Ruin Probabilities and Overshoots for General Lévy Insurance Risk Processes[1]

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Abstract

We formulate the insurance risk process in a general Lévy process setting, and give general theorems and associated applications for the ruin probability and the asymptotic distribution of the overshoot of the process above a high level, when the process drifts to $-\infty$ a.s. and the positive tail of the Lévy measure, or of the ladder height measure, is subexponential or "close to exponential". Recent results of Asmussen and Klüppelberg [3] and Bertoin and Doney [7] for ruin probabilities and the overshoot in random walk and compound Poisson models are shown to have analogues in the general setup. The identities we derive open the way to further investigation of general renewal-type properties of Lévy processes.

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1 Introduction

Various recent studies of insurance risk processes and associated random walks and Lévy processes have paid particular attention to the heavy-tailed case, when downward jumps of the process – claims – may be very large. Such models are now thought to be quite realistic, especially in view of a recent tendency to large-claim events in the insurance industry.

To give some intuition for the much more general framework of this paper, we briefly recall the classical insurance risk model, where all quantities are explicit. In the classical model, the claims arriving within the interval $(0,t]$, $t>0$, are modelled as a compound Poisson process, yielding the risk process

$$R_t = u + \gamma t - \sum_{i=1}^{N_t} Y_i\,, \quad t \geq 0\,, \tag{1.1}$$

where u is the initial risk reserve and $\gamma > 0$ the premium rate (as usual we set $\sum_{i=1}^{0} a_i = 0$). Denote by F the claim size distribution function (df), i.e., the df of the independent and identically distributed (iid) almost surely (a.s.) non-negative random variables (rvs) Y_i, assumed to have finite mean $\mu > 0$. Let $\lambda > 0$ be the intensity of the Poisson process, assume $\gamma > \lambda\mu$, and let $\rho = \lambda\mu/\gamma$. The probability of ultimate ruin is then

$$\begin{aligned}
\psi(u) &= P\left(R_t < 0 \text{ for some } t > 0\right) \\
&= P\left(\sum_{i=1}^{N_t} Y_i - \gamma t > u \text{ for some } t > 0\right) \\
&= P\left(\sum_{i=1}^{n} (Y_i - \gamma T_i) > u \text{ for some } n \in \mathbb{N}\right) \qquad (1.2)\\
&= (1-\rho)\sum_{n=1}^{\infty} \rho^n \overline{F_I^{*n}}(u)\,. \qquad (1.3)
\end{aligned}$$

We have used the following notation and facts:

(i) In this model, ruin can occur only when a claim occurs. This, jointly with the fact that the interarrival times $\{T_i : i \in \mathbb{N}\}$ of a Poisson process are iid exponential rvs, leads to equation (1.2).

(ii) (1.3) follows from a ladder height analysis; in this classical case the *integrated tail distribution*

$$F_I(x) := \frac{1}{\mu}\int_0^x \overline{F}(y)dy\,, \quad x \geq 0\,, \tag{1.4}$$

is the increasing ladder height df (up to a positive constant).

(iii) The condition $\rho < 1$ guarantees that the process $X_t = \sum_{i=1}^{N_t} Y_i - \gamma t$, $t \geq 0$, has negative drift.

(iv) The infinite sum in (1.3) constitutes a renewal measure, which is defective with killing rate ρ.

All this standard theory can be found in various textbooks, e.g. in Feller [18] and Asmussen [1], to mention just the classic and the most recent one.

In analysing (1.3), two regimes can be recognised. The first is called the *Cramér case*, when there exists a $\kappa > 0$ satisfying $\rho \int_0^\infty e^{\kappa u} F_I(dx) = 1$. The defect ρ in the renewal function (1.3) can then be removed by an exponential tilting, and, using Smith's key renewal lemma (see e.g. Embrechts, Klüppelberg and Mikosch [15], Section 1.2) the ruin probability can be shown to decrease exponentially fast, in fact, proportional to $e^{-\kappa u}$, as $u \to \infty$. This result has been extended to a Lévy process setting by Bertoin and Doney [6].

If such a Lundberg coefficient κ does not exist, as is the case for subexponential and "close to exponential" distributions (see Section 2), estimates of the ruin probability have been derived by Embrechts, Goldie and Veraverbeke [14], Embrechts and Veraverbeke [16] and Veraverbeke [29]; see Embrechts et al. [15], Section 1.4.

It is also of prime interest to understand the way ruin happens. This question has been answered by Asmussen [2] for the Cramér case, and more recently by Asmussen and Klüppelberg [3] for the subexponential case. They describe the sample path behaviour of the process along paths leading to ruin via various kinds of conditional limit theorems. As expected, the Cramér case and the non-Cramér case are qualitatively quite different, see e.g. Asmussen [1] and Embrechts and al. [15], Section 8.3.

Our aim is to investigate the non-Cramér case in a general Lévy process setting, which clearly reveals the roles of the various assumptions. Our Lévy process X will start at 0 and be assumed to drift to $-\infty$ a.s., but otherwise is quite general. Upward movement of X represents "claim payments", and the drift to $-\infty$ reflects the fact that "premium income" should outweight claims. "Ruin" will then correspond to passage of X above a specified high level, u, say. In this scenario, heavy-tailedness of the positive side of the distribution of upward jumps models the occurence of large, possibly ruinous, claims, and has previously been studied in connection with the assumption of a finite mean for the process. But in general we do not want to restrict the process in this way. A higher rate of decrease of the process to $-\infty$ is more desirable from the insurer's point of view, while allowing a heavier tail for the positive part is in keeping with the possibility of even more extreme events, which indeed are observed in recent insurance data.

This leads to the idea of considering processes for which the only assumption is of a drift to $-\infty$ a.s., possibly at a linear rate, as is the case when the mean is finite and negative, but possibly much faster. This kind of analysis is aided by results going back to

Erickson [17] which allow us to quantify such behaviour, as is done, for example via easily verified conditions for drift given in Doney and Maller [9]. We also make essential use of important fluctuation identities derived in Bertoin [4] and Vigon [30]. Our results can thus be seen as adding to an understanding of renewal and fluctuation properties of Lévy processes which drift to $-\infty$, with application to passage time and overshoot behaviour at high levels.

The paper is organised as follows. In the next section we introduce some standard notation and definitions for later use in our study. For the most part this consists of the basic aspects of Lévy processes together with definitions and properties of *subexponential* and *close to exponential* distributions. In Section 3 we give an account of some renewal theory for Lévy processes (drifting to $-\infty$), linking the behaviour of the Lévy measure to the drift, renewal measure and Lévy measure of the upward ladder height process. In Section 4 we present our main results which concern the asymptotic analysis of first passage times, asymptotic relations between the Lévy measures of the Lévy process and its ladder height process, an asymptotic conditional overshoot distribution and some ladder height and ladder time considerations. Section 5 offers some examples of the results presented in Section 4. Finally the proofs of the main results are given in Section 6.

2 Setting the Stage

Let us suppose that $X = \{X_t : t \geq 0\}$ is a general Lévy process with law P and Lévy measure Π_X. That is to say, X is a Markov process with paths that are right continuous with left limits such that the increments are stationary and independent and whose characteristic function at each time t is given by the Lévy-Khinchine representation

$$E(e^{i\theta X_t}) = e^{-t\Psi(\theta)}, \quad \theta \in \mathbb{R},$$

where

$$\Psi(\theta) = i\theta a + \sigma^2\theta^2/2 + \int_{(-\infty,+\infty)} (1 - e^{i\theta x} + i\theta x \mathbf{1}_{\{|x|<1\}})\Pi_X(dx), \tag{2.1}$$

and we have $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and $\int_{\mathbb{R}}(x^2 \wedge 1)\Pi_X(dx) < \infty$ (see Bertoin [4], p. 13, Sato [27], Chapter 2). Take $\Pi_X(\{0\}) = 0$. We denote furthermore by $\mathbb{F} := \{\mathcal{F}_t : t \geq 0\}$ the natural filtration generated by X satisfying the usual assumptions of right continuity and completeness.

Throughout we impose two essential restrictions:
(i) $X_0 = 0$ and the process drifts to $-\infty$: $\lim_{t\to\infty} X_t = -\infty$ a.s;
(ii) $\Pi_X\{(0,\infty)\} > 0$, so the process is not spectrally negative.

Further discussion of these points is given below. Otherwise, the only requirement will be the subexponential or, alternatively, "close to exponential" behaviour (see Definition 2.4 below) we assume for the right tail of Π_X, and these too are necessary, in essence, for the kinds of behaviour we study, as we show.

Definition 2.1. The following are standard tools of fluctuation theory for Lévy processes; see for example Bertoin [4], Ch. VI.

(Supremum) Let $\overline{X} = \{\overline{X}_t = \sup_{u\in[0,t]} X_u : t \geq 0\}$ be the process of the last supremum.

(Local Time and Inverse Local Time) Let $L = \{L_t : t \geq 0\}$ denote the *local time* in the time period $[0,t]$ that $\overline{X} - X$ spends at zero. Then $L^{-1} = \{L_t^{-1} : t \geq 0\}$ is the *inverse local time* such that

$$L_t^{-1} = \inf\{s \geq 0 : L_s > t\}.$$

We shall also understand

$$L_{t-}^{-1} = \inf\{s \geq 0 : L_s \geq t\}.$$

In both cases and in the following text we use the standard notation $\inf\emptyset = \infty$. Note that the previous two inverse local times are both stopping times with respect to the natural filtration of X. Since X drifts to minus infinity it follows that, with probability one, $L_\infty < \infty$ and hence there exists a $t > 0$ such that $L_t^{-1} = \infty$, again with probability one.

(Increasing and Decreasing Ladder Height Processes) The process H defined by $\{H_t = X_{L_t^{-1}} : t \geq 0\}$ is the *increasing ladder height process*, that is to say, the process of new maxima indexed by local time at the maximum. We call L^{-1} the *(upwards) ladder time process*. The processes L^{-1} and H are both defective subordinators. It is understood that $H_t = \infty$ when $L_t^{-1} = \infty$.

We shall define $\hat{H} = \{\hat{H}_t : t \geq 0\}$ to be the *decreasing ladder height process* in an analogous way. Note that this means that $\hat{H}$ is a process which is negative valued (this is unconventional, as the usual definition of decreasing ladder height process would correspond to $-\hat{H}$ here).

(Bivariate Ladder Process) On the event $\{t < L_\infty\}$, the joint process (L^{-1}, H) behaves like a bivariate subordinator which is independent of L_∞. Also there exists a constant $q > 0$ such that $L_\infty \stackrel{d}{=} \mathbf{e}_q$, where $\mathbf{e}_q$ is an exponential variable with mean $1/q$; cf. Bertoin [4], Lemma VI.2. Throughout the paper we shall distinguish between the non-defective processes, denoted by $\mathcal{L}$ (with $\mathcal{L}_\infty = \infty$), $\mathcal{L}^{-1}$ and $\mathcal{H}$, and their defective versions

L, L^{-1} and H. The corresponding non-defective bivariate ladder process is then $(\mathcal{L}^{-1}, \mathcal{H})$. It is a bivariate subordinator, independent of $\mathbf{e}_q$, with the property

$$\{(L_t^{-1}, H_t) : t < L_\infty\} \stackrel{Law}{=} \{(\mathcal{L}_t^{-1}, \mathcal{H}_t) : t < \mathbf{e}_q\}.$$

Note that by contrast the decreasing ladder height process is not defective in this sense because we have assumed that X drifts to minus infinity. □

Definition 2.2 (Moment Generating Functions). For a finite measure μ on $[0,\infty)$ the *moment generating function* (mgf) is defined as

$$\delta_\alpha(\mu) = \int_{[0,\infty)} e^{\alpha u}\mu(du), \quad \alpha \geq 0.$$

Definition 2.3 (Lévy Measures). In addition to the measure Π_X we shall call Π_H, $\Pi_{\mathcal{H}}$ and $\Pi_{\hat{H}}$ the Lévy measures of H, $\mathcal{H}$ and $\hat{H}$, with supports in $(0,\infty)$, $(0,\infty)$ and $(-\infty,0)$, respectively. Further, for $u > 0$,

$$\overline{\Pi}_X^+(u) = \Pi_X\{(u,\infty)\}, \; \overline{\Pi}_X^-(u) = \Pi_X\{(-\infty,-u)\} \text{ and } \overline{\Pi}_X = \overline{\Pi}_X^+(u) + \overline{\Pi}_X^-(u)$$

represent the positive, negative and combined tails of Π_X. We use analogous notation for the tails of $\Pi_{\mathcal{H}}$ and $\Pi_{\hat{H}}$.

Each infinitely divisible df generates a Lévy process in the sense that it may serve as the df of X_1. For the most part we shall restrict ourselves to those infinitely divisible dfs which belong to one of the following classes.

