BIRKHOFF INTERPOLATION WITH RECTANGULAR SETS OF NODES

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Abstract. Although it is important both in theory as well as in applications, a theory of Birkhoff interpolation with main emphasis on the shape of the set of nodes is still missing. Here we explain the problem, and then we concentrate on one of the simplest shapes ("rectangular") in the case of bivariate homogeneous Birkhoff interpolation schemes. The ultimate goal is to obtain a geometrical understanding of the solvability. We achieve this by describing several regularity criteria, illustrated by many examples. At the end we discuss several conjectures.
Keywords: multivariate interpolation, Birkhoff interpolation, regular schemes, Polya condition, Hermite polynomials

Contents

1. Introduction

Apart from its historical importance (see e.g. Newton's interpolation formula, Gauss' quadrature formula), interpolation by polynomials still plays a central role in the local construction of various approximation schemes (splines, finite elements, cubature formulas, etc). The Birkhoff-Hermite problem is one of the most general multivariate polynomial interpolation problems. In general, it depends on a set $Z \subset \mathbb{R}^n$ of nodes, a "lower set" S defining the "interpolation space" $\mathcal{P}_S$ of polynomials (to which the solution is required to belong to), and, for each $z \in Z$, a set $A(z) \subset \mathbb{Z}_+^n$ of derivatives which appear in the interpolation equations at the point z. To simplify

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the notations, we will consider only the bivariate case (i.e. $n = 2$). One then looks for polynomials $P \in \mathcal{P}_S$ satisfying the equations

$$\frac{\partial^{i+j} P}{\partial x^i \partial y^j}(z) = c_{i,j}(z), \ \forall z \in Z, (i,j) \in A(z),$$

where $c_{i,j}(z)$ are given arbitrary constants. When the conditions at each point involve the same set A of derivatives (i.e. the sets $A(z) = A$ do not depend on z), one talks about the homogeneous problem. In the literature one usually finds two types of results on the Birkhoff interpolation problem:

1. On one hand, there are regularity results which provide criteria whose final conclusion is that the problem does have solutions for *some* choices of Z. One of the best examples is the work of R.A. Lorentz, who investigates the pairs (A, S) which are almost regular (i.e. with the property that (Z, A, S) is regular for at least one choice of Z). First of all, the determinant $D(Z, A, S)$ associated to the interpolation equations is viewed as a polynomial on the coordinates of the nodes. By taking partial derivatives of $D(Z, A, S)$, one brakes the determinant into determinants of lower complexity, and one finds a beautiful (and powerful!) interpretation of this process. The outcome consists of several geometric criteria which are very useful in deciding the almost regularity of pairs (A, S). Although this is very satisfactory for generic Z's, it does not provide much information when the nodes are in a particular (not generic) position.

2. On the other hand, there are more constructive results which use ideas (or just results) from the univariate case to produce explicit solutions for some special classes of interpolation schemes. A good example is the work of Gasca and Maeztu [2] (for more complete list of references, and historical comments, we refer to the same [2, 3]). In such results, although the shape of Z is very special, it is very rare that it is the starting point. More precisely, the shape comes on the second place: it is the one for which the method works.

Clearly, the patterns of the general problem are far too complex to be understood in an unified manner. One of the few common features of all approaches is that the shape of Z (and sometimes that of A too) plays an essential role. On the other hand, when it comes to applications, it often happens that the set of nodes Z is given, has a very particular shape (think e.g. of cubature formulas), and there are only special cases/aspects of the problem that need to be solved/understood. With all these in mind, there is a clear need of a more systematic study of the interpolation schemes where the main emphasis is put on the shape of Z (or/and of A). In this paper we investigate the (bivariate, homogeneous) Birkhoff interpolation problem with the main emphasis on the shape of Z. Although we will consider several shapes, we will mostly concentrate on the simplest possible ones, namely rectangular shapes (see Definition 2.2).

Let us now give an outline of the paper and of the main results. We start by giving/recalling the main definitions, and presenting one example (Section 2), and then we bring together some of the main constructions on lower sets and shapes (Section 3) to be used in the later sections. We discuss cartesian shapes associated to lower sets (see 3.4) which later in Section 5 show up as the only shapes for the sets of nodes with the property that the associated Lagrange problem is solvable in only one interpolation space (see Theorem 5.1). The shapes which are central to this paper are the (p,q)-rectangular shapes, i.e. shapes associated to the lower set $R(p,q) = \{(i,j) : 0 \leq i \leq p, 0 \leq j \leq q\}$. A set Z will have such a shape if its points are at the intersection of $(p+1)$ vertical lines with $(q+1)$ horizontal lines. In this section we also introduce the process of blowing up points of a lower set S by rectangles $R(p,q)$, which defines a new lower set $S^{p,q}$. This process turns out to be central to the theory.

In Section 4 we discuss Polya type conditions, both the well known one [3], as well as a new stronger version for rectangular sets of nodes (Theorem 4.6). See also Examples 4.7.

In the next sections, we establish some general properties of schemes (A,S) which are (almost) regular with respect to (p,q)-rectangular sets of nodes. However, the way the sections are organized (and entitled) depends on the consequences we derive from each of these properties. More precisely

Section 6: Here we prove that if A contains a lower set R, then R must contain $R^{p,q}$ (Proposition 6.1). As consequence, when A is itself lower, this property determines S uniquely; then we go on and describe explicitly the fundamental interpolation polynomials (Theorem 6.2). The title of the section is.

Section 7: Here we find the condition that the number $|A_x|$ of elements of A situated on the OX axes, and similarly $|A_y|$, must satisfy (Theorem 7.1). These conditions restrict the number of possibilities for the set A (for instance, in combination with the other properties, one can describe all the possible regular schemes with A having at most five elements). As consequence, when A sits on s vertical lines, and on each of the lines it contains exactly t points (e.g. if A has a rectangular shape), this property determines S uniquely; then we go on and we reduce the regularity of the original scheme to the regularity of certain univariate schemes (see Theorem 7.2 and its corollary).

Section 8: Here we investigate the influence of A on S around the coordinate axes. First of all, we show that S is made up (around the axes) of several copies of the rectangle $R(p,q)$, indexed by A_x and A_y (Theorem 8.1). Next, we show that some elements of A can be moved, or even removed, giving rise to simpler schemes (Theorem 8.2). As consequence, when A contains only elements on the axes, S will be uniquely determined, and then we reduce the regularity of the original scheme to the regularity of two univariate schemes (Proposition 8.3).

Section 9: In this section we look at the case $p = q = 1$, and the main theorem here can be viewed as a proof of several of the conjectures that we will make later (for general p and q's).

Section 10: This section contains several examples which illustrate the use the general results in deciding the regularity with respect to rectangular sets of nodes.

All the results we prove, and all the examples we give, point out some simple general properties of regular interpolation schemes (Z,A,S) with (p,q)- rectangular sets of nodes, that we formulate as conjectures. More precisely, it appears that S must be obtained from a lower set R by the process of blowing up (Conjecture 1) , (A,S) must satisfy some simple inequalities (Conjecture 2), A and S are related geometrically, by means of shifts (Conjecture 3), and R is obtainable from A by backwards shifts. In Section 11 we discuss these properties, we remark that Conjecture 2 is an improvement of the known Polya condition (Corollary 11.3), and implies Conjecture 1 (Corollary 11.4), we prove that Conjectures 2 and 3 are equivalent (Theorem 11.7), and we explain the relevance of Conjecture 4 to some of the examples previously presented.

Note also that all the results proven in the previous sections, and all the examples we present, show that the conjectures are true in many unrelated cases. In particular, the Conjectures 1-3 are verified when $p = q = 1$, and also for arbitrary p and q if one assumes regularity with respect to *complex* sets of nodes.

And, finally, one of the regularity criteria of [3] that is used in several examples is summarized in the Appendix.

At this point we strongly believe that the aspects of the interpolation problem that we discuss here (theorems, example and conjectures) are important to the study of the Birkhoff interpolation with the main emphasis on the shape of the set of nodes. After all, all we do is look at the simplest schemes (bivariate, homogeneous) with sets of nodes having one of the simplest possible shapes (rectangular).

2. Main definitions

2.1. Main definitions/terminology: In this paper we will use the following notations/terminology:

- Z denotes a finite set of points in $\mathbb{R}^2$, which plays the role of the *set of nodes* of the interpolation problem.

- A denotes a finite set of pairs of non-negative integers, describing the (order of the) *derivatives* appearing in the interpolation problem. Typical examples are the rectangles $R(u, v)$ and the triangles $T(n)$:

$$R(u, v) = \{(i, j) \in \mathbb{Z}_+^2 : i \leq u, j \leq v\},$$

$$T(n) = \{(i, j) \in \mathbb{Z}_+^2 : i + j \leq n\}.$$

- S is a lower subset, that is, a finite subset $S \subset \mathbb{Z}_+^2$ with the property that

$$(u, v) \in S \Longrightarrow R(u, v) \subset S.$$

Corresponding to S is the space

$$\mathcal{P}_S = \{P \in \mathbb{R}[x, y] : P = \sum_{(i,j)\in S} a_{i,j} x^i y^j \},$$

and $\mathcal{P}_S$ plays the role of the *interpolation space* (where the interpolation is taking place).

- such triples (Z, A, S) are called *interpolation schemes*, and the associated problem is: for given constants $\{c_{i,j}(z) : (i, j) \in A, z \in Z\}$, find polynomials $P \in \mathcal{P}_S$ satisfying the equations

$$\frac{\partial^{i+j} P}{\partial x^i \partial y^j}(z) = c_{i,j}(z),\ \forall\ z \in Z,\ (i, j) \in A,$$

One says that (Z, A, S) is *solvable* if the problem has solutions for any choice of the constants. One says that (Z, A, S) is *regular* if the problem has unique solution. The simplest necessary condition for regularity is normality of (Z, A, S), that is, $|S| = |Z||A|$. All schemes in this paper are assumed to be normal.

- the interpolation equations are in fact a system of linear equations, and one denotes by $D(Z, A, S)$ its determinant (well defined because of the normality condition). This is a polynomial on the coordinates of the nodes ($2n$ such coordinates, where $n = |Z|$). Hence, fixing (A, S), if (Z, A, S) is regular for one choice of Z, then it is regular for almost all choices of Z (where "almost all" refers to the Lebesgue measure on $\mathbb{R}^{2n}$). In this case one says that (A, S) is *almost regular* with respect to sets of n nodes.

As mentioned in the introduction, we concentrate on the effect that the shape of the sets involved has on the regularity of the interpolation problem. Although we will consider several types of shapes, in this paper we will concentrate mainly on the simplest shapes:

Definition 2.2. *We say that the set of nodes Z has a rectangular shape, or that is (p, q)-rectangular, if*

$$Z = \{(x_i, y_j) : 0 \leq i \leq p, 0 \leq j \leq q\}$$

where the x_i's are distinct real numbers, and similarly the y_j's.

In this case, the associated determinant $D(Z, A, S)$ is a polynomial on only $t = p + q + 2$ variables (namely the x_i's, and the y_j's). We then say that a pair (A, S) is almost regular with respect to (p, q)-rectangular sets of nodes if there exists such a set of nodes Z such that (Z, A, S) is regular. As above, this implies that (Z, A, S) is regular for almost all choices of the (p, q)-rectangular set Z (this time, of course, the "almost" part refers to the Lebesgue measure on $\mathbb{R}^t$). The regularity is defined similarly. Of course, one can also allow complex nodes, and talk about (almost) regularity with respect to (p, q)-rectangular sets of complex nodes.

There are several "counting numbers" that can be associated with the interpolation problem, depending on which one of the sets Z, A, S we put more emphasis on, and which is the shape we are looking at. In this paper we will consider the numbers $s(Z, A)$ which, for given Z and A, is defined as the number of lower sets S with the property that (Z, A, S) is regular. We also consider the generic case, and define $s(Z_n, A)$ as the number of lower sets S with the property that (A, S) is almost regular with respect to sets of n nodes ("Z_n" here should be viewed just as a symbol for the "generic case" corresponding to sets of n nodes). And, similarly, define $s(Z_{p,q}, A)$ as the number of lower sets S with the property that (A, S) is almost regular with respect to (p, q)-rectangular sets of nodes. Computing these numbers is probably quite hard (with combinatorial difficulties which go beyond the interpolation problem), but their study should bring some insight into the interpolation problem.

Example 2.3. Before going into a more systematic study, let us illustrate by one example the effect that the shape of Z has on the regularity of the problem. We consider the interpolation problem which involves only evaluation of the polynomials and of their derivative with respect to the first variable, i.e.

$$A = \{(0, 0), (1, 0)\}$$

First of all, if Z is a (p, q)-rectangular set of nodes, then there is at most one lower set for which (Z, A, S) is regular, namely $S = R(2p + 1, q)$. This will be an immediate consequence of the forthcoming sections. In particular, $s(Z, A) \leq 1$, and $s(Z_{p,q}, A) = 1$. In general however (i.e. when Z is not necessary rectangular), there are many more such lower sets. Let us summarize the generic case:

Proposition 2.4. *Given $A = \{(0, 0), (1, 0)\}$ and a lower set S,*

(i) *the pair* (A, S) *is almost regular with respect to sets of* n *nodes if and only if* $|S| = 2n$, *and* S *contains at most* $(n+1)$ *elements on the* OY *axes.*
(ii) *the pair* (A, S) *is almost regular with respect to* (p, q)*-rectangular sets of nodes if and only if* $S = R(2p+1, q)$.

Proof. That the condition must be satisfied follows from the Polya condition [3] (see also our Section 4) applied to $L = S \cap OY$. To prove the converse, we use the ideas of [3] that we summarize in the appendix. We will show that, the conditions on S imply that S admits a pavement of unique type by n copies of A (for terminology, see our Appendix). We use induction on n.

We now consider the shift Λ_1 that moves $(1, 0)$ to $(3, 0)$, and then the origin to $(2, 0)$. Hence Λ_1 is the shift relative to A, moving A minimally to the right. Next, Λ_2 is the shift relative to $A \cup \Lambda_1(A)$ moving A minimally to the right (hence it sends $(1, 0)$ to $(5, 0)$, and then the origin to $(4, 0)$). We continue like this, and, when the first line is filled up, we go to the next line, which we fill in the same way, from the left to the right. In this way we build up S by copies of A in the "obvious way": from the left to the right, row by row. If this process does not stop before S is filled up, then it gives a pavement of S (of unique type) by copies of A, and we are done without even using the induction hypothesis. On the other hand, the process can stop only because the points on the OY axes can only be covered by moving the origin ($(1, 0) \in A$ cannot be moved backwards). In other words, when the process stops, the only points of S that have not been filled are on the OY axes, and there is at least one such point. Hence, in this case, if β is the largest number such that $(0, \beta) \in S$, then $(1, \beta) \notin S$. We then consider the shift Λ which moves A maximally upwards, and then maximally to the right. Then $(0, \beta) \in \Lambda(A)$, and we see that $S \setminus \Lambda(A)$ is a new lower set which has on the OY axes one less element then S has, hence at most n elements. By the induction hypothesis, we find a pavement of $S \setminus \Lambda(A)$ by $(n-1)$ copies of A, which, together with Λ, gives the desired pavement. □

3. More on lower sets, and shapes

In this section we bring together some basic constructions on lower sets and sets of nodes, that will be needed in the next sections.

