# Eisenstein integrals and induction of relations

E. P. van den Ban

In Honor of Jacques Carmona

**Abstract**

In this article I will give a survey of joint work with Henrik Schlichtkrull on the induction of certain relations among (partial) Eisenstein integrals for the minimal principal series of a reductive symmetric space. I will discuss the application of this principle of induction to the proof of the Fourier inversion formula in [11] and to the proof of the Paley-Wiener theorem in [15]. Finally, the relation with the Plancherel decomposition will be discussed.

## 1 Introduction

Let $\mathrm{X} = G/H$ be a reductive symmetric space, with $G$ a real reductive group of Harish-Chandra's class and $H$ an open subgroup of the group $G^\sigma$ of fixed points for an involution $\sigma$ of $G$. Thus, $G_e^\sigma \subset H \subset G^\sigma$, with $G_e^\alpha$ the identity component of $G^\sigma$.

There exists a Cartan involution $\theta$ of $G$ that commutes with $\sigma$. The associated maximal compact subgroup $K := G^\theta$ is invariant under $\sigma$.

There are two important classes of examples of reductive symmetric spaces. The first class, with $H$ compact, consists of the Riemannian symmetric spaces. Here we take $\theta = \sigma$ and $K = H$. The second consists of the real reductive groups of Harish-Chandra's class. Given such a group $^\backprime G$, let $G = {}^\backprime G \times {}^\backprime G$, let $\sigma\colon G \to G$, $(x, y) \mapsto (y, x)$, and let $H = G^\sigma = \mathrm{diag}\,({}^\backprime G)$. Then X equals $^\backprime G$, equipped with the left times right action of $G$. We may take $\theta = {}^\backprime\theta \times {}^\backprime\theta$, with $^\backprime\theta$ a Cartan involution of $^\backprime G$. Accordingly, $K = {}^\backprime K \times {}^\backprime K$, with $^\backprime K$ maximal compact in $^\backprime G$.

We are interested in the analysis of $K$-finite functions on X. For this it is convenient to fix a finite dimensional unitary representation $(\tau, V_\tau)$ of $K$ and to consider the space

$$C^\infty(\mathrm{X}\colon \tau) := [C^\infty(\mathrm{X}) \otimes V_\tau]^K \tag{1.1}$$

of smooth $\tau$-spherical functions on X. Alternatively, we view $C^\infty(\mathrm{X}\colon \tau)$ as the space of smooth functions $f\colon \mathrm{X} \to V_\tau$ transforming according to the rule $f(kx) = \tau(k)f(x)$, for $x \in \mathrm{X}$ and $k \in K$. The subspace of compactly supported functions in (1.1) is denoted by $C_c^\infty(\mathrm{X}\colon \tau)$.

**2000 Mathematics Subject Classification:** Primary 22E30; Secondary 43A85, 22E45.

Keywords: Symmetric space, Eisenstein integral, induction, Paley-Wiener theorem, Arthur-Campoli relations, Plancherel theorem.

As usual, we denote Lie groups by Roman capitals, and the associated Lie algebras by the corresponding German lowercase letters. The involutions $\sigma$ and $\theta$ of $G$ give rise to involutions of the Lie algebra $\mathfrak{g}$, which are denoted by the same symbols. Accordingly, we write

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$$

for the decompositions of $\mathfrak{g}$ into the $+1$ and $-1$ eigenspaces for $\theta$ and $\sigma$, respectively. Let $\mathfrak{a}_{\mathrm{q}}$ be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. The set $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ of restricted roots of $\mathfrak{a}_{\mathrm{q}}$ in $\mathfrak{g}$ is a (possibly non-reduced) root system. Let $\Sigma^+$ be a positive system, $\mathfrak{a}_{\mathrm{q}}^+$ the associated positive chamber, $A_{\mathrm{q}}^+ := \exp \mathfrak{a}_{\mathrm{q}}^+$, and $\Delta$ the associated collection of simple roots. The Weyl group $W$ of $\Sigma$ is canonically isomorphic with $N_K(\mathfrak{a}_{\mathrm{q}})/Z_K(\mathfrak{a}_{\mathrm{q}})$. Each subset $F \subset \Delta$ determines a standard parabolic subgroup $P_F$ of $G$ as follows. Let $\mathfrak{a}_{F\mathrm{q}}$ be the intersection of the root hyperplanes $\ker \alpha$, for $\alpha \in F$, and let $M_{1F}$ be the centralizer of $\mathfrak{a}_{F\mathrm{q}}$ in $G$. Moreover, let

$$\mathfrak{n}_F = \bigoplus_{\alpha \in \Sigma^+ \setminus \mathbb{Z}F} \mathfrak{g}_\alpha, \quad \text{and} \quad N_F := \exp \mathfrak{n}_F.$$