Definition 2.4 (Classes $\mathcal{L}^{(\alpha)}$ and $\mathcal{S}^{(\alpha)}$). Let $\alpha \geq 0$. We shall say that a df G on $[0,\infty)$ with tail $\overline{G} := 1 - G$ belongs to class $\mathcal{L}^{(\alpha)}$ if $\overline{G}(x) > 0$ for each $x \geq 0$ and

$$\lim_{u\to\infty} \frac{\overline{G}(u-x)}{\overline{G}(u)} = e^{\alpha x} \text{ for each } x \in \mathbb{R}, \text{ if } G \text{ is nonlattice;} \tag{2.2}$$

$$\lim_{n\to\infty} \frac{\overline{G}(n-1)}{\overline{G}(n)} = e^{\alpha} \text{ if } G \text{ is lattice (then assumed of span 1).} \tag{2.3}$$

The limit relation (2.2) holds locally uniformly.

If, further, for some $M = M(\alpha) < \infty$ (with $*$ denoting convolution), we have

$$\lim_{u\to\infty} \frac{\overline{G^{*2}}(u)}{\overline{G}(u)} = 2M, \tag{2.4}$$

then we say that G belongs to $\mathcal{S}^{(\alpha)}$.

The class $\mathcal{S}^{(0)}$ is called the *subexponential distributions*. The classes $\mathcal{S}^{(\alpha)}$ with $\alpha > 0$ are *close to exponential*.

We will often write $\overline{G} \in \mathcal{L}^{(\alpha)}$ rather than $G \in \mathcal{L}^{(\alpha)}$, and similarly for $\mathcal{S}^{(\alpha)}$.

We will understand a notation like $\overline{G} \in \mathcal{S}^{(\alpha)}$ to mean that any df F which is tail equivalent to $G \in \mathcal{S}^{(\alpha)}$ (i.e. $\overline{F}(x) \sim \overline{G}(x)$ as $x \to \infty$, equivalently $\lim_{x\to\infty} \overline{F}(x)/\overline{G}(x) = 1$) is in $\mathcal{S}^{(\alpha)}$. In Embrechts et al. [14] it is shown that, when $G \in \mathcal{S}^{(\alpha)}$, then any df which is tail equivalent to G is also in $\mathcal{S}^{(\alpha)}$. The tail of any (Lévy or other) measure finite on (x_0, ∞), for some $x_0 > 0$, can be renormalised to be the tail of a df, and, by extension, then, is said to be in $\mathcal{L}^{(\alpha)}$ or $\mathcal{S}^{(\alpha)}$ if the appropriate conditions among (2.2) – (2.4) are satisfied.

A number of useful properties flow from these definitions. For these results and others, see for example Embrechts and Goldie [13] and their references. We follow Bertoin and Doney [7] in (2.2) and (2.3). ([7] drew attention to the need, when $\alpha > 0$, to distinguish the lattice and nonlattice cases; under (2.2), the geometric distribution, for example, would not be considered "close to exponential", as we would wish. For $\alpha = 0$ no distinction is necessary. Having noted this distinction for $\alpha > 0$, we will confine our proofs to the nonlattice case by considering (2.2) to be the defining property of $\mathcal{L}^{(\alpha)}$.)

When $G \in \mathcal{S}^{(\alpha)}$ we must have $\delta_\alpha(G) < \infty$, and then the constant M in (2.4) must equal $\delta_\alpha(G)$. Moreover, it follows that $\delta_{\alpha+\varepsilon}(G) = \infty$ for all $\varepsilon > 0$. For the class $\mathcal{S}^{(0)}$ of subexponential dfs this means that the moment generating function does not exist – they are "heavy-tailed" in this sense. Typical examples are Pareto, heavy-tailed Weibull and lognormal dfs. Distributions with regularly varying tails are in this class. The classes $\mathcal{S}^{(\alpha)}$ for $\alpha > 0$ are "close to exponential" in the sense that their tails are only slightly modified exponential; see Klüppelberg [23]. The slight modification, however, results in a moment generating function which is finite for argument α. An important class of dfs belonging to the class $\mathcal{S}^{(\alpha)}$ for $\alpha > 0$ is the generalized inverse Gaussian distribution; see Embrechts [11] and Klüppelberg [24].

Extending (2.4), when $G \in \mathcal{S}^{(\alpha)}$ it is in fact true that, for all $k \in \mathbb{N}$,

$$\lim_{u\to\infty} \frac{\overline{G^{*k}}(u)}{\overline{G}(u)} = k\delta_\alpha^{k-1}(G)\,. \tag{2.5}$$

Also, the following uniform bound holds: for each $\varepsilon > 0$, there is a $K(\varepsilon)$ such that, uniformly in $u > 0$ for each $k \in \mathbb{N}$,

$$\frac{\overline{G^{*k}}(u)}{\overline{G}(u)} \le K(\varepsilon)(\delta_\alpha(G) + \varepsilon)^k\,. \tag{2.6}$$

An important property of $\mathcal{S}^{(\alpha)}$ relates these classes to infinitely divisible distributions, and hence to Lévy processes.

Proposition 2.5. *If G is infinitely divisible with Lévy measure $\Pi_G(\cdot) \neq 0$, and with tail $\overline{\Pi}_G(u) = \Pi_G\{(u,\infty)\}$, $u \ge 0$, then*

$$G \in \mathcal{S}^{(\alpha)} \quad \Longleftrightarrow \quad \overline{\Pi}_G \in \mathcal{S}^{(\alpha)} \quad \Longleftrightarrow \quad \overline{G}(u) \sim \delta_\alpha(G)\overline{\Pi}_G(u)\,, \quad u \to \infty\,. \tag{2.7}$$

For a proof see Embrechts, Goldie and Veraverbeke [14] and Pakes [26]; the former restrict themselves to distributions on $[0, \infty)$, the latter gives general results.

For more detailed information on the classes $\mathcal{S}^{(\alpha)}$, and in particular on the subexponential class, we refer to Embrechts et al. [15] and the review paper by Goldie and Klüppelberg [21].

3 Some Renewal Theory for Lévy Processes

In our applications, the first passage time

$$\tau(u) = \inf\{t \geq 0 : X_t > u\}, \quad u > 0,$$

corresponds to ruin occuring at level u, and major objects of interest are the probability that this occurs in a finite time, and the behaviour of this probability as the reserve level u is increased to ∞. Following Embrechts et al. [14] and Asmussen and Klüppelberg [3], a natural way to proceed is by placing close to exponential or subexponential assumptions on $\overline{\Pi}_{\mathcal{H}}$ or on $\overline{\Pi}_X^+$. We are then able to follow in outline the program of [3], finding the limiting conditional distribution as $u \to \infty$ of the overshoot $X_{\tau(u)} - u$ above level u (when it occurs), and of further quantities in our general setup. This gives quite a clear picture of how and when first passage over a high level happens for general Lévy processes.

The following development is essentially based on appropriate sections of Bertoin [4] and Vigon [30], but adapted and extended in part for our requirements.

Definition 3.1 (Ladder Height Renewal Measure). We define the renewal measure of the defective process H in the usual way. Its connection to the renewal measure of the corresponding non-defective process $\mathcal{H}$ with exponential killing time is given as follows.

$$dV(y) = \int_0^\infty dt \cdot P(H_t \in dy) = \int_0^\infty dt \cdot e^{-qt} P(\mathcal{H}_t \in dy), \quad y \geq 0. \tag{3.1}$$

We shall also be interested in the renewal measure of $\hat{H}$, the downward ladder height process:

$$d\hat{V}(y) = \int_0^\infty dt \cdot P(\hat{H}_t \in dy), \quad y \leq 0.$$

The next theorem gives an identity from which we can calculate the distributions of $X_{\tau(u)}$, $L^{-1}_{L_{\tau(u)-}}$ and $X_{L^{-1}_{L_{\tau(u)-}}}$. Although notationally rather complicated, the latter two objects are nothing more than the time corresponding to the ladder time prior to the first passage and the position of this ladder height respectively. The proof of Theorem 3.2 is based on calculations appearing in Proposition III.2, p. 76, of Bertoin [4] (see also Winkel

[31]). The seemingly curious condition $g(u)=0$ functions as a way of excluding from the calculation the fact that there is possibly an atom at u in the distribution of $X_{\tau(u)}$ which is a result of crossing u continuously or "creeping upwards" (see the discussion following Definition 3.4).

Theorem 3.2. *Fix $u>0$. Suppose that f, g and h are bounded, positive and measurable such that $g(u)=0$. Define*

$$dV^h(y)=\int_0^\infty dt\cdot e^{-qt}\int_{[0,\infty)} h(\phi)P(\mathcal{H}_{t^-}\in dy,\mathcal{L}_t^{-1}\in d\phi),\quad y\geq 0.$$

Then

$$\begin{aligned}&E\left(f(X_{L^{-1}_{L_{\tau(u)-}}})\,g(X_{\tau(u)})\,h(L^{-1}_{L_{\tau(u)-}});\ \tau(u)<\infty\right)\\ &\quad=\int_{(0,u]} dV^h(y)\,f(y)\int_{(u-y,\infty)} g(y+s)\,\Pi_{\mathcal{H}}(ds).\end{aligned}\tag{3.2}$$

Proof. Define $T(u)=\inf\{t\geq 0: H_t>u\}$ and recall that X experiences first passage at $\tau(u)$ if and only if H experiences first passage at $T(u)$. The quantity $X_{L^{-1}_{L_{\tau(u)-}}}$ can alternatively be written as $H_{T(u)-}$. On $\{T(u)<\infty\}$, H is a pure jump subordinator and $L_\infty\sim \mathrm{Exp}(q)$. Start from the lefthand side of the statement of the theorem, and decompose according to $\{T(u)=t\}$ to get

$$\begin{aligned}&E\left(f(X_{L^{-1}_{T(u)-}})\,g(H_{T(u)})\,h(L^{-1}_{T(u)-});\ T(u)<L_\infty\right)\\ &=\ E\sum_{0<t<L_\infty}\left(f(H_{t^-})g(H_{t^-}+\Delta H_t)h(L^{-1}_{t^-});\ H_{t^-}\leq u<H_{t^-}+\Delta H_t\right)\\ &=\ E\int_0^\infty dy\cdot qe^{-qy}\sum_{0<t<y}\left(f(\mathcal{H}_{t^-})g(\mathcal{H}_{t^-}+\Delta\mathcal{H}_t)h(\mathcal{L}^{-1}_{t^-});\ \mathcal{H}_{t^-}\leq u<\mathcal{H}_{t^-}+\Delta\mathcal{H}_t\right)\\ &=\ E\sum_{t>0}e^{-qt}\left(f(\mathcal{H}_{t^-})g(\mathcal{H}_{t^-}+\Delta\mathcal{H}_t)h(\mathcal{L}^{-1}_{t^-});\ \mathcal{H}_{t^-}\leq u<\mathcal{H}_{t^-}+\Delta\mathcal{H}_t\right).\end{aligned}$$

Use the compensation formula for the Poisson point process $\{\triangle\mathcal{H}_t: t\geq 0\}$ (Bertoin [4], p. 7) to get that the last expression is equal to

$$\begin{aligned}&\int_0^\infty dt\cdot e^{-qt}E\left(f(\mathcal{H}_{t^-})h\left(\mathcal{L}^{-1}_{t^-}\right)\mathbf{1}_{\{\mathcal{H}_{t^-}\leq u\}}\int_{(0,\infty)}\Pi_{\mathcal{H}}(ds)g\left(\mathcal{H}_{t^-}+s\right)\mathbf{1}_{\{\mathcal{H}_{t^-}+s>u\}}\right)\\ &=\ \int_{(0,u]}\left\{\int_0^\infty dt\cdot e^{-qt}\int_{(0,\infty)}h(\phi)\,P\left(\mathcal{H}_{t^-}\in dy,\mathcal{L}^{-1}_{t^-}\in d\phi\right)\right\}\\ &\qquad\qquad\times f(y)\int_{(u-y,\infty)}\Pi_{\mathcal{H}}(ds)g(y+s)\\ &=\ \int_{(0,u]}dV^h(y)\,f(y)\int_{(u-y,\infty)}\Pi_{\mathcal{H}}(ds)g(y+s).\end{aligned}$$

$\square$

The following result, giving a formula for the ruin probability, is the continuous time version of the Pollacek- Khinchine formula (see also [6], p. 364, [4], p. 172).

Proposition 3.3. $P(\tau(u) < \infty) = qV(u, \infty) := q\overline{V}(u)\,,\ u > 0.$

Proof. Simply note that, using (3.1) for the last equality,

$$\begin{aligned} P(\tau(u) < \infty) &= P\left(\overline{X}_\infty > u\right) = P(\mathcal{H}_{\mathbf{e}_q} > u) \\ &= \int_0^\infty qe^{-qt}dt \cdot P(\mathcal{H}_t > u) \\ &= qV(u, \infty)\,. \end{aligned}$$

$\square$

Definition 3.4 (Wiener-Hopf Factors). The Wiener-Hopf factorization theorem (see, e.g., Bertoin [4], p. 166) together with the downward drift assumption on our Lévy process, tells us that we can write

$$k\Psi(\theta) = -k\log Ee^{i\theta X_1} = [-\log Ee^{i\theta H_1}][-\log Ee^{i\theta \hat{H}_1}] = \kappa(\theta) \times \hat{\kappa}(\theta)\,, \quad \theta \in \mathbb{R}\,, \tag{3.3}$$

where (recalling that $\hat{H}$ is negative) we have, for $\nu > 0$,

$$\kappa(i\nu) = -\log Ee^{-\nu H_1} = \int_{(0,\infty)} (1 - e^{-\nu y})\Pi_{\mathcal{H}}(dy) + \nu c + q\,, \tag{3.4}$$

$$\hat{\kappa}(-i\nu) = -\log Ee^{\nu \hat{H}_1} = \int_{(-\infty,0)} (1 - e^{\nu y})\Pi_{\hat{H}}(dy) + \nu\hat{c}\,. \tag{3.5}$$

The constant k is determined by the choice of normalization of the local time L. We may and will assume without loss of generality that $k = 1$. A different value of k would simply modify the choices of $\mathcal{L}$, $\mathcal{H}$ and q.