3.1. Representing lower sets: First of all, for any non-negative integers s, $n_0 \geq n_1 \geq \ldots \geq n_s$,

$$S_y(n_0, \ldots, n_s) = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i \leq s, 0 \leq j \leq n_i\}$$

is clearly a lower set. Conversely, any lower set S can be uniquely written in such a way. Given S, we denote by $S_y[i]$ the set of all j's with the property that $(i, j) \in S$. Then s is the maximal number i with the property that $S_y[i] \neq \emptyset$, while n_i is determined by $S_y[i]$:

$$S_y[i] = \{0, 1, \ldots, n_i\}$$

A similar discussion is obtained by interchanging the role of x and y: for $m_0 \geq m_1 \geq \ldots \geq m_t$, one defines

$$S_x(m_0, m_1, \ldots, m_t) = \{(i, j) \in \mathbb{Z}^2 : 0 \leq j \leq t, 0 \leq i \leq m_j\}.$$

Note that

$$R(p,q) = S_y(\underbrace{q,\ldots q}_{p+1}) = S_x(\underbrace{p,\ldots p}_{q+1}).$$

In general, if one writes $S = S_y(n_0, \ldots, n_s)$ also as $S_x(m_0, m_1, \ldots, m_t)$, one immediately sees that the relation between these two sets of numbers is:

$$t = n_0,\ m_j = max\{i : n_i \geq j\}.$$

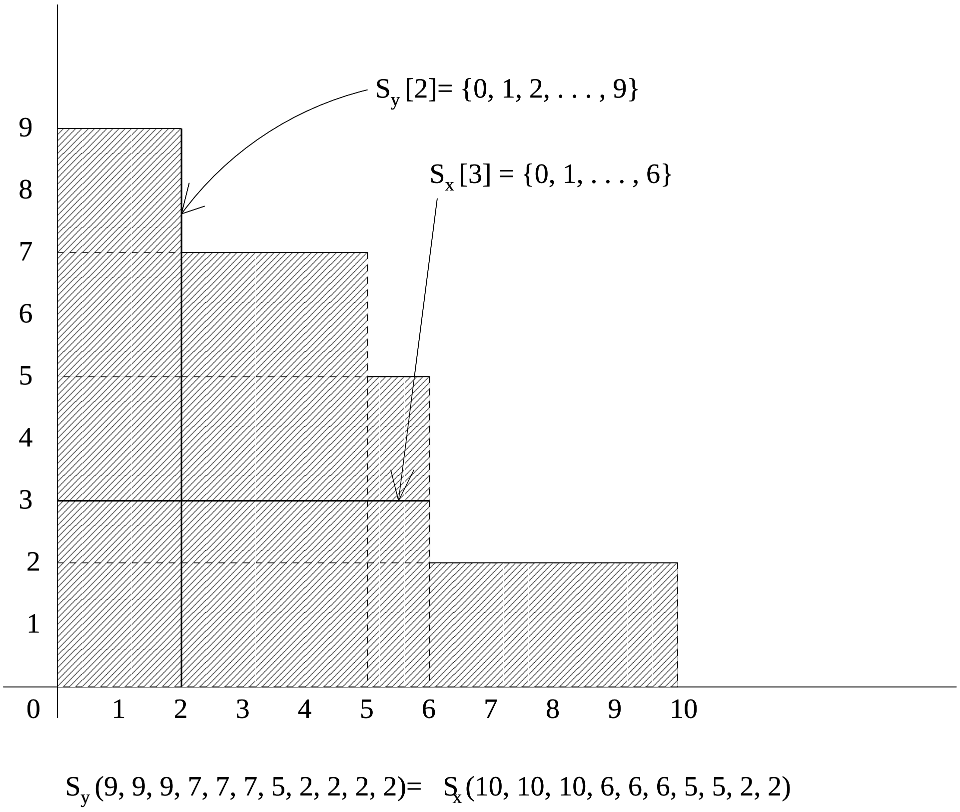


FIGURE 1.

3.2. Boundary points: Given a lower set S, a point $(u,v) \in S$ is called a *boundary point* if $(u+1, v+1) \notin S$. We denote by ∂S the set of such points. We consider the following two possibilities:

(i) $(u, v+1) \in S$;
(ii) $(u+1, v) \in S$.

We denote by $\partial_e S$ the set of boundary points (u,v) for which neither of the two conditions above is satisfied ("exterior boundary points"), by $\partial_i S$ the set of those which satisfy both conditions ("interior boundary points"), by $\partial_x S$ the set of those for which only (ii) holds true ("x-direction boundary points"), and by $\partial_y S$ the set of those for which only (i) holds true ("y-direction boundary points"). These four sets form a partition of the boundary $\partial(S)$ of S.

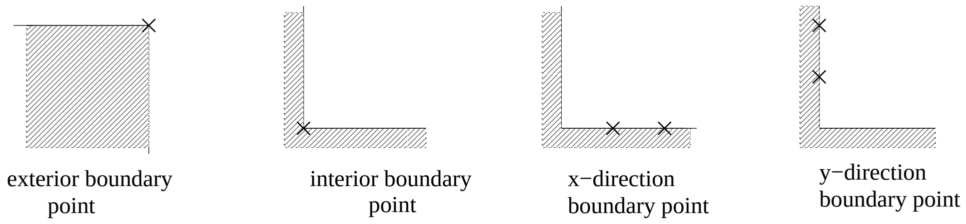


FIGURE 2. Boundary points.

Note that the number of exterior boundary points minus the number of interior boundary points is always one. Also, if S_x is the intersection of S with the OX axis, and similarly S_y, then $|\partial_x S| = |S_x| - |\partial_e S|$, $|\partial_y S| = |S_y| - |\partial_e S|$ (in particular, $|\partial S| = |S_x| + |S_y| - 1$).

Note also that the set $\partial_e S$ of exterior boundary points determines S uniquely, since

$$S = \bigcup_{(u,v)\in \partial_e S} R(u,v).$$

It will actually be useful to express everything more explicitly. Let us denote by (a_k, b_1), (a_{k-1}, b_2), $\ldots$, (a_1, b_k) the elements of $\partial_e S$, indexed so that $a_1 < a_2 < \ldots < a_k$, and $b_1 < b_2 < \ldots < b_k$.

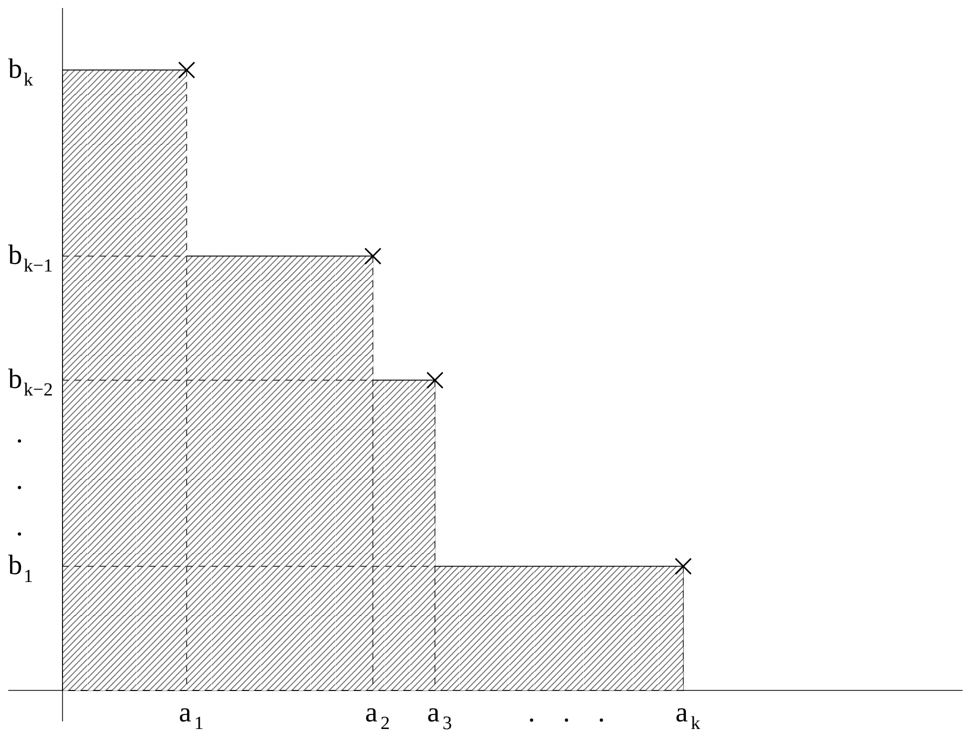


FIGURE 3. Exterior boundary points.

Then an element (u, v) is in S if and only if

$$a_{l-1} < u \leq a_l,\ 0 \leq v \leq b_{k-l+1}$$

for some $l \in \{0, 1, \ldots k\}$. In relation with the description of lower sets given in 3.1 above, we have $S = S_y(n_0, \ldots, n_t)$ where

$$n_0 = \ldots = n_{a_1} = b_k < n_{a_1+1} = \ldots = n_{a_2} = b_{k-1} < \\ \ldots < n_{a_{k-1}+1} = \ldots = n_{a_k} = b_1,$$

and these relations can also be viewed as a way to express the a_i's in terms of the n_i's. Similar formulas holds for the b_j's and the m_j's. In particular, we see that

$$\begin{aligned} |S| &= (a_1+1)(b_k - b_{k-1}) + (a_2+1)(b_{k-1} - b_{k-2}) + \ldots \\ &\quad +(a_{k-1}+1)(b_2 - b_1) + (a_k+1)b_1 \end{aligned} \tag{1}$$

Note also that

$$\partial_e S = \{(a_i, b_j) : i + j = k + 1\},\ \partial_i L = \{(a_i, b_j) : i + j = k\},$$

$$\partial_x S = \{(u, b_{k-j}) : 1 \leq j \leq k, a_j < u < a_{j+1}\},$$

$$\partial_y S = \{(a_{k-i}, v) : 1 \leq i \leq k, b_i < v < b_{i+1}\}.$$

3.3. Blowing up lowers sets: Given p, q non-negative integers, and a lower set S, we definer a new lower set $S^{p,q}$ which is obtained by "blowing up" each of the points of S to a (p, q)-rectangle. One may think that a copy of $R(p, q)$ is placed on each of the points of S, and then one pushes these rectangles minimally to the right and upwards until they become disjoint. For an example with $p = 1$, $q = 2$, see the picture.

More formally,

$$S^{p,q} = \{(\alpha, \beta) : ([\frac{\alpha}{p+1}], [\frac{\beta}{q+1}]) \in S\}.$$

If $S = S_y(n_0, \ldots, n_s)$, then $S^{p,q} = S_y(n_0', \ldots, n_{s'}')$, where

$$k' = (q+1)(k+1) - 1,\ n_i' = (p+1)(n_j+1) - 1) \text{ where } j = [\frac{i}{q+1}].$$

In terms of the exterior boundary points (cf 3.2 above), $\partial_e S^{p,q}$ consists of the pairs $((p+1)u, (q+1)v)$ with $(u, v) \in \partial_e S$.

3.4. Cartesian sets of nodes: Given a lower set S, one says that a set of nodes Z is S-*cartesian* if it is of type

$$Z = \{(x_i, y_j) : (i, j) \in S\},$$

where the x_i's are distinct real numbers, and similarly the y_j's. We say that Z is cartesian if it is S-cartesian for some lower set S. When $S = R(p, q)$ we recover the (p, q)-rectangular configurations of nodes.

In general, any set of nodes Z induces two lower sets $S_x(Z)$, and $S_y(Z)$, which reflect the shape of Z. To describe $S_y(Z)$, one covers Z by lines $l_0, \ldots, l_k$ parallel to the OY axis, and define the numbers n_i so that on each line l_i there are exactly $n_i + 1$ points of Z. We index the lines so that $n_0 \geq n_1 \geq \ldots \geq n_k$, and we define $S_y(Z) = S_y(n_0, \ldots, n_k)$. The lower set $S_x(Z)$ is defined similarly, by interchanging the role of x and y. The relation with cartesian sets is:

Lemma 3.5. *A set of nodes Z is cartesian if and only if $S_x(Z) = S_y(Z)$.*

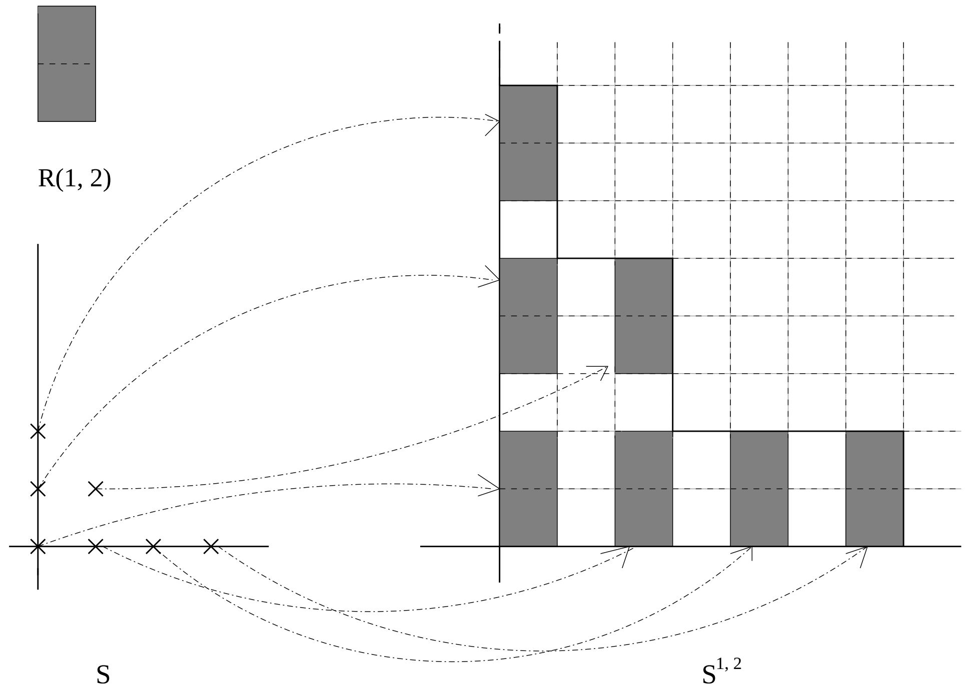


FIGURE 4. Blowing up.

Proof. We use induction on the number of elements n of Z. First of all, since each of the lines l_i does contain at least one point, it follows that $(i, 0) \in S_y(Z)$ for all $0 \leq i \leq k$. Since $S_x(Z) = S_y(Z)$, it then follows that there is a line l, parallel to the OX axis, which intersects each of the lines l_i in a point situated in Z. We then see that $S_y(Z \setminus l) = S_x(Z \setminus l)$, and, by the induction hypothesis, $Z \setminus l$ must be cartesian. This clearly implies that Z must be cartesian too. $\square$

4. Polya-type conditions

Polya type conditions are the regularity conditions that arise because of a very simple reason: a non-zero determinant cannot have "too many zeros". More precisely, one has the following simple observation

Lemma 4.1. *Assume that $M \in M_n(\mathbb{R})$ has a rows and b columns with the property that the ab elements situated at the intersection of these rows and columns are all zero. If $det(M) \neq 0$, then $a + b \leq n$.*

Remark 4.2. By removing the intersection elements (the ab zeros from the statement) from the a rows, one obtains a matrix with a rows and $n-b$ columns, denoted M_1. Similarly, doing the same along the columns, one gets a matrix with $n-a$ rows and b columns, denoted M_2. In the limit case of the lemma (i.e. when $a + b = n$), then both M_1 and M_2 are square matrices, and a simple form of the Laplace formula tells us that $\det(M) = \det(M_1)\det(M_2)$ (up to a sign).

