$ *
t. f. 4: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.19 for $f^{(i)}$, $f^{(j)}$ * - problem 5.23 for $f^{(j)}$ *
t. f. 5: boundary equilibrium of $f^{(i)}$ at $v(1)$ switch to	$H(v(1),\alpha)$ - problem 5.1 - problem 5.3 for $f^{(j)}$ - problem 5.4 for $f^{(j)}$ - problem 5.5 - problem 5.8 for $f^{(i)}$, $f^{(j)}$ - problem 5.12 for $f^{(j)}$ - problem 5.20 for $f^{(i)}$, $f^{(j)}$ - problem 5.22 for $f^{(i)}$, $f^{(j)}$
t. f. 6: branch/limit point switch to	$\mathrm{Re}\nu_p\ :\ p=\arg\min_q\{\lvert\mathrm{Re}\nu_q\rvert\}$ problem 5.4 for $f^{(j)}$ (branch switching) */ continuation of a limit point bifurcation *

5.25 Continuation of an orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium with a tangent point of $f^{(i)}$

$$
\begin{cases}
\dot{u} - Tf^{(i)}(u,\alpha) &= 0, \\
H(u(0),\alpha) &= 0, \\
\lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) &= 0, \quad j \neq i, \\
\lambda_i + \lambda_j - 1 &= 0, \\
H(u(1),\alpha) &= 0, \\
\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle &= 0.
\end{cases} \tag{29}
$$

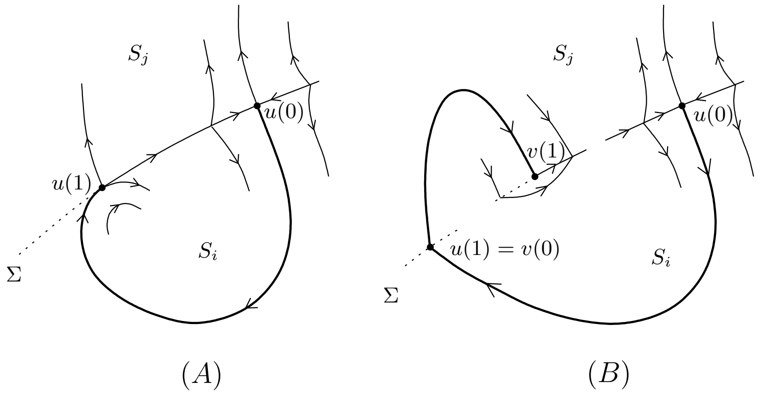

Figure 10: Boundary-value problems corresponding to (A): subsection 5.25; (B): subsection 5.26.

n, m	n, 2
m_d, $(\beta_1, \ldots, \beta_{m_d})$	3, $(T, \lambda_i, \lambda_j)$
n_d, n_b	n, $n+4$
`IPS`, `SCIDIFF`, `NDIM`	4, 1, n
`U(1)`$, \ldots,$ `U(NDIM)`	$u_1, \ldots, u_n$
`F(`k`)`, $k = 1, \ldots, n$	$Tf_k^{(i)}(u,\alpha)$
`NICP`, (`ICP(1)`$, \ldots,$ `ICP(NICP)`)	5, (I_1, I_2, 11(T), 13(λ_i), 14(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.4 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.4 for $f^{(j)}$ - problem 5.12 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n = 2$) - problem 5.7 for $f^{(i)}$ ($n = 2$) - problem 5.15 ($n = 2$) - problem 5.17 for $f^{(i)}$ ($n = 2$)

t. f. 4: boundary equilibrium of $f^{(i)}$ at $u(1)$ ($n=2$) switch to	$\langle H_x^\perp(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$
t. f. 5: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.5 ($n=2$) - problem 5.10 for $f^{(i)}$, $f^{(j)}$ * - problem 5.15 ($n=2$) - problem 5.27 for $f^{(i)}$, $f^{(j)}$

5.26 Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium with a tangent point of $f^{(i)}$

$$\left\{\begin{array}{rcl} \dot{u} - T_i f^{(i)}(u,\alpha) & = & 0, \\ \dot{v} - T_j f^{(j)}(v,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) & = & 0, \\ \lambda_i + \lambda_j - 1 & = & 0, \\ H(u(1),\alpha) & = & 0, \\ u(1) - v(0) & = & 0, \\ H(v(1),\alpha) & = & 0, \\ \langle H_x(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle & = & 0. \end{array}\right. \tag{30}$$

n, m	n, 4
m_d, $(\beta_1,\ldots,\beta_{m_d})$	2, $(T_i, T_j, \lambda_i, \lambda_j)$
n_d, n_b	$2n$, $2n+5$
`IPS`, `SCIDIFF`, `NDIM`	4, 1, $2n$
`U(1)`, $\ldots$, `U(NDIM)`	$u_1,\ldots,u_n$, $v_1,\ldots,v_n$
`F(`k`)`, $k=1,\ldots,n$	$T_i f_k^{(i)}(u,\alpha)$
`F(`k`)`, $k=n+1,\ldots,2n$	$T_j f_k^{(j)}(v,\alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	6, (I_1, I_2, 11(T_i), 12(T_j), 13(λ_i), 14(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.4 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$	λ_i

switch to	- problem 5.2 for $f^{(j)}$ $(n=3)$ - problem 5.3 for $f^{(j)}$ $(n=2)$ - problem 5.4 for $f^{(j)}$ - problem 5.12 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ $(n=3)$ - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ $(n=2)$ - problem 5.7 for $f^{(i)}$ $(n=2)$ * - problem 5.8 for $f^{(i)}$, $f^{(j)}$ $(n=2)$ - problem 5.15 $(n=2)$ - problem 5.18 for $f^{(i)}$, $f^{(j)}$ $(n=2)$
t. f. 4: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ $(n=3)$ - problem 5.3 for $f^{(i)}$ $(n=2)$ - problem 5.25 for $f^{(i)}$ *
t. f. 5: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ $(n=3)$ - problem 5.3 for $f^{(j)}$ $(n=2)$ - problem 5.7 for $f^{(j)}$ $(n=2)$ * - problem 5.19 for $f^{(j)}$, $f^{(i)}$ $(n=2)$ * - problem 5.27 for $f^{(i)}$, $f^{(j)}$ *
t. f. 6: boundary equilibrium of $f^{(i)}$ at $v(1)$ $(n=2)$ switch to	$\langle H_x^{\perp}(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.12 for $f^{(i)}$ - problem 5.30 for $f^{(i)}$, $f^{(j)}$
t. f. 7: tangent point of $f^{(j)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ $(n=3)$ - problem 5.3 for $f^{(j)}$ $(n=2)$ - problem 5.5 $(n=2)$ - problem 5.15 $(n=2)$ - problem 5.28 for $f^{(i)}$, $f^{(j)}$

5.27 Continuation of an orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium with a tangent point of $f^{(j)}$ $(j \neq i)$

$$\left\{\begin{array}{rcl} \dot{u} - Tf^{(i)}(u,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) & = & 0, \\ \lambda_i + \lambda_j - 1 & = & 0, \\ H(u(1),\alpha) & = & 0, \\ \langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle & = & 0. \end{array}\right. \quad (31)$$

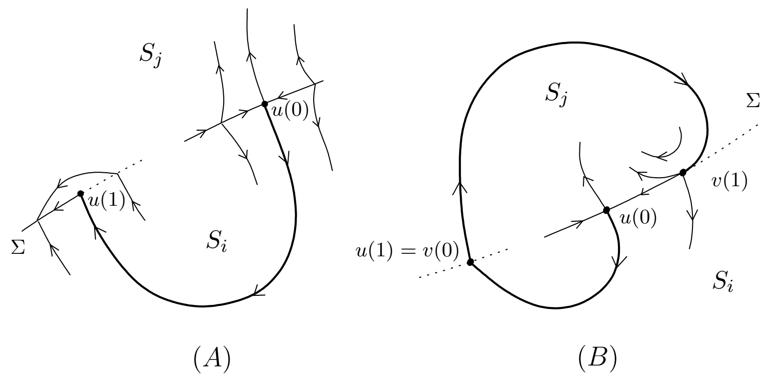

Figure 11: Boundary-value problems corresponding to (A): subsection 5.27; (B): subsection 5.28.

n, m	n, 2
m_d, $(\beta_1, \ldots, \beta_{m_d})$	3, $(T, \lambda_i, \lambda_j)$
n_d, n_b	n, $n+4$
`IPS`, `SCIDIFF`, `NDIM`	4, 1, n
`U(1)`, $\ldots$, `U(NDIM)`	$u_1, \ldots, u_n$
`F(`k`)`, $k = 1, \ldots, n$	$T f_k^{(i)}(u, \alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	5, (I_1, I_2, 11(T), 13(λ_i), 14(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.4 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.4 for $f^{(j)}$ - problem 5.12 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n = 2$) - problem 5.7 for $f^{(i)}$ ($n = 2$) - problem 5.8 for $f^{(i)}$, $f^{(j)}$ ($n = 2$) * - problem 5.15 ($n = 2$) - problem 5.19 for $f^{(i)}$, $f^{(j)}$ ($n = 2$)
t. f. 4: boundary equilibrium of $f^{(j)}$ at $u(1)$ ($n = 2$) switch to	$\langle H_x^{\perp}(u(1), \alpha), f^{(j)}(u(1), \alpha)\rangle$ - problem 5.4 for $f^{(j)}$ - problem 5.5 - problem 5.12 for $f^{(j)}$ - problem 5.29 for $f^{(i)}$
t. f. 5: tangent point of $f^{(i)}$ at $u(1)$	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$

switch to	- problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.5 ($n=2$) - problem 5.15 ($n=2$) - problem 5.25 for $f^{(i)}$

5.28 Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium with a tangent point of $f^{(j)}$

$$\left\{\begin{array}{rcl} \dot{u} - T_i f^{(i)}(u,\alpha) & = & 0, \\ \dot{v} - T_j f^{(j)}(v,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) & = & 0, \\ \lambda_i + \lambda_j - 1 & = & 0, \\ H(u(1),\alpha) & = & 0, \\ u(1) - v(0) & = & 0, \\ H(v(1),\alpha) & = & 0, \\ \langle H_x(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle & = & 0. \end{array}\right. \tag{32}$$

n, m	n, 4
m_d, $(\beta_1,\ldots,\beta_{m_d})$	2, $(T_i, T_j, \lambda_i, \lambda_j)$
n_d, n_b	$2n$, $2n+5$
`IPS`, `SCIDIFF`, `NDIM`	4, 1, $2n$
`U(1)`, $\ldots$, `U(NDIM)`	$u_1,\ldots,u_n$, $v_1,\ldots,v_n$
`F(`k`)`, $k=1,\ldots,n$	$T_i f_k^{(i)}(u,\alpha)$
`F(`k`)`, $k=n+1,\ldots,2n$	$T_j f_k^{(j)}(v,\alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	6, (I_1, I_2, 11(T_i), 12(T_j), 13(λ_i), 14(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.4 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.4 for $f^{(j)}$ - problem 5.12 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$

switch to	- problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n = 2$) - problem 5.7 for $f^{(i)}$ ($n = 2$) * - problem 5.8 for $f^{(i)}$, $f^{(j)}$ ($n = 2$) - problem 5.15 ($n = 2$) - problem 5.20 for $f^{(i)}$, $f^{(j)}$ ($n = 2$)
t. f. 4: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.25 for $f^{(i)}$ *
t. f. 5: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.7 for $f^{(j)}$ ($n = 2$) * - problem 5.17 for $f^{(j)}$ ($n = 2$) * - problem 5.27 for $f^{(i)}$, $f^{(j)}$ *
t. f. 6: boundary equilibrium of $f^{(j)}$ at $v(1)$ ($n = 2$) switch to	$\langle H_x^{\perp}(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle$ - problem 5.4 for $f^{(j)}$ - problem 5.5 - problem 5.12 for $f^{(j)}$
t. f. 7: tangent point of $f^{(i)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.5 ($n = 2$) - problem 5.15 ($n = 2$) - problem 5.26 for $f^{(i)}$, $f^{(j)}$