The factors $\kappa(\cdot)$ and $\hat{\kappa}(\cdot)$ are the Lévy-Khinchine exponents of H and $\hat{H}$, which are increasing and decreasing subordinators, and accordingly the integrals in the definitions of κ and $\hat{\kappa}$ converge. The convention that $e^{i\theta H_1} = 0 = e^{-\nu H_1}$ when $H_1 = \infty$ is used in (3.3) and (3.4). The non-negative constants c and $\hat{c}$ are the drift coefficients of these subordinators and q is the same killing rate that appears in the definition of $\mathcal{H}$. See Bertoin [4], Chapter VI, and Vigon [30] for expositions of these and the following concepts.

Note that (3.4) implies

$$-\log\delta_{-\nu}(\mathcal{H}) = -\log Ee^{-\nu\mathcal{H}_1} = \int_{(0,\infty)} (1 - e^{-\nu y})\Pi_{\mathcal{H}}(dy) + \nu c\,, \quad \nu > 0\,. \tag{3.6}$$

X is said to *creep upwards* if $P(X_{\tau(u)} = u) > 0$ for some (hence every) $u > 0$; equivalently, if $c > 0$. X creeps downwards if $-X$ creeps upwards; equivalently, if $\hat{c} > 0$. When $c > 0$, that is when X creeps upwards, we have

$$P(X_{\tau(u)} = u,\, \tau(u) < \infty) = P(T'(u) < L_\infty) = E(e^{-qT'(u)}), \quad u > 0,$$

where $T'(u) = \inf\{t \geq 0 : H_t = u\}$. According to Corollary II.18, p. 64, of Bertoin [4] it then follows that the derivative $dV(u)/du$ exists and that

$$P(X_{\tau(u)} = u,\, \tau(u) < \infty) = C\frac{dV}{du}(u) =: CV'(u), \quad u > 0,$$

where $C > 0$ is a constant representing the capacity of a single point. The next corollary lists the main formulae, derived from Theorem 3.2, that we will use. Amongst other results it turns out that $C = c$, the drift of the upward ladder height subordinator.

Corollary 3.5. *For $u \in (0, \infty)$ we have the following four convolution identities:*

(i) $P(X_{\tau(u)} - u > x, \tau(u) < \infty) = \int_{[0,u]} dV(y)\, \overline{\Pi}_{\mathcal{H}}(u + x - y)$;

(ii) $P(\tau(u) < \infty) = \int_{(0,u)} dV(y)\, \overline{\Pi}_{\mathcal{H}}(u - y) + cV'(u)$;

(iii) $P(X_{\tau(u)} > u,\, L^{-1}_{L_{\tau(u)-}} > \psi,\, \tau(u) < \infty) = \int_{(0,u)} V(dy; \psi)\overline{\Pi}_{\mathcal{H}}(u - y)$,

where $V(dy; \psi) = \int_0^\infty dt \cdot e^{-qt} P(\mathcal{H}_t \in dy, \mathcal{L}_t^{-1} > \psi)$;

(iv) $P(X_{L^{-1}_{L_{\tau(u)-}}} > \phi,\, \tau(u) < \infty) = \int_{(\phi,u)} V(dy)\overline{\Pi}_{\mathcal{H}}(u - y) + cV'(u), \quad \phi \in [0, u)$.

Proof. In Theorem 3.2, choose $f = h = 1$ and $g = \mathbf{1}_{\{\cdot > x+u\}}$ to get (i). For (ii), multiply each side by $e^{-\lambda u}$ for $\lambda > 0$ and integrate over $u \in [0, \infty)$, making use of Proposition 3.3 and the identities

$$\int_{[0,\infty)} e^{-\lambda y} V(dy) = (q - \log \delta_\alpha(\mathcal{H}))^{-1} \tag{3.7}$$

(obtained by integrating (3.1)), and

$$\lambda c + \lambda \int_{(0,\infty)} e^{-\lambda y}\overline{\Pi}_{\mathcal{H}}(y)dy = \lambda c + \int_{(0,\infty)} (1 - e^{-\lambda y})\Pi_{\mathcal{H}}(dy) = -\log E e^{-\lambda \mathcal{H}_1} \tag{3.8}$$

(from (3.6)) to see that (ii) holds as stated. By taking the limit as x tends to zero in (i), and combining the result with (ii), it is clear that

$$P(X_{\tau(u)} = u, \tau(u) < \infty) = cV'(u),$$

so $C = c$, as asserted.

For (iii), choose $f = 1$, $g = \mathbf{1}_{\{\cdot > x+u\}}$ and $h = \mathbf{1}_{\{\cdot > \psi\}}$ in (3.2), and take the limit as x tends to zero. Next, choosing $f = \mathbf{1}_{\{\cdot > \phi\}}$, $g = \mathbf{1}_{\{\cdot > x+u\}}$ and $h = 1$ in (3.2), then letting x tend to zero, gives an expression for $P(X_{L^{-1}_{L_{\tau(u)-}}} > \phi, \tau(u) < \infty, X_{\tau(u)} > u)$. Since $\{X_{L^{-1}_{L_{\tau(u)-}}} = u\}$ on $\{X_{\tau(u)} = u\}$ almost surely (cf. Proposition III.2 of Bertoin [4]), by adding on cV' we have (iv). □

Further convolution identities that will be of use can be found in Proposition 3.3 of Vigon [30]:

Theorem 3.6 (Vigon). *We have for* $u \in (0, \infty)$

(i) $\overline{\Pi}^{+}_{X}(u) = \int_{(u,\infty)} \overline{\Pi}_{\hat{H}}(u-y)\, d\Pi_{\mathcal{H}}(y) + \hat{c}\,\Pi'_{\mathcal{H}}(u)$, *where* $\Pi'_{\mathcal{H}}$ *is the density of* $\Pi_{\mathcal{H}}$ *which exists if and only if* $\hat{c}$, *the drift coefficient of* $\hat{H}$, *is positive;*

(ii) $\overline{\Pi}_{\mathcal{H}}(u) = -\int_{(-\infty,0)} \overline{\Pi}^{+}_{X}(u-y)\, d\hat{V}(y)$.

Remark 3.7. Note that, by our convention, $\hat{V}(y)$ is positive and nonincreasing on $y \in (-\infty, 0)$, with $\hat{V}(0) = 0$. X drifts to $-\infty$ a.s. in our analysis, so we can and will exclude the case when X is a subordinator. This means that $\hat{H}$, $\hat{V}$ and $\hat{\kappa}$ are not identically zero.

We say that X is *spectrally positive* if $\Pi_X\{(-\infty, 0)\} = 0$. We then have $\hat{H}_t = -t$ and hence $\hat{V}(dy) = -dy$ and $\hat{c} = 1$, and the expressions in Vigon's theorem simplify considerably. In particular (i–ii) both say that

$$\overline{\Pi}_{\mathcal{H}}(u) = \int_{u}^{\infty} \overline{\Pi}^{+}_{X}(y)dy = \int_{u}^{\infty} \overline{\Pi}_{X}(y)dy\,, \quad u > 0 \tag{3.9}$$

(further, implying that the integral in (3.9) is finite). See Bertoin [4], Chapter VII, for other useful results concerning spectrally one-sided processes.

4 Main Results

4.1 First Passage Time Asymptotics

Recall that *throughout the entire paper we will assume*

$$X_0 = 0, \quad \lim_{t\to\infty} X_t = -\infty \text{ a.s.} \quad \text{and} \quad \Pi_X\{(0,\infty)\} > 0. \tag{4.1}$$

(The spectrally negative case, when $\Pi_X\{(0,\infty)\} = 0$, is easily dealt with separately in our context, see Remark 4.15 below.) Moreover, we shall frequently use the following conditions:

$$\overline{\Pi}_{\mathcal{H}} \in \mathcal{S}^{(\alpha)} \quad (\text{where } \alpha \geq 0)\,, \tag{4.2}$$

and the *non-Cramér condition*

$$e^{-q}\delta_\alpha(\mathcal{H}) < 1 \tag{4.3}$$

(only required when $\alpha > 0$; for $\alpha = 0$ condition (4.3) is automatically satisfied under (4.1), since $q > 0$). Here and throughout, we write $\delta_\alpha(\mathcal{H})$ for $\delta_\alpha(\mathcal{H}_1)$.

We start with the asymptotics of the first passage time $\tau(u)$.

Theorem 4.1 (Limiting First Passage Time, $\alpha \geq 0$). *Fix $\alpha \geq 0$ and assume (4.1), (4.2) and (4.3) hold. Then $\overline{C}(u) = P(\tau(u) < \infty)$, $u > 0$, is in $\mathcal{S}^{(\alpha)}$. Furthermore, as $u \to \infty$,*

$$P(\tau(u) < \infty) \sim \frac{q}{(q - \log \delta_\alpha(\mathcal{H}))^2}\overline{\Pi}_{\mathcal{H}}(u) \sim \frac{q}{(q - \log \delta_\alpha(\mathcal{H}))^2 \delta_\alpha(\mathcal{H})} P(\mathcal{H}_1 > u). \tag{4.4}$$

Conversely, suppose that (4.1) holds and $\overline{C}(\cdot)$ is in $\mathcal{S}^{(\alpha)}$. Then (4.2) and (4.3), and hence (4.4), hold.

To be practically useful, we need to replace the quantities depending on the ladder variables in Theorem 4.1 (and similarly in our other results) with quantities defined as far as possible in terms of the marginal distributions of X, or, better, in terms of Π_X. The next subsection is devoted to results like this.

4.2 Relations Between Π_X, $\Pi_{\mathcal{H}}$ and q

Proposition 4.2 (Criterion for (4.3)). *Fix $\alpha > 0$ and assume (4.1). The following equivalences are true:*

$$Ee^{\alpha X_1} < 1 \iff e^{-q}\delta_\alpha(\mathcal{H}) < 1 \iff \delta_\alpha(V) < \infty$$

$$\iff \log \delta_\alpha(\mathcal{H}) = \alpha c + \int_{[0,\infty)} (e^{\alpha y} - 1)\Pi_{\mathcal{H}}(dy) < q$$

$$\iff \alpha a - \sigma^2\alpha^2/2 - \int_{(-\infty,\infty)} (e^{\alpha x} - 1 - \alpha x \mathbf{1}_{|x|<1})\Pi_X(dx) > 0, \tag{4.5}$$

and if any of the conditions holds, then

$$(\delta_\alpha(V))^{-1} = q - \log \delta_\alpha(\mathcal{H}) = \frac{-\log Ee^{\alpha X_1}}{-\log Ee^{\alpha \hat{H}_1}}. \tag{4.6}$$

Remark 4.3. Note that in general, as a consequence of (3.3), (3.4) and (3.6) we have, for $\nu > 0$,

$$q - \log \delta_{-\nu}(\mathcal{H}) = \frac{\Psi(i\nu)}{\hat{\kappa}(i\nu)}, \tag{4.7}$$

and hence

$$q = \lim_{\nu \downarrow 0} \frac{\Psi(i\nu)}{\hat{\kappa}(i\nu)}.$$

Remark 4.4. In the case $\alpha > 0$, Proposition 4.2 shows that our results apply to the class of Lévy processes for which $Ee^{\alpha X_1} < 1$. By contrast, suppose there is a $\nu > 0$ such that $Ee^{\nu X_1} = 1$. This forces X to drift to $-\infty$ a.s., and, without further assumptions, [6] and [4], p. 183, then prove Cramér's estimate: $P(\tau(u) < \infty) \sim Ce^{-\nu u}$, as $u \to \infty$, where $C < \infty$ and $C > 0$ if and only if the Lévy process $X^{\#}$ with exponent $\Psi^{\#}(\lambda) = \Psi(\lambda - i\nu)$ has $E|X_1^{\#}| < \infty$.

Proposition 4.5 (Relation Between Π_X and $\Pi_{\mathcal{H}}$, $\alpha > 0$). *Fix $\alpha > 0$ and assume (4.1). Then $\overline{\Pi}_X^+$ belongs to $\mathcal{L}^{(\alpha)}$ if and only $\overline{\Pi}_{\mathcal{H}}$ does, in which case $\overline{\Pi}_X^+(u) \sim \hat{\kappa}(-i\alpha)\overline{\Pi}_{\mathcal{H}}(u)$, as $u \to \infty$.*

Define

$$A_-(x) = \overline{\Pi}_X^-(1) + \int_1^x \overline{\Pi}_X^-(y)dy\,, \quad x \geq 1\,,$$

and let "$\asymp$" in a relationship denote that ratio of the two sides is bounded away from zero and infinity, over the indicated range of the variable.

