A Paley-Wiener theorem for reductive symmetric spaces

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Abstract

Let $X = G/H$ be a reductive symmetric space and K a maximal compact subgroup of G. The image under the Fourier transform of the space of K-finite compactly supported smooth functions on X is characterized.

Contents

1 Introduction

One of the central theorems of harmonic analysis on $\mathbb{R}$ is the Paley-Wiener theorem which characterizes the class of functions on $\mathbb{C}$ which are Fourier transforms of C^∞-functions on $\mathbb{R}$ with compact support (also called the Paley-Wiener-Schwartz theorem, see [18], p. 249). We consider the analogous question for the Fourier transform of a reductive symmetric space $X = G/H$, that is, G is a real reductive Lie group of Harish-Chandra's class and H is an open subgroup of the group G^σ of fixed points for an involution σ of G.

The paper is a continuation of [4] and [6], in which we have shown that the Fourier transform is injective on $C_c^\infty(X)$, and established an inversion formula for the K-finite functions in this space, with K a σ-stable maximal compact subgroup of G. A conjectural image of the space of K-finite functions in $C_c^\infty(X)$ was described in [4], Remark 21.8, and will be confirmed in the present paper (the conjecture was already confirmed for symmetric spaces of split rank one in [4]).

If G/H is a Riemannian symmetric space (equivalently, if H is compact), there is a well established theory of harmonic analysis (see [17]), and the Paley-Wiener theorem that we obtain generalizes a well known theorem of Helgason and Gangolli ([15], see also [17], Thm. IV,7.1). Furthermore, the reductive group G is a symmetric space of its own right, for the left times right action of $G \times G$. Also in this 'case of the group' there is an established theory of harmonic analysis, and our theorem generalizes the theorem of Arthur [1] (and Campoli [11] for groups of split rank one).

The Fourier transform $\mathcal{F}$ that we are dealing with is defined for functions in the space $C_c^\infty(X\colon\tau)$ of τ-spherical C_c^∞-functions on X. Here τ is a finite dimensional representation of K, and a τ-spherical function on X is a function that has values in the representation space V_τ and satisfies $f(kx) = \tau(k)f(x)$ for all $x \in X$, $k \in K$. This space is a convenient tool for the study of K-finite (scalar) functions on X. Related to τ and the (minimal) principal series for X, there is a family $E^\circ(\psi\colon\lambda)$ of normalized Eisenstein integrals on X (cf. [2], [3]). These are (normalized) generalizations of the elementary spherical functions for Riemannian symmetric spaces, as well as of Harish-Chandra's Eisenstein integrals associated with a minimal parabolic subgroup of a semisimple Lie group. The Eisenstein integral is a τ-spherical smooth function on X. It is linear in the parameter ψ, which belongs to a finite dimensional Hilbert space ${}^\circ\mathcal{C}$, and meromorphic in λ, which belongs to the complex linear dual $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ of a maximal abelian subspace $\mathfrak{a}_\mathrm{q}$ of $\mathfrak{p}\cap\mathfrak{q}$. Here $\mathfrak{p}$ is the orthocomplement of $\mathfrak{k}$ in $\mathfrak{g}$, and $\mathfrak{q}$ is the orthocomplement of $\mathfrak{h}$ in $\mathfrak{g}$, where $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{h}$ are the Lie algebras of G, K and H. The Fourier transform $\mathcal{F}f$ of a function $f \in C_c^\infty(X\colon\tau)$ is essentially defined by integration of f against E° (see (2.1)), it is a ${}^\circ\mathcal{C}$-valued meromorphic function of $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. The fact that $\mathcal{F}f(\lambda)$ is meromorphic in λ, rather than holomorphic, represents a major complication not present in the mentioned special cases.

The Paley-Wiener theorem (Thm. 3.6) asserts that $\mathcal{F}$ maps $C_c^\infty(X\colon\tau)$ onto the Paley-Wiener space $\mathrm{PW}(X\colon\tau)$ (Def. 3.4), which is a space of meromorphic functions $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \to {}^\circ\mathcal{C}$ characterized by an exponential growth condition and so-called *Arthur-Campoli relations*, which are conditions coming from relations of a particular type among the Eisenstein integrals. These relations generalize the relations used in [11] and [1]. Among the relations are conditions for transformation under the Weyl group (Lemma 3.10). In the Riemannian case, no other relations are needed, but this is not so in general.

The proof is based on the inversion formula $f = \mathcal{T}\mathcal{F}f$ of [6], through which a function $f \in C_c^\infty(X\colon\tau)$ is determined from its Fourier transform by an operator $\mathcal{T}$. The same operator can be applied to an arbitrary function φ in the Paley-Wiener space $\mathrm{PW}(X\colon\tau)$. The resulting function $\mathcal{T}\varphi$ on X, called a pseudo wave packet, is then shown to have φ as its Fourier transform. A priori $\mathcal{T}\varphi$ is defined and smooth on a certain dense open subset X_+ of X, and the main difficulty in the proof is to show that it admits a smooth extension to X (Thm. 4.4). In fact, it was shown already in [6], that if a smooth extension of $\mathcal{T}\varphi$ exists, then this extension has compact support and is mapped onto φ by $\mathcal{F}$.

The proof that $\mathcal{T}\varphi$ extends smoothly relies on the residue calculus of [5] and on results of [7]. By means of the residue calculus we write the pseudo wave packet $\mathcal{T}\varphi$ in the form

$$\mathcal{T}\varphi = \sum_{F\subset\Delta} \mathcal{T}_F\varphi$$

(see eq. (8.3)) in which Δ is a set of simple roots for the root system of $\mathfrak{a}_\mathrm{q}$, and in which the individual terms for $F \neq \emptyset$ are defined by means of residue operators. The term $\mathcal{T}_\emptyset\varphi$ is the wave packet given by integration over $\mathfrak{a}^*_\mathrm{q}$ of φ against the normalized Eisenstein integral. The smooth extension of $\mathcal{T}\varphi$ is established by showing that each term $\mathcal{T}_F\varphi$ extends smoothly. The latter fact is obtained by identification of $\mathcal{T}_F\varphi$ with a wave packet formed by generalized Eisenstein integrals. The generalized Eisenstein integrals we use were introduced in [6], they are smooth functions on X. It is shown in [9] that they are matrix coefficients of non-minimal principal series representations and that they agree with the generalized Eisenstein integrals of [12]. However, these facts play no role here.

It is for the identification of $\mathcal{T}_F\varphi$ as a wave packet that the Arthur-Campoli relations are needed when $F \neq \emptyset$. An important step is to show that Arthur-Campoli relations for lower dimensional symmetric spaces, related to certain parabolic subgroups in G, can be induced up to Arthur-Campoli relations for X (Thm. 6.2). For this step we use a result from [7].

As mentioned, our Paley-Wiener theorem generalizes that of Arthur [1] for the group case. Arthur also uses residue calculus in the spirit of [19], but apart from that our approach differs in a number of ways, the following two being the most significant. Firstly, Arthur relies on Harish-Chandra's Plancherel theorem for the group, whereas we do not need the analogous theorem for X, which has been established by Delorme [14] and the authors [8]-[9]. Secondly, Arthur's result involves unnormalized Eisenstein integrals, whereas our involves normalized ones. This facilitates comparison between the Eisenstein integrals related to X and those related to lower rank symmetric spaces coming from parabolic subgroups. For the similar comparison of the unnormalized Eisenstein integrals, Arthur relies on a lifting principle of Casselman, the proof of which has not been published. In [7] we have established a normalized version of Casselman's principle which plays a crucial role in the present work. One can show, using [16], Lemma 2, p. 156, [1], Lemma I.5.1 and [13], that our Paley-Wiener theorem, specialized to the group case, implies Arthur's. In fact, it implies a slightly stronger result, since here only Arthur-Campoli relations for real-valued parameters λ are needed, whereas the Paley-Wiener theorem of [1] requires also the relations at the complex-valued λ.

The Paley-Wiener space $\mathrm{PW}(X\colon\tau)$ is defined in Section 3 (Definition 3.4), and the proof outlined above that it equals the Fourier image of $C_c^\infty(X\colon\tau)$ takes up the following Sections 4-8. A priori the given definition of $\mathrm{PW}(X\colon\tau)$ does not match that of [4], but it is shown in the final Sections 9-10 that the two spaces are equal.

The main result of this paper was found and announced in the fall of 1995 when both authors were visitors of the Mittag-Leffler Institute in Djursholm, Sweden. We are grateful to the organizers of the program and the staff of the institute for providing us with this opportunity, and to Mogens Flensted-Jensen for helpful discussions during that period.

2 Notation

We use the same notation and basic assumptions as in [4], Sect. 2-3, 5-6, and [6], Sect. 2. Only the most essential notions will be recalled, and we refer to the mentioned locations for unexplained notation.

We denote by Σ the root system of $\mathfrak{a}_\mathrm{q}$ in $\mathfrak{g}$, where $\mathfrak{a}_\mathrm{q}$ is a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$, as mentioned in the introduction. Each positive system Σ^+ for Σ determines a parabolic subgroup $P = M_1N$, where M_1 is the centralizer of $\mathfrak{a}_\mathrm{q}$ in G and N is the exponential of $\mathfrak{n}$, the sum of the positive root spaces. In what follows we assume that such a positive system Σ^+ has been fixed. Moreover, notation with reference to Σ^+ or P, as given in [4] and [6], is supposed to refer to this fixed choice, if nothing else is mentioned. For example, we write $\mathfrak{a}_\mathrm{q}^+$ for the corresponding positive open Weyl chamber in $\mathfrak{a}_\mathrm{q}$, denoted $\mathfrak{a}_\mathrm{q}^+(P)$ in [4], and A_q^+ for its exponential $A_\mathrm{q}^+(P)$ in G. We write $P = MAN$ for the Langlands decomposition of P.

Throughout the paper we fix a finite dimensional unitary representation (τ, V_τ) of K, and we denote by ${}^\circ\mathcal{C} = {}^\circ\mathcal{C}(\tau)$ the finite dimensional space defined by [4], eq. (5.1). The Eisenstein integral $E(\psi\colon\lambda) = E(P\colon\psi\colon\lambda)\colon X \to V_\tau$ is defined as in [4], eq. (5.4), and the normalized Eisenstein integral $E^\circ(\psi\colon\lambda) = E^\circ(P\colon\psi\colon\lambda)$ is defined as in [4], p. 283. Both Eisenstein integrals belong to $C^\infty(X\colon\tau)$ and depend linearly on $\psi \in {}^\circ\mathcal{C}$ and meromorphically on $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. For $x \in X$ we denote the linear map ${}^\circ\mathcal{C} \ni \psi \mapsto E^\circ(\psi\colon\lambda\colon x) \in V_\tau$ by $E^\circ(\lambda\colon x)$, and we define $E^*(\lambda\colon x) \in \mathrm{Hom}(V_\tau, {}^\circ\mathcal{C})$ to be the adjoint of $E^\circ(-\bar{\lambda}\colon x)$ (see [6], eq. (2.3)). The Fourier transform that we investigate, maps $f \in C^\infty_c(X\colon\tau)$ to the meromorphic function $\mathcal{F}f$ on $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ given by

$$\mathcal{F}f(\lambda) = \int_X E^*(\lambda\colon x) f(x)\, dx \in {}^\circ\mathcal{C}. \tag{2.1}$$

The open dense set $X_+ \subset X$ is given by

$$X_+ = \cup_{w\in\mathcal{W}}\, KA^+_{\mathrm{q}}wH,$$

see [6], eq. (2.1). It naturally arises in connection with the study of asymptotic expansions of the Eisenstein integrals, see [6], p. 32-33. As a result of this theory, the normalized Eisenstein integral is decomposed as a finite sum

$$E^\circ(\lambda\colon x) = \sum_{s\in W} E_{+,s}(\lambda\colon x), \qquad E_{+,s}(\lambda\colon x) = E_+(s\lambda\colon x)\circ C^\circ(s\colon\lambda) \tag{2.2}$$

for $x \in X_+$, all ingredients being meromorphic in $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. The partial Eisenstein integral $E_+(\lambda\colon x)$ is a $\mathrm{Hom}({}^\circ\mathcal{C}, V_\tau)$-valued function in $x \in X_+$, given by a converging series expansion, and $C^\circ(s\colon\lambda) \in \mathrm{End}({}^\circ\mathcal{C})$ is the (normalized) c-function associated with τ. In general, $x \mapsto E_+(\lambda\colon x)$ is singular along $X \setminus X_+$. The c-function also appears in the following transformation law for the action of the Weyl group

$$E^*(s\lambda\colon x) = C^\circ(s\colon\lambda)\circ E^*(\lambda\colon x) \tag{2.3}$$

for all $s \in W$ and $x \in X$ (see [6], eq. (2.11)), from which it follows that

$$\mathcal{F}f(s\lambda) = C^\circ(s\colon\lambda)\circ\mathcal{F}(\lambda). \tag{2.4}$$

The structure of the singular set for the meromorphic functions $E^\circ(\,\cdot\,\colon x)$ and $E_+(\,\cdot\,\colon x)$ on $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ plays a crucial role. To describe it, we recall from [7], Sect. 10, that a Σ-configuration in $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ is a locally finite collection of affine hyperplanes H of the form

$$H = \{\lambda \mid \langle\lambda, \alpha_H\rangle = s_H\} \tag{2.5}$$

where $\alpha_H \in \Sigma$ and $s_H \in \mathbb{C}$. Furthermore, we recall from [7], Sect. 11, that if $\mathcal{H}$ is a Σ-configuration in $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ and d a map $\mathcal{H} \to \mathbb{N}$, we define for each bounded set $\omega \subset \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ a polynomial function $\pi_{\omega,d}$ on $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ by

$$\pi_{\omega,d}(\lambda) = \prod_{H\in\mathcal{H}, H\cap\omega\neq\emptyset} (\langle\lambda, \alpha_H\rangle - s_H)^{d(H)}, \tag{2.6}$$

where α_H, s_H are as above. The linear space $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \mathcal{H}, d)$ is defined to be the space of meromorphic functions $\varphi\colon \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \to \mathbb{C}$, for which $\pi_{\omega,d}\varphi$ is holomorphic on ω for all bounded open sets $\omega \subset \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$, and the linear space $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \mathcal{H})$ is defined by taking the union of $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \mathcal{H}, d)$ over $d \in \mathbb{N}^{\mathcal{H}}$. If $\mathcal{H}$ is real, that is, $s_H \in \mathbb{R}$ for all H, we write $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}, d)$ and $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H})$ in place of $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \mathcal{H}, d)$ and $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \mathcal{H})$.

Lemma 2.1. *There exists a real Σ-configuration $\mathcal{H}$ such that the meromorphic functions $E^\circ(\,\cdot : x)$ and $E_{+,s}(\,\cdot : x')$ belong to $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}) \otimes \mathrm{Hom}({}^\circ\mathcal{C}, V_\tau)$ for all $x \in X$, $x' \in X_+$, $s \in W$, and such that $C^\circ(s\colon \cdot\,) \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}) \otimes \mathrm{End}({}^\circ\mathcal{C})$ for all $s \in W$.*

Proof. The statement for $E^\circ(\,\cdot : x)$ is proved in [6], Prop. 3.1, and the statement for $E_{+,1}(\,\cdot : x) = E_+(\,\cdot : x)$ is proved in [6], Lemma 3.3. The statement about $C^\circ(s\colon \cdot\,)$ follows from [3], eqs. (68), (57), by the argument given below the proof of Lemma 3.2 in [6]. The statement for $E_{+,s}(\,\cdot : x)$ in general then follows from its definition in (2.2). $\square$

Let $\mathcal{H} = \mathcal{H}(X, \tau)$ denote the collection of the singular hyperplanes for all $\lambda \mapsto E^*(\lambda\colon x)$, $x \in X$ (this is a real Σ-configuration, by the preceding lemma). Moreover, for $H \in \mathcal{H}$ let $d(H) = d_{X,\tau}(H)$ be the least integer $l \geq 0$ for which $\lambda \mapsto (\langle \lambda, \alpha_H \rangle - s_H)^l E^*(\lambda\colon x)$ is regular along $H \setminus \cup\{H' \in \mathcal{H} \mid H' \neq H\}$, for all $x \in X$. Then $E^*(\,\cdot : x) \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}, d) \otimes \mathrm{Hom}(V_\tau, {}^\circ\mathcal{C})$ and d is minimal with this property. It follows that $\mathcal{F}f \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}, d) \otimes {}^\circ\mathcal{C}$ for all $f \in C^\infty_c(X\colon \tau)$.