5.29 Continuation of an orbit of vector field $f^{(i)}$ connecting two pseudo-equilibria

$$\left\{ \begin{array}{rcll} \dot{u} - Tf^{(i)}(u,\alpha) & = & 0, & \\ H(u(0),\alpha) & = & 0, & \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) & = & 0, & j \neq i, \\ \lambda_i + \lambda_j - 1 & = & 0, & \\ H(u(1),\alpha) & = & 0, & \\ \mu_i f^{(i)}(u(1),\alpha) + \mu_j f^{(j)}(u(1),\alpha) & = & 0, & \\ \mu_i + \mu_j - 1 & = & 0. & \end{array} \right. \qquad (33)$$

n, m	2, 2
m_d, $(\beta_1,\ldots,\beta_{m_d})$	5, $(T,\lambda_i,\lambda_j,\mu_i,\mu_j)$
n_d, n_b	2, 8
`IPS`, `SCIDIFF`, `NDIM`	4, 0, 2

`U(1)`$,\ldots,$`U(NDIM)`	u_1, u_2
`F(`k`)`$, k=1,2$	$Tf_k^{(i)}(u,\alpha)$
`NICP`, (`ICP(1)`$,\ldots,$`ICP(NICP)`)	7, (I_1, I_2, 11(T), 13(λ_i), 14(λ_j), 15(μ_i), 16(μ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.3 for $f^{(i)}$ - problem 5.4 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.3 for $f^{(j)}$ - problem 5.4 for $f^{(j)}$ - problem 5.12 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ - problem 5.7 for $f^{(i)}$ - problem 5.15 - problem 5.21 for $f^{(i)}$
t. f. 4: boundary equilibrium of $f^{(i)}$ at $u(1)$ switch to	μ_j - problem 5.3 for $f^{(i)}$ - problem 5.4 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$
t. f. 5: boundary equilibrium of $f^{(j)}$ at $u(1)$ switch to	μ_i - problem 5.3 for $f^{(j)}$ - problem 5.4 for $f^{(j)}$ - problem 5.10 for $f^{(i)}$, $f^{(j)}$ * - problem 5.12 for $f^{(j)}$ - problem 5.27 for $f^{(i)}$, $f^{(j)}$
t. f. 6: singular sliding point at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ - problem 5.10 for $f^{(i)}$, $f^{(j)}$ * - problem 5.15 - problem 5.25 for $f^{(i)}$ - problem 5.27 for $f^{(i)}$, $f^{(j)}$

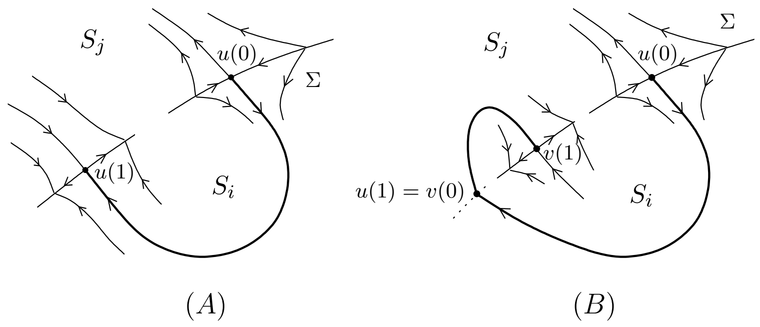

Figure 12: Boundary-value problems corresponding to (A): subsection 5.29; (B): subsection 5.30.

5.30 Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting two pseudo-equilibria

$$\left\{\begin{array}{rcl} \dot{u} - T_i f^{(i)}(u,\alpha) & = & 0, \\ \dot{v} - T_j f^{(j)}(v,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) & = & 0, \\ \lambda_i + \lambda_j - 1 & = & 0, \\ H(u(1),\alpha) & = & 0, \\ u(1) - v(0) & = & 0, \\ H(v(1),\alpha) & = & 0, \\ \mu_i f^{(i)}(v(1),\alpha) + \mu_j f^{(j)}(v(1),\alpha) & = & 0, \\ \mu_i + \mu_j - 1 & = & 0. \end{array}\right. \tag{34}$$

n, m	2, 2
m_d, $(\beta_1, \ldots, \beta_{m_d})$	6, $(T_i, T_j, \lambda_i, \lambda_j, \mu_i, \mu_j)$
n_d, n_b	4, 11
`IPS`, `SCIDIFF`, `NDIM`	4, 0, 4
`U(1)`, $\ldots$, `U(NDIM)`	u_1, u_2, v_1, v_2
`F(`k`)`, $k = 1, 2$	$T_i f_k^{(i)}(u,\alpha)$
`F(`k`)`, $k = 3, 4$	$T_j f_k^{(j)}(v,\alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	8, (I_1, I_2, 11(T_i), 12(T_j), 13(λ_i), 14(λ_j), 15(μ_i), 16(μ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.3 for $f^{(i)}$ - problem 5.4 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.3 for $f^{(j)}$ - problem 5.4 for $f^{(j)}$ - problem 5.12 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$	$\langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$

switch to	- problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ - problem 5.7 for $f^{(i)}$ * - problem 5.8 for $f^{(i)}$, $f^{(j)}$ - problem 5.15 - problem 5.22 for $f^{(i)}$, $f^{(j)}$
t. f. 4: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.25 for $f^{(i)}$ *
t. f. 5: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.7 for $f^{(j)}$ * - problem 5.21 for $f^{(j)}$ * - problem 5.27 for $f^{(i)}$, $f^{(j)}$ *
t. f. 6: boundary equilibrium of $f^{(i)}$ at $v(1)$ switch to	μ_j - problem 5.3 for $f^{(i)}$ - problem 5.4 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$ - problem 5.26 for $f^{(i)}$, $f^{(j)}$
t. f. 7: boundary equilibrium of $f^{(j)}$ at $v(1)$ switch to	μ_i - problem 5.3 for $f^{(j)}$ - problem 5.4 for $f^{(j)}$ - problem 5.12 for $f^{(j)}$
t. f. 8: singular sliding point at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ - problem 5.15 - problem 5.26 for $f^{(i)}$, $f^{(j)}$ - problem 5.28 for $f^{(i)}$, $f^{(j)}$

5.31 Continuation of an orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium with a saddle

$$\left\{\begin{array}{rcll} \dot{u} - Tf^{(i)}(u,\alpha) &=& 0, & \\ H(u(0),\alpha) &=& 0, & \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) &=& 0, & j \neq i, \\ \lambda_i + \lambda_j - 1 &=& 0, & \\ f^{(i)}(y,\alpha) &=& 0, & \\ \left[f_x^{(i)}(y,\alpha)\right]^T w - \nu_k w &=& 0, & \nu_k > 0, \\ \langle w, w\rangle - 1 &=& 0, & \\ \langle w, y - u(1)\rangle &=& 0. & \end{array}\right. \tag{35}$$

n, m	n, 2
m_d, $(\beta_1,\ldots,\beta_{m_d})$	$2n+3$, $(\lambda_i,\lambda_j,y_1,\ldots,y_n,w_1,\ldots,w_n,\nu_k)$

n_d, n_b	n, $3n+4$
`IPS`, `SCIDIFF`, `NDIM`	4, 1, n
`U(1)`, $\ldots$, `U(NDIM)`	$u_1, \ldots, u_n$
`F(`k`)`, $k = 1, \ldots, n$	$Tf_k^{(i)}(u, \alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	$2n+5$, (I_1, I_2, 13(λ_i), 14(λ_j), 15(y_1), $\ldots$, $(15+n-1)(y_n)$, $(15+n)(w_1)$, $\ldots$, $(15+2n-1)(w_n)$, $(15+2n)(\nu_k)$)
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.4 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.4 for $f^{(j)}$ - problem 5.12 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n=2$) - problem 5.15 ($n=2$) - problem 5.23 for $f^{(i)}$ ($n=2$)
t. f. 4: boundary equilibrium of $f^{(i)}$ at $u(1)$ switch to	$H(u(1), \alpha)$ - problem 5.1 ($n=2$) - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$ - problem 5.25 for $f^{(i)}$ - problem 5.29 for $f^{(i)}$ ($n=2$)
t. f. 5: Hopf switch to	$\mathrm{Re}\nu_p \,:\, p = \arg\min_q\{\lvert\mathrm{Re}\nu_q\rvert \,:\, \mathrm{Im}\nu_q \neq 0\}$ continuation of a Hopf bifurcation *
t. f. 6: branch/limit point switch to	$\mathrm{Re}\nu_p \,:\, p = \arg\min_q\{\lvert\mathrm{Re}\nu_q\rvert \,:\, \mathrm{Im}\nu_q = 0\}$ problem 5.4 for $f^{(i)}$ (branch switching) */ continuation of a limit point bifurcation *
note	the saddle y is assumed to have a one-dimensional unstable manifold

5.32 Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium with a saddle

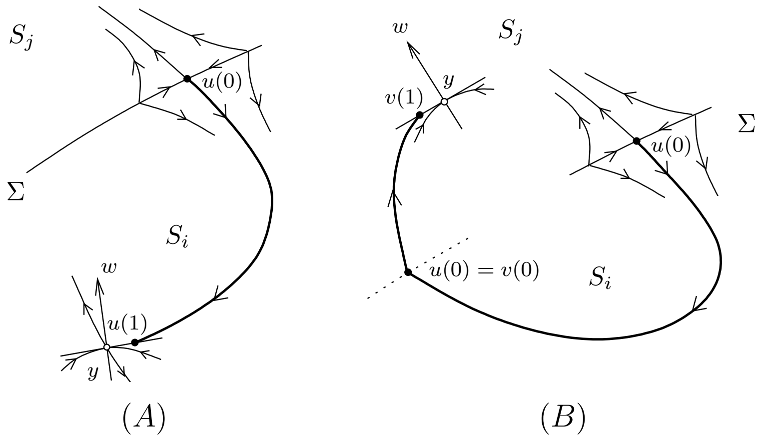

Figure 13: Boundary-value problems corresponding to (A): subsection 5.31; (B): subsection 5.32.