Proposition 4.6 (Relation Between Π_X and $\Pi_{\mathcal{H}}$, $\alpha = 0$). *Assume (4.1) and $\overline{\Pi}_X^+ \in \mathcal{L}^{(0)}$.*

(i) If $\displaystyle\int_1^\infty \overline{\Pi}_X^-(y)dy = \infty$ then $\displaystyle\overline{\Pi}_{\mathcal{H}}(u) \asymp \int_{(1,\infty)} \left(\frac{y}{A_-(y)}\right) \Pi_X(u+dy)\,,\ u \to \infty\,;$ (4.8)

(ii) if $\displaystyle\int_1^\infty \overline{\Pi}_X^-(y)dy < \infty$ then $\displaystyle\overline{\Pi}_{\mathcal{H}}(u) \asymp \int_{(u,\infty)} \overline{\Pi}_X^+(y)dy\,, \quad u \to \infty\,.$ (4.9)

Remark 4.7. By Doney and Maller [9], $\lim_{t\to\infty} X_t = -\infty$ a.s. if and only if

$$\int_{(1,\infty)} \left(\frac{y}{A_-(y)}\right) \Pi_X(dy) < \infty = \int_1^\infty \overline{\Pi}_X^-(y)dy \quad \textit{or} \quad 0 < -EX_1 \leq E|X_1| < \infty. \tag{4.10}$$

Thus the integral on the righthand side of (4.8) is finite when $\lim_{t\to\infty} X_t = -\infty$ a.s.

4.3 Conditional Limit Theorems

Theorem 4.8 (Limiting Overshoot Distribution, $\alpha \geq 0$). *Suppose $\alpha \geq 0$ and assume (4.1), (4.2) and (4.3) hold. Then*

$$\lim_{u\to\infty} P(X_{\tau(u)} - u > x | \tau(u) < \infty) = \overline{G}(x)\,, \quad x > 0\,,$$

where $\overline{G}$ is the tail of a (possibly improper) distribution function:

$$\overline{G}(x) = e^{-\alpha x}\left(1 - \frac{1}{q}\log\delta_\alpha(\mathcal{H}) + \frac{1}{q}\int_{(x,\infty)}(e^{\alpha y} - e^{\alpha x})\Pi_{\mathcal{H}}(dy)\right)\,, \quad x \geq 0\,. \tag{4.11}$$

When $\alpha > 0$, let $x \to 0+$ in (4.11) to see that $\overline{G}(0^+) = 1 - \alpha c/q$, thus we can also conclude that the asymptotic conditional probability of creeping over the barrier u, as $u \to \infty$, is equal to $\alpha c/q$.

On the other hand, when $\alpha = 0$ the distribution G in Theorem 4.8 is degenerate, placing all its mass at infinity. For this case we have the following sharper result:

Theorem 4.9 (Sharper Limiting Overshoot Distribution, $\alpha = 0$). *Suppose that (4.1) holds and $\overline{\Pi}_{\mathcal{H}} \in \mathcal{S}^{(0)}$. Then for all $x > 0$*

$$\lim_{u\to\infty}\left| P(X_{\tau(u)} - u > x|\tau(u) < \infty) - \frac{\overline{\Pi}_{\mathcal{H}}(u+x)}{\overline{\Pi}_{\mathcal{H}}(u)}\right| = 0, \tag{4.12}$$

and the convergence is uniform in $x \geq \eta$ for each $\eta > 0$.

Remark 4.10. (i) We can apply (4.8) and (4.12) as follows. Denote the right-hand side of (4.8) or (4.9) by $\overline{B}_0(u)$, a finite, non-increasing function on $(0,\infty)$. Suppose there are functions $a(u) \to \infty$ as $u \to \infty$ and $b(x) \to 0$ as $x \to \infty$ such that, for each $x > 0$,

$$\frac{\overline{B}_0(u + xa(u))}{\overline{B}_0(u)} \asymp b(x)\,, \quad u \to \infty. \tag{4.13}$$

Note that $a(u)$ and $b(x)$ are defined in terms of the Lévy characteristics of X, rather than of $\mathcal{H}$. Assume $\overline{\Pi}_X^+ \in \mathcal{L}^{(0)}$. Then by (4.8) or (4.9) and (4.12), and using the uniformity of convergence in (4.12), we have

$$P(X_{\tau(u)} - u > xa(u)|\tau(u) < \infty) \asymp b(x)\,, \quad u \to \infty. \tag{4.14}$$

This gives the asymptotic order of magnitude of the overshoot, when normalised by $a(u)$; $(X_{\tau(u)} - u)/a(u)$ is tight as $u \to \infty$, conditional on $\tau(u) < \infty$. It is the counterpart of the corresponding result in equation (1.5) of Asmussen and Klüppelberg [3], except that [3] obtains a limit rather than an order of magnitude estimate, as a result of its more restrictive (finite mean and maximum domain of attraction) but more informative assumptions. We can likewise strengthen (4.8), replacing "$\asymp$" by "$\sim$", under stronger assumptions, using methods such as those of Teugels [28], for example, but we omit further details of this here. When $\overline{\Pi}_X^+ \in \mathcal{L}^{(0)}$, it is shown in the proof of Proposition 4.6 that $\overline{\Pi}_{\mathcal{H}}(u)/\overline{\Pi}_X^+(u) \to \infty$ as $u \to \infty$, so we cannot replace the hypothesis $\overline{\Pi}_{\mathcal{H}} \in \mathcal{L}^{(0)}$ by $\overline{\Pi}_X^+ \in \mathcal{L}^{(0)}$, in Theorem 4.9.

(ii) Theorems 4.1, 4.8 and 4.9 generalise corresponding results of Asmussen and Klüppelberg [3] concerning the classical insurance risk process. In this case the limit df G reduces to a generalized Pareto distribution: for $\alpha = 0$, the normalising function a from (4.14) is the well known *auxiliary function* in extreme value theory (see Embrechts et al. [15], Chapter 3), and G is a Pareto distribution; for $\alpha > 0$ the normalizing function degenerates to the constant α, and G is the standard exponential df.

The remaining three theorems in this section likewise consider analogues in the Lévy process setting of practically interesting aspects of the insurance process.

Theorem 4.11 (Local Time at Ruin, $\alpha \geq 0$). *Suppose that $\alpha \geq 0$ and (4.1), (4.2) and (4.3) hold. Then for all $t \geq 0$*

$$\lim_{u\to\infty} P\left(L_{\tau(u)} > t | \tau(u) < \infty\right)$$

$$= e^{-(q-\log\delta_\alpha(\mathcal{H}))t}\left(1 + t(q - \log\delta_\alpha(\mathcal{H}))\log\delta_\alpha(\mathcal{H})/q\right). \tag{4.15}$$

When $\alpha = 0$, Theorem 4.11 tells us that the local time at ruin has an exponential(q) distribution, asymptotically. Translated to the classical risk process of the introduction, the quantity $L_{\tau(u)}$ denotes the number of claims until ruin happens.

Theorem 4.12 (Last Ladder Height Before Ruin, $\alpha \geq 0$). *Suppose that $\alpha \geq 0$ and (4.1), (4.2) and (4.3) hold. Then for all $\phi \geq 0$*

$$\lim_{u\to\infty} P(X_{L^{-1}_{L_{\tau(u)-}}} \leq \phi | \tau(u) < \infty) = \frac{(q - \log\delta_\alpha(\mathcal{H}))^2}{q}\left(\int_{(0,\phi]} e^{\alpha y} V(dy)\right). \tag{4.16}$$

Remark 4.13. In this last theorem, when $\alpha = 0$ the limiting distribution is proper. This follows by virtue of the fact that

$$V(\infty) = \int_0^\infty e^{-qt} dt = 1/q.$$

On the other hand, when $\alpha > 0$, the limiting distribution is improper having mass

$$1 - \delta_\alpha(V)\frac{(q - \log\delta_\alpha(\mathcal{H}))^2}{q} = 1 - \frac{1}{q\delta_\alpha(V)} < 1$$

at infinity. (See Proposition 4.2 for a justification of the last equality).

For the last ladder time before ruin, we only show tightness:

Theorem 4.14 (Last Ladder Time Before Ruin, $\alpha = 0$). *Assume (4.1) and $\overline{\Pi}_{\mathcal{H}} \in \mathcal{S}^{(0)}$. Then*

$$\lim_{\phi\to\infty} \limsup_{u\to\infty} P(L^{-1}_{L_{\tau(u)-}} > \phi | \tau(u) < \infty) = 0.$$

Remark 4.15. In the spectrally negative case, $\Pi_X\{(0,\infty)\} = 0$, and there are no upwards jumps, so we have $X_{\tau(u)} = u$ on $\tau(u) < \infty$, for all $u > 0$, X creeps up, and the overshoot is a.s. zero at all levels. The ladder height process $\mathcal{H}_t$ is simply the unit drift t (Bertoin [4], p. 191). The passage time $\tau(u)$ has Laplace transform

$$E(e^{-\lambda\tau(u)}; \tau(u) < \infty) = e^{-u\Phi(\lambda)}$$

where Φ is the inverse function to $-\Psi(-i\lambda)$ ([4], p. 189). Thus the ruin probability is $P(\tau(u) < \infty) = e^{-au}$, where $a > 0$ satisfies $\Phi(a) = 0$. In the classical risk model, this setup is taken to describe annuities in life insurance (Grandell [22], p. 9).

5 Examples

In this section we shall consider examples, all of which have the feature that X is spectrally positive, that is to say, we assume $\overline{\Pi}_X^- = 0$. As usual, X also drifts to $-\infty$ a.s. For such processes, from Remark 3.7 we have that the downward ladder height process is simply a negative linear drift, $\hat{H}_t = -t$, and $\overline{\Pi}_{\mathcal{H}}(u) = \int_u^\infty \overline{\Pi}_X^+(y)\,dy$ (finite), for $u > 0$.

Assumption 5.1. *When $\alpha > 0$ we assume that $\overline{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ and when $\alpha = 0$ assume that $\overline{\Pi}_{\mathcal{H}}$ belongs to $\mathcal{S}^{(0)}$. Thus in either case we have $\overline{\Pi}_{\mathcal{H}} \in \mathcal{S}^{(\alpha)}$.*

Suppose that X has Laplace exponent $\phi(\theta)$ for $\theta \in \mathbb{R}$ such that

$$E\left(e^{\theta X_t}\right) = e^{\phi(\theta)t}.$$

The introduction of ϕ makes it convenient to connect with existing literature on one-sided Lévy processes. Note that ϕ is different from Ψ, but, when finite, they are related through the identity $\phi(\theta) = -\Psi(-i\theta)$ (see (2.1) for $\Psi(\cdot)$). From Assumption 5.1 is clear that $\phi(\theta)$ is finite at least for all $\theta \leq \alpha$ (but may be infinite for $\theta > \alpha$). Noting that $-X$ is spectrally negative, we can extract the following facts from Bertoin [4, 5] who has studied spectrally negative processes extensively. The function $\phi(\theta)$ is convex, passes though the origin and tends to $-\infty$ $(+\infty)$ as $\theta \to +\infty$ $(\to -\infty)$. The drift of X is given by the (lefthand) derivative $\phi'(0-) < 0$.

Using Remark 4.3 and taking advantage of the fact that the downward ladder height process is simply a linear drift, we can identify

$$q = \lim_{\theta\downarrow 0} \frac{\Psi(i\theta)}{\hat{\kappa}(i\theta)} = \lim_{\theta\downarrow 0} \frac{-\phi(-\theta)}{-\log E(e^{-\theta \hat{H}_1})} = \lim_{\theta\downarrow 0} \frac{\phi(-\theta)}{\theta}.$$

We thus deduce that $q = -\phi'(0-)$. Next note from (4.6) that when $\alpha > 0$

$$q - \log \delta_\alpha(\mathcal{H}) = \frac{-\phi(\alpha)}{\alpha}$$

and hence the condition $e^{-q}\delta_\alpha(\mathcal{H}) < 1$ reduces to the requirement that $\phi(\alpha) < 0$. (Recall that when $\alpha = 0$ the requirement $e^{-q}\delta_\alpha(\mathcal{H}) < 1$ is automatically satisfied).

We can now draw the following conclusions.

Theorem 5.2. *Suppose that X is spectrally positive, drifts to $-\infty$ a.s., satisfies Assumption 5.1, and $\phi(\alpha) < 0$ if $\alpha > 0$. With the understanding that $-\phi(\alpha)/\alpha = -\phi'(0-)$ when $\alpha = 0$, we have*

(i) $P(\tau(u) < \infty) \sim -\phi'(0-) \left(\frac{\alpha}{\phi(\alpha)}\right)^2 \int_u^\infty \overline{\Pi}_X^+(y)\, dy$, *as* $u \to \infty$;

(ii) $\lim_{u\to\infty} P(X_{\tau(u)} - u > x | \tau(u) < \infty) = \overline{G}(x)$, *where*

$$\overline{G}(x) = \frac{e^{-\alpha x}}{-\phi'(0-)} \left(\frac{-\phi(\alpha)}{\alpha} + \int_{(x,\infty)} (e^{\alpha y} - e^{\alpha x})\, \overline{\Pi}_X^+(y)\, dy \right);$$

(iii) $\lim_{u\to\infty} P(L_{\tau(u)} > t | \tau(u) < \infty) = e^{\phi(\alpha)t/\alpha} \left(1 - \frac{t\phi(\alpha)}{\alpha}\left(1 - \frac{\phi(\alpha)}{\alpha\phi'(0-)}\right)\right)$.