There is more to say about these singular sets. For $R \in \mathbb{R}$ we define

$$\mathfrak{a}^*_{\mathrm{q}}(P, R) = \{\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \mid \forall \alpha \in \Sigma^+ : \mathrm{Re}\langle \lambda, \alpha \rangle < R\} \tag{2.7}$$

and denote by $\bar{\mathfrak{a}}^*_{\mathrm{q}}(P, R)$ the closure of this set. Then it also follows from [6], Prop. 3.1 and Lemma 3.3, that $E^*(\,\cdot : x)$ and $E_+(\,\cdot : x)$ both have the property that for each R only finitely many singular hyperplanes meet $\mathfrak{a}^*_{\mathrm{q}}(P, R)$.

In particular, the set of affine hyperplanes

$$\mathcal{H}_0 = \{H \in \mathcal{H}(X, \tau) \mid H \cap \bar{\mathfrak{a}}^*_{\mathrm{q}}(P, 0) \neq \emptyset\}, \tag{2.8}$$

is finite. Let π be the real polynomial function on $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ given by

$$\pi(\lambda) = \prod_{H \in \mathcal{H}_0} (\langle \lambda, \alpha_H \rangle - s_H)^{d_{X,\tau}(H)} \tag{2.9}$$

where α_H and s_H are chosen as in (2.5). The polynomial π coincides, up to a constant non-zero factor, with the polynomial denoted by the same symbol in [4], eq. (8.1), and in [6], p. 34. It has the property that there exists $\epsilon > 0$ such that $\lambda \mapsto \pi(\lambda) E^*(\lambda\colon x)$ is holomorphic on $\mathfrak{a}^*_{\mathrm{q}}(P, \epsilon)$ for all $x \in X$.

3 The Paley–Wiener space. Main theorem

We define the Paley–Wiener space $\mathrm{PW}(X : \tau)$ for the pair (X, τ) and state the main theorem, that the Fourier transform maps $C_c^\infty(X : \tau)$ onto this space.

First we set up the condition that reflects relations among Eisenstein integrals. In [11] and [1] similar relations are used in the definition of the Paley–Wiener space. However, as we are dealing with functions that are in general meromorphic rather than holomorphic, our relations have to be specified somewhat differently. This is done by means of Laurent functionals, a concept introduced in [7], Def. 10.8, to which we refer (see also the review in [8], Sect. 4). In [4], Def. 21.6, the required relations are formulated differently, we compare the definitions in Lemma 10.4 below.

Definition 3.1. We call a Σ-Laurent functional $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^* \otimes {}^\circ\mathcal{C}^*$ an *Arthur–Campoli functional* if it annihilates $E^*(\,\cdot : x)v$ for all $x \in X$ and $v \in V_\tau$. The set of all Arthur–Campoli functionals is denoted $\mathrm{AC}(X : \tau)$, and the subset of the Arthur–Campoli functionals with support in $\mathfrak{a}_\mathrm{q}^*$ is denoted $\mathrm{AC}_\mathbb{R}(X : \tau)$.

It will be shown below in Lemma 3.8 that the elements of $\mathrm{AC}(X : \tau)$ are natural objects, from the point of view of characterizing $\mathcal{F}(C_c^\infty(X : \tau))$.

Let $\mathcal{H}$ be a real Σ-configuration in $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, and let $d \in \mathbb{N}^\mathcal{H}$. By $\mathcal{P}(\mathfrak{a}_\mathrm{q}^*, \mathcal{H}, d)$ we denote the linear space of functions $\varphi \in \mathcal{M}(\mathfrak{a}_\mathrm{q}^*, \mathcal{H}, d)$ with polynomial decay in the imaginary directions, that is

$$\sup_{\lambda \in \omega + i\mathfrak{a}_\mathrm{q}^*} (1 + |\lambda|)^n |\pi_{\omega,d}(\lambda)\varphi(\lambda)| < \infty \tag{3.1}$$

for all compact $\omega \subset \mathfrak{a}_\mathrm{q}^*$ and all $n \in \mathbb{N}$. The space $\mathcal{P}(\mathfrak{a}_\mathrm{q}^*, \mathcal{H}, d)$ is given a Fréchet topology by means of the seminorms in (3.1). The union of these spaces over all $d : \mathcal{H} \to \mathbb{N}$, equipped with the limit topology, is denoted $\mathcal{P}(\mathfrak{a}_\mathrm{q}^*, \mathcal{H})$.

Definition 3.2. Let $\mathcal{H} = \mathcal{H}(X, \tau)$ and $d = d_{X,\tau}$. We define

$$\mathcal{P}_{\mathrm{AC}}(X : \tau) = \{\varphi \in \mathcal{P}(\mathfrak{a}_\mathrm{q}^*, \mathcal{H}, d) \otimes {}^\circ\mathcal{C} \mid \mathcal{L}\varphi = 0, \forall \mathcal{L} \in \mathrm{AC}_\mathbb{R}(X : \tau)\},$$

and equip this subspace of $\mathcal{P}(\mathfrak{a}_\mathrm{q}^*, \mathcal{H}, d) \otimes {}^\circ\mathcal{C}$ with the inherited topology.

Lemma 3.3. *The space $\mathcal{P}_{\mathrm{AC}}(X : \tau)$ is a Fréchet space.*

Proof. Indeed, $\mathcal{P}_{\mathrm{AC}}(X : \tau)$ is a closed subspace of $\mathcal{P}(\mathfrak{a}_\mathrm{q}^*, \mathcal{H}, d) \otimes {}^\circ\mathcal{C}$, since Laurent functionals are continuous on $\mathcal{P}(\mathfrak{a}_\mathrm{q}^*, \mathcal{H}, d)$ (cf. [5], Lemma 1.11). □

In Definition 3.2 it is required that the elements of $\mathcal{P}_{\mathrm{AC}}(X : \tau)$ belong to $\mathcal{P}(\mathfrak{a}_\mathrm{q}^*, \mathcal{H}, d) \otimes {}^\circ\mathcal{C}$ where $\mathcal{H} = \mathcal{H}(X, \tau)$ and $d = d_{X,\tau}$ are specifically given in terms of the singularities of the Eisenstein integrals. It will be shown in Lemma 3.11 below that this requirement is unnecessarily strong (however, it is convenient for the definition of the topology).

Definition 3.4. The *Paley–Wiener space* $\mathrm{PW}(X : \tau)$ is defined as the space of functions $\varphi \in \mathcal{P}_{\mathrm{AC}}(X : \tau)$ for which there exists a constant $M > 0$ such that

$$\sup_{\lambda \in \bar{\mathfrak{a}}_\mathrm{q}^*(P,0)} (1 + |\lambda|)^n e^{-M\,|\mathrm{Re}\,\lambda|} \|\pi(\lambda)\varphi(\lambda)\| < \infty \tag{3.2}$$

for all $n \in \mathbb{N}$. The subspace of functions that satisfy (3.2) for all n and a fixed $M > 0$ is denoted $\mathrm{PW}_M(X\colon\tau)$. The space $\mathrm{PW}_M(X\colon\tau)$ is given the relative topology of $\mathcal{P}_{\mathrm{AC}}(X\colon\tau)$, or equivalently, of $\mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}, d) \otimes {}^\circ\mathcal{C}$ where $\mathcal{H} = \mathcal{H}(X,\tau)$ and $d = d_{X,\tau}$. Finally, the Paley-Wiener space $\mathrm{PW}(X\colon\tau)$ is given the limit topology of the union

$$\mathrm{PW}(X\colon\tau) = \cup_{M>0}\,\mathrm{PW}_M(X\colon\tau). \tag{3.3}$$

The functions in $\mathrm{PW}(X\colon\tau)$ are called *Paley-Wiener functions*. By the definition just given they are the functions in $\mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}, d) \otimes {}^\circ\mathcal{C}$ for which the estimates (3.1) and (3.2) hold, and which are annihilated by all Arthur-Campoli functionals with real support.

Remark 3.5. It will be verified later that $\mathrm{PW}_M(X\colon\tau)$ is a closed subspace of $\mathcal{P}_{\mathrm{AC}}(X\colon\tau)$, see Remark 4.2. Hence $\mathrm{PW}_M(X\colon\tau)$ is a Fréchet space, and $\mathrm{PW}(X\colon\tau)$ a strict LF-space (see [20], p. 291). Notice that the Paley-Wiener space $\mathrm{PW}(X\colon\tau)$ is not given the relative topology of $\mathcal{P}_{\mathrm{AC}}(X\colon\tau)$. However, the inclusion map $\mathrm{PW}(X\colon\tau) \to \mathcal{P}_{\mathrm{AC}}(X\colon\tau)$ is continuous.

We are now able to state the Paley–Wiener theorem for the pair (X,τ).

Theorem 3.6. *The Fourier transform $\mathcal{F}$ is a topological linear isomorphism of $C_M^\infty(X\colon\tau)$ onto $\mathrm{PW}_M(X\colon\tau)$, for each $M > 0$, and it is a topological linear isomorphism of $C_c^\infty(X\colon\tau)$ onto the Paley–Wiener space $\mathrm{PW}(X\colon\tau)$.*

Here we recall from [6], p. 36, that $C_M^\infty(X\colon\tau)$ is the subspace of $C^\infty(X\colon\tau)$ consisting of those functions that are supported on the compact set $K\exp B_M H$, where $B_M \subset \mathfrak{a}_{\mathrm{q}}$ is the closed ball of radius M, centered at 0. The space $C_M^\infty(X\colon\tau)$ is equipped with its standard Fréchet topology, which is the relative topology of $C^\infty(X\colon\tau)$. Then

$$C_c^\infty(X\colon\tau) = \cup_{M>0} C_M^\infty(X\colon\tau) \tag{3.4}$$

and $C_c^\infty(X\colon\tau)$ carries the limit topology of this union.

The final statement in the theorem is an obvious consequence of the first, in view of (3.3) and (3.4). The proof of the first statement will be given in the course of the next 5 sections (Theorems 4.4, 4.5, proof in Sect. 8). It relies on several results from [6], which are elaborated in the following two sections. At present, we note the following

Lemma 3.7. *The Fourier transform $\mathcal{F}$ maps $C_M^\infty(X\colon\tau)$ continuously and injectively into $\mathrm{PW}_M(X\colon\tau)$ for each $M > 0$.*

Proof. The injectivity of $\mathcal{F}$ is one of the main results in [4], see Thm. 15.1. It follows from [6], Lemma 4.4, that $\mathcal{F}$ maps $C_M^\infty(X\colon\tau)$ continuously into the space $\mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}, d) \otimes {}^\circ\mathcal{C}$, where $\mathcal{H} = \mathcal{H}(X,\tau)$ and $d = d_{X,\tau}$, and that (3.2) holds for $\varphi = \mathcal{F}f \in \mathcal{F}(C_M^\infty(X\colon\tau))$. Finally, it follows from Lemma 3.8 below that $\mathcal{F}$ maps into $\mathcal{P}_{\mathrm{AC}}(X\colon\tau)$. $\square$

Lemma 3.8. *Let $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^* \otimes {}^\circ\mathcal{C}^*$. Then $\mathcal{L} \in \mathrm{AC}(X\colon\tau)$ if and only if $\mathcal{L}\mathcal{F}f = 0$ for all $f \in C_c^\infty(X\colon\tau)$.*

Proof. Recall that $\mathcal{F}f$ is defined by (2.1) for $f \in C_c^\infty(X\colon\tau)$. We claim that

$$\mathcal{L}\mathcal{F}f = \int_X \mathcal{L}E^*(\,\cdot\,\colon x) f(x)\,dx, \tag{3.5}$$

that is, the application of $\mathcal{L}$ can be taken inside the integral.

The function $\lambda \mapsto E^*(\lambda\colon x)$ on $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ belongs to $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}, d) \otimes {}^\circ\mathcal{C}$ for each $x \in X$, where $\mathcal{H} = \mathcal{H}(X, \tau)$ and $d = d_{X,\tau}$. The space $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}, d) \otimes {}^\circ\mathcal{C}$ is a complete locally convex space, when equipped with the initial topology with respect to the family of maps $\varphi \mapsto \pi_{\omega,d}\varphi$ into $\mathcal{O}(\omega)$, and $x \mapsto E^*(\,\cdot\,\colon x)$ is continuous (see [3] Lemma 14). The integrals in (2.1) and (3.5) may be seen as integrals with values in this space. Since Laurent functionals are continuous, (3.5) is justified.

Assume now that $\mathcal{L} \in \mathrm{AC}(X\colon\tau)$ and let $f \in C^\infty_c(X\colon\tau)$. Then $\mathcal{L}E^*(\,\cdot\,\colon x)f(x) = 0$ for each $x \in X$, and the vanishing of $\mathcal{L}\mathcal{F}f$ follows immediately from (3.5).

Conversely, assume that $\mathcal{L}$ annihilates $\mathcal{F}f$ for all $f \in C^\infty_c(X\colon\tau)$. From (3.5) and [4], Lemma 7.1, it follows easily that $\mathcal{L}$ annihilates $E^*(\,\cdot\,\colon a)v$ for $v \in V_\tau^{K\cap H\cap M}$ and $a \in A^+_{\mathrm{q}}(Q)$, with $Q \in \mathcal{P}^{\min}_\sigma$ arbitrary. Let $v \in V_\tau$. Since $E^*(\lambda\colon kah) = E^*(\lambda\colon a) \circ \tau(k)^{-1}$ for $k \in K$, $a \in A_{\mathrm{q}}$ and $h \in H$, it is seen that $E^*(\lambda\colon kah)v = E^*(\lambda\colon a)P(\tau(k)^{-1}v)$ where P denotes the orthogonal projection $V_\tau \to V_\tau^{K\cap H\cap M}$. Hence $\mathcal{L}$ annihilates $E^*(\,\cdot\,\colon x)v$ for all $x \in X_+$, $v \in V$. By continuity and density the same conclusion holds for all $x \in X$. $\square$

Remark 3.9. In Definition 3.2 we used only Arthur-Campoli functionals with real support. Let $\mathcal{P}_{\mathrm{AC}}(X\colon\tau)^\sim$ denote the space obtained in that definition with $\mathrm{AC}_{\mathbb{R}}(X\colon\tau)$ replaced by $\mathrm{AC}(X\colon\tau)$, and let $\mathrm{PW}(X\colon\tau)^\sim$ denote the space obtained in Definition 3.4 with $\mathcal{P}_{\mathrm{AC}}(X\colon\tau)$ replaced by $\mathcal{P}_{\mathrm{AC}}(X\colon\tau)^\sim$. Then clearly $\mathcal{P}_{\mathrm{AC}}(X\colon\tau)^\sim \subset \mathcal{P}_{\mathrm{AC}}(X\colon\tau)$ and $\mathrm{PW}(X\colon\tau)^\sim \subset \mathrm{PW}(X\colon\tau)$. However, it follows from Lemma 3.8 that $\mathcal{F}(C^\infty_c(X\colon\tau)) \subset \mathrm{PW}(X\colon\tau)^\sim$, and hence as a consequence of Theorem 3.6 we will have

$$\mathrm{PW}(X\colon\tau)^\sim = \mathrm{PW}(X\colon\tau).$$

In general, the Arthur-Campoli functionals are not explicitly described. Some relations of a more explicit nature can be pointed out, these are the relations (2.4) that express transformations under the Weyl group. In the following lemma it is shown that these relations are of Arthur-Campoli type, which explains why they are not mentioned separately in the definition of the Paley-Wiener space.