Then $P_F = M_{1F} N_F$. Let $M_{1F} = M_F A_F$ according to the Langlands decomposition of $P_F$, then $\mathfrak{a}_{F\mathrm{q}}$ is the intersection of $\mathfrak{a}_F$, the Lie algebra of $A_F$, with $\mathfrak{q}$. In particular, $\mathfrak{a}_{\mathrm{q}}$ is the intersection of $\mathfrak{a} := \mathfrak{a}_\emptyset$ with $\mathfrak{q}$. The group $M_F$ is a reductive group of Harish-Chandra's class; accordingly, the homogeneous space $\mathrm{X}_F := M_F / M_F \cap H$ belongs to the class of symmetric spaces considered.

Since $\sigma$ and $\theta$ commute, the composition $\sigma\theta$ is an involution of $G$. Its derivative restricts to the identity on $\mathfrak{a}_{\mathrm{q}}$; therefore, the involution $\sigma\theta$ leaves each of the standard parabolic subgroups $P_F$ invariant. Let $\mathcal{P}_\sigma$ denote the collection of $\sigma\theta$-stable parabolic subgroups of $G$ containing $A_{\mathrm{q}}$. Then $W$ acts on $\mathcal{P}_\sigma$ in a natural way. Each element of $\mathcal{P}_\sigma$ is $W$-conjugate to a unique standard parabolic subgroup $P_F$. Finally, each minimal element of $\mathcal{P}_\sigma$ is $W$-conjugate to $P_\emptyset$.

In this article we will discuss relations of a certain type between (partial) normalized Eisenstein integrals for $P_\emptyset$. These Eisenstein integrals, denoted $E^\circ(\lambda : \cdot)$, are essentially sums of matrix coefficients of induced representations of the form $\mathrm{Ind}_{P_\emptyset}^G(\xi \otimes \lambda \otimes 1)$, with $\xi$ an irreducible finite dimensional unitary representation of $M_\emptyset$ and with $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. These induced representations form the minimal principal series of X. Induction of relations describes how relations of a certain type between the Eisenstein integrals $E^\circ(\lambda)$ (or more generally between partial Eisenstein integrals) are induced by similar relations between the similar integrals for $\mathrm{X}_F$.

In terms of the mentioned Eisenstein integrals we define a Fourier transform $\mathcal{F}_\emptyset$. Applied to a function $f \in C_c^\infty(\mathrm{X} : \tau)$ the Fourier transform gives an element of $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*) \otimes {}^\circ\mathcal{C}$, where $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*)$ denotes the space of meromorphic functions on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ and where ${}^\circ\mathcal{C} = {}^\circ\mathcal{C}(\tau)$ is a certain finite dimensional Hilbert space. It is a main result of [9] that the Fourier transform $\mathcal{F}_\emptyset$ is injective on $C_c^\infty(\mathrm{X} : \tau)$. Accordingly, two natural problems arise.

(a) To retrieve $f$ from its Fourier transform $\mathcal{F}_\emptyset f$; this is the problem of **Fourier inversion**.

(b) To characterize the image of $\mathcal{F}_\emptyset(C_c^\infty(\mathrm{X} : \tau))$ in a way that generalizes the **Paley-Wiener theorem** of J. Arthur, [1].

In the answers to these related questions, given in [11] and [15], respectively, the principle of induction of relations plays a fundamental role.

## 2 Eisenstein integrals

As said, Eisenstein integrals for $P_\emptyset$ are essentially sums of $K$-finite matrix coefficients of principal series representations of the form $\mathrm{Ind}_{P_\emptyset}^G(\xi \otimes \lambda \otimes 1)$. We will now give their precise definition. To keep the exposition as light as possible, we make the following

**Simplifying assumption** *The manifold $G/H$ has precisely one open $P_\emptyset$-orbit.*

This assumption is only made for purposes of exposition, it is not necessary for the development of the theory. In the general situation, there are finitely many open $P_\emptyset$-orbits, naturally parametrized by $W/W_{K\cap H}$, where $W_{K\cap H}$ denotes the subgroup of $W$ consisting of elements that are contained in the natural image of $N_K(\mathfrak{a}_\mathrm{q}) \cap H$. The simplifying assumption is satisfied in the Riemannian case as well as in case of the group.