$$\left\{\begin{array}{rcll} \dot{u} - T_i f^{(i)}(u,\alpha) & = & 0, & \\ \dot{v} - T_j f^{(j)}(v,\alpha) & = & 0, & \\ H(u(0),\alpha) & = & 0, & \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) & = & 0, & \\ \lambda_i + \lambda_j - 1 & = & 0, & \\ H(u(1),\alpha) & = & 0, & \\ u(1) - v(0) & = & 0, & \\ f^{(i)}(y,\alpha) & = & 0, & \\ \left[f_x^{(i)}(y,\alpha)\right]^T w - \nu_k w & = & 0, & \nu_k > 0, \\ \langle w, w\rangle - 1 & = & 0, & \\ \langle w, y - v(1)\rangle & = & 0. & \end{array}\right. \tag{36}$$

n, m	n, 2
m_d, $(\beta_1,\ldots,\beta_{m_d})$	$2n$+4, ($T_i, \lambda_i, \lambda_j, y_1, \ldots, y_n, w_1, \ldots, w_n, \nu_k$)
n_d, n_b	$2n$, $4n+5$
IPS, SCIDIFF, NDIM	4, 1, $2n$
U(1), $\ldots$, U(NDIM)	$u_1, \ldots, u_n, v_1, \ldots, v_n$
F(k), $k = 1, \ldots, n$	$T_i f_k^{(i)}(u,\alpha)$
F(k), $k = n+1, \ldots, 2n$	$T_j f_k^{(j)}(v,\alpha)$
NICP, (ICP(1), $\ldots$, ICP(NICP))	$2n+6$, (I_1, I_2, 11(T_i), 13(λ_i), 14(λ_j), 15(y_1), $\ldots$, $(15+n-1)(y_n)$, $(15+n)(w_1)$, $\ldots$, $(15+2n-1)(w_n)$, $(15+2n)(\nu_k)$)
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.4 for $f^{(i)}$ - problem 5.12 for $f^{(i)}$ - problem 5.31 for $f^{(j)}$ *
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$	λ_i

switch to	- problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.4 for $f^{(j)}$ - problem 5.12 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n = 2$) - problem 5.7 for $f^{(i)}$ ($n = 2$) * - problem 5.15 ($n = 2$) - problem 5.24 for $f^{(i)}$, $f^{(j)}$ ($n = 2$)
t. f. 4: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.25 for $f^{(i)}$ *
t. f. 5: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(j)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.23 for $f^{(j)}$ ($n = 2$) * - problem 5.27 for $f^{(i)}$, $f^{(j)}$ *
t. f. 6: boundary equilibrium of $f^{(i)}$ at $v(1)$ switch to	$H(v(1), \alpha)$ - problem 5.1 ($n = 2$) - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.10 for $f^{(i)}$, $f^{(j)}$ - problem 5.12 for $f^{(i)}$ - problem 5.28 for $f^{(i)}$, $f^{(j)}$ - problem 5.30 for $f^{(i)}$, $f^{(j)}$ ($n = 2$)
t. f. 7: Hopf switch to	$\mathrm{Re}\nu_p \,:\, p = \arg\min_q\{\lvert\mathrm{Re}\nu_q\rvert \,:\, \mathrm{Im}\nu_q \neq 0\}$ continuation of a Hopf bifurcation *
t. f. 8: branch/limit point switch to	$\mathrm{Re}\nu_p \,:\, p = \arg\min_q\{\lvert\mathrm{Re}\nu_q\rvert \,:\, \mathrm{Im}\nu_q = 0\}$ problem 5.4 for $f^{(j)}$ (branch switching) */ continuation of a limit point bifurcation *
note	the saddle y is assumed to have a one-dimensional unstable manifold

6 USER GUIDE

A user guide for the software SLIDECONT (version 1.0) is presented in this section. The reader should refer to the previous sections, as well as to Kuznetsov *et al.* (2002) and references therein for the nomenclature and the theory behind the methods.

6.1 Disclaimer

SLIDECONT is freely available for non-commercial use on an "as is" basis. In no circumstances can the authors be held liable for any deficiency, fault or other mishappening with regard to the use or performance of SLIDECONT.

6.2 System requirements

SLIDECONT requires that AUTO97 is installed under UNIX. A pre-defined maximum value (`SCNDIMX`) of the user problem dimension (`SCNDIM`) is set in the header file `slidecont.h`. This maximum affects the run-time memory requirements and should not be set to unnecessarily large values. The restriction can be changed by editing `slidecont.h` followed by recompilation. Notice that the effective dimension of the defining systems and the number of active parameters depend upon the user problem dimension and they may exceed AUTO97 restrictions on problem size, requiring AUTO97 recompilation (see AUTO97 documentation). If the SLIDECONT maximum user problem dimension or the AUTO97 restrictions on problem size are exceeded in a SLIDECONT-run, then the computation halts with an error message.

6.3 Installation

SLIDECONT is freely available for download at:

`http://www.math.uu.nl/people/kuznet/cm`

The compressed tar-file `slidecont.tar.gz` at this location contains SLIDECONT.

The softwaree can be extracted by running `tar -xzvf slidecont.tar.gz` in a directory, where the driver should be installed. Then the (sub)directories `bin`, `cmds`, `examples`, `include`, `lib`, and `src` are created. The SLIDECONT header and source files are contained in `include` and `src` directories, respectively, `cmds` contains SLIDECONT commands (see Subsection 6.4) and the environment file `slidecont.env`, while `examples` contains several tutorial examples, one of which (`examples/hppc`) is used in Section 7. The directories `bin` and `lib` are empty initially.

The environment variables AUTO_DIR and SC_DIR must be set to the absolute paths of the AUTO97 and SLIDECONT directories and the directories `$AUTO_DIR/cmds`, `$AUTO_DIR/bin`, `$SC_DIR/cmds`, and `$SC_DIR/bin` must be added to the system search path list. For this, the environment file `slidecont.env` should be appropriately edited and sourced before running SLIDECONT.

After that SLIDECONT should be compiled by typing `make` in the `$SC_DIR/src` directory (`f77` is assumed to be the Fortran compiler command name). This would produce SLIDECONT preprocessor and libraries and place them into `bin` and `lib`, repectively.

6.4 Running SLIDECONT

The user must provide three files. The equations file `<name>.f` containing the Fortran subroutines `SCFUNC`, `SCBOUND`, `SCSTPNT`, `SCPVLS`, `SCBCND`, `SCICND`, and `SCFOPT`, the constants file `sc.<name>`, and, possibly, the data file `<name>.dat` which specifies numerically the starting solution for boundary-value problems (`<name>` is a user-selected name).

The user-supplied subroutines may be regarded as higher-level input routines that are called by the standard AUTO97 routines contained in the SLIDECONT library. The purpose of the user-supplied subroutines is the following:

`SCFUNC` Fortran prototype

```
      SUBROUTINE SCFUNC(SCNDIM,X,PAR,SCIDIFF,FI,
     +   DFIDX,DFIDP,DFIDXDX,DFIDXDP,I)
```

defines the vector fields $f^{(1)}(x,\alpha)$ and $f^{(2)}(x,\alpha)$; on input, SCNDIM, X(k), PAR(k), SCIDIFF, and I contain the vector fields dimension n, the actual state and parameter values x and α, the order up to which derivatives must be analytically specified, and the index i of the vector field of interest, respectively; as in AUTO97, only parameters PAR(1), ..., PAR(9) can be used by the user; on output, FI(k) contains the k-th component of $f^{(i)}$, while DFIDX(k,p), DFIDP(k,p), DFIDXDX(k,p,q), and DFIDXDP(k,p,q) contain the derivatives of $f_k^{(i)}$, if provided, with respect to x_p, α_p, (x_p, x_q), and (x_p, α_q), respectively;

SCBOUND Fortran prototype

```
      SUBROUTINE SCBOUND(SCNDIM,X,PAR,SCIDIFF,
     +  H,DHDX,DHDP,DHDXDX,DHDXDP)
```

defines the discontinuity boundary function $H(x,\alpha)$. Input and output arguments are analogous to those of SCFUNC;

SCSTPNT Fortran prototype

```
      SUBROUTINE SCSTPNT(SCNDIM,X,PAR,T,I)
```

defines the starting solution (x,α); on input, SCNDIM and I contain the vector fields dimension n and the index i of the vector field of interest, respectively; on output, X(k) and PAR(k) contain state and parameters starting values; for boundary-value problems, only an analytically known solution X(k)=x_k(T) can be specified, where T denotes the independent time variable which, on input, takes values in the interval $[0,1]$; in such cases the time length of the solution must be specified in PAR(11) or PAR(12), depending upon the problem (see Section 5, and AUTO97 documentation for more details);

SCPVLS Fortran prototype

```
      SUBROUTINE SCPVLS(SCNDIM,X,PAR,I)
```

defines user functions and solution measures (see AUTO97 documentation);

SCBCND Fortran prototype

```
      SCBCND(NDIM,PAR,ICP,NBC,U0,U1,FB,IJAC,DBC)
```

is meaningful only for AUTO97 problem types;

`SCICND` Fortran prototype

```
SCICND(NDIM,PAR,ICP,NINT,U,UOLD,UDOT,
+       UPOLD,FI,IJAC,DINT)
```

is meaningful only for AUTO97 problem types;

`SCFOPT` Fortran prototype

```
SCFOPT(NDIM,U,ICP,PAR,IJAC,FS,DFDU,DFDP)
```

is meaningful only for AUTO97 problem types.

For a fully documented equations file see `$SC_DIR/examples/hppc/hppc.f`.

The format of the constants file is the same for all problems. For example, the file `$SC_DIR/examples/hppc/sc.hppc.1` is listed below.

```
2 102 0 1                     NDIM,IPS,IRS,ILP
1 2                           NICP,(ICP(I),I=1,NICP)
0 0 0 1 1 0 0 0               NTST,NCOL,IAD,ISP,ISW,IPLT,NBC,NINT
50 0.0 1.0 0.0 100.0          NMX,RL0,RL1,A0,A1
10 0 2 8 7 5 0                NPR,MXBF,IID,ITMX,ITNW,NWTN,JAC
1.e-8 1.e-8 1.e-7             EPSL,EPSU,EPSS
-1.e-3 1.e-10 0.1 1           DS,DSMIN,DSMAX,IADS
0                             NTHL,((ITHL(I),THL(I)),I=1,NTHL)
0                             NTHU,((ITHU(I),THU(I)),I=1,NTHU)
1                             NUZR,((IUZR(I),UZR(I)),I=1,NUZR)
2 0.33
0 2                           SCISTART,SCIDIFF
1 1                           SCNPSI,(SCIPSI(I),I=1,SCNPSI)
0                             SCNFIXED,(SCIFIXED(I),I=1,SCNFIXED)
```

The first part (ending with `NUZR`) is an AUTO97 compliant constants file, thus we refer to the AUTO97 documentation for the meaning of single constants. In particular, `IPS` is the problem type and `DS` is the starting step size of the continuation algorithm. If `DS` is positive (negative) then the computation is performed forward (backward) with respect to the first active parameter of the `ICP` list. A list of SLIDECONT problem types (with alphabetical labels useful for source code inspection) is reported in Table 1.

Label	Problem type	Problem description
`T3D`i	i`00`	Computation of a curve of tangent points of vector field $f^{(i)}$ in three-dimensional systems
`TAN`i	i`01`	Continuation of a tangent point of vector field $f^{(i)}$ Continuation of a double tangency bifurcation of vector field $f^{(i)}$
`EQL`i	i`02`	Continuation of a standard equilibrium of vector field $f^{(i)}$
`BEQ`i	i`03`	Continuation of a boundary equilibrium of vector field $f^{(i)}$
`CYC`i	i`10`	Continuation of a standard cycle of vector field $f^{(i)}$
`TGB`i	i`11`	Continuation of an orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with the boundary Σ
`TCB`i	i`12`	Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a tangent point of $f^{(i)}$ with the boundary Σ
`PEB`i	i`13`	Continuation of an orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium with the boundary Σ
`PCB`i	i`14`	Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium with the boundary Σ
`TCH`i	i`15`	Continuation of a touching bifurcation of vector field $f^{(i)}$
`TGT`i	i`16`	Continuation of an orbit of vector field $f^{(i)}$ connecting two tangent points of $f^{(i)}$
`TCT`i	i`17`	Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting two tangent points of $f^{(i)}$
`TTD`i	i`18`	Continuation of an orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with a tangent point of $f^{(j)}$ ($j \neq i$)
`TCD`i	i`19`	Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a tangent point of $f^{(i)}$ with a tangent point of $f^{(j)}$
`TGP`i	i`20`	Continuation of an orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with a pseudo-equilibrium
`TCP`i	i`21`	Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a tangent point of $f^{(i)}$ with a pseudo-equilibrium
`TGS`i	i`22`	Continuation of an orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with a saddle
`TCS`i	i`23`	Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a tangent point of $f^{(i)}$ with a saddle of $f^{(j)}$
`PET`i	i`24`	Continuation of an orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium with a tangent point of $f^{(i)}$
`PCT`i	i`25`	Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium with a tangent point of $f^{(i)}$
`PTD`i	i`26`	Continuation of an orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium with a tangent point of $f^{(j)}$ ($j \neq i$)
`PCD`i	i`27`	Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium with a tangent point of $f^{(j)}$
`PEP`i	i`28`	Continuation of an orbit of vector field $f^{(i)}$ connecting two pseudo-equilibria

Table 1: (continue)

Label	Problem type	Problem description
PCPi	i29	Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting two pseudo-equilibria
PESi	i30	Continuation of an orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium with a saddle
PCSi	i31	Continuation of a crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium with a saddle
BND	300	Computation of the discontinuity boundary
PEQ	301	Continuation of a pseudo-equilibrium Continuation of a pseudo-saddle-node bifurcation
CTG	302	Continuation of coinciding tangent points
CCY	310	Continuation of a crossing cycle

Table 1: SLIDECONT problem labels, types, and descriptions.