Lemma 3.10. *Let $\varphi \in \mathcal{P}_{\mathrm{AC}}(X\colon\tau)$. Then $\varphi(s\lambda) = C^\circ(s\colon\lambda)\varphi(\lambda)$ for all $s \in W$ and $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ generic.*

Proof. The relation $\varphi(s\lambda) = C^\circ(s\colon\lambda)\varphi(\lambda)$ is meromorphic in λ, so it suffices to verify it for $\lambda \in \mathfrak{a}^*_{\mathrm{q}}$. Let $\mathcal{H} = \mathcal{H}(X, \tau)$. Fix $s \in W$ and $\lambda \in \mathfrak{a}^*_{\mathrm{q}}$ such that $C^\circ(s\colon\lambda)$ is non-singular at λ, and such that λ and $s\lambda$ do not belong to any of the hyperplanes from $\mathcal{H}$. Let $\psi \in {}^\circ\mathcal{C}$ and consider the linear form $\mathcal{L}_\psi\colon \varphi \mapsto \langle\varphi(s\lambda) - C^\circ(s\colon\lambda)\varphi(\lambda)|\psi\rangle$ on $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}) \otimes {}^\circ\mathcal{C}$. It follows from [7], Remark 10.6, that for each $\nu \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ there exists a Σ-Laurent functional which, when applied to the functions that are regular at ν, yields the evaluation in ν. Obviously, the support of such a functional is $\{\nu\}$. Hence there exists $\mathcal{L} \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \Sigma)^*_{\mathrm{laur}} \otimes {}^\circ\mathcal{C}^*$ with support $\{\lambda, s\lambda\}$ such that $\mathcal{L}\varphi = \mathcal{L}_\psi\varphi$ for all $\varphi \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}) \otimes {}^\circ\mathcal{C}$. It follows from (2.3) and Definition 3.1 that $\mathcal{L} \in \mathrm{AC}_{\mathbb{R}}(X\colon\tau)$. The lemma follows immediately. $\square$

Lemma 3.11. *Let $\mathcal{H}$ be a real Σ-configuration in $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ and let $\varphi \in \mathcal{P}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}) \otimes {}^\circ\mathcal{C}$. Assume $\mathcal{L}\varphi = 0$ for all $\mathcal{L} \in \mathrm{AC}_{\mathbb{R}}(X\colon\tau)$. Then $\varphi \in \mathcal{P}_{\mathrm{AC}}(X\colon\tau)$.*

Proof. Let $d \in \mathbb{N}^{\mathcal{H}}$ be such that $\varphi \in \mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}, d) \otimes {}^\circ\mathcal{C}$. We may assume that $\mathcal{H} \supset \mathcal{H}(X, \tau)$ and that $d \succeq d_{X,\tau}$ (that is, $d(H) \geq d_{X,\tau}(H)$ for all $H \in \mathcal{H}$), where $d_{X,\tau}$ is trivially extended to $\mathcal{H}$. Let $H \in \mathcal{H}$ be arbitrary and let l be the least non-negative integer for which $\lambda \mapsto (\langle \lambda, \alpha_H \rangle - s_H)^l \varphi(\lambda)$ is regular along $H_{\mathrm{reg}} := H \setminus \cup\{H' \in \mathcal{H} \mid H' \neq H\}$. Then $l \leq d(H)$, and the statement of the lemma amounts to $l \leq d_{X,\tau}(H)$.

Assume that $l > d_{X,\tau}(H)$; we will show that this leads to a contradiction. Let $d' \in \mathbb{N}^{\mathcal{H}}$ be the element such that $d'(H) = l$ and which equals d on all other hyperplanes in $\mathcal{H}$. Then $\varphi \in \mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}, d') \otimes {}^\circ\mathcal{C}$ and $d' \succ d_{X,\tau}$. Let $\lambda_0 \in H_{\mathrm{reg}} \cap \mathfrak{a}_{\mathrm{q}}^*$. It follows from [7], Lemmas 10.4-10.5, that there exists $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^*$ such that $\mathcal{L}\phi$ is the evaluation in λ_0 of $(\langle \lambda, \alpha_H \rangle - s_H)^l \phi(\lambda)$ for all $\phi \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}, d')$. Obviously, $\operatorname{supp} \mathcal{L} = \{\lambda_0\} \subset \mathfrak{a}_{\mathrm{q}}^*$. Since $l > d_{X,\tau}(H)$, the functional $\mathcal{L} \otimes \eta$ annihilates $\mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}, d_{X,\tau}) \otimes {}^\circ\mathcal{C}$ for all $\eta \in {}^\circ\mathcal{C}^*$ and hence belongs to $\mathrm{AC}_{\mathbb{R}}(X\colon \tau)$. Then it also annihilates φ, that is, the function $(\langle \lambda, \alpha_H \rangle - s_H)^l \varphi(\lambda)$ vanishes at λ_0, which was arbitrary in $H_{\mathrm{reg}} \cap \mathfrak{a}_{\mathrm{q}}^*$. By meromorphic continuation this function vanishes everywhere. This contradicts the definition of l. $\square$

4 Pseudo wave packets

In the Fourier inversion formula $\mathcal{T}\mathcal{F}f = f$ the *pseudo wave packet* $\mathcal{T}\mathcal{F}f$ is defined by

$$\mathcal{T}\mathcal{F}f(x) = |W| \int_{\eta + i\mathfrak{a}_{\mathrm{q}}^*} E_+(\lambda\colon x) \mathcal{F}f(\lambda)\, d\lambda, \qquad x \in X_+, \tag{4.1}$$

for $f \in C_c^\infty(X\colon \tau)$ and for $\eta \in \mathfrak{a}_{\mathrm{q}}^*$ sufficiently antidominant (the function is then independent of η). Here $d\lambda$ is the translate of Lebesgue measure on $i\mathfrak{a}_{\mathrm{q}}^*$, normalized as in [6], eq. (5.2). A priori, $\mathcal{T}\mathcal{F}f$ belongs to the space $C^\infty(X_+\colon \tau)$ of smooth τ-spherical functions on X_+, but the identity with f shows that it extends to a smooth function on X.

The pseudo wave packets are also used for the proof of the Paley–Wiener theorem: Given a function in the Paley–Wiener space, the candidate for its Fourier preimage is constructed as a pseudo wave packet on X_+. In this section we reduce the proof of the Paley–Wiener theorem to one property of such pseudo wave packets. This property, that they extend to global smooth functions on X, will be established in Section 8.

We first recall some spaces defined in [6], and relate them to the spaces given in Definitions 3.2 and 3.4.

Definition 4.1. Let $\mathcal{P}(X\colon \tau)$ be the space of meromorphic functions $\varphi\colon \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to {}^\circ\mathcal{C}$ having the following properties (i)-(iii) (see (2.9) for the definition of π):

(i) $\varphi(s\lambda) = C^\circ(s\colon \lambda)\varphi(\lambda)$ for all $s \in W$ and generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.
(ii) There exists $\epsilon > 0$ such that $\pi\varphi$ is holomorphic on $\mathfrak{a}_{\mathrm{q}}^*(P, \epsilon)$.
(iii) For some $\epsilon > 0$, for every compact set $\omega \subset \mathfrak{a}_{\mathrm{q}}^*(P, \epsilon) \cap \mathfrak{a}_{\mathrm{q}}^*$ and for all $n \in \mathbb{N}$,

$$\sup_{\lambda \in \omega + i\mathfrak{a}_{\mathrm{q}}^*} (1 + |\lambda|)^n \|\pi(\lambda)\varphi(\lambda)\| < \infty.$$

Moreover, for each $M > 0$ let $\mathcal{P}_M(X\colon \tau)$ be the subspace of $\mathcal{P}(X\colon \tau)$ consisting of the functions $\varphi \in \mathcal{P}(X\colon \tau)$ with the following property (iv).

(iv) For every strictly antidominant $\eta \in \mathfrak{a}_\mathrm{q}^*$ there exists a constant $t_\eta \geq 0$ such that

$$\sup_{t \geq t_\eta, \lambda \in t\eta + i\mathfrak{a}_\mathrm{q}^*} (1 + |\lambda|)^{\dim \mathfrak{a}_\mathrm{q} + 1} e^{-M\,|\mathrm{Re}\,\lambda|} \|\varphi(\lambda)\| < \infty. \tag{4.2}$$

Notice that (ii) and (iii) are satisfied by any function $\varphi \in \mathcal{P}(\mathfrak{a}_\mathrm{q}^*, \mathcal{H}(X,\tau), d_{X,\tau}) \otimes {}^\circ\mathcal{C}$, by the definition of π. If φ belongs to the subspace $\mathcal{P}_{\mathrm{AC}}(X\colon\tau)$ it also satisfies (i), by Lemma 3.10, and hence

$$\mathrm{PW}(X\colon\tau) \subset \mathcal{P}_{\mathrm{AC}}(X\colon\tau) \subset \mathcal{P}(X\colon\tau). \tag{4.3}$$

Moreover, the estimate in (3.2) is stronger than (iv), and hence

$$\mathrm{PW}_M(X\colon\tau) \subset \mathcal{P}_{\mathrm{AC}}(X\colon\tau) \cap \mathcal{P}_M(X\colon\tau). \tag{4.4}$$

Remark 4.2. It will be shown later by Euclidean Fourier analysis, see Lemma 9.3, that the stronger estimate (3.2) holds for all $\varphi \in \mathcal{P}_M(X\colon\tau)$. In particular, it follows that in fact

$$\mathrm{PW}_M(X\colon\tau) = \mathcal{P}_{\mathrm{AC}}(X\colon\tau) \cap \mathcal{P}_M(X\colon\tau). \tag{4.5}$$

It will also follow from Lemma 9.3 that $\mathrm{PW}_M(X\colon\tau)$ is a closed subspace of $\mathcal{P}_{\mathrm{AC}}(X\colon\tau)$, hence a Fréchet space. Alternatively, the latter property of $\mathrm{PW}_M(X\colon\tau)$ follows directly from Theorem 3.6, in the proof of which it is never used. In fact, (4.5) will be established in the course of that proof.

Remark 4.3. It will also be shown, see Lemma 10.2, that there exists a real Σ-configuration $\mathcal{H}^\sim$ and a map $d^\sim\colon \mathcal{H}^\sim \to \mathbb{N}$ such that $\mathcal{P}(X\colon\tau) \subset \mathcal{P}(\mathfrak{a}_\mathrm{q}^*, \mathcal{H}^\sim, d^\sim) \otimes {}^\circ\mathcal{C}$. In combination with Lemma 3.11 this implies that

$$\mathcal{P}_{\mathrm{AC}}(X\colon\tau) = \{\varphi \in \mathcal{P}(X\colon\tau) \mid \mathcal{L}\varphi = 0, \forall \mathcal{L} \in \mathrm{AC}_\mathbb{R}(X\colon\tau)\}.$$

The present remark is not used in the proof of Theorem 3.6.

Recall from [6], Sect. 4, that the *pseudo wave packet* of (4.1) can be formed with $\mathcal{F}f$ replaced by an arbitrary function $\varphi \in \mathcal{P}(X\colon\tau)$. The resulting function $\mathcal{T}\varphi \in C^\infty(X_+\colon\tau)$ is given by

$$\mathcal{T}\varphi(x) = |W| \int_{\eta + i\mathfrak{a}_\mathrm{q}^*} E_+(\lambda\colon x)\varphi(x)\, d\lambda, \qquad x \in X_+, \tag{4.6}$$

for $\eta \in \mathfrak{a}_\mathrm{q}^*$ sufficiently antidominant, so that the function is independent of η. The following theorem represents the main step in the proof of the Paley-Wiener theorem.

Theorem 4.4. *Let $\varphi \in \mathcal{P}_{\mathrm{AC}}(X\colon\tau)$. Then $\mathcal{T}\varphi$ extends to a smooth τ-spherical function on X (also denoted by $\mathcal{T}\varphi$). The map $\mathcal{T}$ is continuous from $\mathcal{P}_{\mathrm{AC}}(X\colon\tau)$ to $C^\infty(X\colon\tau)$.*

We will prove this result in Section 8 (see below Theorem 8.3). However, we first use it to derive the following Theorem 4.5, from which Theorem 3.6 is an immediate consequence.

Theorem 4.5. *Let $M > 0$. Then $\mathcal{T}\varphi \in C_M^\infty(X\colon\tau)$ for all $\varphi \in \mathrm{PW}_M(X\colon\tau)$, and $\mathcal{T}$ is a continuous inverse to the Fourier transform $\mathcal{F}\colon C_M^\infty(X\colon\tau) \to \mathrm{PW}_M(X\colon\tau)$.*

Proof. Let $\mathcal{P}'_M(X\colon\tau)$ denote the set of functions $\varphi \in \mathcal{P}_M(X\colon\tau)$ for which $\mathcal{T}\varphi$ has a smooth extension to X. We have seen in [6], Cor. 4.11, that $\mathcal{F}$ maps $C_M^\infty(X\colon\tau)$ bijectively onto $\mathcal{P}'_M(X\colon\tau)$ with $\mathcal{T}$ as its inverse. It follows from Theorem 4.4 that $\mathcal{P}_{\mathrm{AC}}(X\colon\tau)\cap\mathcal{P}_M(X\colon\tau)$ is contained in $\mathcal{P}'_M(X\colon\tau)$. Combining this with Lemma 3.7 and (4.4) we obtain the following chain of inclusions

$$\mathcal{F}(C_M^\infty(X\colon\tau)) \subset \mathrm{PW}_M(X\colon\tau) \subset \mathcal{P}_{\mathrm{AC}}(X\colon\tau)\cap\mathcal{P}_M(X\colon\tau) \subset \mathcal{P}'_M(X\colon\tau) = \mathcal{F}(C_M^\infty(X\colon\tau)).$$

It follows that these inclusions are equalities (in particular, (4.5) is then established). Thus $\mathcal{F}$ is bijective $C_M^\infty(X\colon\tau) \to \mathrm{PW}_M(X\colon\tau)$ with inverse $\mathcal{T}$.

Since $\mathcal{T}\colon \mathcal{P}_{\mathrm{AC}}(X\colon\tau) \to C^\infty(X\colon\tau)$ is continuous by Theorem 4.4 and since $\mathrm{PW}_M(X\colon\tau)$ and $C_M^\infty(X\colon\tau)$ carry the restriction topologies of these spaces, we conclude that the restriction map $\mathcal{T}\colon \mathrm{PW}_M(X\colon\tau) \to C_M^\infty(X\colon\tau)$ is continuous. □

5 Generalized Eisenstein integrals

In [6], Sect. 10, we defined generalized Eisenstein integrals for X. These will be used extensively in the following. In this section we recall their definition and derive some properties of them. For further properties (not to be used here), we refer to [8], [9].

Let $t \in \mathrm{WT}(\Sigma)$ be an even and W-invariant residue weight (see [5], p. 60) to be fixed throughout the paper. Let $f \mapsto \mathrm{T}_\Delta^t f$, $C_c^\infty(X\colon\tau) \to C^\infty(X\colon\tau)$, be the operator defined by [6], eq. (5.5), with $F = \Delta$. The fact that it maps into $C^\infty(X\colon\tau)$ is a consequence of [6], Cor. 10.11. Moreover, if the vectorial part of X vanishes, that is, if $\mathfrak{a}_{\Delta\mathrm{q}} = \{0\}$, then

$$\mathrm{T}_\Delta^t f(x) = |W| \int_X \mathrm{K}_\Delta^t(x\colon y) f(y)\, dy \tag{5.1}$$

for $x \in X$, cf. [6], eq. (5.10) and proof of Cor. 10.11, where $\mathrm{K}_\Delta^t(x\colon y)$ is the residue kernel defined by [6], eq. (5.7), with $F = \Delta$.