The extreme (and obvious) cases of this lemma show that if (A, S) is almost regular, then A must be contained in S, and it must also contain the origin. Staying in the context of generic Z's, one immediately obtains the known Polya condition [3] which appears as the most general necessary condition for the almost regularity of pairs (A, S) that one can obtain "by counting zeros":

Corollary 4.3. *If the pair (A, S) is almost regular with respect to sets of n nodes, then, for any lower set $L \subset S$, $n|L \cap A| \geq |L|$.*

Indeed, the monomials in $D(Z, A, S)$ which sit in the columns corresponding to L become zero when taking derivatives coming from $A \setminus L$. These derivatives define $n|A \setminus L|$ rows, hence the previous lemma implies that $|L| + n|A \setminus L| \leq |S|$. Since $|S| = n|A|$, and $|A \setminus L| = |A| - |A \cap L|$, the result follows. Also, the limit case described by Remark 4.2 immediately implies

Corollary 4.4. *If (Z, A, S) is a regular scheme, and $L \subset S$ is a lower set satisfying $|L| = n|A \cap L|$ (where $n = |Z|$), then $(Z, A \cap L, L)$ must be regular too.*

Remark 4.5. This corollary applies to the univariate case as well. Writing $A = \{a_0, a_1, \ldots, a_s\}$ with $a_0 < a_1 < \ldots < a_s$, the Polya conditions become:

$$a_i \leq n \cdot i, \quad \forall\ 0 \leq i \leq s.$$

Moreover, this condition actually insures regularity for almost all sets of nodes Z. More precisely, given (A, S) with $|S| = n|A|$, (A, S) is almost regular if and only if it satisfies the Polya conditions. Moreover, if $n = 2$, then the Polya conditions are sufficient also for regularity. For details, see [1].

Depending on the shape of Z, there may be various other conditions of this type. Let us now describe a general Polya condition for rectangular sets of nodes (other examples will appear in the forthcoming sections).

Theorem 4.6. *If (A, S) is almost regular with respect to (p, q)-rectangular sets of nodes, $n = (p+1)(q+1)$, then*

$$n|A \cap L| \geq |L| + pq|A \cap \partial L| + (p+q)|A \cap \partial_e L| + p|A \cap \partial_y L| + q|A \cap \partial_x L|.$$

Proof. We consider the columns corresponding to L, and we look for those rows which intersected with these columns produce zeros (possibly after some elementary operations). The rows are indexed by the pairs $(i, j) \in R(p, q)$ (which give the nodes (x_i, y_j)), and elements $(\alpha, \beta) \in A$, and consist of the derivatives of order (α, β) of the monomials in $\mathcal{P}_S$, evaluated at (x_i, y_j). We distinguish four types of derivatives, depending on the position of (α, β) relative to A

(i) $(\alpha, \beta) \in A \setminus L$: this produces $n|A \setminus L|$ rows of type we are looking for.

(ii) $(\alpha, \beta) \in A \cap \partial_e L$. If we subtract one of these rows (say the one corresponding to (x_0, y_0)) from all others, we obtain $n-1$ new rows that intersected with the columns corresponding to L give zeros. In total, $(n-1)|A \cap \partial_e L|$ new rows.

(iii) $(\alpha, \beta) \in A \cap \partial_x L$. Looking at the corresponding intersections of a row defined by such a derivative (and by a pair $(i, j) \in R(p, q)$) with the columns defined by L, the only possible non-zero elements are powers of x.

Then, for each x_i, we subtract the row corresponding to (x_i, y_0) from the rows corresponding to (x_i, y_j), $j \geq 1$ to get rid of the non-zero elements containing x. This produces q new rows which do have zero at the intersection with the L-columns. We do this for each $0 \leq i \leq p$ and for each derivative

$(\alpha, \beta) \in A \cap \partial_x L$, hence we end up with $(p+1)q|A \cap \partial_x L|$ new rows of the type we are looking for.

(iv) $(\alpha, \beta) \in A \cap \partial_y L$ is similar to (iii) and produces $p(q+1)|A \cap \partial_y L|$ rows.

(v) $(\alpha, \beta) \in A \cap \partial_i L$. We basically apply twice the subtraction that we did in the previous two cases. Looking at the corresponding intersections of a row defined by such a derivative (and by a pair $(i, j) \in R(p, q)$) with the columns defined by L, the only possible non-zero elements are powers of x or powers of y (evaluated at (x_i, y_j)). Then, for each x_i, we subtract the row corresponding to (x_i, y_0) from the rows corresponding to (x_i, y_j), $j \geq 1$ to get rid of the non-zero elements containing x, and then we do the same with to get rid of y's. The outcome consists of $pq|A \cap \partial_i L|$ new rows of the type we are looking for.

Adding up, and using the lemma, we get:

$$|L| + n|A \setminus L| + (n-1)|A \cap \partial_e L| +$$
$$(p+1)q|A \cap \partial_x L| + p(q+1)|A \cap \partial_y L| + pq|A \cap \partial_i L| \leq n|A|,$$

and, since $\partial L = \partial_e L \cup \partial_x L \cup \partial_y L \cup \partial_i L$, this can easily be transformed into the inequality in the statement. □

4.7. Examples:

Let us illustrate the use of Polya criteria by looking at two types of examples. For rectangular sets of nodes, and the simplest possible A, we get

Corollary 4.8. *If Z is (p, q)-rectangular set of nodes, then the Lagrange problem for (Z, S) has solutions if and only if $S = R(p, q)$.*

Proof. Choosing $L = S_x$ (the intersection of S with the OX axis) in the previous theorem we get $(p+1)(q+1) \geq |S_x| + pq + q$, hence $|S_x| \leq p+1$. Similarly, $|S_y| \leq q+1$. Since S is lower, we have $|S| \leq |S_x||S_y|$, and, since $|S| = (p+1)(q+1)$, all the previous inequalities must become equalities, and this happens precisely when $S = R(p, q)$. That $(Z, R(p, q))$ is indeed regular is part of the folklore (or use Lemma 2 below). □

For non-rectangular sets of nodes, we look at one of the simplest possible A's for which the Polya condition gives us non-trivial information:

$$A = \{(0, 0), (1, 1)\},$$

Proposition 4.9. *Let A containing only $(0, 0)$ and $(1, 1)$. Then a scheme (A, S) is almost regular with respect to sets of six nodes if and only if $S = R(2, 3)$ or $S = R(3, 2)$. In particular, $s(Z_6, A) = 2$.*

Proof. Assume first that (A, S) is almost regular with respect to sets of six nodes. Let a be the maximal integer with the property that $(a, 1) \in S$, let b be the maximal integer with the property that $(1, b) \in S$, and let L be the set of the elements of S on the coordinate axes. Since S is lower and $(1, 1)$ must be in S, it follows that $a, b \geq 1$, and $|L| \geq a + b + 1$. But Corollary 4.3 forces $|L| \leq 6$, hence $a + b \leq 5$. On the other hand, $S \setminus L$ is contained on the rectangle with vertexes $(1, 1)$, $(a, 1)$, $(1, b)$ and (a, b), hence $12 - |L| = |S \setminus L| \leq ab$. Since $|L| \leq 6$, we must have $ab \geq 6$. But this, together with $a + b \leq 5$ can only hold when (a, b) is either $(2, 3)$ or $(3, 2)$. Moreover, in both cases equality holds, hence all the inclusions used on deriving those inequalities must become equalities. In particular, L must contain $a + 1$ elements on OX, $b + 1$ elements on OY, and $S \setminus L$ must coincide with the rectangle mentioned above. This forces $S = R(a, b)$ in each of the cases.

Let us now prove that the scheme is almost regular when $S = R(2,3)$. We will use again the regularity criteria described in the appendix, and show that $R(2,3)$ admits a pavement of unique type by six copies of A. The first shift Λ_1 moves A is the unique shift of type U^5, i.e. the one which moves A maximally upwards. The next one, Λ_2, is the maximal upward shift relative to $A \cap \Lambda_1(A)$ (and has type U^3). Next, there is an unique shift of A relative to $A \cap \Lambda_1(A) \cap \Lambda_2(A)$ of type RU^4 (and it moves the origin to $(0,1)$, and $(1,1)$ to $(2,3)$). This is Λ_3. Λ_4 is the one which moves A maximally to the right, and Λ_5 is the only remaining shift relative to $A \cap \Lambda_1(A) \cap \ldots \Lambda_4(A)$ (and it moves the origin to $(1,0)$, and $(1,1)$ to $(2,1)$). Hence Theorem 12.1 can be applied, proving the last part of the proposition. □

5. The Lagrange case

In this section we investigate the solvability of the Lagrange problem. We have already seen in Corollary 4.8 that the Lagrange problem with rectangular sets of nodes have solutions only for one choice of S. In other words, $s(Z) = 1$ for rectangular Z's. In this section we clarify the extreme cases:

Theorem 5.1. *For any set of n nodes Z, one has*

$$1 \leq s(Z) \leq p(n).$$

Moreover,

(i) $s(Z) = 1$ *if and only if Z is cartesian.*
(ii) $s(Z) = p(n)$ *for almost all Z's.*

Here $p(n)$ denotes the number of solutions of the equation $k_0 + k_1 + \ldots + k_s = n$, with $k_0 \geq k_1 \geq \ldots \geq k_s \geq 1$, $s \geq 0$ integers.

Regarding the cartesian case, we will actually show the following:

Proposition 5.2. *Given two lower set S, and S_0, the following are equivalent*

(i) (Z, S_0) *is regular for some S-cartesian set of nodes Z.*
(ii) (Z, S_0) *is regular for all S-cartesian sets of nodes Z.*
(iii) $S_0 = S$.

The remaining part of this section is devoted to the proof of these results. We will need the following

Lemma 5.3. *Let S be a lower set, and write $S = S_y(n_0, \ldots, n_s)$. If*

$$Z = \{(x_i, y^i_j) : 0 \leq i \leq s, 0 \leq j \leq n_i\} \tag{2}$$

where all the x_i's are distinct, as well as all the y^i_j's for each i, then the Lagrange problem for (Z, S) is regular.

Proof. This follows from the univariate case. Denote by $l^x_i(x; x_0, \ldots, x_p)$ the fundamental interpolation polynomial at the node x_i, with respect to the Lagrange problem with the nodes $x_0, \ldots, x_p$; that is:

$$l^y_i(x; x_0, \ldots, x_p) = \frac{(x - x_0)\ldots\widehat{(x - x_i)}\ldots(x - x_p)}{(x_i - x_0)\ldots\widehat{(x_i - x_i)}\ldots(x_i - x_p)}.$$

Then $l^{x,y}_{i,j} = l^x_i(x; x_0, \ldots, x_p) l^y_j(y; y^i_0, \ldots, y^i_{n_i})$ will be the fundamental interpolation polynomials for our problem. □

Now, clearly any set of nodes Z can be written in the form (2), for some lower set S. Actually $S = S_y(Z)$ does the job, by its very definition (cf. 3.4). Hence $(Z, S_y(Z))$ is regular. And similarly $(Z, S_x(Z))$. This shows that $s(Z)$ is always greater then one, and that $s(Z) = 1$ implies that $S_y(Z) = S_x(Z)$, which, in turn (cf. Lemma 3.5), forces Z to be cartesian. Before proving that $s(Z)$ does equal one for cartesian Z's, let us first clarify the other side of the inequality. First of all, from the representation of lower sets given in 3.1, we see that $p(n)$ equals to the number of lower sets of cardinality n (the relation between the n_i's in 3.1 and the k_i's in the definition of $p(n)$ above is $k_i = n_i + 1$). This clearly implies $s(Z) \leq p(n)$. Now, given S, the problem (Z, S) is regular for at least one choice of Z (e.g. use the previous lemma). Hence $D(Z, S)$, viewed as a polynomial in the coordinates of the points of Z, is not identical zero. This implies that the zero set of this polynomial has measure zero (in $\mathbb{R}^{2n}$). On the other hand, $s(Z) = p(n)$ holds precisely when Z is not in the zero set of any of these polynomials. In particular, the complement (in $\mathbb{R}^{2n}$), consisting of those Z with $s(Z) \neq p(n)$, is the union of $p(n)$ sets of measure zero, hence has measure zero. We now turn to the proof of Proposition 5.2. We will make use of the following

Lemma 5.4. *Given a lower set S and a (p, q)-rectangular set of nodes Z, the Lagrange interpolation problem for (Z, S) is solvable if and only if $R(p, q) \subset S$.*

Proof. Since $(Z, R(p, q))$ is regular, we only have to prove that solvability implies $R(p, q) \subset S$. The solvability condition ensures the existence of a polynomial $P \in \mathcal{P}_S$ with the property that $P(x_i, y_j) = 0$ for all $(i, j) \in R(p, q)$, except for $P(x_p, y_q) = 1$. We consider the polynomials $\phi_0^x = 1$,

$$\phi_i^x = (x - x_0) \dots (x - x_{i-1})/(x_i - x_0) \dots (x_i - x_{i-1}),$$

where we extend the sequence $x_0, \dots, x_p$ to an infinite sequence of distinct numbers. Similarly we define the polynomials ϕ_j^y in the variable y. Since $\{\phi_i^x \phi_j^y : i, j \geq 0\}$ is a basis for the space of bivariate polynomials, we can write $P = \sum_{i,j} a_{i,j} \phi_i^x \phi_j^y$, and, since $P \in \mathcal{P}_S$, we see that $(i, j) \in S$ whenever $a_{i,j} \neq 0$. Now, since $\phi_i^x(x_k) = 0$ for all $k < i$, $\phi_i^x(x_i) = 1$, and similarly for the ϕ_j^y's, one has the following implication:

$$a_{i,j} = 0 \;\; \forall \;\; (i, j) \in R(u, v) \setminus \{(u, v)\} \Longrightarrow P(x_u, y_v) = a_{u,v}\,.$$

Hence, by a simple induction, we deduce that $a_{i,j} = 0$ for $(i, j) \in R(p, q) \setminus \{p, q\}$, and $a_{p,q} = P(x_p, y_q) = 1$. Since $a_{p,q} \neq 0$, it follows that $(p, q) \in S$, hence $R(p, q) \subset S$. ☐

We can now prove proposition 5.2. We show that $(i) \Longrightarrow (iii) \Longrightarrow (ii)$. Let us assume that Z is S-cartesian set. Writing $S = S_y(n_0, \dots, n_s)$, this means that $Z = \{(x_i, y_j) : 0 \leq i \leq s, 0 \leq j \leq n_i\}$, with the x_i's, as well as the y_j's, distinct. We now assume that (Z, S_0) is regular. This clearly implies that, for any subset Z_0 of Z, the problem (Z_0, S_0) is solvable. For any k, we choose $Z_0 = Z_0(k)$, where $Z_0(k) = \{x_i, y_j) : 0 \leq i \leq k, 0 \leq j \leq n_k\}$. Since $Z_0(k)$ is (k, n_k)-rectangular, it follows from the Lemma 5.4 that $R(k, n_k) \subset S_0$. Since the sets $R(k, n_k)$ with $0 \leq k \leq s$ cover S entirely, it follows that $S \subset S_0$, and since S and S_0 have the same number of elements, it follows that $S = S_0$. When $S = S_0$, that (Z, S) is regular for any S-cartesian set Z follows from Lemma 2. This concludes the proof.