Let us proceed to examine some specific spectrally positive models in more detail.

5.1 Jump Diffusion Process

Suppose Assumption 5.1 is in force, and, further, that X_t is of the form

$$X_t = \sigma B_t + \sum_{i=1}^{N_t} Y_i - \gamma t, \quad t \ge 0, \tag{5.1}$$

where $\gamma > 0$ and $\sigma > 0$ are constants, B_t is a standard Brownian motion, N_t is a Poisson process of rate λ, the Y_i are a.s. positive iid rvs with df F, and all processes are independent. In the context of insurance risk theory this process is called a *risk process perturbed by Brownian motion*; see Dufresne and Gerber [10] and Furrer and Schmidli [20].

X_t can drift to $-\infty$ only if $EY_1 = \mu < \infty$, so assume this, and then that $\rho = \mu\lambda/\gamma < 1$, so indeed X_t drifts to $-\infty$. For this process we have

$$\phi(\theta) = -\theta\gamma + \sigma^2\theta^2/2 + \lambda \int_{[0,\infty)} (e^{\theta x} - 1)F(dx), \tag{5.2}$$

and this is finite at least for $\theta \le \alpha$. Also, $\overline{\Pi}_X^+(x) = \lambda\overline{F}(x)$, and

$$\overline{\Pi}_{\mathcal{H}}(u) = \lambda \int_u^\infty \overline{F}(y)dy = \lambda\mu\overline{F}_I(u)\,, \quad u \ge 0\,,$$

where F_I is the integrated tail df as defined in (1.4).

(i) Take $\alpha > 0$. Assume that $F \in \mathcal{S}^{(\alpha)}$. Then $F \in \mathcal{L}^{(\alpha)}$, or, equivalently, $\overline{F} \circ \log$ is regularly varying with index $-\alpha$, so by Karamata's Theorem (see Bingham, Goldie and Teugels [8], p. 28) $\lim_{u\to\infty} \overline{F}(u)/\int_u^\infty \overline{F}(y)dy = \alpha$, hence by tail equivalence also $\overline{F}_I \in \mathcal{S}^{(\alpha)}$, and $\delta_\alpha(F_I) = (\delta_\alpha(F) - 1)/(\mu\alpha) < \infty$. Then from (5.2) we have

$$\begin{aligned} -\phi(\alpha) &= \gamma\alpha - \sigma^2\alpha^2/2 - \lambda \int_{[0,\infty)} (e^{\alpha x} - 1)F(dx) \\ &= \gamma\alpha - \sigma^2\alpha^2/2 - \lambda(\delta_\alpha(F) - 1) \\ &= \gamma\alpha - \sigma^2\alpha^2/2 - \lambda\mu\alpha\delta_\alpha(F_I), \end{aligned} \tag{5.3}$$

and this is positive if and only if

$$\rho\delta_\alpha(F_I) + \frac{\sigma^2}{2}\frac{\alpha}{\gamma} < 1,$$

which we will assume to be the case. (Note that this implies $\rho < 1$.) Differentiation shows that $q = -\phi'(0-) = \gamma - \lambda\mu = \gamma(1-\rho)$.

Finally we have via simple substitution in Theorem 5.2 (i) with the help of (5.3) that as $u \to \infty$,

$$P(\tau(u) < \infty) \sim \frac{(1-\rho)\rho}{(1 - \sigma^2\alpha/(2\gamma) - \rho\delta_\alpha(F_I))^2}\overline{F}_I(u) \tag{5.4}$$

$$\sim \frac{(1-\rho)\rho}{\mu\alpha(1 - \sigma^2\alpha/(2\gamma) - \rho\delta_\alpha(F_I))^2}\overline{F}(u)\,. \tag{5.5}$$

Similarly, we can obtain a quite explicit expression for the overshoot limit distribution from Theorem 5.2 (ii), calculable once the (incomplete) moment generating function is calculated.

(ii) Take $\alpha = 0$. Assume that

$$\overline{\Pi}_{\mathcal{H}}(u) = \lambda \int_u^\infty \overline{F}(y)dy \in \mathcal{S}^{(0)}.$$

We may still calculate $-\phi'(0-)$ as a left derivative equal to $\gamma(1-\rho)$. Again filling in Theorem 5.2 (i) we get

$$P(\tau(u) < \infty) \sim \frac{\rho}{1-\rho}\overline{F}_I(u)\,, \quad u \to \infty\,, \tag{5.6}$$

in which the effect of the Brownian component has washed out. ((5.6) is the same as for the classical case, see Embrechts and Veraverbeke [16] and Embrechts et al. [15], Section 1.4.) In this case $\alpha = 0$, (4.11) simply tells us that the overshoot above u tends to ∞ in probability as $u \to \infty$, as we expect from the heavy-tailedness in the positive direction. To sharpen the result we use Theorem 4.12 and the arguments in Remark 4.10. With $\overline{B}_0(u) = \lambda \int_u^\infty \overline{F}(y)dy$, choose $a(u)$ to satisfy (4.13), so that (4.14) holds. This parallels the development of Asmussen and Klüppelberg [3] for essentially the same model (without the Brownian component). They require a maximum domain of attraction condition, which gives "=" in (4.14), whereas our more general analysis only gives "$\asymp$".

Remark 5.3. The last result can be viewed as a robustness result in the sense of suggesting how far we can move away from a specific model without changing the asymptotic ruin probability: to a random walk with subexponential claims, we can add a diffusion term without changing the ruin probability. This effect has been investigated in a more general framework by Embrechts and Samorodnitsky [12]. See also Mikosch and Samorodnitsky [25]. Our next example has also an interpretation in this sense.

5.2 Stable Process with Jumps and Drift

In this example we suppose X_t is of the form

$$X_t = S_t^{(p)} + \sum_{i=1}^{N_t} Y_i - \gamma t\,, \quad t \geq 0\,, \tag{5.7}$$

where $\gamma > 0$, $S_t^{(p)}$ is a stable Lévy motion with index $p \in (1,2)$, and the variables $\{Y_i : i \geq 1\}$ are as before (thus, with $EY_1 = \mu < \infty$). It follows from Zolotarev [32] that

$$\begin{aligned}
\phi(\theta) &= -\gamma\theta + \int_{[0,\infty)} (e^{\theta x} - 1 - \theta x)\frac{p(p-1)}{\Gamma(2-p)x^{1+p}}dx + \lambda \int_{[0,\infty)} (e^{\theta x} - 1)F(dx) \\
&= -\gamma\theta + (-\theta)^p + \lambda \int_{[0,\infty)} (e^{\theta x} - 1)F(dx), \quad \theta \leq 0.
\end{aligned}$$

For this example the mgf $\phi(\theta)$ is finite only if $\theta \leq 0$, so we only consider the case $\alpha = 0$, i.e., we assume $\overline{\Pi}_{\mathcal{H}} \in \mathcal{S}^{(0)}$. The process has no downwards jumps ($\beta = 1$ in the notation of Bertoin [4] Chapter VIII, p. 217). This model has been considered by Furrer [19] and is in the ruin context called a *risk process perturbed by p-stable Lévy motion.*

Again assume $\rho = \mu\lambda/\gamma < 1$, and again by an easy differentiation, $q = \gamma - \lambda\mu$. The Lévy measure of X is given by

$$\overline{\Pi}_X^+(x) = \frac{(p-1)}{\Gamma(2-p)} x^{-p} + \lambda \overline{F}(x) \quad \text{and} \quad \overline{\Pi}_X^-(x) = 0, \; x > 0,$$

and, further,

$$\overline{\Pi}_{\mathcal{H}}(u) = \int_u^\infty \overline{\Pi}_X^+(x)dx = \frac{u^{-(p-1)}}{\Gamma(2-p)} + \lambda \int_u^\infty \overline{F}(x)dx\,, \quad u > 0\,.$$

We distinguish three different cases:

$$\lim_{x\to\infty} x^p \overline{F}(x) = \begin{cases} 0\,, \\ c \in (0,\infty)\,, \\ \infty\,. \end{cases}$$

From l'Hôpital we get, corresponding to the above cases,

$$\overline{\Pi}_{\mathcal{H}}(u) \sim \begin{cases} \dfrac{u^{-(p-1)}}{\Gamma(2-p)}\,, \\ \left(\dfrac{1}{\Gamma(2-p)} + \dfrac{\lambda c}{p-1}\right) u^{-(p-1)}\,, \\ \lambda\mu \overline{F}_I(u)\,. \end{cases}$$

This means that we are in the same situation as in the classical subexponential case (i.e., when $\alpha = 0$), but have two different regimes depending on whether the tail of the claim size distribution is heavier or lighter than that of the stable perturbation.

Consequently, for F with tails lighter than or similar to x^{-p} (i.e., Cases 1 and 2),

$$P(\tau(u) < \infty) \sim \frac{C}{(\gamma - \lambda\mu)} u^{-(p-1)}\,, \quad u \to \infty\,,$$

with $C = 1/\Gamma(2-p)$ or $C = 1/\Gamma(2-p) + \lambda c/(p-1)$. If F is heavier tailed than x^{-p} (Case 3), we again get (5.6), with $\rho = \mu\lambda/\gamma$.

Remark 5.4. We remark that all the previous results (Sections 2 – 5) have exact analogues for random walks, in fact, the results can be strengthened slightly in some cases. Since the proofs for the discrete time case use the same ideas, and are even a little simpler, we omit the details.

6 Proofs

Lemma 6.1. *Suppose $\alpha \ge 0$ and $P(\mathcal{H}_1 > u) \in \mathcal{S}^{(\alpha)}$, or, equivalently, by Proposition 2.5, (4.2) holds. Then for each $t > 0$,*

$$P(\mathcal{H}_t > u) \;\sim\; t\,\delta_\alpha^t(\mathcal{H})\,\overline{\Pi}_{\mathcal{H}}(u) \;\sim\; t\,\delta_\alpha^{t-1}(\mathcal{H})\,P(\mathcal{H}_1 > u)\,, \quad u \to \infty\,, \tag{6.1}$$

and hence $P(\mathcal{H}_t > u) \in \mathcal{S}^{(\alpha)}$ for each $t > 0$. Additionally, for each $\varepsilon > 0$, there is a $K(\varepsilon)$ such that for all $t > 0$ and $u > 0$,

$$P(\mathcal{H}_t > u) \;\le\; K(\varepsilon)(\delta_\alpha(\mathcal{H}) + \varepsilon)^{t+1} P(\mathcal{H}_1 > u). \tag{6.2}$$

Suppose further that (4.3) holds. Then, as $u \to \infty$,

$$\frac{P(\tau(u) < \infty)}{\overline{\Pi}_{\mathcal{H}}(u)} \;=\; \frac{q\overline{V}(u)}{\overline{\Pi}_{\mathcal{H}}(u)} \;\to\; \frac{q}{(q - \log\delta_\alpha(\mathcal{H}))^2} \;=\; q\delta_\alpha^2(V). \tag{6.3}$$

Proof. Apply (2.7) to the infinitely divisible rv $\mathcal{H}_t$ with Lévy measure $\Pi_{\mathcal{H}_t}(\cdot) = t\Pi_{\mathcal{H}}(\cdot)$ to get, for each $t > 0$,

$$P(\mathcal{H}_t > u) \;\sim\; \delta_\alpha(\mathcal{H}_t)\overline{\Pi}_{\mathcal{H}_t}(u) \;=\; t\delta_\alpha^t(\mathcal{H})\overline{\Pi}_{\mathcal{H}}(u)\,, \quad u \to \infty\,,$$

then apply (2.7) again to complete (6.1). For (6.2), just use the fact that $P(\mathcal{H}_t > u)$ does not decrease in t (for each $u > 0$) and the discrete uniform bound (2.6) to get

$$P(\mathcal{H}_t > u) \le P(\mathcal{H}_{\lfloor t\rfloor+1} > u) \le K(\varepsilon)(\delta_\alpha(\mathcal{H}) + \varepsilon)^{\lfloor t\rfloor+1} P(\mathcal{H}_1 > u)\,, \quad u > 0\,.$$

Finally, Proposition 3.3 gives

$$\frac{P(\tau(u) < \infty)}{\overline{\Pi}_{\mathcal{H}}(u)} = \frac{q\overline{V}(u)}{\overline{\Pi}_{\mathcal{H}}(u)} = \frac{q}{\overline{\Pi}_{\mathcal{H}}(u)} \int_0^\infty e^{-qt} P(\mathcal{H}_t > u)dt\,, \quad u > 0\,.$$

Then (6.1) and the uniform bound (6.2), together with dominated convergence, gives

$$\lim_{u\to\infty} \frac{P(\tau(u) < \infty)}{\overline{\Pi}_{\mathcal{H}}(u)} = q\int_0^\infty e^{-qt} t\delta_\alpha^t(\mathcal{H})dt \;=\; \frac{q}{(q - \log\delta_\alpha(\mathcal{H}))^2}.$$

□

The following lemma is a minor modification of some working in Embrechts and Goldie [13].