If the vectorial part of X vanishes, then we follow [6], Remark 10.5, and define a finite dimensional space by

$$\mathcal{A}^t(X\colon\tau) = \mathrm{Span}\{\mathrm{K}_\Delta^t(\,\cdot\,\colon y)u \mid y \in X_+, u \in V_\tau\} \subset C^\infty(X\colon\tau). \tag{5.2}$$

The space is denoted $\mathcal{C}_\Delta$ in [6], whereas the present notation is in agreement with [8], Sect. 9. By continuity of K_Δ^t and finite dimensionality of $\mathcal{A}^t(X\colon\tau)$, $\mathrm{K}_\Delta^t(\,\cdot\,\colon y)u$ belongs to this space for $y \in X \setminus X_+$ as well.

Lemma 5.1. *Assume $\mathfrak{a}_{\Delta\mathrm{q}} = \{0\}$. Then $\mathrm{T}_\Delta^t f \in \mathcal{A}^t(X\colon\tau)$ for all $f \in C_c^\infty(X\colon\tau)$, and the map $\mathrm{T}_\Delta^t\colon C_c^\infty(X\colon\tau) \to \mathcal{A}^t(X\colon\tau)$ is surjective.*

Proof. The map $y \mapsto \mathrm{K}_\Delta^t(\,\cdot\,\colon y)f(y)$ belongs to $C_c^\infty(X\colon\tau) \otimes \mathcal{A}^t(X\colon\tau)$. Hence its integral (5.1) over X belongs to $\mathcal{A}^t(X\colon\tau)$. The surjectivity follows from (5.1), see [8], Lemma 9.1. □

Remark 5.2. It is seen in [8], Thm. 21.2, Def. 12.1 and Lemma 12.6, that $\mathcal{A}^t(X\colon\tau)$ equals the discrete series subspace $L^2_d(X\colon\tau)$ of $L^2(X\colon\tau)$ and that $\mathrm{T}^t_\Delta\colon C^\infty_c(X\colon\tau)\to\mathcal{A}^t(X\colon\tau)$ is the restriction of the orthogonal projection $L^2(X\colon\tau)\to L^2_d(X\colon\tau)$. In particular, the objects $\mathcal{A}^t(X\colon\tau)$ and T^t_Δ are independent of the choice of the residue weight t. In the present paper t is fixed throughout and we do not need these properties. However, to simplify notation let $\mathrm{T}_\Delta := \mathrm{T}^t_\Delta$ and $\mathcal{A}(X\colon\tau) := \mathcal{A}^t(X\colon\tau)$.

Fix $F\subset\Delta$ and let $\mathfrak{a}_{F\mathrm{q}}\subset\mathfrak{a}_\mathrm{q}$ be defined as in [6], p. 41. For each $v\in\mathcal{W}$ let

$$X_{F,v} = M_F/M_F\cap vHv^{-1}$$

be the reductive symmetric space defined as in [6], p. 51. We use notation of [6], pp. 51-52, related to this space. Put $\tau_F = \tau|_{M_F\cap K}$ and let the finite dimensional space

$$\mathcal{A}(X_{F,v}\colon\tau_F) = \mathcal{A}^{*t}(X_{F,v}\colon\tau_F)\subset C^\infty(X_{F,v}\colon\tau_F)$$

be the analog for $X_{F,v}$ of the space $\mathcal{A}(X\colon\tau)$ of (5.2), cf. [6], eq. (10.7), where the space is denoted $\mathcal{C}_{F,v}$. The assumption made before (5.2), that the vectorial part of X vanishes, holds for $X_{F,v}$. For $\psi\in\mathcal{A}(X_{F,v}\colon\tau_F)$ we have defined the generalized Eisenstein integral $E^\circ_{F,v}(\psi\colon\nu)\in C^\infty(X\colon\tau)$ in [6], Def. 10.7; it is a linear function of ψ and a meromorphic function of $\nu\in\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. Let us recall the definition.

The space $\mathcal{A}(X_{F,v}\colon\tau_F)$ is spanned by elements $\psi\in C^\infty(X_{F,v}\colon\tau_F)$ of the form

$$\psi(m) = \psi_{y,u}(m) = \mathrm{K}^{*t}_F(X_{F,v}\colon m\colon y)u \tag{5.3}$$

for some $y\in X_{F,v,+}$, $u\in V_\tau$. Here $\mathrm{K}^{*t}_F(X_{F,v}\colon\cdot\colon\cdot)$ is the analog for $X_{F,v}$ of the kernel K^t_Δ, the residue weight ${}^*t\in\mathrm{WT}(\Sigma_F)$ is defined in [5], eq. (3.16). By definition

$$E^\circ_{F,v}(\psi_{y,u}\colon\nu\colon x) = \sum_{\lambda\in\Lambda(X_{F,v},F)} \mathrm{Res}^{*P,*t}_\lambda\big[E^\circ(\nu-\cdot\colon x)\circ \mathrm{i}_{F,v}\,E^*_+(X_{F,v}\colon -\cdot\colon y)u\big] \tag{5.4}$$

for $x\in X$. Here $E^*_+(X_{F,v}\colon\lambda\colon y) = E_+(X_{F,v}\colon -\bar\lambda\colon y)^*$ and $\Lambda(X_{F,v},F)\subset\mathfrak{a}^{*\perp}_{F\mathrm{q}}$ is the set defined in [6], eq. (8.7). The generalized Eisenstein integral $E^\circ_{F,v}(\psi\colon\nu\colon x)$ is defined for $\psi\in\mathcal{A}(X_{F,v}\colon\tau_F)$ by (5.4) and linearity; the fact that it is well defined is shown in [6], Lemma 10.6, using the induction of relations of [7]. Let

$$\psi = \sum_v \psi_v \in \mathcal{A}_F := \oplus_{v\in{}^F\mathcal{W}}\ \mathcal{A}(X_{F,v}\colon\tau_F), \tag{5.5}$$

where ${}^F\mathcal{W}$ is as in [6], above Lemma 8.1. Define

$$E^\circ_F(\psi\colon\nu\colon x) = \sum_{v\in{}^F\mathcal{W}} E^\circ_{F,v}(\psi_v\colon\nu\colon x). \tag{5.6}$$

Remark 5.3. A priori the generalized Eisenstein integral $E^\circ_F(\psi\colon\nu\colon x)$ depends on the choice of the residue weight t. In fact, already the parameter space $\mathcal{A}(X_{F,v}\colon\tau_F)$ for ψ

depends on t through the residue weight ${}^*t$. However, according to Remark 5.2 (applied to the symmetric space $X_{F,v}$) the latter is actually not the case. Once the independence of $\mathcal{A}(X_{F,v}\colon \tau_F)$ on ${}^*t$ has been established, it follows from the characterization in [8], Thm. 9.3, that $E_F^\circ(\psi\colon \nu\colon x)$ is independent of t. Therefore, this parameter is not indicated in the notation. The independence of t is not used in the present paper.

Lemma 5.4. *Let $\psi = \psi_{y,u} \in \mathcal{A}(X_{F,v}\colon \tau_F)$ be given by (5.3) with $y \in X_{F,v}$, $u \in V_\tau$. Then*

$$E_{F,v}^\circ(\psi_{y,u}\colon \nu\colon x) = \sum_{\lambda \in \Lambda(X_{F,v},F)} \mathrm{Res}_\lambda^{{}^*P,{}^*t} \Big[\sum_{s \in W^F} E_{+,s}(\nu + \cdot\colon x) \circ \mathrm{i}_{F,v}\, E^*(X_{F,v}\colon \cdot\colon y) u \Big] \quad (5.7)$$

for $x \in X_+$ and generic $\nu \in \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$.

Proof. If $y \in X_{F,v,+}$ then (5.4) holds and (5.7) follows from [6], eq. (8.9). The map $y \mapsto \psi_{y,u}$, $X_{F,v} \to \mathcal{A}(X_{F,v}\colon \tau_F)$ is continuous, and $E_{F,v}^\circ(\psi\colon \nu\colon x)$ is linear in ψ, hence the left side of (5.7) is continuous in $y \in X_{F,v}$. The other side is continuous as well, so (5.7) follows by the density of $X_{F,v,+}$ in $X_{F,v}$. $\square$

Let

$$f \mapsto \mathrm{T}_F(X_{F,v}\colon f), \quad C_c^\infty(X_{F,v}\colon \tau) \to \mathcal{A}(X_{F,v}\colon \tau_F) \subset C^\infty(X_{F,v}\colon \tau)$$

be the analog for $X_{F,v}$ of the operator T_Δ of (5.1) (with respect to some choice of invariant measure dy on $X_{F,v}$). The operator $\mathrm{T}_F(X_{F,v}\colon f)$ should not be confused with the operator T_F^t of [6], eq. (5.5), which maps between function spaces on X. In the following lemma we examine the generalized Eisenstein integral $E_{F,v}^\circ(\mathrm{T}_F(X_{F,v}\colon f)\colon \nu)$. Let the Fourier transform associated with $X_{F,v}$ be denoted $f \mapsto \mathcal{F}(X_{F,v}\colon f)$. It maps $C_c^\infty(X_{F,v}\colon \tau)$ into $\mathcal{M}(\mathfrak{a}_{F\mathrm{q}\mathbb{C}}^{*\perp}, \Sigma_F) \otimes {}^\circ\mathcal{C}_{F,v}$ and is given by (see (2.1))

$$\mathcal{F}(X_{F,v}\colon f)(\nu) = \int_{X_{F,v}} E^*(X_{F,v}\colon \nu\colon y) f(y)\, dy, \quad (\nu \in \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^{*\perp}). \quad (5.8)$$

Lemma 5.5. *Let $f \in C_c^\infty(X_{F,v}\colon \tau)$ and let $\psi = |W_F|^{-1} \mathrm{T}_F(X_{F,v}\colon f) \in \mathcal{A}(X_{F,v}\colon \tau_F)$. Then*

$$E_{F,v}^\circ(\psi\colon \nu\colon x) = \sum_{\lambda \in \Lambda(X_{F,v},F)} \mathrm{Res}_\lambda^{{}^*P,{}^*t} \Big[\sum_{s \in W^F} E_{+,s}(\nu + \cdot\colon x) \circ \mathrm{i}_{F,v}\, \mathcal{F}(X_{F,v}\colon f)(\,\cdot\,) \Big] \quad (5.9)$$

for $x \in X_+$ and generic $\nu \in \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$.

Proof. For each $y \in X_{F,v}$ let $\psi_y \in C^\infty(X_{F,v}\colon \tau)$ be defined by $\psi_y(m) = \psi_{y,f(y)}(m) = \mathrm{K}_F^{*t}(X_{F,v}\colon m\colon y) f(y)$, cf. (5.3). Then $\psi_y \in \mathcal{A}(X_{F,v}\colon \tau_F)$ and $y \mapsto \psi_y$ is continuous into this space. We conclude from (5.1), applied to $X_{F,v}$, that $\psi = \int_{X_{F,v}} \psi_y\, dy$ pointwise on $X_{F,v}$, and hence also as a $\mathcal{A}(X_{F,v}\colon \tau_F)$-valued integral. The Eisenstein integral $E_{F,v}^\circ(\psi\colon \nu\colon x)$ is linear in the first variable, hence we further conclude that

$$E_{F,v}^\circ(\psi\colon \nu\colon x) = \int_{X_{F,v}} E_{F,v}^\circ(\psi_y\colon \nu\colon x)\, dy. \quad (5.10)$$

It follows from Lemma 5.4 that

$$E^\circ_{F,v}(\psi_y\colon \nu\colon x) = \sum_{\lambda\in\Lambda(X_{F,v},F)} \mathrm{Res}^{*P,*t}_{\lambda}\big[\sum_{s\in W^F} E_{+,s}(\nu+\,\cdot : x)\circ \mathrm{i}_{F,v}\, E^*(X_{F,v}\colon \cdot : y) f(y)\big]$$

for $x\in X_+$. We insert this relation into (5.10) and take the residue operator outside the integral over $y\in \operatorname{supp} f\subset X_{F,v}$. The justification is similar to that given in the proof of Lemma 3.8. Using (5.8) we then obtain (5.9). □

Lemma 5.6. *The expressions* (5.4), (5.7), (5.9) *remain valid if the set of summation* $\Lambda(X_{F,v},F)$ *is replaced by any finite subset* Λ *of* $\mathfrak{a}^{*\perp}_{F\mathrm{q}}$ *containing* $\Lambda(X_{F,v},F)$.

Proof. It follows from [6], Lemma 10.6, that the sum in (5.4) remains unchanged if $\Lambda(X_{F,v},F)$ is replaced by Λ. That the same conclusion holds for (5.7) and (5.9) is then seen as in the proofs of Lemmas 5.4 and 5.5. □

6 Induction of Arthur-Campoli relations

In this section we prove in Theorem 6.2 a result that will play a crucial role for the Paley–Wiener theorem. It shows that Arthur–Campoli functionals on the smaller symmetric space $X_{F,v}$ induce Arthur–Campoli functionals on the full space X. The result is established by means of the theory of induction of relations developed in [7], Cor. 16.4. The corresponding result in the group case is [1], Lemma III.2.3, however, for the unnormalized Eisenstein integrals. Let $F\subset\Delta$, and let $S\subset \mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}$ be finite.

Lemma 6.1. *Let* $\mathcal{H}$ *be a* Σ*-configuration in* $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$*, and let* $\mathcal{L}\in\mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}},\Sigma_F)^*_{\mathrm{laur}}$ *with* $\operatorname{supp}\mathcal{L}\subset S$.

(i) *The set of affine hyperplanes in* $\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$*,*

$$\mathcal{H}_F(S) = \cup_{a\in S}\,\{H' \mid \exists H\in\mathcal{H}\colon\ a+H' = (a+\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}})\cap H \subsetneq a+\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}\},$$

is a $\Sigma_r(F)$*-configuration, which is real if* $\mathcal{H}$ *is real and* $S\subset\mathfrak{a}^{*\perp}_{F\mathrm{q}}$*. The corresponding set of regular points is*

$$\mathrm{reg}(\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}},\mathcal{H}_F(S)) = \{\nu\in\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}} \mid \forall a\in S, H\in\mathcal{H} : a+\nu\in H \Rightarrow a+\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}\subset H\}.$$

(ii) *For each* $\varphi\in\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}},\mathcal{H})$ *and each* $\nu\in\mathrm{reg}(\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}},\mathcal{H}_F(S))$ *there exists a neighborhood* Ω *of* S *in* $\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}$ *such that the function* $\varphi^\nu\colon\lambda\mapsto\varphi(\lambda+\nu)$ *belongs to* $\mathcal{M}(\Omega,\Sigma_F)$.

(iii) *Fix* $\nu\in\mathrm{reg}(\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}},\mathcal{H}_F(S))$*. There exists a Laurent functional (in general not unique)* $\mathcal{L}'\in\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}},\Sigma)^*_{\mathrm{laur}}$*, supported by the set* $\nu+S$*, such that* $\mathcal{L}'\varphi=\mathcal{L}\varphi^\nu$ *for all* $\varphi\in\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}},\mathcal{H})$.

(iv) *The function* $\mathcal{L}_*\varphi\colon\nu\mapsto\mathcal{L}\varphi^\nu$ *belongs to* $\mathcal{M}(\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}},\mathcal{H}_F(S))$ *for each* $\varphi\in\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}},\mathcal{H})$.

(v) *The map* $\mathcal{L}_*$ *maps* $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}},\mathcal{H})$ *continuously into* $\mathcal{M}(\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}},\mathcal{H}_F(S))$ *and if* $\mathcal{H}$ *is real,* $\mathcal{P}(\mathfrak{a}^*_{\mathrm{q}},\mathcal{H})$ *continuously into* $\mathcal{P}(\mathfrak{a}^*_{F\mathrm{q}},\mathcal{H}_F(S))$.