6. Lower sets of derivatives on each node

Here we discuss the interpolation problem with rectangular sets of nodes, where the set A of derivatives appearing in the problem is a lower set. First of all, we will prove the following more general result:

Proposition 6.1. *If (A,S) is almost regular with respect to (p,q)-rectangular sets of nodes, and if A contains a lower set R, then S must contain the lower set $R^{p,q}$.*

If A is lower, this clearly shows that S must be equal to $A^{p,q}$. In this case we can explicitly describe the interpolating polynomials. Assume that the set of nodes is

$$Z = \{(x_i, y_j) : 0 \le i \le p, 0 \le j \le q\}.$$

Consider first the (univariate) set of nodes $\{x_i : 0 \le i \le p\}$. For non-negative integers u, a, and a node x_s, we consider the univariate Hermite polynomial

$$H_a^{u,s}(x) = \phi_s(x) \sum_{k=u}^{a} \frac{(x-x_s)^k}{u!(k-u)!} (\frac{1}{\phi_s})^{(k-u)}(x_s),$$

where

$$\phi_s(x) = (x-x_0)^{a+1} \dots \widehat{(x-x_s)^{a+1}} \dots (x-x_p)^{a+1}.$$

We make the convention that $H_{a,s}^{u}(x) = 0$ if $a < u$. Similarly, we denote by $H_b^{v,t}$ the univariate Hermite polynomials associated to the nodes $\{y_j : 0 \le j \le q\}$. As in 3.2, we denote by (a_k, b_1), (a_{k-1}, b_2), ... , (a_1, b_k) the exterior boundary points of A, indexed so that $a_1 < a_2 < \dots < a_k$, and $b_1 < b_2 < \dots < b_k$. We then have (compare with formula 1 in 3.2)

Theorem 6.2. *An interpolation scheme (Z, A, S) with a (p,q)-rectangular set of nodes Z, and a lower set of derivatives A is regular if and only if $S = A^{p,q}$. In this case moreover, for any $(u,v) \in A$, and any node $(x_s, y_t) \in Z$, the polynomial*

$$\begin{gathered} H_{u,v}^{s,t}(x,y) = \\ H_u^{a_1,s}(x)(H_v^{b_k,t}(y) - H_v^{b_{k-1},t}(y)) + H_u^{a_2,s}(x)(H_v^{b_{k-1},t}(y) - H_v^{b_{k-2},t}(y)) + \\ \dots + H_u^{a_{k-1},s}(x)(H_v^{b_2,t}(y) - H_v^{b_1,t}(y)) + H_u^{a_k,s}(x) H_v^{b_1,t}(y) \end{gathered}$$

is the unique solution of

$$\frac{\partial^{\alpha+\beta} H_{u,v}^{s,t}}{\partial x^\alpha \partial y^\beta}(x_i, y_j) = \begin{cases} 1 & \text{if } \alpha = u, \beta = v, i = s, j = t \\ 0 & \text{otherwise} \end{cases},$$

for all $(\alpha, \beta) \in A, 0 \le i \le p, 0 \le j \le q$.

Clearly, one has similar formulas for the fundamental interpolation polynomials at each of the nodes (x_i, y_j), which combined give the explicit interpolating polynomials for the general problem associated to $(Z, A, A^{p,q})$.

Proof. The remaining part of this section is devoted to the proof of these results. The case $A = \{(0,0)\}$ of Proposition 6.1 has been treated in the previous section. We use the same idea as there, and, similar to Lemma 5.4, we have

Lemma 6.3. *Given a lower set S and a (p,q)-rectangular set of nodes Z, if the interpolation problem associated to $(Z, R(s,t), S)$ is solvable, then $R(s,t)^{p,q} \subset S$.*

The proof is very similar to the proof of Lemma 5.4. This time however, we use a different basis for the space of bivariate polynomials. We put $l_{-1}^x(x) = 1$, $l_i^x(x) = (x - x_0)\dots(x - x_i)$ for $0 \leq i \leq p$, while for $i \geq p+1$, write $i = (p+1)c + i_0$ with $0 \leq i_0 \leq p$, and put $l_i^x = l_{i_0}^x (l_p^x)^c$. Similarly one defines the polynomials l_j^y, and we use $\{l_i^x(x) l_j^y(y) : i, j \geq -1\}$ as our basis. Then, as in the proof of Lemma 5.4, one chooses $P \in \mathcal{P}_S$ satisfying

$$\frac{\partial^{u+v} P}{\partial x^u \partial y^v}(x_i, y_j) = \begin{cases} 1 & \text{if } u = s, v = t, (i,j) = (p,q) \\ 0 & \text{otherwise} \end{cases},$$

for all $(u, v) \in R(s,t)$, and $(i, j) \in R(p,q)$, and one proves that the coefficient in P of $x^{(s+1)(p+1)-1} y^{(t+1)(q+1)-1}$ is non-zero (hence $((s+1)(p+1)-1, (t+1)(q+1)-1) \in S$, which, in turn, forces $R(s,t)^{p,q} \subset S$). The details are left to the reader.

We now turn to the proof of Theorem 6.2. To avoid too many indices, we assume $s = t = 0$, and we omit them from the notations (hence the old $H_a^{u,0}$ is now denoted H_a^u). We will make use of the basic equations that the Hermite polynomials $H_u^a(x)$ satisfy:

$$\frac{\partial^\alpha H_u^a}{\partial x^\alpha}(x_i) = \begin{cases} 1 & \text{if } \alpha = u, i = 0 \\ 0 & \text{otherwise} \end{cases},$$

for all $0 \leq \alpha \leq a$, $0 \leq i \leq p$. We only have to check that $H_{u,v}$ does satisfy the equations in the statement. For simplicity, we shorten the notation $H_{u,v}$ to H. Since $(u, v) \in A$, we find (cf. 3.2) an unique $l \in \{0, 1, \dots, k\}$ such that

$$a_{l-1} < u \leq a_l,\ 0 \leq v \leq b_{k-l+1} \tag{3}$$

Note that this implies that, for any $\phi \in \{0, 1, \dots, k\}$, one has

$$a_\phi \geq u \Longleftrightarrow \phi \geq l. \tag{4}$$

In particular,

$$\begin{gathered} H(x,y) = \\ H_u^{a_l}(x)(H_v^{b_{k-l+1}}(y) - H_v^{b_{k-l}}(y)) + H_u^{a_{l+1}}(x)(H_v^{b_{k-l}}(y) - H_v^{b_{k-l-1}}(y)) + \\ \dots + H_u^{a_{k-1}}(x)(H_v^{b_2}(y) - H_v^{b_1}(y)) + H_u^{a_k}(x) H_v^{b_1}(y) \end{gathered}$$

We fix $(\alpha, \beta) \in A$, and we want to decide when

$$\frac{\partial^{\alpha+\beta} H_{u,v}}{\partial x^\alpha \partial y^\beta}(x_i, y_j) \tag{5}$$

is non-zero. For this we look at the derivative of each of the individual terms

$$H_u^{a_\phi}(x)(H_v^{b_{k-\phi+1}}(y) - H_v^{b_{k-\phi}}(y)) \text{ with } l \leq \phi \leq k.$$

We distinguish the following cases

(Case 1) $\alpha \neq u$ and $\beta \neq v$, or $(i, j) \neq (0, 0)$. Since $(\alpha, \beta) \in A$, and $(a_\phi, b_{k-\phi+1})$ is a boundary point, then either $\alpha \leq a_\phi$ or $\beta \leq b_{k-\phi}$. In the first case one has

$$\frac{\partial^\alpha H_u^{a_\phi}}{\partial x^\alpha}(x_i) = 0 \text{ if } \alpha \neq u \text{ or } i \neq 0,$$

while in the second,

$$\frac{\partial^\beta H_v^{b_{k-\phi}}}{\partial y^\beta}(y_j) = \frac{\partial^\beta H_v^{b_{k-\phi+1}}}{\partial y^\beta}(y_j) = 0 \text{ if } \beta \neq v \text{ or } j \neq 0.$$

Hence in this case all the terms, hence also (5), are zero. Hence we must have $i = j = 0$, and either $\alpha = u$ or $\beta = v$.

(Case 2) $\alpha = u$, $i = j = 0$. Since the u^{th} derivative of $H_u^{a_\phi}$ at x_0 is 1, the only part that is left of our (5) is the β^{th} derivative at $y = y_0$ of

$$(H_v^{b_{k-l+1}} - H_v^{b_{k-l}}) + \ldots + (H_v^{b_2} - H_v^{b_1}) + H_v^{b_1} = H_v^{b_{k-l+1}}.$$

On the other hand, since $(u, \beta) \in A$, the first inequality in (3) (and the main properties of the exterior boundary points, see 3.2) implies that $\beta \leq b_{k-l+1}$. Hence our derivative is zero if $\beta \neq v$, and is 1 if $\beta = v$.

(Case 3) $\beta = v$, $\alpha \neq u$, $i = j = 0$. Then the derivative of order α at $x = x_0$ of $H_u^{a_\phi}$ may be non-zero only when $\alpha > a_\phi$. But then, since $(\alpha, v) \in A$, it follows that $v \leq b_{k-\phi}$, and the v^{th} derivatives at $y = y_0$ of both $H_v^{b_{k-\phi}}(y)$ and $H_v^{b_{k-\phi+1}}(y)$ equal zero. Hence, for all ϕ's, the (α, v)-derivative at (x_0, y_0) of $H_u^{a_\phi}(x)(H_v^{b_{k-\phi+1}}(y) - H_v^{b_{k-\phi}}(y))$ equal zero. In particular, (5) is zero.

This concludes the proof of Theorem 6.2.

□

7. Rectangular sets of derivatives, and beyond

In this section we investigate the interpolation problem with rectangular sets of nodes, in the case where the set A of derivatives has a rectangular shape. However, the results that we prove only make use of certain properties of rectangular sets. (Hence, as the title of the section suggests, they actually hold much more generally.) Let us first describe these properties, and state the results. The first such property is that $|A| = |A_x||A_y|$ for rectangular A's. We will prove

Theorem 7.1. *If (A, S) is almost regular with respect to (p, q)-rectangular sets of nodes, then $|A| \leq |A_x||A_y|$. If $|A| = |A_x||A_y|$, then S must be*

$$S = R(p', q'), \quad p' = (p+1)|A_x| - 1, \ q' = (q+1)|A_y| - 1.$$

The next property of rectangular sets is that $S_y(A)$ is rectangular too. The property that $S_y(A)$ is $R(s,t)$-rectangular is equivalent to saying that

$$A_x = \{\alpha : (\alpha, 0) \in A\}$$

has $(s+1)$ distinct elements, and that the sets

$$A_y[\alpha] = \{\beta : (\alpha, \beta) \in A\}, \quad \alpha \in A_x$$

all have the $(t+1)$ elements. In other words, $S_y(A)$ is rectangular if and only if it is of form

$$A = \{(\alpha_i, \beta_i^j) : 0 \leq i \leq s, 0 \leq j \leq t\},$$

where all the α_i's, as well as all the β_i^j for each i, are distinct. Then $A_x = \{\alpha_0, \ldots, \alpha_s\}$, and $A_y[\alpha_i] = \{\beta_i^0, \ldots, \beta_i^t\}$. We will show that

Theorem 7.2. *If Z is a (p, q)-rectangular set of nodes, and A has the property that $S_y(A)$ is a $R(s,t)$- rectangular (in particular, if A is $R(s,t)$- rectangular), then the scheme (Z, A, S) is regular if and only if*

(i) $S = R(s,t)^{p,q}$.
(ii) *The univariate scheme (Z_x, A_x, S_x) is regular.*
(iii) *All univariate schemes $(Z_y, A_y[\alpha], S_y)$, with $\alpha \in A_x$, are regular.*

Before turning to the proofs of these results, let us mention the particular case:

Corollary 7.3. *If Z is a (p,q)-rectangular set of nodes, and A is (s,t)-rectangular, then the interpolation scheme (Z,A,S) is regular if and only if*

(i) $S = R(s,t)^{p,q}$.
(ii) *the univariate schemes (Z_x, A_x, S_x), and (S_y, A_y, S_y) are regular.*

Proof. (of Theorem 7.1): We first claim that $|S_x| \leq (p+1)|A_x|$. If $S_x = \emptyset$, this is obvious. On the other hand if $S_x \neq \{0\}$, then choosing $L = S_x$ in Theorem 4.6, we get $(p+1)(q+1)|A_x| \geq |S_x| + pq|A_x| + q|A_x|$, which is precisely the desired inequality. Similarly, $|S_y| \leq (q+1)|A_y|$. Multiplying these two inequalities we get $|S_x||S_y| \leq (p+1)(q+1)|A_x||A_y|$. On the other hand, since S is a lower set, $|S_x||S_y| \geq |S|$ (with equality if and only if S is a rectangle). Also, since the scheme is assumed to be regular, we have $|S| = (p+1)(q+1)|A|$. Combining all these, the desired inequality follows immediately, and the equality holds only when all the previous inequalities become equalities (hence, in particular, S must be the rectangle $R(p', q')$). □

Proof. (of Theorem 7.2). From Theorem 7.1 we know that the regularity implies (i) of our theorem. Hence we have to prove that, if $S = R(s,t)^{p,q}$, and A is as in the statement, then the regularity of (Z,A,S) is equivalent to (ii) and (iii). For this we compute the determinant $D(Z,A,S)$. Again, we will do this by using some more general properties of determinants. Let us first fix some notations. First of all, for any matrix A, we denote by $c(A)$, and $l(A)$, the number of its columns, and of its rows, respectively. For any two matrices M, and $A = (a_{i,j})$, we consider the "tensor product" matrix $M \otimes A$ which equals to

$$M \otimes A = \begin{pmatrix} a_{1,1}M & a_{1,2}M & \dots & a_{1,c(A)}M \\ a_{2,1}M & a_{2,2}M & \dots & a_{2,c(A)}M \\ \dots & \dots & \dots & \dots \\ a_{l(A),1}M & a_{l(A),2}M & \dots & a_{l(A),c(A)}M \end{pmatrix}$$

Note that $l(M \otimes A) = l(M)l(A)$, $c(M \otimes A) = c(M)c(A)$. With this notation, remark that the matrix $M(Z,A,S)$ of the interpolation problem (and whose determinant is $D(Z,A,S)$) is, up to a re-arrangement of its lines and columns, of type

$$M(Z,A,S) = \begin{pmatrix} M_0 \otimes A_0 \\ M_1 \otimes A_1 \\ \dots \\ M_s \otimes A_s \end{pmatrix}$$

where the matrices M_i and A_i are defined as follows. To describe A_i, we consider the row

$$l_x(x) = (1 \ x \ \dots \ x^{p'})$$

and the rows of A_i will be

$$\frac{\partial^\alpha l_x}{\partial x^\alpha}(x_0), \dots, \frac{\partial^\alpha l_x}{\partial x^\alpha}(x_p),$$

with $\alpha = \alpha_i$. Hence

$$c(A_i) = p' + 1, l(A_i) = p + 1.$$

To describe M_i, we consider the row $l_y(y) = (1\ y\ \dots\ y^{q'})$, and the rows of M_i will be

$$\begin{array}{c} \frac{\partial^{\beta_i^0} l_y}{\partial y^{\beta_i^0}}(y_0), \quad \dots \quad, \frac{\partial^{\beta_i^0} l_y}{\partial y^{\beta_i^0}}(y_q) \\ \dots \quad \dots \quad \dots \\ \frac{\partial^{\beta_i^t} l_y}{\partial y^{\beta_i^t}}(y_0), \quad \dots \quad, \frac{\partial^{\beta_i^t} l_y}{\partial y^{\beta_i^t}}(y_q) \end{array}$$