Lemma 6.2. *Let $\alpha \ge 0$ and let the df $\nu(\cdot) \in \mathcal{S}^{(\alpha)}$. Then for each $x \ge 0$*

$$\lim_{a\to\infty} \limsup_{u\to\infty} \int_{(a,u+x-a]} \frac{\overline{\nu}(u + x - y)}{\overline{\nu}(u)} \nu(dy) = 0. \tag{6.4}$$

Further, the convergence in (6.4) is uniform in $x \ge 0$.

Proof of Lemma 6.2. Write

$$\overline{\nu^{*2}}(z) = \overline{\nu}(z) + \overline{\nu} * \nu(z) = \overline{\nu}(z) + \int_{(0,z]} \overline{\nu}(z-y)\nu(dy)\,, \quad z \geq 0\,.$$

For $a > 0$ and $z > 2a$ split up the convolution integral into integrals over $(0,a], (a, z-a), [z-a, z)$ and use partial integration on the last integral. This gives the identity

$$\overline{\nu}^{*2}(z) = \left(2\int_{(0,a]} + \int_{(a,z-a)}\right) \overline{\nu}(z-y)\nu(dy) + \overline{\nu}(a)\overline{\nu}(z-a) \tag{6.5}$$

from which we see that

$$\int_{(a,u+x-a)} \overline{\nu}(u+x-y)\nu(dy) = \overline{\nu}^{*2}(u+x) - 2\int_{(0,a]} \overline{\nu}(u+x-y)\nu(dy) - \overline{\nu}(a)\overline{\nu}(u+x-a). \tag{6.6}$$

Now $\nu(\cdot) \in \mathcal{S}^{(\alpha)}$, so $\overline{\nu^{*2}}(u)/\overline{\nu}(u) \to 2\delta_\alpha(\nu)$ and $\overline{\nu}(u-z) \sim e^{\alpha z}\overline{\nu}(u)$ as $u \to \infty$. Divide by $\overline{\nu}(u)$ in (6.6) and let $u \to \infty$, using dominated convegence, to get the limit

$$\begin{aligned} &2e^{-\alpha x} \textstyle\int_{(0,\infty)} e^{\alpha y}\nu(dy) - 2\int_{(0,a]} e^{\alpha(y-x)}\nu(dy) - e^{\alpha(a-x)}\overline{\nu}(a) \\ &\quad = 2e^{-\alpha x} \textstyle\int_{(a,\infty)} e^{\alpha y}\nu(dy) - e^{-\alpha x}e^{\alpha a}\overline{\nu}(a). \end{aligned}$$

The last tends to 0 as $a \to \infty$.

For the uniformity, note that (6.4) with $x = 0$ gives $\int_{(a,u-a]} \overline{\nu}(u-y)\nu(dy) \leq \varepsilon\overline{\nu}(u)$ once $a \geq a_0(\varepsilon)$ and $u \geq u_0(a)$. Then if $x \geq 0$ and $u_x = u + x$, $\int_{(a,u_x-a]} \overline{\nu}(u_x - y)\nu(dy) \leq \varepsilon\overline{\nu}(u)$ if $a \geq a_0(\varepsilon)$ and $u_x \geq u_0(a)$, certainly if $a \geq a_0(\varepsilon)$ and $u \geq u_0(a)$. $\square$

We shall use the non-lattice part of the following lemma; for completeness we also treat the lattice case. It is simply a re-representation of the defining properties for $\mathcal{L}^{(\alpha)}$, and we omit the proof.

Lemma 6.3. *For $\alpha > 0$, $G \in \mathcal{L}^{(\alpha)}$ is equivalent to*

$$\lim_{u\to\infty} \frac{G(u, u+h)}{G(u, u+1)} = \frac{1-e^{-\alpha h}}{1-e^{-\alpha}},$$

where the limit is through values u in $\mathbb{R}$ or $\mathbb{N}$, and for all $h > 0$, or for all $h \in \mathbb{N}$, for the non-lattice and lattice case, respectively. This in turn is equivalent to saying that $G(u+dy)/\overline{G}(u)$ converges weakly to an exponential distribution with parameter α or to a geometric distribution with parameter $e^{-\alpha}$, respectively.

Proof of Theorem 4.1. Fix $\alpha \geq 0$, assume (4.1) – (4.3), and suppose that $\overline{\Pi}_{\mathcal{H}} \in \mathcal{S}^{(\alpha)}$. The forward part of the theorem follows from (6.3), together with the use of (2.7).

For the converse part, let $\overline{C}(u) = P(\tau(u) < \infty) = q\overline{V}(u)$, $u > 0$, so that $\overline{C}(u)$ is the tail of a df C. By hypothesis, $C \in \mathcal{S}^{(\alpha)}$, thus $\delta_\alpha(C) = q\delta_\alpha(V) < \infty$ and so by Proposition 4.2, (4.3) holds. Now $\overline{C}(u)$ satisfies

$$\overline{C}(u) = \overline{C}_q(u) := q\int_0^\infty e^{-qt}P\left(\mathcal{H}_t > u\right)dt = P(\mathcal{H}_{\mathbf{e}_q} > u)\,, \quad u > 0\,,$$

where $\mathbf{e}_q$ is an independent exponential variable with parameter q, and, since $C_q \in \mathcal{S}^{(\alpha)}$, we have by (2.5), for each $k \in \mathbb{N}$,

$$\lim_{u\to\infty} \frac{\overline{C_q^{*k}}(u)}{\overline{C}_q(u)} = k\delta_\alpha^{k-1}(C_q)\,. \tag{6.7}$$

Using the fact that $\mathcal{H}$ has stationary independent increments we have that $\overline{C_q^{*k}}(u) = P(\mathcal{H}_{\mathbf{e}_q^k} > u)$ where $\mathbf{e}_q^k$ is the sum of k independent exponential rvs, each with parameter q. That is to say,

$$\overline{C_q^{*k}}(u) = \frac{q^k}{(k-1)!}\int_0^\infty t^{k-1}e^{-qt}P(\mathcal{H}_t > u)dt\,, \quad u > 0\,.$$

Thus by (6.7)

$$\lim_{u\to\infty} \frac{q^k}{(k-1)!\overline{C}_q(u)}\int_0^\infty t^{k-1}e^{-qt}P(\mathcal{H}_t > u)dt = k\delta_\alpha^{k-1}(C_q).$$

Multiplying both sides of this by $(1-\lambda/q)^{k-1}$, with $q(1-1/\delta_\alpha(C_q)) < \lambda < q(1+1/\delta_\alpha(C_q))$ (so that $|1-\lambda/q| < 1/\delta_\alpha(C_q)$), and summing over $k \in \mathbb{N}$, gives

$$\lim_{u\to\infty} \frac{1}{\overline{C}_q(u)}\int_0^\infty e^{-\lambda t}P(\mathcal{H}_t > u)dt = \frac{1/q}{(1-(1-\lambda/q)\delta_\alpha(C_q))^2} = \frac{(q-\log\delta_\alpha(\mathcal{H}))^2}{q(\lambda-\log\delta_\alpha(\mathcal{H}))^2}, \tag{6.8}$$

because $\delta_\alpha(C_q) = q/(q-\log\delta_\alpha(\mathcal{H}))$. Relation (6.8) is valid for $\lambda \in (q(1-1/\delta_\alpha(C_q))\,,\, q(1+1/\delta_\alpha(C_q)))$. It means that

$$\overline{C}_\lambda(u) = P(\mathcal{H}_{\mathbf{e}_\lambda} > u) = \lambda\int_0^\infty e^{-\lambda t}P(\mathcal{H}_t > u)dt \sim const.\ \overline{C}_q(u),$$

and hence C_λ is in $\mathcal{S}^{(\alpha)}$ for λ in the indicated range. So by repeating the above argument with q replaced by a $\lambda_0 \in (q, q(1+1/\delta_\alpha(C_q)))$ (for which one should note that $\delta_\alpha(C_{\lambda_0}) < q^{-1}\lambda_0\delta_\alpha(C_q)$ and hence that $\lambda_0(1+1/\delta_\alpha(C_{\lambda_0})) > q(1+1/\delta_\alpha(C_q))$), we can extend the upper limit of the range of applicability of (6.8). Continuing in this way we see that (6.8) holds for all $\lambda > q(1-1/\delta_\alpha(C_q))$. So we can write, for all large λ,

$$\lim_{u\to\infty} \frac{1}{\overline{C}_q(u)}\int_0^\infty e^{-\lambda t}P(\mathcal{H}_t > u)dt = \frac{(q-\log\delta_\alpha(\mathcal{H}))^2}{q}\int_0^\infty te^{-(\lambda-\log\delta_\alpha(\mathcal{H}))t}dt\,.$$

Then by the continuity theorem for Laplace transforms (Feller [18], p. 433) we get

$$\lim_{u\to\infty} \frac{P(\mathcal{H}_t > u)}{\overline{C}_q(u)} = \frac{(q-\log\delta_\alpha(\mathcal{H}))^2}{q}t\delta_\alpha^t(\mathcal{H})\,.$$

By tail equivalence this means that $P(\mathcal{H}_1 > u) \in \mathcal{S}^{(\alpha)}$. $\square$

6.1 The Relation Between $\Pi_X, \Pi_{\mathcal{H}}$ and q

Proof of Proposition 4.2. Fix $\alpha > 0$ and assume (4.1). Put $\theta = -i\alpha$ in the Wiener-Hopf factorisation (3.3) to get

$$-\log E(e^{\alpha X_1}) = \log(e^{-q} E(e^{\alpha \mathcal{H}_1})) \log E(e^{\alpha \hat{H}_1}). \tag{6.9}$$

Now since $\lim_{t\to\infty} X_t = -\infty$ a.s., $E(e^{\alpha \hat{H}_1})$ is always (finite and) less than 1 (recall our convention that $\hat{H}$ is negative), so $E(e^{\alpha X_1}) < 1$ if and only if $E(e^{-q} e^{\alpha \mathcal{H}_1}) < 1$. Thus the first equivalence in (4.5) holds, and the second equality in (4.6) holds.

Next note that

$$\begin{aligned}
\delta_\alpha(V) &= \int_{[0,\infty)} e^{\alpha y} \int_0^\infty e^{-qt} dt \cdot P\left(\mathcal{H}_t \in dy\right) \\
&= \int_0^\infty e^{-qt} dt \cdot E(e^{\alpha \mathcal{H}_t}) \\
&= \int_0^\infty e^{-qt} dt \cdot \delta_\alpha^t(\mathcal{H}) \\
&= (q - \log \delta_\alpha(\mathcal{H}))^{-1},
\end{aligned} \tag{6.10}$$

giving the second equivalence in (4.5) and the first equality in (4.6). Also, from (3.6), which is valid for $\alpha > 0$ when $\delta_\alpha(\mathcal{H}) < \infty$, we get

$$\delta_\alpha(\mathcal{H}) = E(e^{\alpha \mathcal{H}_1}) = \exp\left(\alpha c + \int_{[0,\infty)} (e^{\alpha y} - 1)\Pi_{\mathcal{H}}(dy)\right),$$

giving the third equivalence in (4.5). The fourth equivalence in (4.5) follows from $Ee^{\alpha X_1} < 1$ and (2.1). □

Proof of Proposition 4.5. Suppose first that $\overline{\Pi}_X^+ \in \mathcal{L}^{(\alpha)}$. (This part is based on the analogous version for random walks which appears in Bertoin and Doney [7].) Using Theorem 3.6(ii) we have that

$$\Pi_{\mathcal{H}}(u, u+h) = \int_{(-\infty,0)} \Pi_X^+(u-y, u+h-y) d\hat{V}(y).$$

It is not difficult to justify integrating by parts to get

$$\frac{\Pi_{\mathcal{H}}(u, u+h)}{\overline{\Pi}_X^+(u)} = \int_{(0,\infty)} [\hat{V}(-y) - \hat{V}(-(y-h)^+)] \frac{\Pi_X(u+dy)}{\overline{\Pi}_X^+(u)}.$$

Since $\overline{\Pi}_X^+ \in \mathcal{L}^{(\alpha)}$ it follows that $\Pi_X(u+dy)/\overline{\Pi}_X^+(u)$ converges for $u \to \infty$ weakly to the exponential distribution. This, together with the fact that $\hat{V}$ is a renewal measure and hence the integrand in the last equality is uniformly bounded, implies that

$$\lim_{u\to\infty} \frac{\Pi_{\mathcal{H}}(u, u+h)}{\overline{\Pi}_X^+(u)} = \alpha(1 - e^{-\alpha h}) \int_{(0,\infty)} \hat{V}(-y) e^{-\alpha y} dy\,, \quad h > 0\,.$$

Since the righthand side is nonzero (recall Remark 3.7), Lemma 6.3 suffices to conclude that $\overline{\Pi}_{\mathcal{H}} \in \mathcal{L}^{(\alpha)}$.