Proof. See [7], Cor. 11.6 and Lemma 11.7. The continuity in (v) between the $\mathcal{M}$ spaces is proved in [7] Cor. 11.6(b), the continuity between the $\mathcal{P}$ spaces is similar, see also [5], Lemma 1.10. □

Let $\mathcal{H} = \mathcal{H}(X, \tau)$ and let $\nu \in \mathrm{reg}(\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}, \mathcal{H}_F(S))$. Let $v \in {}^F\mathcal{W}$ and let $\mathrm{pr}_{F,v}\colon {}^\circ\mathcal{C} \to {}^\circ\mathcal{C}_{F,v}$ be the projection operator defined by [7], (15.3).

Theorem 6.2. *For each $\mathcal{L} \in \mathrm{AC}(X_{F,v}\colon \tau_F)$ with $\mathrm{supp}\,\mathcal{L} \subset S$ there exists a Laurent functional (in general not unique) $\mathcal{L}' \in \mathrm{AC}(X\colon\tau)$, supported by the set $\nu + S$, such that*

$$\mathcal{L}[\mathrm{pr}_{F,v}\,\varphi(\nu + \,\cdot\,)] = \mathcal{L}'\varphi, \tag{6.1}$$

for all $\varphi \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}) \otimes {}^\circ\mathcal{C}$. In particular, if in addition $S \subset \mathfrak{a}^{*\perp}_{F\mathrm{q}}$ then

$$\mathcal{L}[\mathrm{pr}_{F,v}\,\varphi(\nu + \,\cdot\,)] = 0 \tag{6.2}$$

for all $\varphi \in \mathcal{P}_{\mathrm{AC}}(X\colon\tau)$.

Proof. The existence of $\mathcal{L}' \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \Sigma)^*_{\mathrm{laur}} \otimes {}^\circ\mathcal{C}^*$ such that (6.1) holds follows from Lemma 6.1(iii). We will show that every such element $\mathcal{L}'$ belongs to $\mathrm{AC}(X\colon\tau)$. If $\nu \in \mathrm{reg}(\mathfrak{a}^*_{F\mathrm{q}}, \mathcal{H}_F(S))$ the statement (6.2) is then straightforward from the definition of $\mathcal{P}_{\mathrm{AC}}(X\colon\tau)$, and in general it follows by meromorphic continuation.

That $\mathcal{L} \in \mathrm{AC}(X_{F,v}\colon\tau_F)$ means by definition that it belongs to $\mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}} \otimes {}^\circ\mathcal{C}^*_{F,v}$ and satisfies

$$\mathcal{L}[E^*(X_{F,v}\colon\,\cdot\,\colon m)u] = 0 \tag{6.3}$$

for every $m \in X_{F,v}$, $u \in V_\tau$. By (6.1) the claim that $\mathcal{L}' \in \mathrm{AC}(X\colon\tau)$ amounts to

$$\mathcal{L}[\mathrm{pr}_{F,v}\,E^*(X\colon\nu + \,\cdot\,\colon x)u] = 0 \tag{6.4}$$

for all $x \in X$. This claim will now be established by means of [7], Cor. 16.4.

If $\psi \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)$, then the function $\psi^\vee\colon \lambda \mapsto \overline{\psi(-\bar\lambda)}$ belongs to $\mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)$ as well. If $\mathcal{L} \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}}$, then it is readily seen that there exists a unique $\mathcal{L}^\vee \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}}$ such that

$$\mathcal{L}^\vee\psi = (\mathcal{L}\psi^\vee)^* \tag{6.5}$$

for all $\psi \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)$; here the superscript $*$ indicates that the complex conjugate is taken. The maps $\psi \mapsto \psi^\vee$ and $\mathcal{L} \to \mathcal{L}^\vee$ are antilinear. More generally, if H is a Hilbert space and $v \in H$, then by v^* we denote the element of the dual Hilbert space H^* defined by $v^* : w \mapsto \langle w, v\rangle$. The maps $(\psi, v) \mapsto \Psi^\vee \otimes v^*$ and $(\mathcal{L}, v) \mapsto \mathcal{L}^\vee \otimes v^*$ induce antilinear maps from $\mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F) \otimes H$ to $\mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F) \otimes H^*$, and from $\mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}} \otimes H$ to $\mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}} \otimes H^*$, which we denote by $\psi \mapsto \psi^\vee$ and $\mathcal{L} \mapsto \mathcal{L}^\vee$ as well. With this notation formula (6.5) is valid for all $\psi \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F) \otimes H \otimes V_\tau$ and all $\mathcal{L} \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}} \otimes H$. It is then an identity between members of V_τ.

Notice that by definition of $E^*(X_{F,v}\colon\,\cdot\,\colon m)$ it is the $\psi^\vee$ of $\psi = E^\circ(X_{F,v}\colon\,\cdot\,\colon m) \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F) \otimes {}^\circ\mathcal{C}^*_{F,v} \otimes V_\tau$. It now follows from (6.5) and (6.3) that

$$\mathcal{L}^\vee(E^\circ(X_{F,v}\colon\,\cdot\,\colon m)) = 0 \tag{6.6}$$

for all $m \in X_{F,v}$, with $\mathcal{L}^\vee \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}} \otimes {}^\circ\mathcal{C}_{F,v}$ defined as above. Let

$$\mathcal{L}_2 = (1 \otimes \mathrm{i}_{F,v})\mathcal{L}^\vee \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}} \otimes {}^\circ\mathcal{C},$$

then $\mathcal{L}_2(E^\circ(X_{F,u}\colon \cdot : m) \circ \mathrm{pr}_{F,u}) = 0$ for all $u \in {}^F\mathcal{W}$, by (6.6) and [7], (16.2). In view of [7] Cor. 16.4 with $\mathcal{L}_1 = 0$ this implies that

$$\mathcal{L}_2[E^\circ(X\colon \nu + \cdot : x)] = 0 \tag{6.7}$$

for $x \in X_+$, hence by continuity also for $x \in X$. Since $\mathcal{L}_2 = (\mathcal{L}(1 \otimes \mathrm{pr}_{F,v}))^\vee$ we readily obtain (6.4) by application of (6.5) to (6.7). $\square$

7 A property of the Arthur–Campoli relations

The aim of this section is to establish a result, Lemma 7.4, which elaborates on the definition of the space $\mathrm{AC}(X\colon\tau)$ by means of some simple linear algebra.

For any finite set $S \subset \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ we denote by $\mathcal{O}_S$ the space of germs at S of functions $\phi \in \mathcal{O}(\Omega)$, holomorphic on some open neighborhood Ω of S. Moreover, if Ω is an open neighborhood of S and $d\colon \Sigma \to \mathbb{N}$ a map, then by $\mathcal{M}(\Omega, S, \Sigma, d)$ we denote the space of meromorphic functions ψ on Ω, whose germ at a belongs to $\pi_{a,d}^{-1}\mathcal{O}_a$ for each $a \in S$. Here

$$\pi_{a,d}(\lambda) = \Pi_{\alpha\in\Sigma}\, \langle \alpha, \lambda - a\rangle^{d(\alpha)}$$

for $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ (cf. [7], eq. (10.1)). Finally, we put $\mathcal{M}(\Omega, S, \Sigma) = \cup_d \mathcal{M}(\Omega, S, \Sigma, d)$.

Lemma 7.1. *Let $\mathbf{L} \subset \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \Sigma)^*_{\mathrm{laur}} \otimes {}^\circ\mathcal{C}^*$ be a finite dimensional linear subspace, and let S denote the finite set $\mathrm{supp}\,\mathbf{L} := \cup_{\mathcal{L}\in\mathbf{L}}\, \mathrm{supp}\,\mathcal{L} \subset \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. Then there exists a finite dimensional linear subspace $V \subset C_c^\infty(X\colon\tau)$ with the following properties:*

- (i) *Let $\Omega \subset \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ be an open neighborhood of S and let $\psi \in \mathcal{M}(\Omega, S, \Sigma) \otimes {}^\circ\mathcal{C}$ be annihilated by $\mathbf{L}\cap\mathrm{AC}(X\colon\tau)$. Then there exists a unique function $f = f_\psi \in V$ such that $\mathcal{L}\mathcal{F}f = \mathcal{L}\psi$ for all $\mathcal{L} \in \mathbf{L}$.*
- (ii) *The map $\psi \mapsto f_\psi$ has the following form. There exists a $\mathrm{Hom}({}^\circ\mathcal{C}, V)$-valued Laurent functional $\mathcal{L}' \in \mathbf{L} \otimes V \subset \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \Sigma)^*_{\mathrm{laur}} \otimes \mathrm{Hom}({}^\circ\mathcal{C}, V)$ such that $f_\psi = \mathcal{L}'\psi$ for all ψ.*

We first formulate a result in linear algebra, and then deduce the above result.

Lemma 7.2. *Let A, B and C be linear spaces with $\dim C < \infty$, and let $\alpha \in \mathrm{Hom}(A, B)$ and $\beta \in \mathrm{Hom}(B, C)$ be given. Put $C' = \beta(\alpha(A))$. Then there exists a finite dimensional linear subspace $V \subset A$ with the property that, for each $\psi \in \beta^{-1}(C')$, there exists a unique element $f_\psi \in V$ such that $\beta(\alpha(f_\psi)) = \beta(\psi)$. Moreover, there exists an element $\mu \in \mathrm{Hom}(C, V)$ such that $f_\psi = \mu(\beta(\psi))$ for all ψ.*

Proof. The proof is shorter than the statement. Since $\beta \circ \alpha$ maps A onto C' we can choose $V \subset A$ such that the restriction of $\beta \circ \alpha$ to it is bijective $V \to C'$. Then $f_\psi \in V$ is uniquely determined by $\beta \circ \alpha(f_\psi) = \beta(\psi)$, and if $\mu\colon C \to V$ is any linear extension of $(\beta \circ \alpha)^{-1}\colon C' \to V$, the relation $f_\psi = \mu(\beta(\psi))$ holds for all ψ. $\square$

Proof of Lemma 7.1. It is easily seen by using a basis for $\mathbf{L}$ that S is a finite set.

We shall apply Lemma 7.2 with $A = C_c^\infty(X\colon\tau)$, $B = \mathcal{M}(\Omega, S, \Sigma) \otimes {}^\circ\mathcal{C}$ and $C = \mathbf{L}^*$, the linear dual of $\mathbf{L}$. Furthermore, as $\alpha\colon A \to B$ we use the Fourier transform $\mathcal{F}$ followed by taking restrictions to Ω, and as $\beta\colon B \to C = \mathbf{L}^*$ we use the map induced by the pairing $(\mathcal{L}, \psi) \mapsto \mathcal{L}\psi$, $\mathcal{L} \in \mathbf{L}$, $\psi \in B$.

We now determine the image $C' = \beta(\alpha(A))$. By definition it consists of all the linear forms on $\mathbf{L}$ given by the application of $\mathcal{L} \in \mathbf{L}$ to a function in $\mathcal{F}(C_c^\infty(X\colon\tau))$. Hence the polar subset $C'^\perp \subset \mathbf{L}$ is exactly the set of $\mathcal{L} \in \mathbf{L}$ that annihilate $\mathcal{F}(C_c^\infty(X\colon\tau))$. By Lemma 3.8, an element $\mathcal{L} \in \mathbf{L}$ annihilates $\mathcal{F}(C_c^\infty(X\colon\tau))$ if and only if it belongs to $\mathrm{AC}(X\colon\tau)$. Hence $C'^\perp = \mathbf{L} \cap \mathrm{AC}(X\colon\tau)$. Thus $\beta^{-1}(C')$ consists precisely of those elements $\psi \in B = \mathcal{M}(\Omega, S, \Sigma) \otimes {}^\circ\mathcal{C}$ that are annihilated by $\mathbf{L} \cap \mathrm{AC}(X\colon\tau)$

The lemma now follows immediately from Lemma 7.2. $\square$

Lemma 7.3. *Let $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^*$ and let $\phi \in \mathcal{O}_S$ where $S = \operatorname{supp}\mathcal{L}$. The map $\mathcal{L}_\phi\colon \psi \mapsto \mathcal{L}(\phi\psi)$ is a Laurent functional in $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^*$, supported at S.*

Proof. (See also [7], eq. (10.7).) For each $a \in S$, let $u_a = (u_{a,d})$ be the string that represents $\mathcal{L}$ at a. Let Ω be an open neighborhood of S. Fix $d\colon \Sigma \to \mathbb{N}$. For $\psi \in \mathcal{M}(\Omega, S, \Sigma, d)$ we have $\mathcal{L}_\phi\psi = \sum_{a\in S} u_{a,d}[\pi_{a,d}\phi\psi](a)$. Hence by the Leibniz rule we can write

$$\mathcal{L}_\phi\psi = \sum_{a\in S}\sum_i u_{a,i}^1[\phi](a)\, u_{a,i}^2[\pi_{a,d}\psi](a) \tag{7.1}$$

for finitely many $u_{a,i}^1, u_{a,i}^2 \in S(\mathfrak{a}_{\mathrm{q}}^*)$. Thus $\mathcal{L}_\phi$ has the form required of a Laurent functional with support in S. $\square$

Lemma 7.4. *Let $\mathcal{L}_0 \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^*$ and let $d\colon \Sigma \to \mathbb{N}$. There exists a finite dimensional linear subspace $V \subset C_c^\infty(X\colon\tau)$ with the following properties:*

(i) *Let $\Omega \subset \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ be an open neighborhood of $S := \operatorname{supp}\mathcal{L}_0$ and let $\psi \in \mathcal{M}(\Omega, S, \Sigma, d) \otimes {}^\circ\mathcal{C}$. Assume that $\mathcal{L}\psi = 0$ for all $\mathcal{L} \in \mathrm{AC}(X\colon\tau)$ with $\operatorname{supp}\mathcal{L} \subset S$. Then there exists a unique function $f = f_\psi \in V$ such that $\mathcal{L}_0(\phi\mathcal{F}f) = \mathcal{L}_0(\phi\psi)$ for all $\phi \in \mathcal{O}_S \otimes {}^\circ\mathcal{C}^*$.*

(ii) *The map $\psi \mapsto f_\psi$ has the following form. There exists a $Hom({}^\circ\mathcal{C}, V)$-valued germ $\phi' \in \mathcal{O}_S \otimes \mathrm{Hom}({}^\circ\mathcal{C}, V)$ such that $f_\psi = \mathcal{L}_0(\phi'\psi)$ for all ψ.*

Proof. We may assume that the given $d \in \mathbb{N}^\Sigma$ satisfies that $\mathcal{F}f|_\Omega$ belongs to $\mathcal{M}(\Omega, \Sigma, d) \otimes {}^\circ\mathcal{C}$ for all $f \in C_c^\infty(X\colon\tau)$, for some neighborhood Ω of S (otherwise we just replace d by a suitable successor in $\mathbb{N}^\Sigma$).