We now need one more notation. For square matrices $\mathcal{M}_i$, $0 \leq i \leq k$ (k is any non-negative integer), we consider

$$\operatorname{diag}(\mathcal{M}_0, \dots, \mathcal{M}_k) = \begin{pmatrix} \mathcal{M}_0 & 0 & \dots & 0 \\ 0 & \mathcal{M}_1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathcal{M}_k \end{pmatrix},$$

while for any square matrix $\mathcal{M}$ we put

$$\operatorname{diag}_k(\mathcal{M}) = \operatorname{diag}(\underbrace{\mathcal{M}, \dots, \mathcal{M}}_{k}).$$

One clearly has $\det(\operatorname{diag}(\mathcal{M}_0, \dots, \mathcal{M}_k)) = \prod_i \det(\mathcal{M}_i)$. With these, the tensor product of a square matrix M (with m lines and m columns), with an arbitrary matrix A is

$$M \otimes A = \operatorname{diag}_{l(A)}(M)(I_m \otimes A).$$

Also, for any matrices $\mathcal{M}_i$, $\mathcal{A}_i$, $0 \leq i \leq s$, with

$$c(\mathcal{M}_i) = l(\mathcal{A}_i),\ \sum l(\mathcal{M}_i) = c(\mathcal{A}_0) = \dots = c(\mathcal{A}_s),$$

one has

$$\begin{pmatrix} \mathcal{M}_0 \otimes \mathcal{A}_0 \\ \mathcal{M}_1 \otimes \mathcal{A}_1 \\ \dots \\ \mathcal{M}_s \otimes \mathcal{A}_s \end{pmatrix} = \operatorname{diag}(\mathcal{M}_0, \dots, \mathcal{M}_s) \begin{pmatrix} \mathcal{A}_0 \\ \mathcal{A}_1 \\ \dots \\ \mathcal{A}_s \end{pmatrix}$$

Coming back to our determinant, we apply the previous formula to $\mathcal{M}_i = diag_{p+1}(M_i)$, $\mathcal{A}_i = I_{q'+1} \otimes A_i$, and we get(up to a sign)

$$D(Z, A, S) = \prod_i \det(\operatorname{diag}_{p+1}(M_i)) det \begin{pmatrix} I_{q'+1} \otimes A_0 \\ \dots \\ I_{q'+1} \otimes A_s \end{pmatrix}$$

After a rearrangement of the lines and of the columns, the last matrix is precisely $I_{q'+1} \otimes M(Z_x, A_x, S_x)$, hence has determinant $D(Z_x, A_x, S_x)^{q'+1}$. Also, since $M_i = M(Z_y, A_y(\alpha_i), S_y)$, we deduce that, up to a sign,

$$D(Z, A, S) = (\prod_i D(Z_y, A_y(\alpha_i), S_y))^{p+1} D(Z_x, A_x, S_x)^{q'+1}.$$

This clearly implies the assertion in the statement. □

8. No mixed derivatives, and beyond

Here we discuss the interpolation problem with rectangular sets of nodes, where no mixed derivatives appear in the interpolation conditions (that is, A is has only points on the coordinate axes). As in the previous section, we will actually prove several general results which, in the case under discussion, give a complete answer. Let us first recall that, for any S, and any j, $S_x[j]$ is the set of those i's with the property that $(i,j) \in S$. The following shows that the lowers sets of interest are, near the axes, uniquely determined by the number of derivatives on the axes.

Theorem 8.1. *If Z is a (p,q)-rectangular set of nodes, and (Z,A,S) is regular, then*

$$\begin{aligned} S_x[0] = S_x[1] = \ldots = S_x[q] &= \{0,1,\ldots,p'\}, \\ S_y[0] = S_y[1] = \ldots = S_y[p] &= \{0,1,\ldots,q'\}, \end{aligned}$$

where, as in Theorem 7.1, $p' = (p+1)|A_x| - 1$, $q' = (q+1)|A_y| - 1$.

Next, we remove from S the lowest (p,q)-rectangle on its most right (that is, remove the last $(p+1)$ elements of each of the sets $S_x[i]$, $0 \leq i \leq q$), to get a new set S', which is lower if and only if $|S_x(q+1)| \leq p' - (p+1)$. We also denote by A_{min} the set obtained from A by replacing the elements of A_x (i.e. on the OX axes) with the set $\{0,1,2,\ldots,|A_x|-1\}$, and by A_{max} the one obtained by replacing the elements of A_x with the set $\{0, p+1, 2(p+1), \ldots, (p+1)(|A_x|-1)\}$ (note that A_{max} depends on p). Removing from A_{max} its last element on the the OX axes gives a new set denoted A'_{max}, and similarly one defines A'_{min}. Note that (Z, A'_{max}, S') are new (smaller) interpolation schemes. The following is a first result which shows that some points in A can be moved (or even removed), transforming the given scheme into a simpler one.

Theorem 8.2. *Let Z be a (p,q)-rectangular set of nodes, and let A and S be given. Then the following are equivalent:*

(i) *(Z,A,S) is regular.*
(ii) *(Z,A_{max},S), and the univariate scheme (Z_x,A_x,S_x), are regular.*
(iii) *(Z,A_{min},S), and the univariate scheme (Z_x,A_x,S_x), are regular.*

Moreover, if S' is a lower set, then (i)-(iii) above are equivalent also to

(iv) *(Z,A'_{max},S'), and the univariate scheme (Z_x,A_x,S_x), are regular.*
(v) *(Z,A'_{min},S'), and the univariate scheme (Z_x,A_x,S_x), are regular.*

This theorem obviously has a version with the coordinate y instead of x. Using this result we immediately see that

Proposition 8.3. *If Z is a (p,q)-rectangular set of nodes, and A contains no mixed derivatives, then the scheme (Z,A,S) is regular if and only if*

(i) *$S = R(s,t)^{p,q}$, where $s = |A_x| - 1$, $t = |A_y| - 1$.*
(ii) *The univariate schemes (Z_x,A_x,S_x), (Z_y,A_y,S_y) are regular.*

Proof. (of Theorem 8.1, and Theorem 8.2) Write S as

$$S = S_x(m_0, m_1, \ldots, m_k),$$

and let $\alpha_0, \ldots, \alpha_s$ be the derivatives on the OX axes. We then have $|S| = \sum(m_i + 1)$, $|S_x| = m_0 + 1$, $|A_x| = s$. In particular, from the proof of Theorem 7.1 we know that

$$(m_0+1) \leq (p+1)(s+1) \tag{6}$$

We rewrite the determinant $D(Z, A, S)$. For this we introduce the notation $l_m(x)$ (where m is any positive integer) for the row matrix

$$l_m(x) = (1\ x\ \dots x^m),$$

and define $\mathcal{M}(m)$ as the matrix (depending on $(x_0, x_1, \dots, x_p)$) which has as rows

$$\begin{array}{ccc} \frac{\partial^{\alpha_0} l_m}{\partial x^{\alpha_0}}(x_0), & \dots & , \frac{\partial^{\alpha_0} l_m}{\partial x^{\alpha_0}}(x_p) \\ \dots & \dots & \dots \\ \frac{\partial^{\alpha_s} l_m}{\partial x^{\alpha_s}}(x_0), & \dots & , \frac{\partial^{\alpha_s} l_m}{\partial x^{\alpha_s}}(x_p) \end{array}$$

Note that $\mathcal{M}(m)$ has $l(\mathcal{M}(m)) = (p+1)(s+1)$ rows, and $c(\mathcal{M}(m)) = m+1$ columns, and $\mathcal{M}(m_0)$ is precisely the matrix associated to the univariate interpolation scheme (Z_x, A_x, S_x). On the other hand, after eventually rearranging some rows and columns, the matrix $M(Z, A, S)$ has the form

$$M(Z, A, S) = \begin{pmatrix} \mathcal{M}(m_0) & y_0\mathcal{M}(m_1) & \dots & y_0^k\mathcal{M}(m_k) \\ \mathcal{M}(m_0) & y_1\mathcal{M}(m_1) & \dots & y_1^k\mathcal{M}(m_k) \\ \dots & \dots & \dots & \dots \\ \mathcal{M}(m_0) & y_q\mathcal{M}(m_1) & \dots & y_q^k\mathcal{M}(m_k) \\ \mathcal{X}_0 & \mathcal{X}_1 & \dots & \mathcal{X}_k \end{pmatrix}$$

where $\mathcal{X}_u$ consists of the $A \setminus A_x$-derivatives of $y^u l_{m_u}(x)$, evaluated at the nodes. Next, after elementary transformations we can kill the first row of matrices except for $\mathcal{M}(m_0)$ (one multiplies the first (m_1+1) rows by y_0 and subtract them from the rows sitting below $y_0\mathcal{M}(m_1)$, then one multiplies the first (m_2+1) rows by y_0^2 and subtract them from the rows sitting below $y_0^2\mathcal{M}(m_2)$, etc). The resulting matrix will have zero elements at all the intersections of the last

$$(m_1+1) + \dots + (m_k+1) = |S| - (m_0+1)$$

columns with the first $(p+1)(s+1)$ rows. Applying Lemma 4.1 we deduce that $(m_0+1) \geq (p+1)(s+1)$, which shows that (6) must become equality. In particular, $\mathcal{M}(m_0)$ is a square matrix. Now we can write $M(Z, A, S)$ as a product between

$$\begin{pmatrix} \mathcal{M}(m_0) & 0 & \dots & 0 & 0 \\ 0 & \mathcal{M}(m_0) & \dots & 0 & 0 \\ 0 & 0 & \dots & \mathcal{M}(m_0) & 0 \\ 0 & 0 & \dots & 0 & I \end{pmatrix}$$

(where $\mathcal{M}(m_0)$ appears $(q+1)$-times, and I is the unit matrix of dimension $|S| - (q+1)(m_0+1)$), and

$$M'(Z, A \setminus OX, S) = \begin{pmatrix} I(m_0, m_0) & y_0 I(m_0, m_1) & \dots & y_0^k I(m_0, m_k) \\ I(m_0, m_0) & y_1 I(m_0, m_1) & \dots & y_1^k I(m_0, m_k) \\ \dots & \dots & \dots & \dots \\ I(m_0, m_0) & y_q I(m_0, m_1) & \dots & y_q^k I(m_0, m_k) \\ \mathcal{X}_0 & \mathcal{X}_1 & \dots & \mathcal{X}_k \end{pmatrix}$$

where $I(m, n)$ stands for the matrix with $m+1$ rows and $n+1$ columns which has 1's on the positions (i, i) and zero in rest. Note that, as the notation suggests, the last matrix only depends on Z, S and $A \setminus OX$ and not on A_x. We denote its determinant by $D'(Z, A \setminus OX, S)$. Hence

$$D(Z, A, S) = D(Z_x, A_x, S_x)^{q+1} D'(Z, A \setminus OX, S)$$

Applying this to $D(Z, A_{min}, S)$ we then see that

$$D(Z, A, S) = D(Z, A_{min}, S) \frac{D(Z_x, A_x, S_x)^{q+1}}{D(Z_x, \{0, 1, \dots, k\}, S_x)^{q+1}} ,$$

and similarly for A_{max}, and also for x and y interchanged. This proves the equivalence of (i)-(iii) in Theorem 8.2. To prove theorem 8.1, we denote by a the largest element with the property that $m_0 = m_1 = \dots = m_a$, and we will prove that $a \geq q$. We will do this by looking at the zeros of $M'(Z, A \setminus OX, S)$, and using Lemma 4.1 again. First of all, the rows that we consider are the ones on the positions $j(m_0+1)$ with $0 \leq j \leq q$, i.e. the last row of each of the matrices

$$(I(m_0, m_0) \quad y_j I(m_0, m_1) \quad \dots \quad y_j^k I(m_0, m_k)).$$

In total, there are $(q + 1)$ rows. The columns that we consider are all but the columns situated on the positions $((m_0 + 1) + \dots + (m_u + 1))$ with $0 \leq u \leq a$, i.e. all but the last column of each of the matrices

$$\begin{pmatrix} y_0^u I(m_0, m_u) \\ y_1^u I(m_0, m_u) \\ \dots \\ y_q^u I(m_0, m_u) \\ \mathcal{X}_u \end{pmatrix}$$

In total, there are

$$m_0 + \dots + m_a + (m_a + 1) + \dots (m_k + 1) = |S| - (a + 1)$$

columns. Clearly, at the intersections of all these rows and columns there are only zeros, hence, by Lemma 4.1, we must have $(q+1) + |S| - (a+1) \leq |S|$, hence $a \geq q$. We still have to show the last part of Theorem 8.2. For this we use the remark 4.2 which describes the limit situation of Lemma 4.1. We then see that $D'(Z, A \setminus OX, S)$ is the product of two determinants. One of them is the Vandermonde determinant $V(y_0, \dots, y_q)$, while to the other one we can apply remark 4.2 again. Continuing this process, after applying remark 4.2 $(p+1)$ times, we see that $D'(Z, A \setminus OX, S)$ is equal to the product between $V(y_0, \dots, y_q)^{p+1}$ and the determinant of the matrix

$$\begin{pmatrix} I(m'_0, m'_0) & y_0 I(m'_0, m'_1) & \dots & y_0^k I(m'_0, m'_k) \\ I(m'_0, m'_0) & y_1 I(m'_0, m'_1) & \dots & y_1^k I(m'_0, m'_k) \\ \dots & \dots & \dots & \dots \\ I(m'_0, m'_0) & y_q I(m'_0, m'_1) & \dots & y_q^k I(m'_0, m'_k) \\ \mathcal{X}'_0 & \mathcal{X}'_1 & \dots & \mathcal{X}'_k \end{pmatrix}$$

where $m'_j = m_j - (p + 1)$ for $0 \leq j \leq q$, and $m'_j = m_j$ for $j > q$. Clearly, $S' = S_x(m'_0, \dots, m'_k)$. Also, $\mathcal{X}'_j = \mathcal{X}_j$ for $j > q$, while for $j \leq q$, $\mathcal{X}'_j$ is obtained from $\mathcal{X}_j$ be removing its last $(p+1)$ columns. Hence this matrix is precisely $M'(Z, A'_{min} \setminus OX, S')$. Using the formulas above, we get

$$D(Z, A, S) = D(Z, A'_{min}, S') V(y_0, \dots, y_q)^{p+1} \frac{D(Z_x, A_x, S_x)^{q+1}}{D(Z_x, \{0, 1, \dots, k\}, S_x)^{q+1}} ,$$

and this clearly proves the last part of Theorem 8.2. $\square$

Note that, from the previous proof, we have

Corollary 8.4. *If Z is a (p,q)-rectangular set of nodes, and A contains no mixed derivatives, then (up to a sign)*

$$D(Z,A,S) = D(Z_x,A_x,S_x)^{q+1} D(Z_y,A_y,S_y)^{p+1}.$$

9. The case $p = q = 1$

We now shortly look at the case where $p = q = 1$, i.e. where the set of nodes is of type

$$Z = \{(x_0,y_0),(x_0,y_1),(x_1,y_0),(x_1,y_1)\},$$

with $x_0 \neq x_1$, $y_0 \neq y_1$. Special to this case is that regularity is equivalent to almost regularity. We will show

Theorem 9.1. *Given A and S, the following are equivalent:*

(i) *The scheme (A,S) is regular for some $(1,1)$-rectangular set of nodes Z.*
(ii) *The scheme (Z,A,S) is regular for any $(1,1)$-rectangular set of nodes Z.*

In this case moreover,

(a) *$S = R^{1,1}$ for some lower set R;*
(b) *For any lower set $L \subset S$, one has $n_{1,1}(L) \leq |A \cap L|$.*

where $n_{1,1}(L)$ denotes the number of elements of L with even coordinates.