We define ${}^\circ\mathcal{C} = {}^\circ\mathcal{C}(\tau)$ by

$$ {}^\circ\mathcal{C} := C^\infty(\mathrm{X}_\emptyset : \tau_\emptyset), $$

the space of smooth $\tau_\emptyset$-spherical functions $\mathrm{X}_\emptyset \to V_\tau$; here $\tau_\emptyset := \tau|_{K\cap M_\emptyset}$. By compactness of $\mathrm{X}_\emptyset$, it follows that ${}^\circ\mathcal{C}$ is finite dimensional. Moreover, ${}^\circ\mathcal{C} = L^2(\mathrm{X}_\emptyset : \tau_\emptyset)$.

Given $F \subset \Delta$ we define $\rho_F \in \mathfrak{a}_{F\mathrm{q}}^*$ by $\rho_F = \frac{1}{2}\mathrm{tr}\,(\mathrm{ad}(\,\cdot\,)|_{\mathfrak{n}_F})$. In particular, we put $\rho = \rho_\emptyset$. Let $\psi \in {}^\circ\mathcal{C}$ and $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. We define the function $\psi_\lambda : G \to V_\tau$ by

$$ \begin{array}{rcll} \psi_\lambda(x) & = & a^{\lambda+\rho}\psi(m) & \text{for} \quad x \in manH, \quad (m,a,n) \in M_\emptyset \times A_\emptyset \times N_\emptyset, \\ & = & 0 & \text{for} \quad x \in G \setminus P_\emptyset H. \end{array} $$

We equip $\mathfrak{g}$ with a non-degenerate $\mathrm{Ad}(G)$-invariant bilinear form $B$ that is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$ and for which $\mathfrak{h}$ and $\mathfrak{q}$ are orthogonal. Then $B$ induces a positive definite inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathfrak{a}_\mathrm{q}^*$ which is extended to a complex bilinear form on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. For $R \in \mathbb{R}$, we define

$$ \mathfrak{a}_\mathrm{q}^*(P_\emptyset, R) := \{\nu \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \mid \langle \mathrm{Re}\,\nu\,,\,\alpha\rangle < R,\ \forall\,\alpha \in \Sigma^+\}. \tag{2.1} $$

For $\lambda \in -\rho + \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*(P_\emptyset, 0)$ we define the **Eisenstein integral** $E(\psi : \lambda : \,\cdot\,)$, also denoted $E(P_\emptyset : \psi : \lambda : \,\cdot\,)$, by

$$ E(\psi : \lambda : x) = \int_K \tau(k)\psi_\lambda(k^{-1}x)\,dk, \tag{2.2} $$

for $x \in \mathrm{X}$. The following result is due to [3], Prop. 10.3.

**Proposition 2.1.** *The integral (2.2) is absolutely convergent for $\lambda \in -\rho + \mathfrak{a}_\mathrm{q}^*(P_\emptyset, 0)$ and defines a holomorphic function of $\lambda$ with values in $C^\infty(\mathrm{X} : \tau)$. Moreover, it extends to a meromorphic function of $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ with values in the space $C^\infty(\mathrm{X} : \tau)$. The singular locus of this meromorphic extension is a locally finite union of hyperplanes of the form $\lambda_0 + (\alpha^\perp)_\mathbb{C}$, with $\lambda_0 \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ and $\alpha \in \Sigma$.*

For generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ with $\langle \mathrm{Re}\,\lambda - \rho\,,\,\alpha\rangle > 0$ for all $\alpha \in \Sigma^+$ we have that

$$ \lim_{\substack{a\to\infty \\ a\in A_\mathrm{q}^+}} a^{-\lambda+\rho}E(\psi : \lambda : a) = [C(1 : \lambda)\psi](e) $$

with $C(1:\lambda) = C_{P_\emptyset|P_\emptyset}(1:\lambda) \in \mathrm{End}({}^\circ\mathcal{C})$ a meromorphic function of $\lambda$ that extends meromorphically to all of $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$; see [3], Sect. 14. Since the function $\lambda \mapsto \det C(1:\lambda)$ is not identically zero, we may define the **normalized Eisenstein integral**

$$E^\circ(\psi:\lambda:x) := E(C(1:\lambda)^{-1}\psi:\lambda:x),$$

see [3], Sect. 16, and [8], Sect. 5, for details. The definition generalizes that of Harish-Chandra, [28], p. 135, in the case of the group.

The normalized Eisenstein integral $E^\circ(\psi:\lambda)$ is a meromorphic function of $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ with values in $C^\infty(\mathrm{X}:\tau)$. Its asymptotic behavior is described by the following theorem. Given $a \in A_{\mathrm{q}}$ we write $z(a)$ for the point in $\mathbb{C}^\Delta$ with components $z(a)_\alpha = a^{-\alpha}$, for $\alpha \in \Delta$. Let $D \subset \mathbb{C}$ denote the complex unit disc. Then $z$ maps $A_{\mathrm{q}}^+$ into $D^\Delta$. If $\Omega$ is a complex analytic manifold, then by $\mathcal{O}(\Omega)$ we denote the algebra of holomorphic functions on $\Omega$. Let $V_\tau^\emptyset$ denote the space of $M_\emptyset \cap K \cap H$-fixed elements in $V_\tau$.