Let $\mathcal{O}_1 = \mathcal{O}_S \otimes {}^\circ\mathcal{C}^*$ and let $\mathcal{O}_0$ denote the subspace of $\mathcal{O}_1$ consisting of the elements $\phi \in \mathcal{O}_1$ for which the Laurent functional $\mathcal{L}_{0\phi}\colon \psi \mapsto \mathcal{L}_0(\phi\psi)$ in $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^* \otimes {}^\circ\mathcal{C}^*$ annihilates $\mathcal{M}(\Omega, S, \Sigma, d) \otimes {}^\circ\mathcal{C}$ (with the fixed element d), for all neighborhoods Ω of S. It follows immediately from (7.1), applied componentwise on ${}^\circ\mathcal{C}$, that an element $\phi \in \mathcal{O}_1$ belongs to $\mathcal{O}_0$ if a finite number of fixed linear forms on $\mathcal{O}_1$ annihilate it; hence $\dim \mathcal{O}_1/\mathcal{O}_0 < \infty$. Fix a complementary subspace $\mathcal{O}'$ of $\mathcal{O}_0$ in $\mathcal{O}_1$, and let

$$\mathbf{L} = \{\mathcal{L}_{0\phi} \mid \phi \in \mathcal{O}'\} \subset \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^* \otimes {}^\circ\mathcal{C}^*.$$

Choose $V \subset C_c^\infty(X\colon\tau)$ according to Lemma 7.1. Then for each $\psi \in \mathcal{M}(\Omega, S, \Sigma, d) \otimes {}^\circ\mathcal{C}$ satisfying $\mathcal{L}\psi = 0$ for all $\mathcal{L} \in \mathbf{L} \cap \mathrm{AC}(X\colon\tau)$, there exists a unique function $f_\psi \in V$ such that $\mathcal{L}\mathcal{F}f_\psi = \mathcal{L}\psi$ for all $\mathcal{L} \in \mathbf{L}$. Thus $\mathcal{L}_0(\phi\mathcal{F}f_\psi) = \mathcal{L}_0(\phi\psi)$ for all $\phi \in \mathcal{O}'$, and this property determines f_ψ uniquely. On the other hand, by the definition of $\mathcal{O}_0$ we have $\mathcal{L}_0(\phi\mathcal{F}f_\psi) = 0 = \mathcal{L}_0(\phi\psi)$ for $\phi \in \mathcal{O}_0$. Thus $\mathcal{L}_0(\phi\mathcal{F}f_\psi) = \mathcal{L}_0(\phi\psi)$ holds for all $\phi \in \mathcal{O}_1$.

The statement (ii) follows immediately from the above and the corresponding statement in Lemma 7.1. □

8 Proof of Theorem 4.4

The inversion formula for the Fourier transform that was obtained in [6], Thm. 1.2, reads

$$f(x) = \mathcal{T}\mathcal{F}f(x) = \sum_{F\subset\Delta} \mathrm{T}^t_F f(x), \qquad x \in X_+, \tag{8.1}$$

where the term in the middle is the pseudo wave packet (4.1) and where the operators on the right-hand side are defined in [6], eq. (5.5). Motivated by the latter definition we define, for $F \subset \Delta$, $\varphi \in \mathcal{P}(X\colon\tau)$ and $x \in X_+$,

$$\mathcal{T}^t_F\varphi(x) = |W|\, t(\mathfrak{a}^+_{F\mathrm{q}}) \int_{\epsilon_F + i\mathfrak{a}^*_{F\mathrm{q}}} \sum_{\lambda\in\Lambda(F)} \mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}} \big[\sum_{s\in W^F} E_{+,s}(\,\cdot : x)\varphi(\,\cdot\,)\big](\lambda+\nu)\, d\mu_{\mathfrak{a}^*_{F\mathrm{q}}}(\nu) \tag{8.2}$$

so that $\mathrm{T}^t_F f = \mathcal{T}^t_F\mathcal{F}f$. The element $\epsilon_F \in \mathfrak{a}^{*+}_{F\mathrm{q}}$, the set $\Lambda(F) \subset \mathfrak{a}^{*\perp}_{F\mathrm{q}}$ and the measure $d\mu_{\mathfrak{a}^*_{F\mathrm{q}}}$ on $i\mathfrak{a}^*_{F\mathrm{q}}$ are as defined in [6], p. 42 (with $\mathcal{H}$ equal to the union of $\mathcal{H}(X,\tau)$ with the set of singular hyperplanes for E_+). It follows from [6], eq. (4.2) and [5], Lemma 1.11, that the integral in (8.2) converges, and that $\mathcal{T}^t_F\varphi \in C^\infty(X_+\colon\tau)$. Moreover,

$$\mathcal{T}\varphi = \sum_{F\subset\Delta} \mathcal{T}^t_F\varphi, \tag{8.3}$$

in analogy with the second equality in (8.1), see the arguments leading up to [6], eq. (5.3).

The existence of a smooth extension of $\mathcal{T}\varphi$ will be proved by showing that $\mathcal{T}^t_F\varphi$ has the same property, for each F. We shall do this by exhibiting it as a wave packet of generalized Eisenstein integrals.

Let $\mathcal{H}$ denote the union of $\mathcal{H}(X,\tau)$ with the set of all affine hyperplanes in $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ along which $\lambda \mapsto E_{+,s}(\lambda\colon x)$ is singular, for some $x \in X_+$, $s \in W$. By Lemma 2.1 this is a real Σ-configuration and there exists $d\colon \mathcal{H} \to \mathbb{N}$ such that $E_{+,s}(\,\cdot : x) \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}, d) \otimes \mathrm{Hom}({}^\circ\mathcal{C}, V_\tau)$ for all $x \in X_+$ and $s \in W$.

Lemma 8.1. *Let $F \subset \Delta$ and $v \in {}^F\mathcal{W}$. Let $\mathcal{L} \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}}$ with $S := \mathrm{supp}\,\mathcal{L} \subset \mathfrak{a}^{*\perp}_{F\mathrm{q}}$. There exist a finite dimensional linear subspace $V \subset C^\infty_c(X_{F,v}\colon\tau)$ and for each $\nu \in \mathrm{reg}(\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}, \mathcal{H}_F(S))$ a linear map $\varphi \mapsto f_{\nu,\varphi}$, $\mathcal{P}_{\mathrm{AC}}(X\colon\tau) \to V$ such that*

$$\begin{aligned}&\mathcal{L}\Big[\sum_{s\in W^F} E_{+,s}(\nu + \,\cdot : x) \circ \mathrm{i}_{F,v} \circ \mathrm{pr}_{F,v}\, \varphi(\nu + \,\cdot\,)\Big] \\ &\quad = \mathcal{L}\Big[\sum_{s\in W^F} E_{+,s}(\nu + \,\cdot : x) \circ \mathrm{i}_{F,v}\, \mathcal{F}(X_{F,v}\colon f_{\nu,\varphi})(\,\cdot\,)\Big]\end{aligned} \tag{8.4}$$

for all $x \in X_+$.

Moreover, the elements $f_{\nu,\varphi} \in V$ *can be chosen of the following form. There exists a Laurent functional* $\mathcal{L}'_v \in \mathcal{M}(\mathfrak{a}^{*\perp}_{Fq\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}} \otimes \mathrm{Hom}({}^\circ\mathcal{C}_{F,v}, V)$*, supported by* S*, such that*

$$f_{\nu,\varphi} = \mathcal{L}'_v[\mathrm{pr}_{F,v}\, \varphi(\nu + \,\cdot\,)] \tag{8.5}$$

for all $\nu \in \mathrm{reg}(\mathfrak{a}^*_{Fq\mathbb{C}}, \mathcal{H}_F(S))$ *and all* $\varphi \in \mathcal{P}_{\mathrm{AC}}(X\colon\tau)$.

Proof. For each $\nu \in \mathrm{reg}(\mathfrak{a}^*_{Fq\mathbb{C}}, \mathcal{H}_F(S))$ and $a \in S$ the element $a + \nu$ is only contained in a given hyperplane from $\mathcal{H}$ if this hyperplane contains all of $a + \mathfrak{a}^*_{Fq\mathbb{C}}$. Let $\mathcal{H}(a + \mathfrak{a}^*_{Fq\mathbb{C}})$ denote the (finite) set of such hyperplanes, and let $\mathcal{H}(S + \mathfrak{a}^*_{Fq\mathbb{C}}) = \cup_{a\in S}\mathcal{H}(a + \mathfrak{a}^*_{Fq\mathbb{C}})$. Let $d\colon \mathcal{H} \to \mathbb{N}$ be as mentioned before the lemma, and let the polynomial function p be given by (2.6) with $\omega = \nu + S$, where $\nu \in \mathrm{reg}(\mathfrak{a}^*_{Fq\mathbb{C}}, \mathcal{H}_F(S))$. Then

$$p(\lambda) = \prod_{H\in\mathcal{H}(S+\mathfrak{a}^*_{Fq\mathbb{C}})} (\langle\alpha_H, \lambda\rangle - s_H)^{d(H)},$$

and thus p is independent of ν. Moreover, since $a + \mathfrak{a}^*_{Fq\mathbb{C}} \subset H$ we conclude that $\alpha_H \in \Sigma_F$ for all $H \in \mathcal{H}(S + \mathfrak{a}^*_{Fq\mathbb{C}})$. Hence $p(\nu + \lambda) = p(\lambda)$ for $\nu \in \mathfrak{a}^*_{Fq\mathbb{C}}$ and $\lambda \in \mathfrak{a}^{*\perp}_{Fq\mathbb{C}}$. The maps

$$\lambda \mapsto p(\lambda)E_{+,s}(\nu + \lambda\colon x), \quad \mathfrak{a}^{*\perp}_{Fq\mathbb{C}} \to \mathrm{Hom}({}^\circ\mathcal{C}, V_\tau),$$

are then holomorphic at S for all $\nu \in \mathrm{reg}(\mathfrak{a}^*_{Fq\mathbb{C}}, \mathcal{H}_F(S))$, $s \in W$ and $x \in X_+$.

Choose $d_0 \in \mathbb{N}$ such that $d_{X,\tau}(H) \le d_0$ for all $H \in \mathcal{H}(S + \mathfrak{a}^*_{Fq\mathbb{C}}) \cap \mathcal{H}(X,\tau)$ and define $d'\colon \Sigma_F \to \mathbb{N}$ by $d'(\alpha) = d_0$ for all α. Then, for each $\nu \in \mathrm{reg}(\mathfrak{a}^*_{Fq\mathbb{C}}, \mathcal{H}_F(S))$ and $\varphi \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}(X,\tau), d_{X,\tau}) \otimes {}^\circ\mathcal{C}$ the function

$$\psi^{\nu,\varphi} := \mathrm{pr}_{F,v} \circ \varphi^\nu\colon \lambda \mapsto \mathrm{pr}_{F,v}\, \varphi(\nu + \lambda)$$

on $\mathfrak{a}^{*\perp}_{Fq\mathbb{C}}$ belongs to $\mathcal{M}(\Omega, \Sigma_F, d') \otimes {}^\circ\mathcal{C}_{F,v}$ for some neighborhood Ω of S (cf. Lemma 6.1). If in addition $\varphi \in \mathcal{P}_{\mathrm{AC}}(X\colon\tau)$ then by Theorem 6.2 this function is annihilated by all elements of $\mathrm{AC}(X_{F,v}\colon\tau)$ supported by S.

Let $\mathcal{L}_0$ be the functional on $\mathcal{M}(\mathfrak{a}^{*\perp}_{Fq\mathbb{C}}, \Sigma_F)$ defined by $\mathcal{L}_0\psi = \mathcal{L}(p^{-1}\psi)$; it is easily seen that $\mathcal{L}_0 \in \mathcal{M}(\mathfrak{a}^{*\perp}_{Fq\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}}$ and that $\mathrm{supp}\,\mathcal{L}_0 \subset S$. Choose $V \subset C^\infty_c(X_{F,v}\colon\tau)$ according to Lemma 7.4, applied to $X_{F,v}$, $\mathcal{L}_0$ and d'. Then there exists for each $\nu \in \mathrm{reg}(\mathfrak{a}^*_{Fq\mathbb{C}}, \mathcal{H}_F(S))$ and $\varphi \in \mathcal{P}_{\mathrm{AC}}(X\colon\tau)$ a unique element $f_{\nu,\varphi} = f_{\psi^{\nu,\varphi}} \in V$ such that

$$\mathcal{L}_0(\phi\mathcal{F}(X_{F,v}\colon f_{\nu,\varphi})) = \mathcal{L}_0(\phi\psi^{\nu,\varphi})$$

for all $\phi \in \mathcal{O}_S \otimes {}^\circ\mathcal{C}^*_{F,v}$. We apply this identity with

$$\phi(\lambda) = p(\lambda) \sum_{s\in W^F} v^* \circ E_{+,s}(\nu + \lambda\colon x) \circ \mathrm{i}_{F,v}$$

for arbitrary $v^* \in V^*_\tau$, and deduce (8.4).

According to Lemma 7.4 (ii) there exists $\phi' \in \mathcal{O}_S \otimes \mathrm{Hom}({}^\circ\mathcal{C}_{F,v}, V)$ such that $f_{\nu,\varphi} = \mathcal{L}_0(\phi'\psi^{\nu,\varphi})$. The map $\mathcal{L}'_v\colon \psi \mapsto \mathcal{L}_0(\phi'\psi)$ is a $\mathrm{Hom}({}^\circ\mathcal{C}_{F,v}, V)$-valued Laurent functional (see Lemma 7.3) satisfying (8.5). The linearity of $\varphi \mapsto f_{\nu,\varphi}$ follows from (8.5). $\square$

Lemma 8.2. *Let $v \in {}^F\mathcal{W}$. There exists a Laurent functional*

$$\mathcal{L}_v \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}} \otimes \mathrm{Hom}({}^\circ\mathcal{C}_{F,v}, \mathcal{A}(X_{F,v}\colon \tau_F)),$$

supported by the set $\Lambda := \Lambda(F) \cup \Lambda(X_{F,v}, F)$, such that

$$\begin{aligned}\sum_{\lambda\in\Lambda(F)} \mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}} \Big[\sum_{s\in W^F} E_{+,s}(\,\cdot : x) \circ \mathrm{i}_{F,v} \circ \mathrm{pr}_{F,v}\, \varphi(\,\cdot\,)\Big](\nu+\lambda) \\ = E^\circ_{F,v}(\mathcal{L}_v[\mathrm{pr}_{F,v}\,\varphi(\nu + \,\cdot\,)]\colon \nu\colon x)\end{aligned} \tag{8.6}$$

for all $\varphi \in \mathcal{P}_{\mathrm{AC}}(X\colon\tau)$, $x \in X_+$ and generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. Here, generic means that $\nu \in \mathrm{reg}(\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}, \mathcal{H}_F(\Lambda))$, where $\mathcal{H}$ is as defined above Lemma 8.1.

Proof. In the expression on the left side of (8.6) we can replace the set $\Lambda(F)$ by Λ (see [6], Lemma 7.5). Moreover, we can replace the residue operator $\mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}$ by $\mathrm{Res}^{*P,*t}_{\lambda}$ (see [6], eq. (8.5)), which, as observed in [6], above eq. (8.5), can be regarded as an element in $\mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}}, \Sigma_F)^*_{\mathrm{laur}}$, supported at λ. We thus obtain on the left of (8.6):

$$\sum_{\lambda\in\Lambda} \mathrm{Res}^{*P,*t}_{\lambda} \Big[\sum_{s\in W^F} E_{+,s}(\nu + \,\cdot : x) \circ \mathrm{i}_{F,v} \circ \mathrm{pr}_{F,v}\, \varphi(\nu + \,\cdot\,)\Big]. \tag{8.7}$$

We obtain from Lemma 8.1 that there exist a finite dimensional space $V \subset C^\infty_c(X_{F,v}\colon\tau)$ and a Laurent functional $\mathcal{L}'_v \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}}, \Sigma_F)^*_{\mathrm{laur}} \otimes \mathrm{Hom}({}^\circ\mathcal{C}_{F,v}, V)$ supported by Λ, such that (8.7) equals

$$\sum_{\lambda\in\Lambda} \mathrm{Res}^{*P,*t}_{\lambda} \Big[\sum_{s\in W^F} E_{+,s}(\nu + \,\cdot : x) \circ \mathrm{i}_{F,v}\, \mathcal{F}(X_{F,v}\colon f_{\nu,\varphi})(\,\cdot\,)\Big]. \tag{8.8}$$

Here $f_{\nu,\varphi} = \mathcal{L}'_v[\mathrm{pr}_{F,v}\,\varphi(\nu + \,\cdot\,)] \in V$ for $\nu \in \mathrm{reg}(\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}, \mathcal{H}_F(\Lambda))$. We apply Lemmas 5.5, 5.6 and obtain that (8.8) equals $E^\circ_{F,v}(\psi\colon \nu\colon x)$ with $\psi = |W_F|^{-1}\mathrm{T}_F(X_{F,v}\colon f_{\nu,\varphi}) \in \mathcal{A}(X_{F,v}\colon\tau_F)$.