SLIDECONT problem types are composed of three digits with the following meaning: the first digit is either 1 or 2 if the problem refers to vector field $f^{(i)}$ with $i = 1$ or $i = 2$ in the defining system (see Section 5), while it is equal to 3 if the problem does not refer to a particular vector field. The remaining two digits form a progressive number. Thus, SLIDECONT problem types do not overlap with AUTO97 ones, so that, all AUTO97 facilities are accessible through SLIDECONT. In fact, if `IPS` is an AUTO97 problem type, then the rest of the constants file (starting with `SCISTART`) is ignored, the vector field $f^{(1)}$ is assumed to be the right-hand side of the AUTO97 problem, and the user-supplied subroutines `SCBCND`, `SCICND`, and `SCFOPT` assume the role of the analogous AUTO97 subroutines.

By contrast, for SLIDECONT problems, the constants `NBC`, `NINT`, `JAC` are ignored while the significance of the remaining SLIDECONT constants is described below:

`SCISTART` type of initial solution when switching from a boundary-value to an algebraic problem (see Section 5):
-1, start from $u(0)$;
-2, start from $u(1)$;
-3, start from $v(1)$;

`SCIDIFF` order up to which derivatives are provided;

`SCNPSI`, `SCIPSI` — number and numerical labels of monitored test functions (see Section 5 for the list of the test functions for each problem); if the test function k is monitored then its value is assigned to `PAR(20+`k`)`, an overspecified parameter that appears in the output; thus, zeros of the test function k are detected as zeros of `PAR(20+`k`)` and classified by AUTO97 as user-specified solution points (alphabetical code `UZ`, numerical code -4);

`SCNFIXED`, `SCIFIXED` — dummy constants.

SLIDECONT can be run only in command mode through the commands described below. However, for applications involving several computations, a more flexible approach is to use a program (e.g. *make*) for directing recompilation. The commands `@scprep` and `@scexe` are specially useful for this purpose.

`@sc` — Type `@sc <name>` to run SLIDECONT. Starting data, if needed, must be in `q.<name>` and SLIDECONT constants in `sc.<name>`. This is the simplest way to run SLIDECONT.

`@scprep` — Type `@scprep <name>` to run the SLIDECONT preprocessor. SLIDECONT constants must be in `sc.<name>`. The corresponding AUTO97 constants file `r.<name>` and the problem specific Fortran file `scprob.f` (see Section 4) are created.

`@scexe` — Type `@scexe <name>` to produce the executable file `<name>.exe` (see Section 4). SLIDECONT constants must be in `sc.<name>`. The corresponding AUTO97 constants file `r.<name>` is also created.

`@scdat` — Type `@scdat <name>` to convert the user-supplied data file `<name>.dat` into the AUTO97 formatted file `q.<name>`. SLIDECONT constants must be in `sc.<name>`.

The command `@scdat` is necessary for each problem whose defining system consists of a boundary-value problem to start the computation from a numerically known solution not obtained by previous computations or as a solution of a different problem from which automatic switch is not supported. In such cases the user must provide a data (text) file containing numerical data representing the starting solution of the boundary-value problem. Each row in the data file contains NDIM$+1$ numbers, namely the time variable $t \in [0, 1]$ and the AUTO97 state components U(1), $\ldots$, U(NDIM) at t. As detailed in Section 5, when only one vector field is involved in the differential equations of the boundary-value problem (i.e. when NDIM = SCNDIM) the time length of the starting solution must be specified in PAR(11). By contrast, when both vector fields are involved (i.e. when NDIM = 2*SCNDIM), the time lengths of the two connecting parts of the starting solution must be specified in PAR(11) and PAR(12).

7 EXAMPLE

This example illustrates a variety of calculations. System (5) with

$$f^{(1)}(x, \alpha) = \begin{pmatrix} x_1(1 - x_1) - \psi(x_1)x_2 \\ \psi(x_1)x_2 - \alpha_3 x_2 \end{pmatrix}, \tag{37}$$

$$f^{(2)}(x, \alpha) = \begin{pmatrix} x_1(1 - x_1) - \psi(x_1)x_2 \\ \psi(x_1)x_2 - \alpha_3 x_2 - \alpha_4 x_2 \end{pmatrix}, \tag{38}$$

$$\psi(x_1) = \frac{\alpha_1 x_1}{\alpha_2 + x_1}, \tag{39}$$

$$H(x, \alpha) = x_2 - \alpha_5. \tag{40}$$

models an harvested prey-predator community where x_1 and x_2 are prey and predator population densities and harvesting of the predator population, at constant effort (α_4), is allowed only if predator are sufficiently abundant (i.e. if $x_2 > \alpha_5$) (see Kuznetsov *et al.*, 2002; Dercole *et al.*, 2002, for more details). The analysis is performed with respect to α_2 and α_5 for the following other parameter values: $\alpha_1 = 0.3556$, $\alpha_3 = 0.0444$, $\alpha_4 = 0.2067$. The results are shown in Figures 14 and 15.

The directory $SC_DIR/examples/hppc contains a constants file sc.hppc.k for each successive computation ($k = 1, \ldots, 40$), several equations files hppc.f.k, for computations requiring parameter and state initialization in subroutine SCSTPNT (computations not requiring initialization use hppc.f.log), and several data files hppc.dat.k, for manually set up the starting solution of boundary-value problems when automatic switch from previous computations is not supported. Browse the file hppc.f.log to see how the equations have been entered (in subroutine SCFUNC and

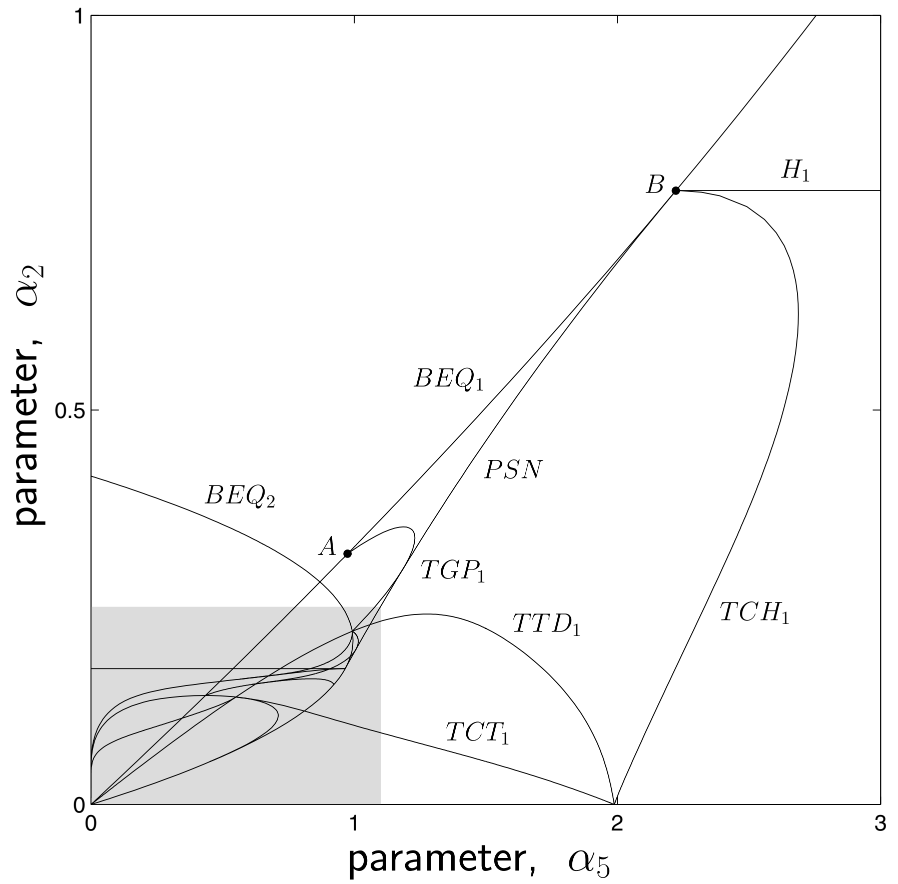

Figure 14: Bifurcation diagram of model (5,37-40) in the (α_5, α_2) plane. Bifurcation curves: $BEQ_{1,2}$ - boundary equilibrium of vector field $f^{(1,2)}$; PSN - pseudo-saddle-node bifurcation; TCH_1 - touching bifurcation of vector field $f^{(1)}$; TCT_1 - crossing orbit of vector fields $f^{(1)}, f^{(2)}$ connecting two tangent points of $f^{(1)}$; TTD_1 - orbit of vector field $f^{(1)}$ connecting a tangent point of $f^{(1)}$ with a tangent point of $f^{(2)}$; TGP_1 - orbit of vector field $f^{(1)}$ connecting a tangent point of $f^{(1)}$ with a pseudo-equilibrium; H_1 - Hopf bifurcation of vector field $f^{(1)}$; Points A and B are codimension-2 bifurcation points detected by SLIDECONT.