Conversely, let $\overline{\Pi}_{\mathcal{H}} \in \mathcal{L}^{(\alpha)}$. Write Theorem 3.6(i) as

$$\overline{\Pi}_X^{+}(u) = \int_{(u,\infty)} \overline{\Pi}_{\hat{H}}\,(u-y)\, d\left(\overline{\Pi}_{\mathcal{H}}\,(u) - \overline{\Pi}_{\mathcal{H}}\,(y)\right) + \hat{c}\; \Pi'_{\mathcal{H}}(u), \tag{6.11}$$

where the derivative is only present if $\hat{c} > 0$. By Fubini's theorem

$$\overline{\Pi}_X^{+}(u) = \int_{(-\infty,0)} \Pi_{\mathcal{H}}\,(u, u-y)\, \Pi_{\hat{H}}\,(dy) + \hat{c}\; \Pi'_{\mathcal{H}}(u)\,, u > 0\,.$$

Take $0 < h_1 < h_2$ and integrate both sides of the last equation to get

$$\begin{aligned}\int_{u+h_1}^{u+h_2} \overline{\Pi}_X^{+}(z)dz \;&=\; \int_{(-\infty,0)} \left(\int_{h_1}^{h_2} \Pi_{\mathcal{H}}\,(u+z, u+z-y)\, dz\right) \Pi_{\hat{H}}\,(dy) \\ &\quad + \hat{c}\left(\overline{\Pi}_{\mathcal{H}}\,(u+h_1) - \overline{\Pi}_{\mathcal{H}}\,(u+h_2)\right).\end{aligned} \tag{6.12}$$

By dominated convergence

$$\begin{aligned}\lim_{u\to\infty} \frac{1}{\overline{\Pi}_{\mathcal{H}}(u)} \int_{h_1}^{h_2} \left(\overline{\Pi}_{\mathcal{H}}\,(u+z) - \overline{\Pi}_{\mathcal{H}}\,(u+z-y)\right) dz &= (1-e^{\alpha y}) \int_{h_1}^{h_2} e^{-\alpha z} dz \\ &= \alpha^{-1}(1-e^{\alpha y})(e^{-\alpha h_1} - e^{-\alpha h_2}),\end{aligned}$$

uniformly in $y \le 0$. Thus, dividing (6.12) by $\overline{\Pi}_{\mathcal{H}}\,(u)$ and letting $u \to \infty$, we get

$$\begin{aligned}\lim_{u\to\infty} \int_{h_1}^{h_2} \left(\frac{\overline{\Pi}_X^{+}\,(u+z)}{\overline{\Pi}_{\mathcal{H}}\,(u)}\right) dz \;&=\; \alpha^{-1}(e^{-\alpha h_1} - e^{-\alpha h_2}) \left(\int_{(-\infty,0)} (1-e^{\alpha y})\, \Pi_{\hat{H}}\,(dy) + \hat{c}\alpha\right) \\ &=\; \alpha^{-1}(e^{-\alpha h_1} - e^{-\alpha h_2}) \hat{\kappa}\,(-i\alpha)\,.\end{aligned} \tag{6.13}$$

Finally, take any sequence $u_n'' \to \infty$ and a subsequence $u_n' \to \infty$ such that, by Helly's theorem,

$$\lim_{u_n'\to\infty} \frac{\overline{\Pi}_X^{+}\,(u_n'+z)}{\overline{\Pi}_{\mathcal{H}}\,(u_n')} = p(z)\,, \quad z > 0\,.$$

Using Fatou's lemma in (6.13) we see that the non-increasing function $p(z)$ is finite for all $z > 0$, and then we can use dominated convergence in (6.13) to deduce that

$$\int_{h_1}^{h_2} p(z)dz = \alpha^{-1}(e^{-\alpha h_1} - e^{-\alpha h_2}) \hat{\kappa}\,(-i\alpha)\,.$$

Differentiating, we see that $p(z) = e^{-\alpha z}\hat{\kappa}\,(-i\alpha)$ for all $z > 0$, true also for all subsequences, and so

$$\overline{\Pi}_X^{+}\,(u+z) \sim e^{-\alpha z}\hat{\kappa}\,(-i\alpha)\, \overline{\Pi}_{\mathcal{H}}(u)\,, \quad u \to \infty\,.$$

Consequently, $\overline{\Pi}_X^{+} \in \mathcal{L}^{(\alpha)}$, and the exact form of the asymptotic is also established. $\square$

Proof of Proposition 4.6. By Remark 3.7, $\overline{\Pi}_{\hat{H}}(y) > 0$ for all $x > 0$. Next, Theorem 3.6(ii) gives

$$\overline{\Pi}_{\mathcal{H}}(u) = \int_{(u,\infty)} \hat{V}(-(y-u))\Pi_X(dy)\,, \quad u > 0\,, \tag{6.14}$$

and Bertoin [4], p. 74, gives for all $y \geq 0$

$$\hat{V}(-y) \asymp \frac{y}{\hat{c} + \hat{A}(y)} \asymp \frac{y}{\hat{A}(y)}\,,$$

where

$$\hat{A}(x) = \int_0^x \overline{\Pi}_{\hat{H}}(y)dy$$

is non-zero for all $x > 0$. Consequently, for all $u > 0$,

$$\overline{\Pi}_{\mathcal{H}}(u) \asymp \int_{(u,\infty)} \left(\frac{y-u}{\hat{A}(y-u)}\right) \Pi_X(dy)\,. \tag{6.15}$$

Note that $\hat{A}(x)/x = \int_0^1 \overline{\Pi}_{\hat{H}}(xy)dy$ is non-increasing and tends to 0 as $x \to \infty$, and has a positive (possibly infinite) limit as $x \to 0+$. Thus $a_0 := \lim_{x\to 0+}(x/\hat{A}(x))$ is finite (possibly 0).

Symmetrically to (6.14), we have

$$\overline{\Pi}_{\hat{H}}(u) = \int_{(u,\infty)} V(y-u)|\overline{\Pi}^-_X(dy)|\,, \quad u > 0.$$

Now $V(y) \leq V(\infty) = 1/q$, thus $\overline{\Pi}_{\hat{H}}(u) \leq \overline{\Pi}^-_X(u)/q$ and it follows that

$$\hat{A}(x) \leq \hat{A}(1) + \frac{1}{q}\int_1^x \overline{\Pi}^-_X(y)dy\,, \quad x > 1\,. \tag{6.16}$$

If $\int_1^x \overline{\Pi}^-_X(y)dy = \infty$ then the right-hand side is asymptotic to $(1/q)\int_1^x \overline{\Pi}^-_X(y)dy$ as $x \to \infty$. For a reverse inequality in this case, choose x_0 so that $V(x_0) \geq 1/(2q)$. Then for $x > x_0$

$$\begin{aligned}
\hat{A}(x) &= \int_0^x \overline{\Pi}_{\hat{H}}(z)dz = \int_0^x \int_{(z,\infty)} V(y-z)|\overline{\Pi}^-_X(dy)|dz \\
&\geq \int_{x_0}^x \int_{(2z,\infty)} V(x_0)|\overline{\Pi}^-_X(dy)|dz \\
&\geq \frac{1}{2q}\int_{x_0}^x \overline{\Pi}^-_X(2z)dz \asymp \int_1^x \overline{\Pi}^-_X(z)dz\,, \quad x \to \infty.
\end{aligned}$$

Together these give

$$\hat{A}(x) \asymp \int_1^x \overline{\Pi}^-_X(y)dy \asymp A_-(x)\,, \quad x \to \infty\,,$$

thus $c_- A_-(x) \le \hat{A}(x) \le c_+ A_-(x)$ for some $0 < c_- \le c_+ < \infty$ whenever $x \ge x_0 > 1$. Integration by parts in (6.15) gives

$$\overline{\Pi}_{\mathcal{H}}(u) \asymp a_0 \overline{\Pi}_X^+(u) + \int_0^\infty \overline{\Pi}_X^+(u+y) d\left(\frac{y}{\hat{A}(y)}\right), \quad u \ge 0 .$$

Now assume $\overline{\Pi}_X^+ \in \mathcal{L}^{(0)}$. Then Fatou's lemma applied to the last equation shows that $\overline{\Pi}_{\mathcal{H}}(u)/\overline{\Pi}_X^+(u) \to \infty$ as $u \to \infty$. Then

$$\int_{(u,u+x_0)} \left(\frac{y-u}{\hat{A}(y-u)}\right) \Pi_X(dy) \le \left(\frac{x_0}{\hat{A}(x_0)}\right) \overline{\Pi}_X^+(u) = o(\overline{\Pi}_{\mathcal{H}}(u)), \quad u \to \infty.$$

Thus as $u \to \infty$

$$\begin{aligned} \overline{\Pi}_{\mathcal{H}}(u) &\asymp \int_{(u+x_0,\infty)} \left(\frac{y-u}{\hat{A}(y-u)}\right) \Pi_X(dy) \\ &\asymp \int_{(u+x_0,\infty)} \left(\frac{y-u}{A_-(y-u)}\right) \Pi_X(dy) \\ &\asymp \int_{(u+1,\infty)} \left(\frac{y-u}{A_-(y-u)}\right) \Pi_X(dy). \end{aligned}$$

This proves (4.8) in case $\int_1^\infty \overline{\Pi}_X^-(y)dy = \infty$. If $\int_1^\infty \overline{\Pi}_X^-(y)dy < \infty$ then (6.16) gives $\hat{A}(\infty) < \infty$ and then (4.9) follows from (6.15). $\square$

6.2 Conditional Limit Theorems

Proof of Theorem 4.8. Fix $\alpha \ge 0$, and suppose that (4.1) – (4.3) hold. Take $x > 0$ and $a > 0$, choose $u > 2a$, and write, from Corollary 3.5(i),

$$\begin{aligned} P(X_{\tau(u)} - u > x, \tau(u) < \infty) &= \left(\int_{(0,a]} + \int_{(a,u]}\right) \overline{\Pi}_{\mathcal{H}}(u+x-y)V(dy) \\ &=: A_u + B_u . \end{aligned} \tag{6.17}$$

By (2.2) we have

$$\lim_{u\to\infty} \frac{\overline{\Pi}_{\mathcal{H}}(u-y)}{\overline{\Pi}_{\mathcal{H}}(u)} = e^{\alpha y}, \quad y \in \mathbb{R}.$$

In A_u, $y \le a$, so the integrand is dominated by

$$\overline{\Pi}_{\mathcal{H}}(u+x-a) \le \overline{\Pi}_{\mathcal{H}}(u-a) \le 2e^{\alpha a}\overline{\Pi}_{\mathcal{H}}(u), \quad u \ge u_0(a),$$

for some $u_0(a)$ large enough. Thus by dominated convergence

$$\lim_{u\to\infty} \frac{A_u}{\overline{\Pi}_{\mathcal{H}}(u)} = \int_{(0,a]} e^{\alpha(y-x)} V(dy),$$

and as the convergence of monotone functions the convergence is uniform in $x \geq 0$. So by (6.3),

$$\begin{aligned}
\lim_{u\to\infty} \frac{A_u}{P(\tau(u)<\infty)} &= \frac{1}{q\delta_\alpha^2(V)} \int_{(0,a]} e^{\alpha(y-x)} V(dy) \\
&= \frac{e^{-\alpha x}}{q\delta_\alpha^2(V)} \left(\delta_\alpha(V) - \int_{(a,\infty)} e^{\alpha y} V(dy)\right).
\end{aligned}$$

Since $\delta_\alpha(V) < \infty$ by Proposition 4.2, when $a \to \infty$ the integral here tends to 0, and, with (6.10), we get the first two terms in (4.11).

Next we deal with B_u, in (6.17). Integration by parts gives

$$\begin{aligned}
B_u &= \overline{\Pi}_{\mathcal{H}}(u+x-a)\overline{V}(a) - \overline{\Pi}_{\mathcal{H}}(x)\overline{V}(u) + \int_{[x,u+x-a)} \overline{V}(u+x-y)\Pi_{\mathcal{H}}(dy) \\
&= \overline{\Pi}_{\mathcal{H}}(u+x-a)\left(\overline{V}(a) - \overline{V}(u)\right) + \int_{[x,u+x-a)} \left(\overline{V}(u+x-y) - \overline{V}(u)\right)\Pi_{\mathcal{H}}(dy) \\
&=: \overline{\Pi}_{\mathcal{H}}(u+x-a)\left(\overline{V}(a) - \overline{V}(u)\right) + C_u\,. \qquad (6.18)
\end{aligned}$$

When divided by $\overline{\Pi}_{\mathcal{H}}(u)$ the first term is dominated by $\overline{\Pi}_{\mathcal{H}}(u-a)\overline{V}(a)/\overline{\Pi}_{\mathcal{H}}(u)$ which tends to $e^{\alpha a}\overline{V}(a)$ as $u \to \infty$, and since $\delta_\alpha(V) < \infty$, we have $e^{\alpha a}\overline{V}(a) \to 0$ as $a \to \infty$.