Proof. For any polynomial P we consider the associated polynomial P_0 defined by

$$P_0(x,y) = P(2\frac{x-x_0}{x_1-x_0} - 1, 2\frac{y-y_0}{y_1-y_0} - 1).$$

Using also a rescaling of the constants appearing in the interpolation problem, we see that the regularity of the scheme (Z,A,S) is equivalent to the regularity of the scheme (Z_0,A,S), where

$$Z = \{(-1,-1),(-1,1),(1,-1),(1,1)\}.$$

This proves the equivalence of (i) and (ii). We now show that the regularity of (Z_0,A,S) implies that $S = R^{1,1}$ for some lower set R. We will show that $R = S_{1,1}$ does the job (for notations, see 3.3 of Section 3). By the very definition of $R = S_{1,1}$, we have $S \subset R^{1,1}$, and we will show that both sets have the same number of elements. Assume that $|S| < |R^{1,1}| = 4|R|$. Since $|S| = 4|A|$, the assumption is $|R| > |A|$. We now consider the space $\mathcal{P}_{2R}$ of polynomials spanned by $x^{2\alpha}y^{2\beta}$ with $(\alpha,\beta) \in R$. Note that $\mathcal{P}_{2R} \subset \mathcal{P}_S$. Next, since the polynomials in $\mathcal{P}_{2R}$ depend on $|R| > |A|$ variables (the coefficients), we find a non-zero polynomial $P \in \mathcal{P}_{2R}$ such that

$$\frac{\partial^{i+j}P}{\partial x^i \partial y^j}(1,1) = 0, \quad \forall \ (i,j) \in A \tag{7}$$

But $P \in \mathcal{P}_{2R}$ implies that $P(-x,-y) = P(-x,y) = P(x,-y) = P(x,y)$. Taking derivatives, this relations still hold true, at least up to a sign. In particular, (7) implies that

$$\frac{\partial^{i+j}P}{\partial x^i \partial y^j}(x,y) = 0, \quad \forall \ (i,j) \in A,$$

holds for all $x,y \in \{-1,1\}$. This produces a non-zero polynomial $P \in \mathcal{P}_{2R} \subset \mathcal{P}_S$ which satisfies the interpolation equation where all the constants zero, which contradicts the regularity of (Z_0, A, S). Assume now that there exists a lower subset $L \subset S$ such that $n_{1,1}(L) > |A \cap L|$. This precisely mens that $|L_{1,1}| > |A \cap L|$. Exactly as above, we find a non-zero polynomial P in $\mathcal{P}_L$ with the property that all its derivatives coming from $A \cap L$ are zero at the points $(1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$. On the other hand, since (L is lower and) $P \in \mathcal{P}_L$, all its derivatives coming from $A \setminus L$ are identically zero. Hence P would be a non-trivial solution of the original problem where all the constants are zero, and that contradicts the regularity. $\square$

Remark 9.2. The extreme cases $p = 0$ or $q = 0$ are treated similarly. Let us mention here what happens when $q = 0$, i.e. $Z = \{(x_i, y_0) : 1 \leq i \leq p\}$ consists of $(p+1)$ points on a line parallel to the OX axes. Then (Z, A, S) is regular if and only if:

(i) $S = R^{p,0}$ for some lower set R.
(ii) on each of the lines $y = constant$, A and R have the same number of elements.
(iii) for each j, the univariate problem $(Z_x, A_x[j], S_x[j])$ is regular.

Recall that $A_x[j] = \{i : (i,j) \in A\}$, and similarly $S_x[j]$. The proof uses the same techniques that we have already used. As in the proof above, one may assume that $y_0 = 0$, and then one sees that the determinant will be the product of the determinants of the univariate problems $(Z_x, A_x[j], S_x[j])$.

10. Examples

In this section we present several examples that illustrate the results of the previous sections. For simplicity, we restrict most of the examples to the case $p = q = 1$, i.e. the case of $(1,1)$-rectangular sets of nodes. One of the simplifications comes from the fact that, in this case, the notions of regular and almost regular coincide (cf. the last section). Hence, unless otherwise specified, the term "regular" in this section stands for "regular with respect to $(1,1)$-rectangular sets of nodes". Passing to the general case of (p,q)-rectangular sets of nodes require some care on almost regularity versus regularity, but many of the arguments remain the same.

Example 10.1. Let A be as in Fig. 5. Then there is no lower set S which makes (A, S) into a regular scheme. Assume there is one. First of all, it must be $R(5,5)$, as implied by Theorem 7.1. In this case however, condition (b) of Theorem 9.1 is violated by the lower set L drawn in the picture. In particular,

$$s(Z_{1,1}, A) = 0.$$

Note also that after moving any of the first two points of A situated on the line $y = 3$ one step downwards, the condition is no longer violated, and one can actually show that the resulting schemes are regular. However, moving the last point on that line (i.e. $(3,3)$) one step downwards, produces a scheme which is still non-regular since the same condition is violated (this time by L' shown in Fig. 6).

Example 10.2. Consider now A as in Fig. 7. As above, regularity forces $S = R(5,5)$. This time however, condition (b) of Theorem 9.1 is satisfied. Let us show that (A, S) is regular. We first remark that $S_y(A)$ is $(3,3)$-rectangular, hence we can use Theorem 7.2 to reduce the regularity of (A, S) to the regularity of several univariate schemes. In turn, the univariate schemes are being taking care of by the Polya condition (see Remark. 4.5). When $p, q \geq 1$ the same arguments apply

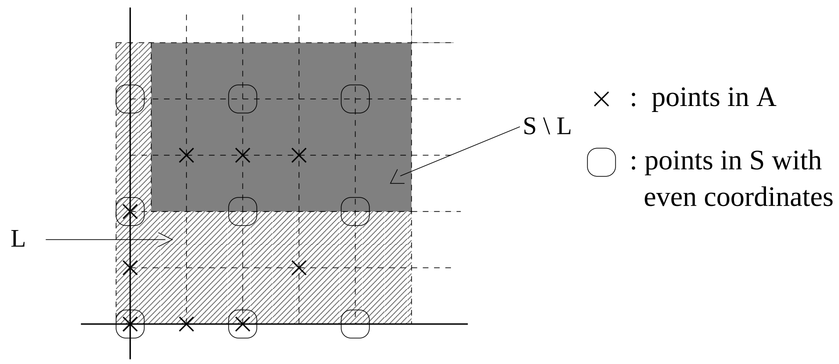


FIGURE 5

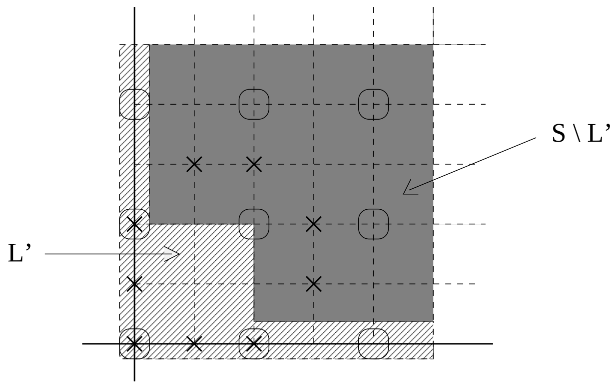


FIGURE 6

to conclude that (A, S) is almost regular (with respect to (p, q)-rectangular sets of nodes) if and only if $S = R(2p + 3, 2q + 3)$. In particular,

$$s(Z_{p,q}, A) = 1.$$

Moreover, given the set of nodes Z, Theorem 7.2 rephrases the regularity of (Z, A, S) in terms of the regularity of certain induced univariate schemes which are easier to handle.

Example 10.3. We consider now the variation of the previous example shown in Fig. 8. One cannot apply Theorem 7.2 directly, but one can first invoke Theorem 8.2 to re-arrange the points of A on OX to occupy the first three positions. Then $S_y(A)$ becomes rectangular, and Theorem 8.2 can be used. Alternatively, one can use Theorem 8.2 twice (once on each of the axes) to and then reduce the problem to the one of the previous example.

Example 10.4. Let A be as in Fig. 9. Using Theorem 8.1, we see that S must contain S_0 shown in the picture, and it must be contained in $R(5, 5)$. Using (a) of Theorem 9.1 we see that S obtained from S_0 together with a copy of the rectangle $R(1, 1)$. But there are only two ways one can add such a rectangle to S_0 to obtain a lower set, and the two possibilities are shown as Case 1 and Case 2 in the picture.

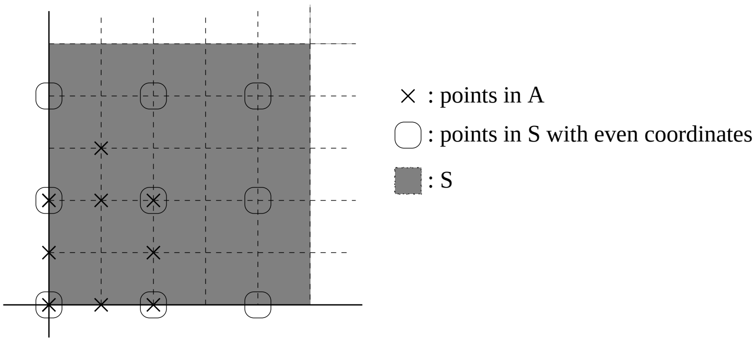


FIGURE 7

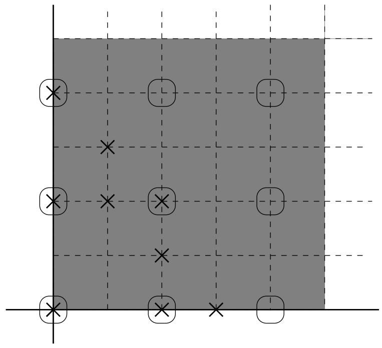


FIGURE 8

In the first case, condition (b) of Theorem 9.1 applied to $L = S \setminus \{(3,1)\}$ is not satisfied. The situation is different in the second case, when we obtain a regular scheme. To see this, one first uses Theorem 8.2 to remove the last point of A situated on OY, and then one treats the remaining scheme as in Example 10.2. In particular, $s(Z_{1,1}, A) = 1$.

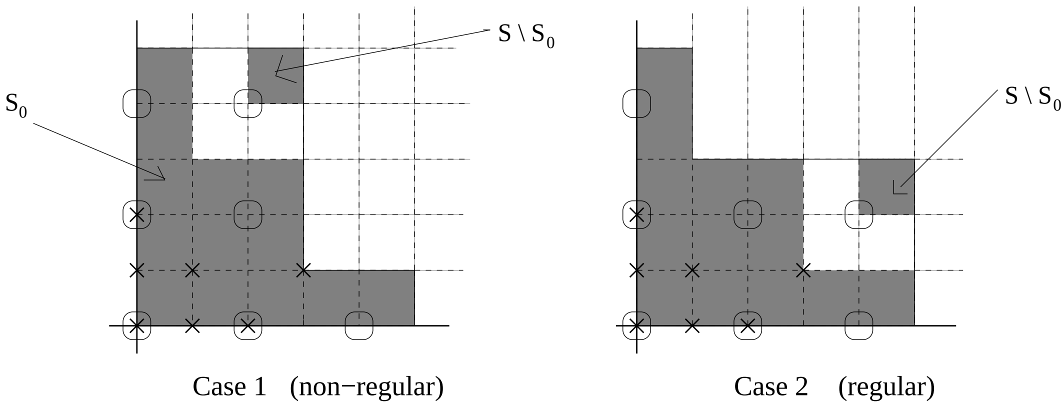
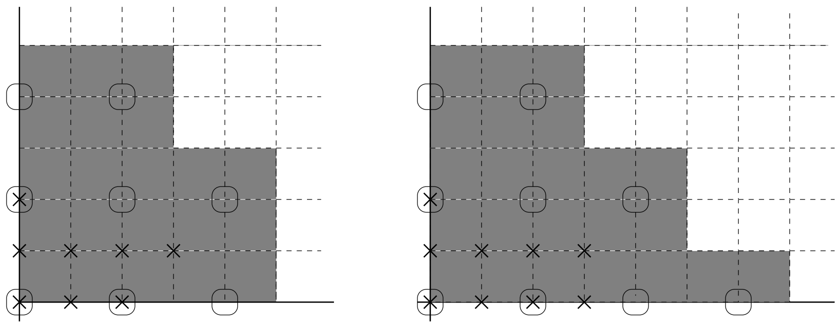


FIGURE 9

Example 10.5. Let us now describe two examples which show that (b) of Theorem 9.1 does not imply regularity with respect to $(1,1)$-rectangular sets of nodes. Consider the scheme (A, S) appearing on the left hand side of Fig. 10. It does satisfy the desired condition, but it is not regular. To see this, we remark that A can be obtained from $\tilde{A}$ (see right hand side of Fig. 10) by removing the last element from OX. Hence we can apply Theorem 8.2 to $(\tilde{A}, \tilde{S})$ to conclude that it is regular if (A, S) is. But this cannot happen because the regularity of $(\tilde{A}, \tilde{S})$ and the fact that $\tilde{A}$ is lower would imply that $\tilde{S} = \tilde{A}^{1,1}$ (cf. Theorem 6.2), which is not the case. We should say here that what causes the non-regularity in this example



The scheme (A, S) (on the left) is obtained from (Ã, S̃) (on the right)
by removing the last element of A on OX

FIGURE 10

is another simple condition that must be satisfied by all regular schemes (and, in this example, it is not): the number of points of A on the line $y = 1$ cannot exceed those on the line $y = 0$.

Example 10.6. A bit more subtle is the example drawn in Fig. 11, which still satisfies condition (b) of Theorem 9.1. We advise the reader to try to guess a "general regularity condition" that is broken in this example (see also our next section). Any way, here is an argument that proves that (A, S) cannot be regular. Assume it is. We first move the last two elements of A on OX to new positions to get the scheme (A', S') of Fig. 11. By Theorem 8.2 (and Remark 4.5), (A', S') is still regular. Now, choose L as in the picture. One has $|L| = 4|A \cap L|$, hence, by Corollary 4.4, $(A \cap L, L)$ must be regular (with respect to $(1,1)$-rectangular sets of nodes). But, by the same arguments as in the example above, $(A \cap L, L)$ cannot be regular.

Example 10.7. Let us return to the set A appearing in the last example in 10.5 (Fig. 11), and look for all S's which make (A, S) regular. As in Example 10.4, there are two possible cases. One of them is precisely the one treated in Example 10.5, while the other one is shown in Fig. 12. This new one is regular. To see this, one first uses Theorem 8.2 to remove the last point of A situated on OY. The resulting scheme (A', S') has $S_x(A)$ rectangular, hence we can use Theorem 7.2 (or, more precisely, the version obtained by interchanging x and y). The subsequent univariate schemes are regular by 4.5 again.