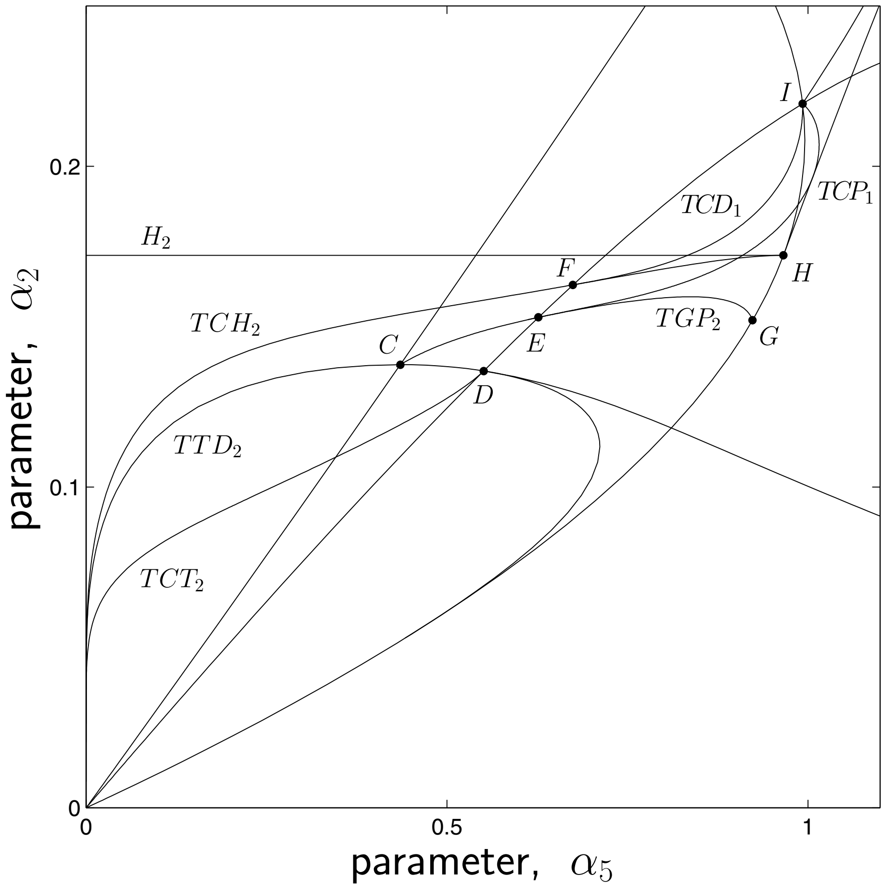

Figure 15: Magnified view of the shaded region in Figure 14. Bifurcation curves: TCH_2 - touching bifurcation of vector field $f^{(2)}$; TCT_2 - crossing orbit of vector fields $f^{(2)}, f^{(1)}$ connecting two tangent points of $f^{(2)}$; TTD_2 - orbit of vector field $f^{(2)}$ connecting a tangent point of $f^{(2)}$ with a tangent point of $f^{(1)}$; TCD_1 - crossing orbit of vector fields $f^{(1)}, f^{(2)}$ connecting a tangent point of $f^{(1)}$ with a tangent point of $f^{(2)}$; TGP_2 - orbit of vector field $f^{(2)}$ connecting a tangent point of $f^{(2)}$ with a pseudo-equilibrium; TCP_1 - crossing orbit of vector fields $f^{(1)}, f^{(2)}$ connecting a tangent point of $f^{(1)}$ with a pseudo-equilibrium; H_2 - Hopf bifurcation of vector field $f^{(2)}$; Points $C - I$ are codimension-2 bifurcation points detected by SLIDECONT.

SCBOUND) and how the starting solution has been set (in subroutine SCSTPNT). In particular, notice that in order to avoid numerical problems when x_1 or x_2 are very small, new state variables (z_1, z_2) with $z_i = \ln(x_i)$, $i = 1, 2$, are introduced.

To execute all prepared computations simply enter `make`. In the following we execute all computations separately by means of SLIDECONT commands (see Section 6). Some differences in the resulting screen outputs are to be expected on different machines. The list of all commands to be typed for each successive computation is reported in Table 2. The four commands of computation 1 are present in all computations, thus, they are reported only once. For $k = 2, \ldots, 40$ the first (last) two commands of computation 1 precede (follow) the listed commands and the equations file hppc.f.k, if not present, must be replaced with hppc.f.log).

k	Command	Action
1	cp sc.hppc.k sc.hppc cp hppc.f.k hppc.f @sc hppc @sv hppc.k	get the constants file get the equations file run SLIDECONT save output to p.hppc.k, q.hppc.k, d.hppc.k
2	cp q.hppc.1 q.hppc	get the starting solution file
3	cp q.hppc.2 q.hppc	get the starting solution file
4	cp q.hppc.2 q.hppc	get the starting solution file
5	cp q.hppc.2 q.hppc	get the starting solution file
6	cp q.hppc.5 q.hppc	get the starting solution file
7	cp q.hppc.5 q.hppc	get the starting solution file
8	cp q.hppc.5 q.hppc	get the starting solution file
9	cp q.hppc.8 q.hppc	get the starting solution file
10	cp q.hppc.8 q.hppc	get the starting solution file
11	cp q.hppc.8 q.hppc	get the starting solution file
12	cp q.hppc.11 q.hppc	get the starting solution file
13	cp hppc.dat.13 hppc.dat @scdat hppc cp q.hppc q.dat.13	get the data file convert the data file into the starting solution file save the starting solution file
14	cp q.dat.13 q.hppc	get the starting solution file
15	cp q.dat.13 q.hppc	get the starting solution file
16	cp hppc.dat.16 hppc.dat @scdat hppc cp q.hppc q.dat.16	get the data file convert the data file into the starting solution file save the starting solution file
17	cp q.dat.16 q.hppc	get the starting solution file
18	cp q.hppc.15 q.hppc	get the starting solution file
19	cp q.hppc.15 q.hppc	get the starting solution file
20	cp hppc.dat.20 hppc.dat @scdat hppc	get the data file convert the data file into the starting solution file

Table 2: (continue)

k	Command	Action
	`cp q.hppc q.dat.20`	save the starting solution file
21	`cp q.hppc.20 q.hppc`	get the starting solution file
22	`cp q.hppc.20 q.hppc`	get the starting solution file
23	`cp q.hppc.1 q.hppc`	get the starting solution file
24	`cp q.hppc.23 q.hppc`	get the starting solution file

25	`cp hppc.dat.25 hppc.dat`
	`@scdat hppc`
	`cp q.hppc q.dat.25`
		get the data file
		convert the data file into the starting solution file
		save the starting solution file

26	`cp q.dat.25 q.hppc`	get the starting solution file
27	`cp q.dat.25 q.hppc`	get the starting solution file

28	`cp hppc.dat.28 hppc.dat`
	`@scdat hppc`
	`cp q.hppc q.dat.28`
		get the data file
		convert the data file into the starting solution file
		save the starting solution file

29	`cp q.dat.28 q.hppc`	get the starting solution file
30	`cp q.hppc.27 q.hppc`	get the starting solution file

31	`cp hppc.dat.31 hppc.dat`
	`@scdat hppc`
	`cp q.hppc q.dat.31`
		get the data file
		convert the data file into the starting solution file
		save the starting solution file

32	`cp q.dat.31 q.hppc`	get the starting solution file

33	`cp hppc.dat.33 hppc.dat`
	`@scdat hppc`
	`cp q.hppc q.dat.33`
		get the data file
		convert the data file into the starting solution file
		save the starting solution file

34	`cp q.hppc.33 q.hppc`	get the starting solution file
35	`cp q.hppc.33 q.hppc`	get the starting solution file
36	`cp q.hppc.35 q.hppc`	get the starting solution file

37	`cp hppc.dat.37 hppc.dat`
	`@scdat hppc`
	`cp q.hppc q.dat.37`
		get the data file
		convert the data file into the starting solution file
		save the starting solution file

38	`cp q.dat.37 q.hppc`	get the starting solution file
39	`cp q.dat.36 q.hppc`	get the starting solution file
40	`cp q.dat.36 q.hppc`	get the starting solution file

Table 2: Command list: the first (last) two commands of computation 1 must precede (follow) the commands listed for $k = 2, \ldots, 40$; the equations file `hppc.f.`k, if not present, must be replaced with `hppc.f.log`; `@sv` is an AUTO97 command.

For $\alpha_2 = 1$ and $\alpha_5 = 3$ (top-right corner of Figure 14) vector field $f^{(1)}$ has a stable focus at $(z_1 = -1.9472, z_2 = 1.0134)$. Starting from this solution, we continue the equilibrium for decreasing values of α_2 (computation 1). We get the following output

```
BR    PT  TY LAB    PAR(2)      ...       U(1)          U(2)        PAR(21)
 1     1  EP   1  1.000000E+00 ... -1.947196E+00  1.013383E+00 -2.450949E-01
 1    10       2  9.642185E-01 ... -1.983634E+00  9.828827E-01 -3.278520E-01
```

```
  1    20       3  8.283327E-01 ... -2.135537E+00  8.532105E-01 -6.528297E-01
  1    23  HB   4  7.780000E-01 ... -2.198225E+00  7.986327E-01 -7.775000E-01
  1    30       5  5.759489E-01 ... -2.498933E+00  5.298372E-01 -1.301344E+00
  1    39  UZ   6  3.300000E-01 ... -3.055859E+00  1.043016E-02 -1.989515E+00
  1    40       7  3.075176E-01 ... -3.126419E+00 -5.676957E-02 -2.055188E+00
  1    50  EP   8  1.512673E-01 ... -3.835903E+00 -7.432052E-01 -2.524413E+00
```

where label `4` (`LAB=4`) indicates a Hopf bifurcation while label `6` is a user output point at $\alpha_2 = 0.33$ (see constants file `sc.hppc.1`). Since vector field $f^{(1)}$ does not depend on α_5, the Hopf bifurcation curve in the (α_5, α_2) plane is a straight line (see curve H_1 in Figure 14), therefore, we skip its numerical computation. By inspection of the output file `q.hppc.1`, one can check that after the Hopf bifurcation the equilibrium is an unstable focus.

Starting from the solution at the user output point (label `6`), we continue the equilibrium for decreasing values of α_5 (computation 2).

```
 BR    PT  TY LAB    PAR(5)     ...      U(1)          U(2)         PAR(21)
  2     1  EP   9  3.000000E+00 ... -3.055859E+00  1.043016E-02 -1.989515E+00
  2    10      10  2.672000E+00 ... -3.055859E+00  1.043016E-02 -1.661515E+00
  2    20      11  1.672000E+00 ... -3.055859E+00  1.043016E-02 -6.615151E-01
  2    27  UZ  12  1.010485E+00 ... -3.055859E+00  1.043016E-02 -3.848495E-09
  2    30      13  7.104847E-01 ... -3.055859E+00  1.043016E-02  3.000000E-01
  2    38  EP  14 -8.951536E-02 ... -3.055859E+00  1.043016E-02  1.100000E+00
```

Label `12` indicates a zero of test function 1, i.e. a boundary equilibrium bifurcation. Starting from this solution, we continue the bifurcation in the plane (α_5, α_2) forward (computation 3) and backward (computation 4), thus obtaining the following outputs (see curve BEQ_1 in Figure 14)

```
 BR    PT  TY LAB    PAR(5)     ...    PAR(2)        PAR(21)       PAR(23)
  3     1  EP  15  1.010485E+00 ...  3.300000E-01  3.594923E-02 -2.088672E-01
  3    10      16  1.049661E+00 ...  3.434875E-01  3.486694E-02 -2.169648E-01
  3    20      17  1.248717E+00 ...  4.129281E-01  2.929477E-02 -2.581099E-01
  3    30      18  1.807860E+00 ...  6.169031E-01  1.292703E-02 -3.736846E-01
  3    36  UZ  19  2.222500E+00 ...  7.780000E-01  4.942303E-11 -4.593907E-01
  3    40      20  2.540487E+00 ...  9.083088E-01 -1.045648E-02 -5.251187E-01
  3    43  UZ  21  2.754905E+00 ...  1.000000E+00 -1.781413E-02 -5.694389E-01
  3    46  EP  22  3.000040E+00 ...  1.109123E+00 -2.657057E-02 -6.201083E-01

 BR    PT  TY LAB    PAR(5)     ...    PAR(2)        PAR(21)       PAR(23)
  3     1  EP  15  1.010485E+00 ...  3.300000E-01  3.594923E-02 -2.088672E-01
  3    20      16  8.341623E-01 ...  2.699916E-01  4.076454E-02 -1.724214E-01
  3    40      17  1.499880E-01 ...  4.699132E-02  5.865893E-02 -3.100253E-02
  3    60      18  2.038495E-03 ...  6.344370E-04  6.237879E-02 -4.213568E-04
  3    80      19  2.328653E-05 ...  7.246702E-06  6.243025E-02 -4.813317E-06
  3   100  EP  20  2.659996E-07 ...  8.277820E-08  6.243083E-02 -5.498200E-08
```

where label `19` in the forward computation identifies a codimension-2 bifurcation, namely a boundary-Hopf bifurcation (see point B in Figure 14), as a zero of test function 1.