The map $f \mapsto |W_F|^{-1}\mathrm{T}_F(X_{F,v}\colon f)$ is linear $V \to \mathcal{A}(X_{F,v}\colon\tau_F)$; composing it with the coefficients of $\mathcal{L}'_v \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}}, \Sigma_F)^*_{\mathrm{laur}} \otimes \mathrm{Hom}({}^\circ\mathcal{C}_{F,v}, V)$ we obtain a Laurent functional $\mathcal{L}_v \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}}, \Sigma_F)^*_{\mathrm{laur}} \otimes \mathrm{Hom}({}^\circ\mathcal{C}_{F,v}, \mathcal{A}(X_{F,v}\colon\tau_F))$. Now $\psi = \mathcal{L}_v[\mathrm{pr}_{F,v}\,\varphi(\nu + \,\cdot\,)]$, and (8.6) follows. $\square$

Theorem 8.3. *Let $F \subset \Delta$. There exists $\mathcal{L} \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}} \otimes \mathrm{Hom}({}^\circ\mathcal{C}, \mathcal{A}_F)$ with support contained in $\Lambda(F) \cup [\cup_{v\in {}^F\mathcal{W}} \Lambda(X_{F,v}, F)]$, such that*

$$\mathcal{T}^t_F\varphi(x) = \int_{\epsilon_F + i\mathfrak{a}^*_{F\mathrm{q}}} E^\circ_F(\mathcal{L}[\varphi(\nu + \,\cdot\,)]\colon \nu\colon x)\, d\mu_{\mathfrak{a}^*_{F\mathrm{q}}}(\nu) \tag{8.9}$$

for all $\varphi \in \mathcal{P}_{\mathrm{AC}}(X\colon\tau)$, $x \in X_+$. In particular, $\mathcal{T}^t_F\varphi \in C^\infty(X\colon\tau)$, and $\varphi \mapsto \mathcal{T}^t_F\varphi$ is continuous $\mathcal{P}_{\mathrm{AC}}(X\colon\tau) \to C^\infty(X\colon\tau)$.

Proof. Recall, see (5.5) and [6], eq. (8.4), that

$$\mathcal{A}_F = \oplus_{v\in {}^F\mathcal{W}}\ \mathcal{A}(X_{F,v}\colon\tau_F), \quad {}^\circ\mathcal{C} = \oplus_{v\in {}^F\mathcal{W}}\ \mathrm{i}_{F,v}\big({}^\circ\mathcal{C}_{F,v}\big).$$

Let $\mathcal{L}_v$ be as in Lemma 8.2 for each $v \in {}^F\mathcal{W}$, and let

$$\mathcal{L} = |W| t(\mathfrak{a}^+_{F\mathrm{q}}) \sum_{v\in {}^F\mathcal{W}} \mathcal{L}_v \circ \mathrm{pr}_{F,v} \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}} \otimes \mathrm{Hom}({}^\circ\mathcal{C}, \mathcal{A}_F).$$

The identity (8.9) then follows immediately from (8.2), (8.6), (5.6). The remaining statements follow from Lemma 6.1(v) combined with the estimate in [6], Lemma 10.8. □

As a corollary we immediately obtain (cf. (8.3)) that $\mathcal{T}\varphi \in C^\infty(X\colon\tau)$ for every $\varphi \in \mathcal{P}_{\mathrm{AC}}(X\colon\tau)$, and that $\mathcal{T}\colon \mathcal{P}_{\mathrm{AC}}(X\colon\tau) \to C^\infty(X\colon\tau)$ is continuous. The proofs of Theorems 4.4, 4.5 and 3.6 are then complete.

9 A comparison of two estimates

The purpose of this section is to compare the estimates (3.2) and (4.2), and to establish the facts mentioned in Remark 4.2. The method is elementary Euclidean Fourier analysis.

Fix $R \in \mathbb{R}$ and let $\mathcal{Q} = \mathcal{Q}(R)$ denote the space of functions $\phi \in \mathcal{O}(\mathfrak{a}^*_\mathrm{q}(P,R))$ (see (2.7)) for which

$$\nu_{\omega,n}(\phi) := \sup_{\lambda\in\omega+i\mathfrak{a}^*_\mathrm{q}} (1+|\lambda|)^n |\phi(\lambda)| < \infty \tag{9.1}$$

for all $n \in \mathbb{N}$ and all bounded sets $\omega \subset \mathfrak{a}^*_\mathrm{q}(P,R) \cap \mathfrak{a}^*_\mathrm{q}$. The space $\mathcal{Q}$, endowed with the seminorms $\nu_{\omega,n}$, is a Fréchet space.

For $M > 0$ we denote by $\mathcal{Q}_M = \mathcal{Q}_M(R)$ the subspace of $\mathcal{Q}$ consisting of the functions $\phi \in \mathcal{Q}$ that satisfy the following: For every strictly antidominant $\eta \in \mathfrak{a}^*_\mathrm{q}$ there exist constants $t_\eta, C_\eta > 0$ such that

$$|\phi(\lambda)| \le C_\eta (1+|\lambda|)^{-\dim\mathfrak{a}_\mathrm{q}-1} e^{M|\operatorname{Re}\lambda|} \tag{9.2}$$

for all $t \ge t_\eta$ and $\lambda \in t\eta + i\mathfrak{a}^*_\mathrm{q}$ (note that $t\eta + i\mathfrak{a}^*_\mathrm{q} \subset \mathfrak{a}^*_\mathrm{q}(P,R)$ for t sufficiently large).

Lemma 9.1. (i) *Let $\lambda_0 \in \mathfrak{a}^*_\mathrm{q}(P,R) \cap \mathfrak{a}^*_\mathrm{q}$ and let $\omega \subset \mathfrak{a}^*_\mathrm{q}(P,R) \cap \mathfrak{a}^*_\mathrm{q}$ be a compact neighborhood of λ_0. Let $M > 0$ and $N \in \mathbb{N}$. There exist $n \in \mathbb{N}$ and $C > 0$ such that*

$$|\phi(\lambda)| \le C(1+|\lambda|)^{-N} e^{M|\operatorname{Re}\lambda|} \nu_{\omega,n}(\phi) \tag{9.3}$$

for all $\lambda \in \lambda_0 + \bar{\mathfrak{a}}^*_\mathrm{q}(P,0)$ and $\phi \in \mathcal{Q}_M$.
(ii) *$\mathcal{Q}_M$ is closed in $\mathcal{Q}$.*
(iii) *Let $\phi \in \mathcal{Q}_M$. Then $p\phi \in \mathcal{Q}_M$ for each polynomial p on $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$.*

Proof. (i) From the estimates in (9.1) it follows that $\mu \mapsto \phi(\lambda_0+\mu)$ is a Schwartz function on the Euclidean space $i\mathfrak{a}^*_\mathrm{q}$; in fact by a straightforward application of Cauchy's integral formula we see that every Schwartz-type seminorm of this function can be estimated from above by (a constant times) $\nu_{\omega,n}(\phi)$ for some n.

Let $f\colon \mathfrak{a}_\mathrm{q} \to \mathbb{C}$ be defined by

$$f(x) = \int_{\lambda_0 + i\mathfrak{a}_\mathrm{q}^*} e^{\lambda(x)} \phi(\lambda)\, d\lambda. \tag{9.4}$$

Then $x \mapsto e^{-\lambda_0(x)} f$ is a Schwartz function on $\mathfrak{a}_\mathrm{q}$, and by continuity of the Fourier transform for the Schwartz topologies every Schwartz-seminorm of this function can be estimated by one of the $\nu_{\omega,n}(\phi)$. Moreover, it follows from the Fourier inversion formula that

$$\phi(\lambda) = \int_{\mathfrak{a}_\mathrm{q}} e^{-\lambda(x)} f(x)\, dx, \tag{9.5}$$

for $\lambda \in \lambda_0 + i\mathfrak{a}_\mathrm{q}^*$, where dx is Lebesgue measure on $\mathfrak{a}_\mathrm{q}$ (suitably normalized).

It follows from (9.4) and an application of Cauchy's theorem, justified by (9.1), that $f(x)$ is independent of the choice of the element λ_0. Since this element was arbitrary in $\mathfrak{a}_\mathrm{q}^*(P,R) \cap \mathfrak{a}_\mathrm{q}^*$ we conclude that (9.5) holds for all $\lambda \in \mathfrak{a}_\mathrm{q}^*(P,R)$.

Let $\mu \in \mathfrak{a}_\mathrm{q}^*(P,0)$ and let $\eta = \operatorname{Re}\mu$. Then η is strictly antidominant. Let $t \geq t_\eta$. Replacing λ_0 by $t\eta$ in (9.4) and applying (9.2) we obtain the estimate

$$|f(x)| \leq C_\eta\, e^{t\eta(x)} e^{tM|\eta|} \int_{i\mathfrak{a}_\mathrm{q}^*} (1 + |\lambda|)^{-\dim \mathfrak{a}_\mathrm{q} - 1}\, d\lambda.$$

By taking the limit as $t \to \infty$ we infer that if $\eta(x) + M|\eta| < 0$ then $f(x) = 0$.

We use (9.5) to evaluate $\phi(\lambda_0 + \mu)$. It follows from the previous statement that we need only to integrate over the set where $-\eta(x) \leq M|\eta|$. On this set the integrand $e^{-(\lambda_0+\mu)(x)} f(x)$ is dominated by $e^{M|\eta|} e^{-\lambda_0(x)} |f(x)|$. Thus we obtain

$$|\phi(\lambda_0 + \mu)| \leq e^{M|\operatorname{Re}\mu|} \int_{\mathfrak{a}_\mathrm{q}} e^{-\lambda_0(x)} |f(x)|\, dx \tag{9.6}$$

for $\mu \in \mathfrak{a}_\mathrm{q}^*(P,0)$; hence, by continuity, also for $\mu \in \bar{\mathfrak{a}}_\mathrm{q}^*(P,0)$. Using (9.5) and partial integration, a similar estimate is obtained for $\mu(x_0)^k \phi(\lambda_0 + \mu)$ for any $x_0 \in \mathfrak{a}_\mathrm{q}$, $k \in \mathbb{N}$; on the right-hand side of (9.6) $e^{-\lambda_0} f$ is then replaced by its k-th derivative in the direction x_0. This shows that for each $N \in \mathbb{N}$, $(1 + |\mu|)^N |\phi(\lambda_0 + \mu)|$ can be estimated in terms of $e^{M|\operatorname{Re}\mu|}$ and a Schwartz-seminorm of $e^{-\lambda_0} f$. The latter seminorm may then be estimated by $\nu_{\omega,n}(\phi)$, for suitable n, and (9.3) follows, but with $\mu = \lambda - \lambda_0$ in place of λ on the right-hand side. Since $1 + |\lambda| \leq 1 + |\lambda_0| + |\mu| \leq (1 + |\lambda_0|)(1 + |\mu|)$ and $|\operatorname{Re}\mu| \leq |\operatorname{Re}\lambda_0| + |\operatorname{Re}\lambda|$, the stated form of (9.3) follows from that.

(ii) Let ϕ be in the closure of $\mathcal{Q}_M$ in $\mathcal{Q}$, then by continuity (i) holds for ϕ as well. Let η be a given strictly antidominant element of $\mathfrak{a}_\mathrm{q}^*$. Choose $t_\eta > 0$ such that $\lambda_0 := t_\eta \eta \in \mathfrak{a}_\mathrm{q}^*(P,R)$. Now (9.2) follows from (9.3) with $N = \dim \mathfrak{a}_\mathrm{q} + 1$. Hence $\phi \in \mathcal{Q}_M$.

(iii) As before, let η be given and choose $t_\eta > 0$ such that $\lambda_0 = t_\eta \eta \in \mathfrak{a}_\mathrm{q}^*(P,R)$. Then by (i), (9.3) holds, and since N is arbitrary (9.2) follows with ϕ replaced by $p\phi$. $\square$

Lemma 9.2. *There exist a real Σ-configuration $\mathcal{H}^\sim$, a map $d^\sim\colon \mathcal{H}^\sim \to \mathbb{N}$ and a number $\epsilon > 0$ with the following property. Let $\varphi\colon \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to {}^\circ\mathcal{C}$ be any meromorphic function such that*

(i) $\varphi(s\lambda) = C^\circ(s\colon\lambda)\varphi(\lambda)$ *for all* $s \in W$ *and generic* $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$,

(ii) $\pi\varphi$ *is holomorphic on a neighborhood of* $\bar{\mathfrak{a}}^*_{\mathrm{q}}(P, 0)$.

Then $\varphi \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}^\sim, d^\sim) \otimes {}^\circ\mathcal{C}$ *and* $\pi\varphi$ *is holomorphic on* $\mathfrak{a}^*_{\mathrm{q}}(P, \epsilon)$.

Notice (cf. (2.3)) that (i)-(ii) hold with $\varphi = E^*(\,\cdot : x)v$, for any $x \in X$, $v \in V_\tau$. It follows that $E^*(\,\cdot : x)v \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}^\sim, d^\sim) \otimes {}^\circ\mathcal{C}$. Hence $\mathcal{H}(X, \tau) \subset \mathcal{H}^\sim$ and $d_{X,\tau} \preceq d^\sim|_{\mathcal{H}(X,\tau)}$.

Proof. Let $\mathcal{H}(X, \tau)$ and $d_{X,\tau}$ be as in Section 2, and for each $s \in W$ let $\mathcal{H}_s$, d_s be such that $C^\circ(s\colon\cdot\,) \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}_s, d_s)$, cf. Lemma 2.1. Let

$$\mathcal{H}^\sim = \cup_{s\in W}\{sH \mid H \in \mathcal{H}(X, \tau) \cup \mathcal{H}_s\}.$$

Furthermore, let $d^\sim \in \mathbb{N}^{\mathcal{H}^\sim}$ be defined as follows. We agree that $d_{X,\tau}(H) = 0$ for $H \notin \mathcal{H}(X, \tau)$ and and $d_s(H) = 0$ for $H \notin \mathcal{H}_s$. For $H \in \mathcal{H}^\sim$ let

$$d^\sim(H) = \max_{s\in W} d_{X,\tau}(s^{-1}H) + d_s(s^{-1}H).$$

We now assume that φ satisfies (i) and (ii). Let $\lambda_0 \in \bar{\mathfrak{a}}^*_{\mathrm{q}}(P, 0)$ and $s \in W$. Let π_0 denote the polynomial determined by (2.6) with $\omega = \{\lambda_0\}$ and with $\mathcal{H} = \mathcal{H}(X, \tau)$ and $d = d_{X,\tau}$. Since $\lambda_0 \in \bar{\mathfrak{a}}^*_{\mathrm{q}}(P, 0)$, we see that π_0 divides π and the quotient π/π_0 is nonzero at λ_0. Hence $\pi_0\varphi$ is holomorphic in a neighborhood of λ_0, by (ii). Likewise, let π_s denote the polynomial determined by (2.6) with $\omega = \{\lambda_0\}$ and with $\mathcal{H} = \mathcal{H}_s$ and $d = d_s$, then $\pi_s C^\circ(s\colon\cdot\,)$ is holomorphic at λ_0. Hence $\pi_0\pi_s C^\circ(s\colon\cdot\,)\varphi$ is holomorphic at λ_0, and by (i) it follows that $\lambda \mapsto \pi_0(s^{-1}\lambda)\pi_s(s^{-1}\lambda)\varphi(\lambda)$ is holomorphic at $s\lambda_0$. Let $\pi^\sim$ be defined by (2.6) with $\omega = \{s\lambda_0\}$ and with $\mathcal{H} = \mathcal{H}^\sim$ and $d = d^\sim$. Then the polynomial $\lambda \mapsto \pi_0(s^{-1}\lambda)\pi_s(s^{-1}\lambda)$ divides $\pi^\sim$, by the definition of $d^\sim$, and hence $\pi^\sim\varphi$ is holomorphic at $s\lambda_0$. Since every point in $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ can be written in the form $s\lambda_0$ with $\lambda_0 \in \bar{\mathfrak{a}}^*_{\mathrm{q}}(P, 0)$ and $s \in W$, it follows that $\varphi \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}}, \mathcal{H}^\sim, d^\sim) \otimes {}^\circ\mathcal{C}$. The statement about the existence of ϵ is now an easy consequence of (ii) and the local finiteness of $\mathcal{H}^\sim$. $\square$

It follows from Lemma 9.2 that a fixed number ϵ can be chosen such that the condition in (ii) of Definition 4.1 holds for all $\varphi \in \mathcal{P}(X\colon\tau)$ simultaneously. In the following lemma, we fix such a number $\epsilon > 0$.