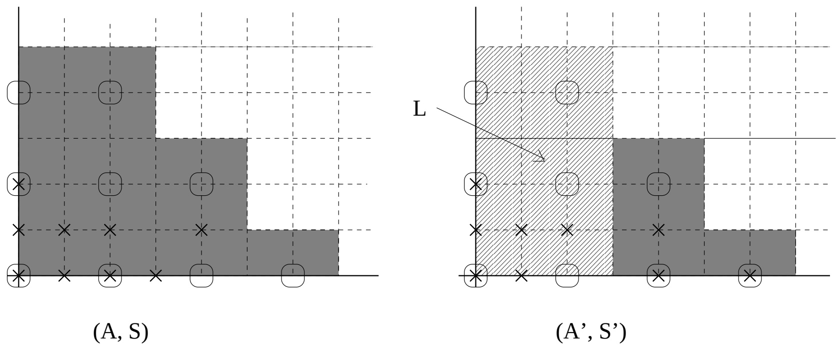


FIGURE 11

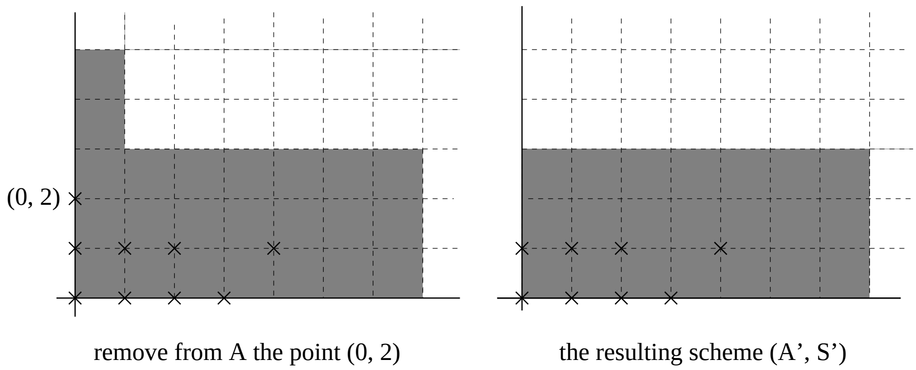


FIGURE 12

Example 10.8. A similar example is obtained by considering A as in Fig. 13. As before, S must be obtained from S_0 in the picture by adding one copy of $R(1,1)$. There are two possible ways to do that, but only the one shown in the picture produces a regular scheme. However, we do not know how to use the general results of the previous sections to prove the non-regularity of the other scheme.

Example 10.9. Consider now A as in Fig. 14. As before, Theorem 8.1 tells us what S must be around the axes. We then have to fit three more (disjoint) copies of $R(1,1)$ inside $R(5,5)$ to get the lower set S. This time, this is possible in only one way (as in the picture). One sees that (b) of Theorem 9.1 is satisfied, and one can actually show that the scheme is regular.

Example 10.10. In all the previous examples, given A, there was at most one S making (A,S) regular. Here is an example where two such S's can be chosen. Consider A as in Fig. 15. As in the previous examples, S is contained in $R(5,5)$ and must be obtained by adding one copy of $R(1,1)$ to the blow up $(T_2)^{1,1}$ of the triangle T_2. This is possible in two ways, with the resulting S's: $S_y(2,1,1)^{1,1}$ and $S_y(2,2,1)^{1,1}$ (see also Fig. 17). That both resulting schemes are regular follows again by removing one point and then using Theorem 7.2. In particular,

$$s(Z_{1,1}, A) = 2.$$

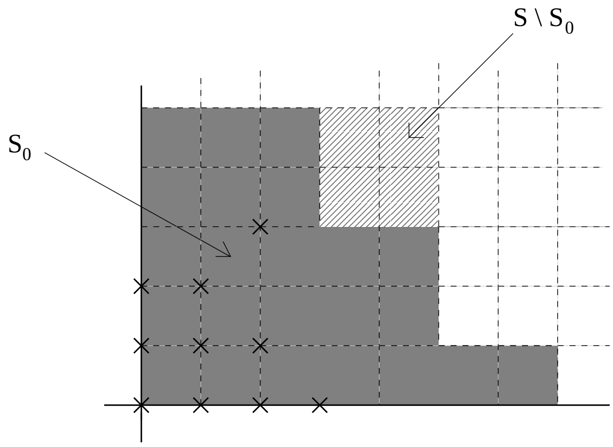


FIGURE 13

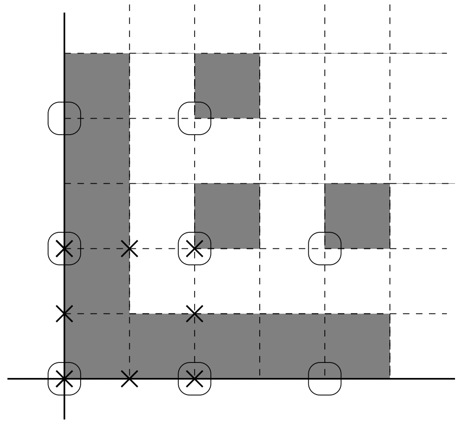


FIGURE 14

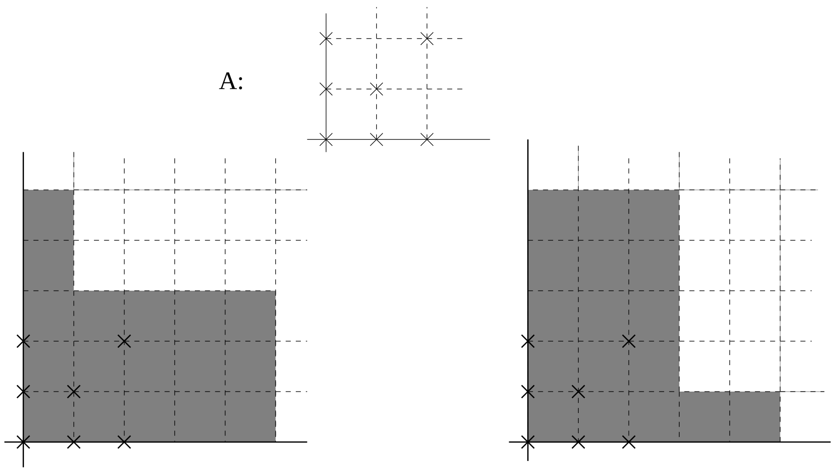
Example 10.11. Using the results of the previous sections one can describe all the possible schemes (Z, A, S) which are regular with respect to (p, q)-rectangular sets of nodes, and which involve at most five derivatives (i.e. $|A| \leq 5$). Details will be given elsewhere.

11. Further regularity conditions, and some open problems

In this section we point out some general features of the interpolation problem with rectangular sets of nodes (as they arise from our results) by formulating and discussing several conjectures. Hence, throughout this section, Z will denote a (p, q)-rectangular set of nodes, and (Z, A, S) will be an interpolation scheme.

11.1. What is the shape of S?

What is probably most striking about all the interpolation schemes we have looked at is the following property that we conjecture



Two different regular schemes with the same set of derivatives A

FIGURE 15

Conjecture 1. *If a scheme (A, S) is almost regular with respect to (p, q)-rectangular sets of nodes, then $S = R^{p,q}$ for some lower set R.*

Note that we have proven this conjecture in many unrelated cases: when $p = q = 1$, when $|A| = |A_x||A_y|$ (in particular, when A is rectangular, or when one of the sets $S_x(A)$ or $S_y(A)$ are rectangular), when A is a lower set, when A has points only on the coordinate axes, as well as in the examples presented in the previous section.

11.2. Stronger Polya conditions?

For a lower set S, we denote by $n_{p,q}(S)$ the number of elements $(\alpha, \beta) \in S$ with the property that α is divisible by $(p+1)$, and β is divisible by $(q+1)$. This is inspired by Theorem 9.1, as is the following

Conjecture 2. *If a scheme (A, S) is almost regular with respect to (p, q)-rectangular sets of nodes, then*

$$|L \cap A| \geq n_{p,q}(L)$$

for all lower sets $L \subset S$.

This conjecture is particularly interesting because, as we will show, it is related to (and relates): Conjecture 1, the Polya conditions of Corollary 4.3, as well as the "geometrical properties" of regular schemes. To explain these relations, we first associate to any lower set L another lower set $L_{p,q}$ obtained from L by "collapsing (p, q)-rectangles to points":

$$L_{p,q} = \{(\alpha, \beta) : ((p+1)\alpha, (q+1)\beta) \in L\}.$$

First of all, it is clear that $n_{p,q}(S) = |S_{p,q}|$. Secondly, it is easy to see that $(L^{p,q})_{p,q} = L$, and $L \subset (L_{p,q})^{p,q}$ for all lower sets L. Passing to cardinalities in the last inequality, we obtain

Corollary 11.3. *For all lower sets L,*

$$n_{p,q}(L) \geq \frac{1}{n}|L|, \qquad (n = (p+1)(q+1)).$$

In particular, Conjecture 2 is a strengthening of the Polya conditions of Corollary 4.3.

On the other hand, one immediately sees that $S_{p,q}$ is the smallest lower set with the property its (p,q)-blow up contains S. In particular, if $S = R^{p,q}$ for some lower set R, then R must coincide with $S_{p,q}$. So, the assertion of Conjecture 1 is equivalent to saying that $S \subset (S_{p,q})^{p,q}$ becomes equality. Passing to cardinalities, we have $|A| \leq n_{p,q}(S)$, and Conjecture 1 says that equality must hold. Hence:

Corollary 11.4. *The following are equivalent*

(i) *$S = R^{p,q}$ for some lower set R;*
(ii) *$n_{p,q}(S) \geq |A|$;*
(ii) *$n_{p,q}(S) = |A|$.*

In particular, Conjecture 2 implies Conjecture 1.

Note again that our results (and examples) can be viewed as a confirmation of this conjecture. Let us mention here one more:

Corollary 11.5. *If (A, S) is regular with respect to (p,q)-rectangular sets of complex nodes, then the conclusion of Conjecture 2 (hence also of Conjecture 1) holds true.*

The proof is completely similar to the proof in the case $p = q = 1$ (i.e. the proof of Theorem 9.1). This time, instead of considering the set of nodes with coordinates 1 or -1, one considers the set consisting of (ξ_i, η_j), where the ξ_i's are the $(p+1)^{th}$ roots of unity, and the η_j's are the $(q+1)^{th}$ roots of unity.

11.6. A and S related geometrically?
Again supported by all our examples and results, is a geometric property that A and S share. Roughly speaking, this property says that S is obtainable by attaching a copy of the rectangle $R(p,q)$ to each point of A, and then moving these rectangles upwards or to the right until they become disjoint. To make this more precise, we need to use "shifts" of A in S, which are transformations which move A step by step, at each step one of its elements being moved upwards or to the right on a new position which is still in S and which is not occupied by any other element of A. For more on shifts, see the Appendix. We also consider the set

$$\mathcal{Z}^{p,q} = \{((p+1)i, (q+1)j) : i, j - \text{positive integers }\},$$

which can be viewed as the blow up of the lattice $\mathcal{Z}$ of integral points situated in the first quadrant. With this preparation, the geometrical property we have mentioned takes the following precise form.

Conjecture 3. *If a scheme (A, S) is almost regular with respect to (p,q)-rectangular sets of nodes, then there exists a shift of A in S which moves A to $S \cap \mathcal{Z}^{p,q}$.*

We advise the reader to check this conjecture in the examples presented in the previous section (recall that, in many of the pictures there, the points of $S \cap \mathcal{Z}^{p,q}$ are precisely the ones rounded by circles). In general, it turns out that the last conjecture is precisely the geometric translation of Conjecture 2, as shown by the following

Theorem 11.7. *Given (A, S) ($|S| = n|A|$, $n = (p+1)(q+1)$), the following are equivalent*

(i) $|L \cap A| \geq n_{p,q}(L)$ *for all lower sets* $L \subset S$.

(ii) *there exists a shift of* A *in* S *which moves* A *to* $S \cap \mathcal{Z}^{p,q}$.

In particular, Conjectures 2 and 3 are equivalent.

Proof. (of Theorem 11.7): Before starting the proof, let us fix some notations. Given an element $e \in S$, we denote by e^r the one step translation of e to the right, and similarly e^u (upwards translation). Also, for any lower set L, we consider two new lower sets $\overset{\circ}{L}$ and $\overline{L}$ such that

$$\overset{\circ}{L} \subset L \subset \overline{L},\ n_{p,q}(\overset{\circ}{L}) = n_{p,q}(L) = n_{p,q}(\overline{L}),$$

and such that $\overset{\circ}{L}$ is the smallest possible one, while $\overline{L}$ is the largest possible. Note that $\overline{L} = (L_{p,q})^{p,q}$. It is easy to see that $\overline{L_1 \cup L_2} = \overline{L_1} \cup \overline{L_2}$, and similarly for intersections and also for the operations $\overset{\circ}{L}$, and

$$n_{p,q}(L) = \frac{1}{n}|\overline{L}|.$$

We now turn to the proof. Assume first that such a shift Λ exists, and let $L \subset S$ be a lower set. Then the points of $L \cap \mathcal{Z}^{p,q}$ will be obtained by moving some of the points of A, and those points must come from $A \cap L$. Hence Λ defines a bijection between a subset of $A \cap L$ and $L \cap \mathcal{Z}^{p,q}$, and this proves the desired inequality. We now keep p and q fixed, we denote by $\mathcal{P}$ the set of pairs (A, S) that satisfy (i) and have $|S| = n|A|$, and we prove that (ii) holds for any $(A, S) \in \mathcal{P}$ by induction on $|S|$. The starting point is $|S| = n$. Then A is forced to be $\{(0,0)\}$ (apply the condition to $L = \{(0,0)\}$), S is forced to be the rectangle $R(p,q)$ (apply the condition to $L = S$), hence $A = S \cap \mathcal{Z}^{p,q} = \{(0,0)\}$. Assume now that (A, S) does satisfy (i), and the implication has been shown for all pairs (A', S') with $|S'| < |S|$. If one of the points of A can be moved one step upwards, or one step to the right, so that the condition (i) is still satisfied (by S and the new set A), then we perform the move. We repeat this if still possible. This process will stop at some point (e.g. because A cannot be moved outside S). Hence, we may assume that A is "maximal", in the sense that none of its elements can be moved any further without violating (i). We will show that $A = S \cap \mathcal{Z}^{p,q}$. We first prove the following:

Claim 1: If $L \subset S$ is a lower subset with $|A \cap L| = n_{p,q}(L)$, then either $\overline{L} = S$, or $A \cap L = \mathcal{Z}^{p,q} \cap L$.

Proof of the claim: We first show that $(A \cap L, \overline{L}) \in \mathcal{P}$. First of all, $|\overline{L}| = n_{p,q}(L) = |A \cap L|$. Secondly, for any lower set $P \subset \overline{L}$ one has $\overline{P} \subset \overline{L}$, hence $\overline{L \cap P} = \overline{L} \cap \overline{P} = \overline{P}$. This, and the fact that (A, S) does satisfy (i), imply that

$$n_{p,q}(P) = n_{p,q}(L \cap P) \leq |A \cap (L \cap P)| = |(A \cap L) \cap P|$$

for all $P \subset \overline{L}$ lower. Hence $(A \cap L, \overline{L}) \in \mathcal{P}$. Assume now that $\overline{L} \neq S$. Then, by the induction hypothesis, there is a shift that moves $A \cap L$ to $\mathcal{Z}^{p,q} \cap \overline{L} = \mathcal{Z}^{p,q} \cap L$. Since this shift stays inside L, it does not touches the elements in $A \setminus L$, hence it can be viewed as a shift Λ of the entire A, which leaves $A \setminus L$ intact. In other words, the image A' of Λ is given by

$$A' \cap L = \mathcal{Z}^{p,q} \cap L,\ A' \setminus L = A \setminus L.$$

Now, for any $P \subset S$ lower, one has

$$\begin{aligned} |A' \cap P| &= |A' \cap (P \cap L)| + |A' \cap (P \setminus L)| \\ &= |A' \cap (P \cap L)| + |A \cap (P \setminus L)| \\ &= |(A' \cap L) \cap (P \cap L)| + |A \cap (P \cup L)| - |A \cap L| \\ &\geq n_{p,q}(P \cap L) + n_{p,q}(P \cup L) - n_{p,q}(L) \\ &= n_{p,q}(R) \end{aligned}$$

In the inequality above we used that $(A' \cap L, \overline{L})$ and (A, S) satisfy (i), and that $|A \cap L| = n_{p,q}(L)$. This shows that $(A', S) \in \mathcal{P}$. From the maximality of A, the shift must be the identity shift, which precisely means that $A \cap L = \mathcal{Z}^{p,q} \cap L$. This concludes the proof of the claim.