Starting again from the solution at label `12`, we continue the pseudo-saddle colliding with the unstable focus at the boundary equilibrium bifurcation for increasing values of α_5 (computation 5). We get the following output

```
BR    PT  TY LAB    PAR(5)      ...    PAR(21)       PAR(22)       PAR(23)
 3     1  EP  15  1.010485E+00 ...  3.873642E-12  1.000000E+00 -2.462115E-13
 3    10      16  1.015678E+00 ...  1.276639E-02  9.872336E-01  2.680182E-03
 3    20      17  1.044230E+00 ...  8.175974E-02  9.182403E-01  1.764722E-02
 3    30      18  1.139885E+00 ...  3.051888E-01  6.948112E-01  7.190681E-02
 3    40  LP  19  1.243602E+00 ...  6.518473E-01  3.481527E-01  1.675591E-01
 3    50  EP  20  9.043294E-01 ...  9.447629E-01  5.523711E-02  1.765997E-01
```

where label `19` indicates a pseudo-saddle-node bifurcation, whose forward and backward continuations (computations 6 and 7) are reported below (see curve PSN in Figure 14).

```
BR    PT  TY LAB    PAR(5)      ...    PAR(2)        PAR(21)       PAR(22)
19    10      21  1.271177E+00 ...  3.446645E-01  6.236340E-01  3.763660E-01
19    20      22  1.387619E+00 ...  4.049019E-01  5.139212E-01  4.860788E-01
19    30      23  1.712144E+00 ...  5.605619E-01  2.696336E-01  7.303664E-01
19    40      24  2.214239E+00 ...  7.746926E-01  3.606508E-03  9.963935E-01
19    41  UZ  25  2.222500E+00 ...  7.780000E-01 -1.877900E-08  1.000000E+00
19    50  EP  26  2.535823E+00 ...  8.991986E-01 -1.234942E-01  1.123494E+00
```

```
BR    PT  TY LAB    PAR(5)      ...    PAR(2)        PAR(21)       PAR(22)
19    10      21  1.216501E+00 ...  3.154283E-01  6.805050E-01  3.194950E-01
19    20      22  1.111669E+00 ...  2.574730E-01  8.010583E-01  1.989417E-01
19    28  UZ  23  9.660815E-01 ...  1.722433E-01  1.000000E+00  4.546613E-10
19    30      24  9.118196E-01 ...  1.388469E-01  1.086073E+00 -8.607326E-02
19    39  UZ  25  7.030371E-01 ... -1.975441E-10  1.505564E+00 -5.055636E-01
19    40      26  6.859732E-01 ... -1.221040E-02  1.548096E+00 -5.480957E-01
19    50  EP  27  5.351082E-01 ... -1.275678E-01  2.008671E+00 -1.008671E+00
```

Notice that point B of Figure 14 is detected again during the forward computation (zero of test function 1 at label `25`), while label `23` in the backward computation identifies another codimension-2 bifurcation, namely a boundary-pseudo-saddle-node bifurcation (see point H in Figure 15), as a zero of test function 2.

Starting from the solution at label `19` of computation 5, we now continue the pseudo-node colliding with the pseudo-saddle at the pseudo-saddle-node bifurcation for decreasing values of α_5 (computation 8)

```
BR    PT  TY LAB    PAR(5)      ...    PAR(21)       PAR(22)       PAR(23)
 4     1  EP  21  1.243602E+00 ...  6.518473E-01  3.481527E-01  1.675591E-01
 4    10      22  1.243015E+00 ...  6.700032E-01  3.299968E-01  1.721447E-01
 4    20      23  1.229312E+00 ...  7.345034E-01  2.654966E-01  1.866364E-01
 4    30      24  1.113411E+00 ...  8.605471E-01  1.394529E-01  1.980480E-01
 4    40      25  8.398532E-01 ...  9.617226E-01  3.827737E-02  1.669528E-01
 4    45  UZ  26  6.538504E-01 ...  1.000000E+00 -2.993796E-13  1.351509E-01
 4    50  EP  27  4.536362E-01 ...  1.030487E+00 -3.048749E-02  9.662532E-02
```

detecting a boundary equilibrium of vector field $f^{(2)}$ (zero of test function 2 at label 26). The forward and backward continuations of the boundary equilibrium bifurcation (computations 9 and 10) give the following outputs (see curve BEQ_2 in Figure 14)

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(21)	PAR(22)
5	1	EP	28	6.538504E-01	...	3.300000E-01	0.000000E+00	-2.455628E-02
5	20		29	7.329652E-01	...	3.149616E-01	0.000000E+00	-3.247275E-02
5	39	LP	30	9.956193E-01	...	2.080844E-01	-7.346738E-02	0.000000E+00
5	40		31	9.945728E-01	...	2.013383E-01	-5.963918E-02	0.000000E+00
5	45	UZ	32	9.660815E-01	...	1.722433E-01	3.272406E-14	0.000000E+00
5	60		33	5.282595E-01	...	6.551756E-02	2.187670E-01	0.000000E+00
5	80		34	1.156572E-02	...	1.212148E-03	0.000000E+00	1.284747E-01
5	100	EP	35	1.323048E-04	...	1.382631E-05	0.000000E+00	1.275443E-01

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(21)	PAR(22)
5	1	EP	28	6.538504E-01	...	3.300000E-01	0.000000E+00	-2.455628E-02
5	20		29	5.809495E-01	...	3.423746E-01	0.000000E+00	-1.929347E-02
5	40		30	2.239442E-01	...	3.912780E-01	0.000000E+00	-4.942120E-03
5	60		31	1.573981E-03	...	4.160043E-01	0.000000E+00	-2.919611E-05
5	80		32	2.820199E-06	...	4.161686E-01	0.000000E+00	-5.225509E-08
5	100	EP	33	5.053125E-09	...	4.161689E-01	0.000000E+00	-9.362848E-11

where label 32 in the forward computation identifies a second boundary-Hopf codimension-2 bifurcation (see again point H in Figure 15), as a zero of test function 1.

Restarting from the solution at label 26 of computation 8, we continue the stable focus of vector field $f^{(2)}$ for decreasing values of α_2 (computation 11)

BR	PT	TY	LAB	PAR(2)	...	U(1)	U(2)	PAR(21)
5	1	EP	28	3.300000E-01	...	-2.319984E-01	-4.248767E-01	3.108069E-12
5	10		29	3.254829E-01	...	-2.457811E-01	-3.875660E-01	2.485643E-02
5	20		30	3.086316E-01	...	-2.989428E-01	-2.702923E-01	1.093060E-01
5	30		31	2.574242E-01	...	-4.803659E-01	-6.225582E-02	2.857921E-01
5	40		32	1.922810E-01	...	-7.721330E-01	-1.017499E-02	3.360262E-01
5	44	HB	33	1.722433E-01	...	-8.821832E-01	-3.450710E-02	3.122311E-01
5	50	EP	34	1.324055E-01	...	-1.145222E+00	-1.462682E-01	2.100755E-01

detecting a Hopf bifurcation (label 33). For the same arguments used before, the Hopf bifurcation curve of vector field $f^{(2)}$ is a straight line (see curve H_2 in Figure 15), therefore, we avoid its numerical computation.

We now continue the stable limit cycle originating from the Hopf bifurcation (which can be shown to be supercritical by means of suitable software, e.g. CONTENT (Kuznetsov & Levitin, 1995-1997); see Kuznetsov (1998) for an analytical proof) for decreasing values of α_2 (computation 12), obtaining the following output

BR	PT	TY	LAB	PAR(2)	...	PERIOD	PAR(21)
33	10		35	1.721114E-01	...	3.025804E+01	2.900965E-01
33	20		36	1.708702E-01	...	3.070288E+01	2.307981E-01
33	30		37	1.674898E-01	...	3.207707E+01	1.330312E-01
33	40	UZ	38	1.622259E-01	...	3.485695E+01	-6.206147E-14
33	50	EP	39	1.578459E-01	...	3.797262E+01	-1.138060E-01

where label 38 indicates a touching bifurcation (zero of test function 1), whose forward and backward continuations (computations 13 and 14) give the following outputs (see curve TCH_2 in Figure 15)

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(11)	PAR(21)
1	10		2	6.595797E-01	...	1.624516E-01	3.471777E+01	-1.222616E-01
1	20		3	7.206470E-01	...	1.648781E-01	3.334167E+01	-1.059870E-01
1	30		4	8.018395E-01	...	1.680604E-01	3.182626E+01	-7.825040E-02
1	40		5	9.175517E-01	...	1.716838E-01	3.040809E+01	-2.658453E-02
1	43	UZ	6	9.660815E-01	...	1.722433E-01	3.021241E+01	1.567003E-08
1	50	EP	7	1.033208E+00	...	1.699106E-01	3.106718E+01	4.517847E-02

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(11)	PAR(21)
1	40		2	5.446540E-02	...	1.119661E-01	6.807163E+01	-3.704335E-02
1	80		3	1.283142E-05	...	3.666983E-02	1.751001E+02	-1.170034E-05
1	120		4	7.380721E-11	...	1.841621E-02	3.376757E+02	-7.054111E-11
1	160		5	3.091705E-17	...	1.149044E-02	5.364043E+02	-3.006343E-17
1	200	EP	6	3.460031E-24	...	8.155765E-03	7.531864E+02	-3.392223E-24

where point H of Figure 14 is detected again during the forward computation (zero of test function 1 at label 6).

Notice that SLIDECONT does not support the automatic switch from cycle to touching continuation. Thus, the user must provide a data file (`hppc.dat.13`) which specifies numerically the starting solution. This file can be easily constructed from the solution at label 38 of the output file (`q.hppc.12`) produced by cycle continuation. Though not explicitly pointed out in the following, a similar remark holds for each non-automatically supported switch between problems.

Starting from the grazing cycle at label 38, we continue the sliding cycle originating from the touching bifurcation (i.e. an orbit of vector field $f^{(2)}$ connecting a tangent point of $f^{(2)}$ with the boundary Σ) for decreasing values of α_2 (computation 15).

BR	PT	TY	LAB	PAR(2)	...	PAR(21)	PAR(24)	PAR(25)
1	10		2	1.621938E-01	...	-1.235773E-01	1.314635E-01	-1.718989E-02
1	20		3	1.600212E-01	...	-1.232510E-01	1.009534E-01	-1.382299E-02
1	26	UZ	4	1.541669E-01	...	-1.218470E-01	6.563355E-02	-3.459552E-12
1	30		5	1.421640E-01	...	-1.162381E-01	2.461673E-02	2.657905E-02
1	34	UZ	6	1.294416E-01	...	-1.053393E-01	-8.433334E-11	5.217604E-02
1	40		7	9.763977E-02	...	-4.288862E-02	-2.107341E-02	1.151668E-01
1	48	UZ	8	8.870509E-02	...	-1.083923E-02	-1.508937E-09	1.154560E-01
1	50	EP	9	8.759474E-02	...	-6.199531E-03	2.765735E-02	9.150117E-02

Label 4 indicates a pseudo-homoclinic bifurcation, i.e. the presence of an orbit of vector field $f^{(2)}$ connecting a tangent point of $f^{(2)}$ with a pseudo-equilibrium (zero of test function 5), while labels 6 and 8 indicate buckling bifurcations, i.e. the presence of an orbit of vector field $f^{(2)}$ connecting a tangent point of $f^{(2)}$ with a tangent point of $f^{(1)}$ (zeros of test function 4).