Take $a > x > 0$ and $u + x > a$ and write

$$\frac{C_u}{\overline{V}(u)} = \left(\int_{[x,a]} + \int_{(a,u+x-a)}\right)\left(\frac{\overline{V}(u+x-y)}{\overline{V}(u)} - 1\right)\Pi_{\mathcal{H}}(dy) =: \frac{D_u}{\overline{V}(u)} + \frac{E_u}{\overline{V}(u)}\,. \qquad (6.19)$$

In the first term the integrand is dominated by

$$\frac{\overline{V}(u-a)}{\overline{V}(u)} \leq 2e^{\alpha a}\,, \quad u \geq u_1(a)\,,$$

for $u_1(a)$ large enough, and a constant is integrable with respect to $\Pi_{\mathcal{H}}(dy)$ over $y \in (x,\infty)$, $x > 0$. Thus by Proposition 3.3,

$$\frac{D_u}{P(\tau(u)<\infty)} = \frac{D_u}{q\overline{V}(u)} \to \frac{1}{q}\int_{[x,a]} \left(e^{\alpha(y-x)} - 1\right)\Pi_{\mathcal{H}}(dy)\,, \quad u \to \infty\,,$$

for each $a > 0$. This convergence of monotone functions is uniform in $x \in [\eta, \infty)$ for each $\eta > 0$. As $a \to \infty$ we get the last term on the right-hand side of (4.11).

The second term on the right-hand side of (6.19) is not negative, and since $\overline{V}(u) \leq$ *const.* $\overline{\Pi}_{\mathcal{H}}(u)$ for $u \geq u_2$, u_2 large enough, by (6.3), E_u is bounded above by a constant multiple of

$$\int_{(a,u+x-a)} \overline{\Pi}_{\mathcal{H}}(u+x-y)\Pi_{\mathcal{H}}(dy) \qquad (6.20)$$

once $u+x-y \geq u_2$, and this is the case when $y < u+x-a$ if we choose $a > u_2$. Now since $\Pi_{\mathcal{H}}(\cdot) \neq 0$ we can choose $z_0 > 0$ such that $\overline{\Pi}_{\mathcal{H}}(z_0) > 0$. Also keep $a > z_0$. Then define

$$\nu(z) = \left(1 - \frac{\overline{\Pi}_{\mathcal{H}}(z)}{\overline{\Pi}_{\mathcal{H}}(z_0)}\right) 1_{\{z \geq z_0\}},$$

which is a (proper) df on $[0,\infty)$ with tail $\overline{\nu}(z) = \overline{\Pi}_{\mathcal{H}}(z)/\overline{\Pi}_{\mathcal{H}}(z_0)$, $z > z_0$. The integral in (6.20) is, apart from a constant multiple,

$$\int_{(a,u+x-a)} \overline{\nu}(u+x-y)\nu(dy). \tag{6.21}$$

The proof is now completed with Lemma 6.2. □

Proof of Theorem 4.9. Suppose that $\overline{\Pi}_{\mathcal{H}} \in \mathcal{S}^{(0)}$. Take $a = 0$ in (6.17) and (6.18) to see that

$$P(X_{\tau(u)} - u > x | \tau(u) < \infty) - \frac{\overline{V}(0)\overline{\Pi}_{\mathcal{H}}(u+x)}{P(\tau(u) < \infty)}$$

is bounded in modulus by

$$\frac{\overline{\Pi}_{\mathcal{H}}(u+x)\overline{V}(u)}{P(\tau(u) < \infty)} + \int_{[x,u+x)} \left(\frac{\overline{V}(u+x-y) - \overline{V}(u)}{P(\tau(u) < \infty)}\right) \Pi_{\mathcal{H}}(dy). \tag{6.22}$$

Since $\overline{V}(0) = 1/q$ and $P(\tau(u) < \infty) \sim \overline{\Pi}_{\mathcal{H}}(u)/q$ by (4.4), the first term in (6.22) converges to 0 (uniformly in $x \geq 0$) as $u \to \infty$, and it suffices to show that the integral converges to 0 as $u \to \infty$, where, in the denominator, we can replace $P(\tau(u) < \infty)$ by $\overline{\Pi}_{\mathcal{H}}(u)$ or by $\overline{V}(u)$. Take $a > 0$ and $u > a$ and write the integral in (6.22) as

$$\left(\int_{[x,u+x-a)} + \int_{[u+x-a,u+x)}\right) \left(\overline{V}(u+x-y) - \overline{V}(u)\right) \Pi_{\mathcal{H}}(dy).$$

The first integral on the right-hand side is the same one we dealt with in (6.18), called C_u, and consequently when divided by $\overline{V}(u)$ has the same limit as $C_u/\overline{V}(u)$ has, but with $\alpha = 0$, namely, 0. As observed there, the convergence is uniform in $x \in [\eta, \infty)$, $\eta > 0$.

Finally

$$\int_{[u+x-a,u+x)} \frac{\overline{V}(u+x-y)}{\overline{\Pi}_{\mathcal{H}}(u)} \Pi_{\mathcal{H}}(dy) \leq \overline{V}(0) \left(\frac{\overline{\Pi}_{\mathcal{H}}((u+x-a)-) - \overline{\Pi}_{\mathcal{H}}(u+x)}{\overline{\Pi}_{\mathcal{H}}(u)}\right)$$

and this tends to 0 as $u \to \infty$, for each $a > 0$, uniformly in $x \geq 0$. □

Proof of Theorem 4.11. Assume that (4.2) and (4.3) hold. Use the Strong Markov Property at the stopping time L_t^{-1} to deduce

$$\begin{aligned}
P(L_{\tau(u)} > t, \tau(u) < \infty) &= P(H_t < u, \tau(u) < \infty) \\
&= E\left(\mathbf{1}_{(H_t<u)} P(\tau(u) < \infty | \mathcal{F}_{L_t^{-1}})\right) \\
&= E\left(\mathbf{1}_{(H_t<u)} P_{H_t}(\tau(u) < \infty)\right) \\
&= E\left(\mathbf{1}_{(H_t<u)} P(\tau(u - H_t) < \infty)\right) \\
&= E\left(\mathbf{1}_{(H_t<u)} P(\tau(u - H_t) < \infty); t < L_\infty\right) \\
&= e^{-qt} E\left(\mathbf{1}_{(\mathcal{H}_t<u)} P(\tau(u - \mathcal{H}_t) < \infty)\right) \\
&= qe^{-qt} \int_{(0,u)} \overline{V}(u-y) P(\mathcal{H}_t \in dy).
\end{aligned}$$

Write the last expression as

$$qe^{-qt}\left(\int_{(0,a]} + \int_{(a,u-a]} + \int_{(u-a,u)}\right) \overline{V}(u-y) P(\mathcal{H}_t \in dy), \tag{6.23}$$

where $u > 2a > 0$, then divide it by $P(\tau(u) < \infty) = q\overline{V}(u)$ and let $u \to \infty$. By dominated convergence the first term tends to

$$e^{-qt} \int_{(0,a)} e^{\alpha y} P(\mathcal{H}_t \in dy) = e^{-qt}\left(\delta_\alpha(\mathcal{H}_t) - \int_{(a,\infty)} e^{\alpha y} P(\mathcal{H}_t \in dy)\right)$$

and this tends to $e^{-qt}\delta_\alpha^t(\mathcal{H})$ as $a \to \infty$. (Recall that t is kept constant in this proof.)

By Lemma 6.1 we can choose a such that

$$\overline{V}(y) \leq const.\ \overline{\Pi}_{\mathcal{H}}(y) \leq const.\ P(\mathcal{H}_t > y)$$

for $y \geq a$, so the second integral in (6.23) is not larger than

$$const. \int_{(a,u-a]} P(\mathcal{H}_t > u-y) P(\mathcal{H}_t \in dy),$$

and, after division by $P(\mathcal{H}_t > u)$, this tends to 0 as $u \to \infty$ then $a \to \infty$ by Lemma 6.2.

Finally

$$\begin{aligned}
&q \int_{(u-a,u)} \overline{V}(u-y) P(\mathcal{H}_t \in dy) \\
&= q\overline{V}(a) P(\mathcal{H}_t > u-a) - q\overline{V}(0) P(\mathcal{H}_t > u) + q \int_{(0,a)} P(\mathcal{H}_t > u-y) V(dy),
\end{aligned} \tag{6.24}$$

while by (6.1) and (4.4) as $u \to \infty$,

$$\begin{aligned}
\frac{P(\mathcal{H}_t > u-y)}{\overline{V}(u)} &= \frac{P(\mathcal{H}_t > u-y)}{P(\mathcal{H}_1 > u-y)} \frac{P(\mathcal{H}_1 > u-y)}{P(\mathcal{H}_1 > u)} \frac{P(\mathcal{H}_1 > u)}{\overline{V}(u)} \\
&\to t\delta_\alpha^t(\mathcal{H}) e^{\alpha y} (q - \log \delta_\alpha(\mathcal{H}))^2 \\
&= c_t e^{\alpha y},
\end{aligned}$$

say. Thus the right-hand side of (6.24), when divided by $q\overline{V}(u)$, tends as $u \to \infty$ then $a \to \infty$ to

$$c_t\left(\delta_\alpha(V) - 1/q\right) = \frac{c_t \log \delta_\alpha(\mathcal{H})}{q(q - \log \delta_\alpha(\mathcal{H}))} = t\delta_\alpha^t(\mathcal{H})(q - \log \delta_\alpha(\mathcal{H})) \log \delta_\alpha(\mathcal{H})/q.$$

Thus the limit is

$$\begin{aligned} &e^{-qt}\left(\delta_\alpha^t(\mathcal{H}) + t\delta_\alpha^t(\mathcal{H})(q - \log \delta_\alpha(\mathcal{H})) \log \delta_\alpha(\mathcal{H})/q\right) \\ &\quad = e^{-(q - \log \delta_\alpha(\mathcal{H}))t}\left(1 + t(q - \log \delta_\alpha(\mathcal{H})) \log \delta_\alpha(\mathcal{H})/q\right). \end{aligned}$$

□

Proof of Theorem 4.12. Use Corollary 3.5 to write

$$P(X_{L^{-1}_{L_{\tau(u)-}}} \geq \phi, \tau(u) < \infty) = P(\tau(u) < \infty) - \int_{[0,\phi)} V(dy)\overline{\Pi}_{\mathcal{H}}(u-y),$$

divide by $P(\tau(u) < \infty)$, and take the limit as u tends to infinity, using (4.4), to get (4.16). □

Proof of Theorem 4.14. From Corollary 3.5 (iii) and the remark in the proof thereof concerning an expression for $P(X_{\tau(u)} = u)$, we can write

$$\begin{aligned} P(L^{-1}_{L_{\tau(u)-}} > \psi,\, \tau(u) < \infty) &\leq \int_{(0,u)} V(dy;\psi)\,\overline{\Pi}_{\mathcal{H}}(u-y) + cV'(u) \\ &= \left\{\int_{(0,a)} + \int_{[a,u)}\right\} V(dy;\psi)\,\overline{\Pi}_{\mathcal{H}}(u-y) + cV'(u), \end{aligned}$$

where $u > 2a > 0$. Also $P(\tau(u) < \infty) \sim \mathit{const.}\ \overline{\Pi}_{\mathcal{H}}(u)$ so it suffices for our purposes to divide by $\overline{\Pi}_{\mathcal{H}}(u)$. For the first integral

$$\lim_{u\to\infty} \int_{(0,a)} V(dy;\psi)\frac{\overline{\Pi}_{\mathcal{H}}(u-y)}{\overline{\Pi}_{\mathcal{H}}(u)} \leq \lim_{u\to\infty} \frac{\overline{\Pi}_{\mathcal{H}}(u-a)}{\overline{\Pi}_{\mathcal{H}}(u)} V(a;\psi) \leq e^{\alpha a}V(a;\psi).$$

However

$$\begin{aligned} V(a;\psi) &= \int_0^\infty dt \cdot e^{-qt}P(\mathcal{H}_t \leq a, \mathcal{L}_t^{-1} > \psi) \\ &\leq \int_0^\infty dt \cdot e^{-qt}P(\mathcal{L}_t^{-1} > \psi) \\ &= \frac{1}{q}P(\mathcal{L}^{-1}_{\mathrm{e}_q} > \psi) \to 0 \end{aligned}$$

as $\psi \to \infty$ because $\mathcal{L}^{-1}_{\mathrm{e}_q} < \infty$ almost surely. For the remaining terms note that

$$\begin{aligned} \int_{[a,u)} V(dy;\psi)\overline{\Pi}_{\mathcal{H}}(u-y) + cV'(u) &\leq \int_{[a,u)} V(dy)\overline{\Pi}_{\mathcal{H}}(u-y) + cV'(u) \\ &= P(X_{L^{-1}_{L_{\tau(u)-}}} \geq a,\, \tau(u) < \infty). \end{aligned}$$

Divide by $P(\tau(u) < \infty)$, take the limsup as u tends to infinity, and then let $a \to \infty$. The result is zero, by Theorem 4.12. $\square$

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