Claim 2: If $e \in A \setminus \mathcal{Z}(p,q)$ is an element with the property that $e^r \notin A$, then

$$e \in \partial_y(\overset{\circ}{S}) \cup \partial_e(\overset{\circ}{S}).$$

Proof of the claim: Denote by A^r_e the new set obtained from A by moving e to e^r. By the maximality of A, (A^r_e, S) does not satisfy (i), hence we find a lower set L such that $|A^r_e \cap L| < n_{p,q}(L)$. But $|A^r_e \cap L|$ is either $|A \cap L| - 1$ (if $e \in L$, $e^r \notin L$), or $|A \cap L|$ (in the remaining cases). Since $|A \cap L| \geq n_{p,q}(L)$, we must have

$$|A \cap L| = n_{p,q}(L), \;\; e \in L, \;\; e^r \notin L.$$

We now use the previous claim. Since $e \in (A \cap L) \setminus \mathcal{Z}(p,q)$, we must have $\overline{L} = S$. In turn, this implies that $\overset{\circ}{L} = \overset{\circ}{S}$, hence, since $e^r \notin L$, we have $e^r \notin \overset{\circ}{S}$. On the other hand, applying (i) to the largest lower set which does not contain e, we immediately see that S contains at least one element in $\mathcal{Z}^{p,q}$ larger then e. In other words, $e \in \overset{\circ}{S}$. But the elements $e \in \overset{\circ}{S}$ with $e^r \notin \overset{\circ}{S}$ are exactly those those situated on $\partial_y(\overset{\circ}{S}) \cup \partial_e(\overset{\circ}{S})$. This concludes the proof of the claim.

Clearly, one can replace e^r by e^u, and arrive to a similar conclusion. In particular, if $e \in A \setminus \mathcal{Z}^{p,q}$, then either e^r or e^u must be in A (otherwise we must have $e \in \partial_e(\overset{\circ}{S})$, which is a contradiction because the extremal boundary points of $\overset{\circ}{S}$ belong to $\mathcal{Z}^{p,q}$).

We are now ready to prove that $A = L \cap \mathcal{Z}^{p,q}$. If this is not so, then we pick up an extremal element e of $A \setminus \mathcal{Z}^{p,q}$. That means that $e \in A \setminus \mathcal{Z}^{p,q}$, and $A \setminus \mathcal{Z}^{p,q}$ contains no other element whose coordinates are greater or equal to the coordinates of e. Then at least one of the positions e^u or e^r are not in A. Otherwise, since e is extremal, e^r and e^u would both be in $\mathcal{Z}^{p,q}$, and that is clearly impossible. Hence, combined with the previous remark, one (and only one) of these two elements are in A. Hence we may assume that

$$e^r \notin A, \quad e^u \in A.$$

From Claim 2, and the extremality of e, we deduce that

$$e^r \in \partial_y(\overset{\circ}{S}) \cup \partial_e(\overset{\circ}{S}), \;\; e^u \in \mathcal{Z}^{p,q}.$$

Let e' be the exterior boundary point of $\overset{\circ}{S}$ with the property that the segment $l = [e, e']$ is parallel to OY (see Fig. 16). We apply condition (i) to the lower set $S \setminus Q_e$, where Q_e is as in the picture. (i.e. $S \setminus Q_e$ is the largest lower subset of S which does not contain e). We immediately get $|l \cap \mathcal{Z}^{p,q}| \geq |l \cap A|$. Since e is the only point of A situated on l but not on $\mathcal{Z}^{p,q}$, we find a point $f = (\alpha, \beta) \in A \cap l \cap \mathcal{Z}^{p,q}$ with the property that $\tilde{f} = (\alpha + p, \beta)$ is in $l \cap \mathcal{Z}^{p,q}$ but it is not in A. We then consider the shift that moves f to $\tilde{f}$, and leaves the rest of A unchanged. The maximality of A implies again the existence of a lower set L with the property that $|A \cap L| = n_{p,q}(L)$, $f \in L$, $\tilde{f} \notin L$. We can now use Claim 1 above. Since $\tilde{f} \notin L$ and $\tilde{f} \in \mathcal{Z}^{p,q}$, one cannot have $\overline{L} = S$. Hence we must have $A \cap L = \mathcal{Z}^{p,q} \cap L$. On the other hand, since L is lower and e sits below f, it follows that $e \in A \cap L = \in L$, hence $e \in \mathcal{Z}^{p,q}$. This contradicts the choice of e, and concludes the proof of the Theorem. $\square$

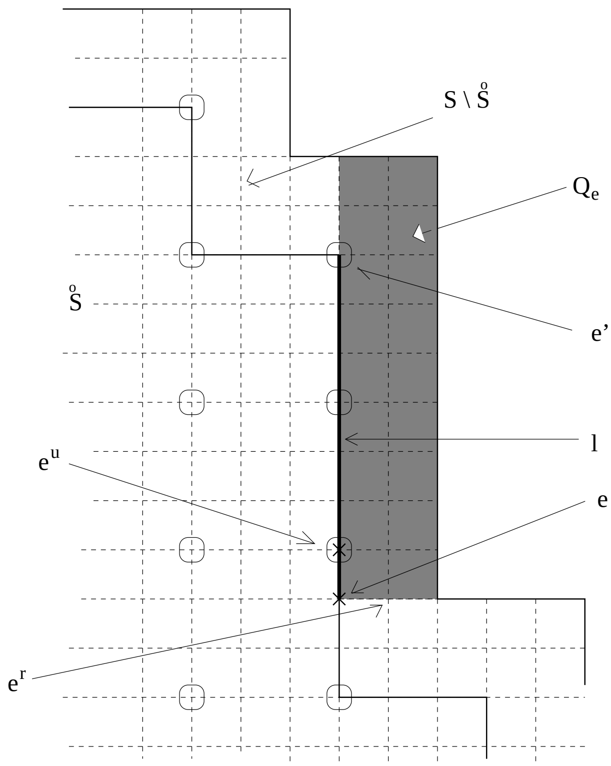


FIGURE 16

Remark that the conjectured properties we have looked at so far still do not explain the non-regularity of the example drawn in Fig. 11. This example (as well as others that we have looked at) suggests a different (but not unrelated) type of conditions, based on *inverse shifts*. This notion is completely analogous to that of shift, with the only difference that, at each step, the moves one allows are downwards or to the left.

Conjecture 4. *If (A, S) is almost regular with respect to (p, q)-rectangular sets of nodes, then there is an inverse shift of A in S, which moves A to a lower set R, and $S = R^{p,q}$.*

As in Theorem 11.7 (and proven using the same ideas), this geometrical condition is equivalent to an algebraic condition, which says that $|A \cap L| \leq |R \cap L|$ for all lower sets L. However, these more geometrical versions are easier to check in explicit examples, and they also suggests that stronger versions have to be considered, where only certain type of shifts are allowed. For instance, one should not allow shifts which move elements on the coordinate axes, or backward shifts which move new elements to the axes (cf. our Theorem 8.1).

Example 11.8. The last conjecture is relevant to several of the examples presented in the previous section. For instance, one can check that the condition in the conjecture is not satisfied neither by the non-regular scheme of Example 10.4 (left hand side of Fig. 9), nor by the example drawn in Fig. 11.

Note also that, in almost all the cases we have considered so far (e.g. no mixed derivatives, or lower sets of derivatives), moving A backwards to a lower set was possible in only one way. Hence the last conjecture would explain the uniqueness of S (proven by us in each case separately).

Let us now look at Example 10.10 which was the first example of a set A for which there is more then one choice for S. The way we constructed this example was by using the last conjecture, hence looking for A's for which the process of "moving it back" to a lower set was not unique. In that example, there are clearly only two such ways, as shown in Fig. 17: the extremal element of A can be moved one step down, or one step to the left. Using the same idea, one can actually produce new examples of sets A, for which the number $s(Z_{p,q}, A, S)$ takes given values.

11.9. A unifying conjecture? Putting Conjecture 3 and Conjecture 4 together, it is now tempting to make a stronger conjecture with the conclusion that $S = R^{p,q}$ for some lower set R, and there exists a shift of R to $S \cap \mathcal{Z}^{p,q}$, which moves (i,j) to $((p+1)i, (q+1)j)$, and which, at some intermediate step, cover A. Again, we do believe that only certain shifts are relevant, namely those which do not violate certain "regularity conditions". What "regularity condition" exactly means is still to be discovered, but it certainly excludes moving the elements on OX upwards, or those on OY to the right (Theorem 8.1).

12. Appendix

In this appendix we recall one of the main (sufficient) criteria for the regularity of interpolation schemes with nodes in general position. This is an example of the criteria of [3] that we have mentioned in the introduction, which we have used in several examples.

Let S be a lower set. The basic idea is to view the determinant $D(Z, A, S)$ as a polynomial on the coordinates of the points of Z, and to analyze the partial derivatives of $D(Z, A, S)$. The non-vanishing of $D(Z, A, S)$ is then a consequence of the non-vanishing of higher order partial derivatives. The point is that these higher order partial derivatives are easier to control. By the well-known rule of expressing the derivative of the determinant as a sum of the determinants obtained by taking the derivative of each of the individual rows, it follows that taking derivatives of $D(Z, A, S)$ involves "moving A upwards or to the right". Once we have this basic idea in mind, the rest is a careful bookkeeping of the derivation rules, which eventually provide us with "regularity criteria" for interpolation schemes. Let us state

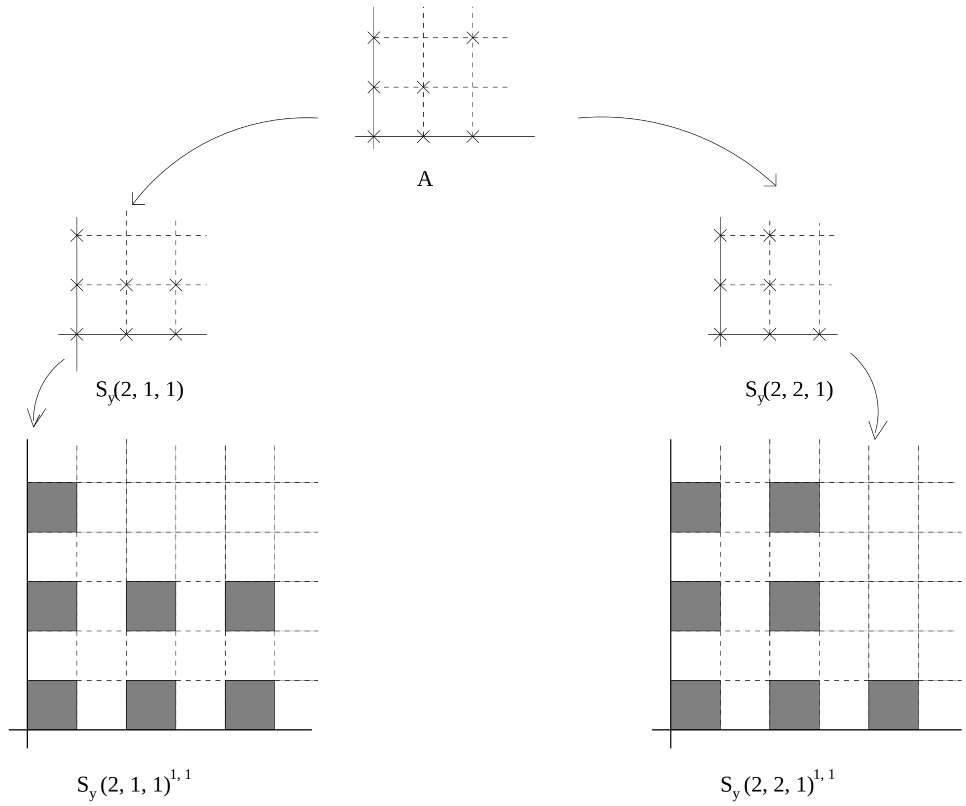


FIGURE 17. First move A backwards to lower sets, and then blow up.

here one of these criteria as it follows from [3] (however, we do urge the reader to prove this criteria himself, using the "general" idea we have just described).

We start by recalling the precise meaning of "moving A upwards or to the right". Given $A \subset S$, a shift Λ of A (in S) is a composition $\Lambda_k \ldots \Lambda_2\Lambda_1$ of "elementary shifts". That means an operation Λ_1 which moves (exactly) one element of A one step to the right or upwards, on a place which is still inside S and is not occupied by any other element of A, to get another set $\Lambda_1 A \subset S$; this is followed by a new such operation Λ_2 applied this time to $\Lambda_1 A$, etc. The type of a shift is a formal (non-commutative) word $L_k \ldots L_2 L_1$ with letters $L_i \in \{R, U\}$, where L_i is R if Λ_i moves the element to the right, and L_i is U if moves it upwards. The result of a shift is a new set $\Lambda A \subset S$, together with a bijection from A to ΛA which assigns to each $(\alpha, \beta) \in A$ its new position in $\Lambda(A)$ after all the elementary shifts. In general, there can be shifts which have the same type (i.e. the same associated word), but different associated functions. One says that Λ is of unique type if, for any other shift Λ' which has the same type as Λ, one has $\Lambda'(A) = \Lambda(A)$, and the two bijections induced by Λ and Λ' are equal. There are straightforward version of these notions relative to a given subset $B \subset S$, where "relative" means "disjoint from B". A shift (in S) of A relative to B is a shift Λ as above, with the property that $\Lambda(A) \cap B = \emptyset$. The notion of "shifts of unique type" relative to B is the same as the one above, with the only exception that the condition is put only for those shifts Λ' which are relative to B. Given $A_0, \ldots, A_m \subset S$ we say that S admits a pavement of unique type by $\{A_0, \ldots, A_m\}$ if there exists

(1) A shift Λ_1 of A_1 (in S) of unique type relative to A_0;
(2) A shift Λ_2 of A_2 of unique type relative to $A_0 \cup \Lambda_1(A_1)$;
...
(m) A shift Λ_m of A_m of unique type relative to $A_0 \cup \Lambda_1(A_1) \cup \ldots \cup \Lambda_{m-1}(A_{m-1})$.

and such that A_0 and $\Lambda_i(A_i)$ with $1 \leq i \leq m$ cover S entirely. One of the consequences of [3] is

Theorem 12.1. *If S admits a pavement of unique type by $\{\underbrace{A, \ldots, A}_{n \text{ times}}\}$, then the homogeneous interpolations problem associated to (A, S) is regular with respect to almost all choices of the set of n nodes Z (i.e. (A, S) is almost regular).*

References

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