Starting from the solutions at labels 4 and 6, we continue the corresponding bifurcations, forward and backward (computations 16-19), thus obtaining the outputs reported below (see curves TGP_2 and TTD_2 in Figure 15).

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(23)	PAR(24)
1	20		2	6.949533E-01	...	1.558770E-01	4.414444E-01	5.585556E-01
1	40		3	7.647598E-01	...	1.582007E-01	3.260036E-01	6.739964E-01
1	60		4	8.372856E-01	...	1.593251E-01	2.081665E-01	7.918335E-01
1	80		5	9.055146E-01	...	1.565728E-01	7.139396E-02	9.286060E-01
1	89	UZ	6	9.232856E-01	...	1.519973E-01	3.402056E-08	1.000000E+00
1	92	LP	7	9.250491E-01	...	1.493514E-01	-2.847298E-02	1.028473E+00
1	100	EP	8	9.086873E-01	...	1.388136E-01	-1.091954E-01	1.109195E+00

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(23)	PAR(24)
1	20		2	6.116167E-01	...	1.521635E-01	5.948873E-01	4.051127E-01
1	40		3	5.579163E-01	...	1.491606E-01	7.070975E-01	2.929025E-01
1	60		4	5.098992E-01	...	1.458120E-01	8.172831E-01	1.827169E-01
1	80		5	4.671524E-01	...	1.419413E-01	9.215330E-01	7.846695E-02
1	97	UZ	6	4.352424E-01	...	1.381713E-01	1.000000E+00	-4.520598E-11
1	100	EP	7	4.299233E-01	...	1.374435E-01	1.012809E+00	-1.280878E-02

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(21)	PAR(23)
1	19	LP	10	7.116378E-01	...	1.125535E-01	-4.898224E-02	7.000173E-01
1	20		11	7.087202E-01	...	1.077686E-01	-3.814412E-02	7.526075E-01
1	40		12	4.192168E-01	...	4.975711E-02	-1.258007E-09	6.829214E-01
1	60		13	4.127401E-01	...	4.887008E-02	-5.921960E-12	6.749382E-01
1	80		14	4.125619E-01	...	4.884575E-02	-4.718599E-12	6.747172E-01
1	100	EP	15	4.125559E-01	...	4.884493E-02	-4.681977E-12	6.747098E-01

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(21)	PAR(23)
1	20		10	5.269382E-01	...	1.369660E-01	-1.416694E-01	1.142382E-01
1	25	UZ	11	4.352424E-01	...	1.381713E-01	-1.474667E-01	-4.553584E-11
1	31	UZ	12	2.000000E-01	...	1.296926E-01	-1.054430E-01	-1.003185E-01
1	40		13	2.858149E-02	...	9.188168E-02	-2.134219E-02	-2.544000E-02
1	60		14	8.186596E-04	...	5.361793E-02	-7.118798E-04	-8.085071E-04
1	80		15	1.437650E-05	...	3.603654E-02	-1.313102E-05	-1.430080E-05
1	100	EP	16	1.410204E-07	...	2.622691E-02	-1.321332E-07	-1.404925E-07

Labels 6 in computations 16 and 17 correspond to the codimension-2 bifurcation points G and C in Figure 15, respectively (zeros of test functions 3 and 4). Point C is also detected during computation 19 (zero of test function 3 at label 11).

Starting from the user output point at label 12 of computation 19, we continue the sliding cycle originating from the buckling bifurcation (i.e. a crossing orbit of vector fields $f^{(2)}, f^{(1)}$ connecting a tangent point of $f^{(2)}$ with the boundary Σ) for decreasing values of α_2 (computation 20)

BR	PT	TY	LAB	PAR(2)	...	PAR(11)	PAR(12)	PAR(25)
1	20		2	1.252957E-01	...	4.505731E+01	2.278452E+00	-3.636810E-02
1	40		3	9.439926E-02	...	3.804779E+01	1.648036E+01	-5.383023E-04
1	41	UZ	4	9.382930E-02	...	3.794630E+01	1.677800E+01	2.346717E-12
1	50	EP	5	5.932347E-02	...	3.288170E+01	4.364691E+01	1.626999E-02

detecting a crossing-crossing bifurcation (zero of test function 5 at label 4), whose forward and backward continuations (computations 21 and 22) give the following outputs (see curve TCT_2 in Figure 15)

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(21)	PAR(24)
1	20		6	2.344789E-01	...	9.748667E-02	-1.206170E-01	-1.027748E-02
1	40		7	3.290559E-01	...	1.074862E-01	-1.387992E-01	-1.391346E-02
1	60		8	4.468088E-01	...	1.208180E-01	-1.444213E-01	-1.516709E-02
1	79	UZ	9	5.508304E-01	...	1.362127E-01	-1.377688E-01	-2.234463E-13
1	80		10	5.542291E-01	...	1.370111E-01	-1.374837E-01	1.534032E-03
1	88	LP	11	5.665272E-01	...	1.423028E-01	-1.371198E-01	1.328356E-02
1	100	EP	12	5.455609E-01	...	1.465457E-01	-1.412111E-01	2.352369E-02

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(21)	PAR(24)
1	20		6	1.489314E-01	...	8.814077E-02	-9.109173E-02	-6.585785E-03
1	40		7	1.079530E-02	...	5.891426E-02	-9.060368E-03	-4.789267E-04
1	60		8	1.593209E-04	...	3.796452E-02	-1.447172E-04	-7.068183E-06
1	80		9	1.261165E-06	...	2.691387E-02	-1.179599E-06	-5.594704E-08
1	100	EP	10	6.527067E-09	...	2.044703E-02	-6.206382E-09	-2.895371E-10

where label 9 of the forward computation corresponds to the codimension-2 bifurcation points D of Figure 15 (zero of test function 4).

We now restart from the (supercritical) Hopf bifurcation of vector field $f^{(1)}$ (label 4 of computation 1) and continue the stable limit cycle originating from it for decreasing values of α_2 (computation 23)

BR	PT	TY	LAB	PAR(2)	...	PERIOD	PAR(21)
4	10		9	7.775373E-01	...	3.382240E+01	-7.304723E-01
4	20		10	7.580640E-01	...	3.452006E+01	-5.106371E-01
4	30		11	6.639275E-01	...	3.863955E+01	-3.228900E-01
4	40		12	4.913923E-01	...	5.074448E+01	-3.763009E-01
4	49	UZ	13	3.300000E-01	...	7.187532E+01	-5.593070E-01
4	50	EP	14	3.149638E-01	...	7.478807E+01	-5.794996E-01

and, starting from the user output at label 13, for decreasing values of α_5 (computation 24)

```
BR    PT  TY LAB    PAR(5)     ...     PERIOD       PAR(21)
 4    10       15  2.489000E+00 ...  7.187532E+01 -4.829326E-02
 4    11  UZ   16  2.440707E+00 ...  7.187532E+01 -4.636845E-08
 4    14  EP   17 -5.592936E-01 ...  7.187532E+01  3.000001E+00
```

thus detecting, at label `16`, a touching bifurcation (zero of test function 1), whose forward and backward continuations (computations 25 and 26) are reported below (see curve TCH_1 in Figure 14).

```
BR    PT  TY LAB    PAR(5)     ...    PAR(2)        PAR(11)       PAR(21)
 1    20        2  2.470800E+00 ...  3.529198E-01  6.786258E+01  3.036789E+00
 1    40        3  2.591390E+00 ...  4.567531E-01  5.418234E+01  2.152826E+00
 1    59  LP    4  2.687617E+00 ...  6.217073E-01  4.097355E+01  1.166444E+00
 1    60        5  2.686736E+00 ...  6.345777E-01  4.022545E+01  1.096527E+00
 1    72  UZ    6  2.222500E+00 ...  7.780000E-01  3.380637E+01  4.727138E-08
 1    80        7  1.557826E+00 ...  7.178726E-01  3.611587E+01 -3.462366E-01
 1   100  EP    8  6.261067E-01 ...  4.943754E-01  5.046802E+01 -3.351821E-01
```

```
BR    PT  TY LAB    PAR(5)     ...    PAR(2)        PAR(11)       PAR(21)
 1    20        2  2.400260E+00 ...  3.001236E-01  7.791814E+01  3.676390E+00
 1    40        3  2.187533E+00 ...  1.492103E-01  1.412397E+02  7.839486E+00
 1    60        4  2.033157E+00 ...  3.680030E-02  5.120719E+02  3.293419E+01
 1    80        5  1.995369E+00 ...  6.283005E-03  2.912472E+03  1.952117E+02
 1   100  EP    6  1.991388E+00 ...  2.558442E-03  7.129293E+03  4.803744E+02
```

As already done for vector field $f^{(2)}$, we continue the sliding cycle originating from the touching bifurcation (label `16` of computation 24) for decreasing values of α_5 (computation 27)

```
BR    PT  TY LAB    PAR(5)     ...    PAR(21)       PAR(24)       PAR(25)
 1    40        2  2.280425E+00 ...  2.731019E+00 -2.574786E-01 -7.654774E-01
 1    80        3  1.817974E+00 ...  1.384364E+00 -9.106789E-02 -1.917820E-01
 1   103  UZ    4  1.500000E+00 ...  6.924415E-01 -6.796604E-02 -6.244388E-02
 1   119  UZ    5  1.227528E+00 ...  2.512482E-01 -1.161229E-01  2.398169E-13
 1   120        6  1.208573E+00 ...  2.257655E-01 -1.234690E-01  1.602614E-03
 1   130  UZ    7  1.029396E+00 ...  1.835839E-02 -2.029555E-01 -4.029265E-13
 1   132  UZ    8  1.010485E+00 ... -7.333084E-09 -2.088672E-01  7.869359E-11
 1   160        9  8.016753E-01 ... -1.578608E-01 -2.007026E-01  1.823302E-02
 1   200  EP   10  6.617239E-01 ... -2.176357E-01 -1.661517E-01  2.597207E-02
```

thus detecting a pseudo-homoclinic bifurcation (zero of test function 5 at labels `5` and `7`), whose forward and backward continuations (computations 28 and 29) are reported below (see curve TGP_1 in Figure 14)

```
BR    PT  TY LAB    PAR(5)     ...    PAR(2)        PAR(23)       PAR(24)
 1    10        2  1.228662E+00 ...  3.318674E-01  5.295557E-01  4.704443E-01
 1    15  LP    3  1.230382E+00 ...  3.377473E-01  4.853467E-01  5.146533E-01
 1    20        4  1.215563E+00 ...  3.492857E-01  3.573893E-01  6.426107E-01
 1    30        5  1.023114E+00 ...  3.286684E-01  4.037209E-02  9.596279E-01
 1    32  UZ    6  9.747807E-01 ...  3.177575E-01 -3.197882E-08  1.000000E+00
 1    40        7  9.223123E-01 ...  3.047806E-01 -3.696894E-02  1.036969E+00
 1    50  EP    8  7.861535E-01 ...  2.670094E-01 -1.087761E-01  1.108776E+00
```

```
BR    PT  TY LAB    PAR(5)      ...     PAR(2)        PAR(23)       PAR(24)
 1    10        2  1.225477E+00 ...  3.272807E-01  5.601155E-01  4.398845E-01
 1    20        3  1.193354E+00 ...  3.030474E-01  6.913037E-01  3.086963E-01
 1    30        4  1.078069E+00 ...  2.503790E-01  9.004122E-01  9.958777E-02
 1    37  UZ    5  9.925992E-01 ...  2.195448E-01  1.000000E+00 -1.173486E-08
 1    40        6  9.496145E-01 ...  2.053591E-01  1.042223E+00 -4.222287E-02
 1    50  EP    7  8.116777E-01 ...  1.639623E-01  1.155483E+00 -1.554826E-01
```

and identify the codimension-2 bifurcation points A and I in Figures 14 and 15 (zeros of test functions 3 and 4 at labels 6 and 5 in the forward and backward computations, respectively).