Lemma 9.3. *Let* $M > 0$ *and let* $\omega \subset \mathfrak{a}^*_{\mathrm{q}}(P, \epsilon)$ *be a compact neighborhood of 0. Let* $N \in \mathbb{N}$. *Then there exist* $n \in \mathbb{N}$ *and* $C > 0$ *such that*

$$\sup_{\lambda\in\bar{\mathfrak{a}}^*_{\mathrm{q}}(P,0)} (1 + |\lambda|)^N e^{-M|\operatorname{Re}\lambda|} \|\pi(\lambda)\varphi(\lambda)\| \leq C\nu_{\omega,n}(\pi\varphi) \tag{9.7}$$

for all $\varphi \in \mathcal{P}_M(X\colon\tau)$ *(see Definition 4.1). Moreover,*

$$\mathrm{PW}_M(X\colon\tau) = \mathcal{P}_M(X\colon\tau) \cap \mathcal{P}_{\mathrm{AC}}(X\colon\tau), \tag{9.8}$$

and this is a closed subspace of $\mathcal{P}_{\mathrm{AC}}(X\colon\tau)$.

Proof. We first show that $\pi\varphi \in \mathcal{Q}_M(\epsilon) \otimes {}^\circ\mathcal{C}$ for all $\varphi \in \mathcal{P}_M(X\colon\tau)$. Let $\varphi \in \mathcal{P}_M(X\colon\tau)$ and let $R_1 \in \mathbb{R}$ sufficiently negative so that φ is holomorphic on $\mathfrak{a}^*_{\mathrm{q}}(P, R_1)$. Then $\varphi \in$

$\mathcal{Q}_M(R_1)\otimes{}^\circ\mathcal{C}$ and hence it follows from Lemma 9.1(iii) with $R = R_1$, applied componentwise to the ${}^\circ\mathcal{C}$-valued function φ, that $\pi\varphi \in \mathcal{Q}_M(R_1) \otimes {}^\circ\mathcal{C}$. Since (9.2) does not invoke R, and since $\pi\varphi$ is already known to satisfy (9.1) with $R = \epsilon$ (see Def. 4.1) it follows that $\pi\varphi \in \mathcal{Q}_M(\epsilon) \otimes {}^\circ\mathcal{C}$ as well. By a second application of Lemma 9.1, this time with $R = \epsilon$ and $\lambda_0 = 0$, we now obtain (9.7). The identity (9.8) follows from (4.4) and (9.7). The map $\varphi \to \pi\varphi$ is continuous $\mathcal{P}_{\mathrm{AC}}(X\colon\tau) \to \mathcal{Q} \otimes {}^\circ\mathcal{C}$ and $\mathcal{P}_M(X\colon\tau) \cap \mathcal{P}_{\mathrm{AC}}(X\colon\tau)$ is the preimage of $\mathcal{Q}_M \otimes {}^\circ\mathcal{C}$. Hence it is closed. $\square$

10 A different characterization of the Paley–Wiener space

In [4], Def. 21.6, we defined the Paley–Wiener space $\mathrm{PW}(X\colon\tau)$ somewhat differently from Definition 3.4, and we conjectured in [4], Remark 21.8, that this space was equal to $\mathcal{F}(C_c^\infty(X\colon\tau))$. The purpose of this section is to establish equivalence of the two definitions of $\mathrm{PW}(X\colon\tau)$ and to confirm the conjecture of [4].

The essential difference between the definitions is that in [4] several properties are required only on $\bar{\mathfrak{a}}_{\mathrm{q}}^*(P,0)$; the identity $\varphi(s\lambda) = C^\circ(s\colon\lambda)\varphi(\lambda)$ (cf. Lemma 3.10) is then part of the definition of the Paley-Wiener space. In the following theorem we establish a property of $C^\circ(s\colon\lambda)$ which is crucial for the comparison of the definitions. Let $\Pi_{\Sigma,\mathbb{R}}$ denote the set of polynomials on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ which are products of functions of the form $\lambda \mapsto \langle\alpha,\lambda\rangle + c$ with $\alpha \in \Sigma$ and $c \in \mathbb{R}$.

Theorem 10.1. *Let $s \in W$ and let $\omega \subset \mathfrak{a}_{\mathrm{q}}^*$ be compact. There exist a polynomial $q \in \Pi_{\Sigma,\mathbb{R}}$ and a number $N \in \mathbb{N}$ such that $\lambda \mapsto (1 + |\lambda|)^{-N} q(\lambda) C^\circ(s\colon\lambda)$ is bounded on $\omega + i\mathfrak{a}_{\mathrm{q}}^*$.*

Proof. See [10]. $\square$

Lemma 10.2. *The space $\mathcal{P}(X\colon\tau)$ of Definition 4.1 is equal to the space of ${}^\circ\mathcal{C}$-valued meromorphic function on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ that have the properties* (i)-(ii) *of Lemma 9.2 together with:*

(iii) *For every compact set $\omega \subset \bar{\mathfrak{a}}_{\mathrm{q}}^*(P,0) \cap \mathfrak{a}_{\mathrm{q}}^*$ and for all $n \in \mathbb{N}$,*

$$\sup_{\lambda\in\omega+i\mathfrak{a}_{\mathrm{q}}^*} (1 + |\lambda|)^n \|\pi(\lambda)\varphi(\lambda)\| < \infty.$$

Moreover, there exist a real Σ-configuration $\mathcal{H}^\sim$ and a map $d^\sim\colon \mathcal{H}^\sim \to \mathbb{N}$ such that

$$\mathcal{P}(X\colon\tau) \subset \mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}^\sim, d^\sim) \otimes {}^\circ\mathcal{C}. \tag{10.1}$$

Proof. Condition (i) in Definition 4.1 is the same as (i) in Lemma 9.2, whereas (ii) is stronger. However, it was seen in Lemma 9.2 that (i)$\wedge$(ii) implies (ii) of Definition 4.1. The condition (iii) in Definition 4.1 is also stronger than (iii) above.

It thus remains to be seen that (i)-(iii) above imply (iii) of Definition 4.1, and that (10.1) holds. We will establish both at the same time. Let $\mathcal{H}^\sim$ and $d^\sim$ be as in Lemma 9.2, and assume that φ satisfies (i)-(iii) above, then $\varphi \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}^\sim, d^\sim) \otimes {}^\circ\mathcal{C}$. Let $\omega \subset \mathfrak{a}_{\mathrm{q}}^*$ be compact. Using Theorem 10.1 we see from (iii) together with (i) that there exists a polynomial $Q \in \Pi_{\Sigma,\mathbb{R}}$ such that

$$\sup_{\lambda\in\omega+i\mathfrak{a}_{\mathrm{q}}^*} (1 + |\lambda|)^n \|Q(\lambda)\varphi(\lambda)\| < \infty$$

for each $n \in \mathbb{N}$. Clearly we may assume that Q is divisible by $\pi_{\omega,d^\sim}(\lambda)$ (see (2.6)). Using [2] Lemma 6.1 and the fact that ω was arbitrary, we can in fact remove all factors of $Q/\pi_{\omega,d^\sim}(\lambda)$ from the estimate, so that we may assume $Q = \pi_{\omega,d^\sim}(\lambda)$. Hence $\varphi \in \mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}^\sim, d^\sim) \otimes {}^\circ\mathcal{C}$. The statement in (iii) of Definition 4.1 follows by the same reasoning, invoking the already established statement (ii) of that definition. $\square$

Lemma 10.3. *The pre-Paley-Wiener space $\mathcal{M}(X\colon\tau)$ defined in [4], Def. 21.2, is identical with $\cup_{M>0}\mathcal{P}_M(X\colon\tau)$, where $\mathcal{P}_M(X\colon\tau)$ is as defined in Definition 4.1.*

Proof. Let $M > 0$ and $\varphi \in \mathcal{P}_M(X\colon\tau)$. Then properties (a) and (b) of [4], Def. 21.2, are obviously fulfilled, and (c), with $R = M$, follows from (9.7). Hence $\varphi \in \mathcal{M}(X\colon\tau)$.

Conversely, let $\varphi \in \mathcal{M}(X\colon\tau)$, then $\varphi \in \mathcal{P}(X\colon\tau)$ by Lemma 10.2. Moreover, condition (iv) in Definition 4.1 results easily from (c) of [4], with $M = R$. Hence $\varphi \in \mathcal{P}_M(X\colon\tau)$. $\square$

In [4] the space $\mathrm{PW}(X\colon\tau)$ is defined as the space of functions $\varphi \in \mathcal{M}(X\colon\tau)$ that satisfy certain relations. These relations will now be interpreted in terms of Laurent functionals by means of the following lemma.

Lemma 10.4. *Let $u_1, \dots, u_k \in S(\mathfrak{a}_{\mathrm{q}}^*)$, $\psi_1, \dots, \psi_k \in {}^\circ\mathcal{C}$, and $\lambda_1, \dots, \lambda_k \in \bar{\mathfrak{a}}_{\mathrm{q}}^*(P, 0)$. Then there exists a Laurent functional $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^* \otimes {}^\circ\mathcal{C}^*$, such that*

$$\mathcal{L}\varphi = \sum_{i=1}^{k} u_i[\pi(\lambda)\langle\varphi(\lambda)|\psi_i\rangle]_{\lambda=\lambda_i} \tag{10.2}$$

for all $\varphi \in \mathcal{M}(X\colon\tau)$. Conversely, given $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^* \otimes {}^\circ\mathcal{C}^*$ there exist k, u_i, ψ_i and λ_i as above such that (10.2) *holds for all $\varphi \in \mathcal{M}(X\colon\tau)$.*

Proof. To prove the existence of $\mathcal{L}$ we may assume that $k = 1$. Let $d = d_{X,\tau}$ and let $\pi_1 = \pi_{\{\lambda_1\},d}$ be determined by (2.6). Then π_1 divides π; let p denote their quotient. It follows from [7], Lemma 10.5, that there exists $\mathcal{L}_1 \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^* \otimes {}^\circ\mathcal{C}^*$ such that

$$\mathcal{L}_1\varphi = u_1[\pi_1(\lambda)\langle\varphi(\lambda)|\psi_1\rangle]_{\lambda=\lambda_1}$$

for all φ such that $\pi_1\varphi$ is holomorphic near λ_1. By Lemma 7.3 the map $\mathcal{L}\colon \varphi \mapsto \mathcal{L}_1(p\varphi)$ belongs to $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^* \otimes {}^\circ\mathcal{C}^*$. It clearly satisfies (10.2).

Conversely, let $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^* \otimes {}^\circ\mathcal{C}^*$ be given. We may assume that the support of $\mathcal{L}$ consists of a single point in $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. This points equals $s\lambda_0$ for suitable $\lambda_0 \in \bar{\mathfrak{a}}_{\mathrm{q}}^*(P, 0)$ and $s \in W$. Let π_0, π_s and $\pi^\sim$ be as in the proof of Lemma 9.2. The restriction of $\mathcal{L}$ to $\mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}^\sim, d^\sim) \otimes {}^\circ\mathcal{C}$ is a finite sum of terms of the form

$$\varphi \mapsto u[\pi^\sim(\lambda)\langle\varphi(\lambda)|\psi\rangle]_{\lambda=s\lambda_0}, \tag{10.3}$$

where $\psi \in {}^\circ\mathcal{C}$ and $u \in S(\mathfrak{a}_{\mathrm{q}}^*)$. For $\varphi \in \mathcal{M}(X\colon\tau)$ we use the Weyl conjugation property and rewrite (10.3) in the form

$$\varphi \mapsto u[\pi^\sim(s\lambda)\langle C^\circ(s\colon\lambda)\varphi(\lambda)|\psi\rangle]_{\lambda=\lambda_0},$$

in which the element u has been replaced by its s-conjugate. Since the polynomial $\pi_0\pi_s$ divides $\pi^\sim(s\lambda)$, and since $\pi_s(\lambda)C^\circ(s\colon\lambda)$ is holomorphic at λ_0 it follows from the Leibniz rule that this expression can be further rewritten as a finite sum of terms of the form

$$\varphi \mapsto u[\pi_0(\lambda)\langle\varphi(\lambda)|\psi\rangle]_{\lambda=\lambda_0} \tag{10.4}$$

where $\psi \in {}^\circ\mathcal{C}$ and $u \in S(\mathfrak{a}_{\mathrm{q}}^*)$. Finally, since π_0 divides π, the following lemma shows that there exists $u' \in S(\mathfrak{a}_{\mathrm{q}}^*)$ such that (10.4) takes the form $\varphi \mapsto u'[\pi(\lambda)\langle\varphi(\lambda)|\psi\rangle]_{\lambda=\lambda_0}$, which is as desired in (10.2). $\square$

Let $\Pi_{\mathbb{R}}$ denote the set of polynomials on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ which are products of functions of the form $\lambda \mapsto \langle\xi,\lambda\rangle + c$ with $\xi \in \mathfrak{a}_{\mathrm{q}}^* \setminus \{0\}$ and $c \in \mathbb{R}$.

Lemma 10.5. *Let $p \in \Pi_{\mathbb{R}}$. There exists, for each $u \in S(\mathfrak{a}_{\mathrm{q}}^*)$, an element $u' \in S(\mathfrak{a}_{\mathrm{q}}^*)$ such that $u'(p\varphi)(0) = u\varphi(0)$ for all germs φ at 0 of holomorphic functions on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.*

Proof. We may assume that the degree of p is one. Then $p(\lambda) = \langle\xi,\lambda\rangle + p(0)$ for some non-zero $\xi \in \mathfrak{a}_{\mathrm{q}}^*$. The case that $p(0) = 0$ is covered by [5], Lemma 1.7 (i). Thus, we may assume that $p(0) = 1$. Let $\xi' = \xi/\langle\xi,\xi\rangle$. Then $\xi' p = 1$, when ξ' is considered as a constant coefficient differential operator acting on the function p. By linearity we may assume that u is of the form $u = u''\xi'^k$ with $k \in \mathbb{N}$ and $u'' \in S(\xi'^\perp)$. Let $u' = u''\sum_{i=0}^k (-1)^{k-i}\frac{k!}{i!}\xi'^i$. A simple calculation with the Leibniz rule shows that $u'(p\varphi)(0) = u\varphi(0)$, as desired. $\square$

Corollary 10.6. *The Paley–Wiener spaces* $\mathrm{PW}(X\colon\tau)$ *in Definition 3.4 and in [4], Def. 21.6, are identical, and both equal to $\mathcal{F}(C_c^\infty(X\colon\tau)))$.*

Proof. In view of (9.7), it is immediate from Lemmas 10.3 and 10.4 that the space $\mathrm{PW}(X\colon\tau)$ of [4] is identical to the space denoted $\mathrm{PW}(X\colon\tau)^\sim$ in Remark 3.9. According to that remark, it follows from Theorem 3.6 that this space is equal to $\mathrm{PW}(X\colon\tau)$ as well as to $\mathcal{F}(C_c^\infty(X\colon\tau))$. $\square$

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