Continuing again the sliding cycle of computation 27, starting from the user output point at label 4 for decreasing values of α_2 (computation 30)

```
BR    PT  TY LAB    PAR(2)      ...     PAR(21)       PAR(24)       PAR(25)
 1    10       11  3.271804E-01 ...  7.101235E-01 -6.525920E-02 -6.392786E-02
 1    20       12  3.002607E-01 ...  8.962322E-01 -4.289021E-02 -7.760583E-02
 1    30       13  2.515214E-01 ...  1.337687E+00 -1.366180E-02 -9.919165E-02
 1    35  UZ   14  2.231573E-01 ...  1.685455E+00  5.252412E-13 -1.109843E-01
 1    40       15  1.846355E-01 ...  2.331851E+00  1.786597E-02 -1.286792E-01
 1    50  EP   16  1.402797E-01 ...  3.521479E+00  4.036113E-02 -1.545451E-01
```

we detect a buckling bifurcation (zero of test function 4 at label 14), whose forward and backward continuations (computations 31 and 32) give the following outputs (see curve TTD_1 in Figure 15)

```
BR    PT  TY LAB    PAR(5)      ...     PAR(2)        PAR(21)       PAR(23)
 1    20        2  1.567530E+00 ...  2.104887E-01  2.112353E+00  4.451766E-01
 1    40        3  1.836869E+00 ...  1.274579E-01  6.434676E+00  1.492042E+00
 1    60        4  1.966666E+00 ...  3.513726E-02  3.229890E+01  9.702340E+00
 1    80        5  1.985015E+00 ...  6.150573E-03  1.973830E+02  6.499090E+01
 1   100  EP    6  1.987208E+00 ...  2.536230E-03  4.825624E+02  1.607349E+02
```

```
BR    PT  TY LAB    PAR(5)      ...     PAR(2)        PAR(21)       PAR(23)
 1    40        2  1.365826E+00 ...  2.385952E-01  1.113818E+00  2.342645E-01
 1    69  UZ    3  9.925992E-01 ...  2.195448E-01  4.350672E-01  6.321302E-13
 1    80        4  5.053154E-01 ...  1.258643E-01  1.350968E-01 -1.404886E-01
 1   120        5  2.603490E-02 ...  6.830941E-03  4.870017E-03 -1.523831E-02
 1   160        6  1.528243E-03 ...  4.016967E-04  2.812101E-04 -9.191889E-04
 1   200  EP    7  9.029147E-05 ...  2.373545E-05  1.659848E-05 -5.439311E-05
```

where label 3 in the backward computation corresponds again to the codimension-2 bifurcation point I in Figure 15 (zero of test function 3).

We now continue the sliding cycle originating from the buckling bifurcation (label 14 of computation 30) for decreasing values of α_2 (computation 33)

BR	PT	TY	LAB	PAR(2)	...	PAR(11)	PAR(12)	PAR(25)
1	20		2	2.186813E-01	...	6.274620E+01	1.548913E-01	3.079685E-01
1	40		3	1.886377E-01	...	7.425575E+01	1.047551E+00	2.926075E-01
1	60		4	1.047014E-01	...	1.361479E+02	2.026112E+00	1.839764E-01
1	72	UZ	5	5.451434E-02	...	2.573628E+02	1.846822E+00	-3.572858E-11
1	80		6	2.362097E-02	...	5.865844E+02	1.617635E+00	-6.612973E-02
1	100	EP	7	4.001519E-03	...	3.435921E+03	1.474053E+00	-6.660000E-02

thus detecting a crossing-crossing bifurcation (zero of test function 5 at label 5), whose forward and backward continuations (computations 34 and 35) are reported below (see curve TCT_1 in Figure 15).

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(21)	PAR(24)
1	20		8	1.529843E+00	...	5.165400E-02	1.258176E+01	1.015954E-01
1	40		9	1.732970E+00	...	3.089305E-02	2.852713E+01	1.197986E-01
1	60		10	1.923859E+00	...	8.446400E-03	1.344470E+02	1.389841E-01
1	80		11	1.966981E+00	...	2.881643E-03	4.158637E+02	1.436932E-01
1	100	EP	12	1.975804E+00	...	1.727216E-03	7.013896E+02	1.446740E-01

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(21)	PAR(24)
1	80		8	1.347257E+00	...	6.866427E-02	6.892341E+00	8.649928E-02
1	160		9	9.969313E-01	...	1.005366E-01	2.093790E+00	5.951787E-02
1	173	UZ	10	9.000000E-01	...	1.098597E-01	1.408597E+00	5.235699E-02
1	197	UZ	11	5.508304E-01	...	1.362127E-01	1.530732E-01	1.377292E-09
1	203	UZ	12	4.259264E-01	...	1.381676E-01	-8.926302E-03	-9.411345E-02
1	240		13	3.182449E-01	...	1.363382E-01	-8.087732E-02	-7.991131E-02
1	320		14	1.766939E-01	...	1.274771E-01	-9.726349E-02	-4.436784E-02
1	400	EP	15	2.852687E-02	...	9.184912E-02	-2.539581E-02	-7.163096E-03

Labels 11 and 12 of the backward computation correspond again to the codimension-2 bifurcation points D and C of Figure 15, respectively (zeros of test functions 4 and 1).

Starting from the user output point at label 10, we continue the sliding cycle originating from the crossing-crossing bifurcation (i.e. a crossing orbit of vector fields $f^{(1)}, f^{(2)}$ connecting a tangent point of $f^{(1)}$ with the boundary Σ) for increasing values of α_2 (computation 36).

BR	PT	TY	LAB	PAR(2)	...	PAR(25)	PAR(26)	PAR(27)
1	20		16	1.148059E-01	...	9.581688E-03	-1.764483E-01	-2.305742E-01
1	40		17	1.296047E-01	...	4.379791E-02	-1.422321E-01	-1.455009E-01
1	60		18	1.480265E-01	...	9.321812E-02	-9.281188E-02	-6.158865E-02
1	80		19	1.696015E-01	...	1.506270E-01	-3.540300E-02	-2.172040E-03
1	82	UZ	20	1.709855E-01	...	1.541726E-01	-3.185738E-02	5.042737E-12
1	99	LP	21	1.803463E-01	...	1.860300E-01	-1.602115E-12	7.560075E-03
1	100	EP	22	1.803463E-01	...	1.860306E-01	6.012778E-07	7.559976E-03

Label 20 indicates a pseudo-homoclinic bifurcation, i.e. the presence of a crossing orbit of vector fields $f^{(1)}, f^{(2)}$ connecting a tangent point of $f^{(1)}$ with a pseudo-equilibrium (zero of test function 7), while label 21 indicates the presence of a crossing orbit of vector fields $f^{(1)}, f^{(2)}$ connecting a tangent point of $f^{(1)}$ with a tangent point of $f^{(2)}$ (zero of test function 6). The forward and backward continuations of these bifurcations (computations 37-40) give the following outputs (see curves TCP_1 and TCD_1 in Figure 15)

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(13)	PAR(24)
1	100		2	7.666273E-01	...	1.596931E-01	3.329570E-01	2.406895E-02
1	200		3	8.497330E-01	...	1.655771E-01	2.263359E-01	2.844972E-02
1	300		4	9.586979E-01	...	1.807751E-01	1.127816E-01	2.686107E-02
1	400		5	1.014589E+00	...	2.036729E-01	4.234355E-02	1.589137E-02
1	500	EP	6	9.959791E-01	...	2.188215E-01	2.743327E-03	1.225193E-03

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(13)	PAR(24)
1	20		2	6.927891E-01	...	1.559834E-01	4.466410E-01	1.515276E-02
1	40		3	6.626477E-01	...	1.545961E-01	4.987071E-01	9.363765E-03
1	59	UZ	4	6.266266E-01	...	1.529062E-01	5.655461E-01	5.180943E-10
1	60		5	6.218126E-01	...	1.526724E-01	5.748701E-01	-1.496514E-03
1	80		6	5.361943E-01	...	1.477800E-01	7.561932E-01	-3.996061E-02
1	100	EP	7	4.857828E-01	...	1.437775E-01	8.757975E-01	-6.951435E-02

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(24)	PAR(25)
1	20		23	9.377489E-01	...	1.877745E-01	2.123627E-02	-2.525118E-02
1	40		24	9.806271E-01	...	2.032837E-01	1.348635E-02	-7.290409E-03
1	60		25	9.909409E-01	...	2.126132E-01	6.600286E-03	-2.048922E-03
1	78	LP	26	9.926568E-01	...	2.184607E-01	1.135814E-03	-2.311605E-04
1	80		27	9.926395E-01	...	2.190535E-01	5.199800E-04	-1.009071E-04
1	82	UZ	28	9.925992E-01	...	2.195448E-01	-1.556742E-10	2.901891E-11
1	100		29	9.911574E-01	...	2.240440E-01	-5.205476E-03	6.448560E-04
1	120		30	9.884717E-01	...	2.278735E-01	-1.036173E-02	8.418056E-04
1	140		31	9.855384E-01	...	2.308456E-01	-1.490730E-02	8.171753E-04
1	160		32	9.827122E-01	...	2.332261E-01	-1.893404E-02	7.166529E-04
1	180		33	9.801038E-01	...	2.351891E-01	-2.253373E-02	5.998603E-04
1	200	EP	34	9.777304E-01	...	2.368485E-01	-2.578377E-02	4.898906E-04

BR	PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(24)	PAR(25)
1	20		23	8.605149E-01	...	1.750106E-01	2.328499E-02	-5.649530E-02
1	40		24	7.589211E-01	...	1.669317E-01	1.650420E-02	-9.395186E-02
1	60		25	6.753262E-01	...	1.630739E-01	2.662421E-04	-1.184496E-01
1	61	UZ	26	6.744013E-01	...	1.630373E-01	6.915083E-10	-1.186810E-01
1	80		27	6.100481E-01	...	1.605792E-01	-2.556144E-02	-1.324710E-01
1	100	EP	28	5.644581E-01	...	1.588235E-01	-5.343912E-02	-1.394071E-01

and identify the codimension-2 bifurcation points E, I and F of Figure 15 (zeros of test functions 4, 5, and 4 at labels 4, 28, and 26 of computations 38, 39, and 40, respectively).

8 Conclusions and future work

In this paper, we have presented the new software SlideCont for sliding bifurcation analysis of Filippov systems, that is based on Auto97. There are several direction in which this work could be extended. First of all, there are interesting global sliding bifurcations in n-dimensional ($n > 2$) Filippov systems, which involve multiple sliding and, therefore, are not supported by SlideCont. As for the planar case ($n = 2$), more systematic detection of codimension 2 local and global bifurcations and branch switching at these bifurcations require further implementation efforts.

Another open field of research and software development is the computation of sliding segments of (periodic) orbits as solutions to differential-algebraic equations (see Ascher & Spiteri (1994)).

It should be noted, that SlideCont does not support time-integration of Filippov systems. Such integration should be based on automatic switching from the computation of an orbit of $f^{(i)}$ to the integrating of the differential-algebraic system (4) on the sliding manifold.

Finally, SlideCont runs only in command mode of Auto97 and has no GUI. An ambitious goal could be the development of a software system that supports all mentioned above computational tasks for n-dimensional Filippov systems in one integrated user-friendly graphic environment.

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