**Proposition 2.2.** *There exists a unique meromorphic function $\lambda \mapsto \Phi_\lambda$ with values in $\mathcal{O}(D^\Delta) \otimes \mathrm{End}(V_\tau^\emptyset)$ and, for $s \in W$, unique meromorphic $\mathrm{End}({}^\circ\mathcal{C})$-valued meromorphic functions $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \ni \lambda \mapsto C^\circ(s:\lambda)$ such that, for all $\psi \in {}^\circ\mathcal{C}$,*

$$E^\circ(\psi:\lambda:a) = \sum_{s\in W} a^{s\lambda-\rho}\Phi_{s\lambda}(z(a))\,[C^\circ(s:\lambda)\psi](e), \tag{2.3}$$

*for $a \in A_{\mathrm{q}}^+$, as a meromorphic identity in the variable $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. The meromorphic functions $\lambda \mapsto \Phi_\lambda$ and $\lambda \mapsto C^\circ(s:\lambda)$, for $s \in W$, all have a singular locus that is a locally finite union of hyperplanes of the form $\lambda_0 + (\alpha^\perp)_{\mathbb{C}}$, with $\lambda_0 \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ and $\alpha \in \Sigma$.*

For a proof of this result, we refer the reader to [7], Sect. 11, and [12], Sect. 14. From the above result it follows in particular that $\lambda \mapsto E^\circ(\psi:\lambda)$ is a meromorphic $C^\infty(\mathrm{X}:\tau)$-valued function with singularities along hyperplanes of the form $\lambda_0 + (\alpha^\perp)_{\mathbb{C}}$, with $\lambda_0 \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ and $\alpha \in \Sigma$.

We note that it follows from the definition of the normalized Eisenstein integral that

$$C^\circ(1:\lambda) = \mathrm{I}_{{}^\circ\mathcal{C}},$$

as a meromorphic identity in the variable $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. Besides in the expansion (2.3) the normalized $c$-functions $C^\circ(s:\,\cdot\,)$ also appear in the following functional equation for the Eisenstein integral

$$E^\circ(s\lambda:x)C^\circ(s:\lambda) = E^\circ(\lambda:x), \tag{2.4}$$

for every $x \in \mathrm{X}$, as an identity of meromorphic functions in the variable $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

The following result is crucial for the further development of the theory.

**Theorem 2.3. (Maass-Selberg relations)** *For each $s \in W$,*

$$C^\circ(s:-\bar{\lambda})^* C^\circ(s:\lambda) = \mathrm{I}_{{}^\circ\mathcal{C}},$$

*as a meromorphic identity in the variable $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.*

In the case of the group, the terminology Maass-Selberg relations was introduced by Harish-Chandra, because of striking analogies with the theory of automorphic forms. In the mentioned setting of the group Harish-Chandra derived

the relations for the $c$-functions associated with arbitrary parabolic subgroups, see [31]. In the present setting the above result is due to [3], Thm. 16.3, see also [4]. The result has been generalized to $c$-functions associated with arbitrary $\sigma\theta$-stable parabolic subgroups by P. Delorme [24], see also [19]. It plays a crucial role in Delorme's proof of the Plancherel formula, see [25], as well in the proof of the Plancherel formula by myself and Schlichtkrull, see [13] and [14]. Recently the last mentioned authors have been able to obtain the Maass-Selberg relations for arbitrary parabolic subgroups from those for the minimal one, see [13]. The proof in the latter paper is thus independent from the one by Delorme.

From the Maass-Selberg relations, combined with the information that the singular locus of the meromorphic $c$-functions is a locally finite union of translates of root hyperplanes, the following result is an easy consequence.

**Corollary 2.4.** *Let $s \in W$. The normalized c-function $C^\circ(s\colon \cdot\,)$ is regular on $i\mathfrak{a}_{\mathrm{q}}^*$. Moreover, for $\lambda \in i\mathfrak{a}_{\mathrm{q}}^*$, the endomorphism $C^\circ(s\colon \lambda) \in \mathrm{End}({}^\circ\mathcal{C})$ is unitary.*

From this result and an asymptotic analysis involving induction with respect to the split rank of X, i.e., $\dim \mathfrak{a}_{\mathrm{q}}$, it can be shown that the normalized Eisenstein integrals are regular for imaginary values of $\lambda$. This is the main motivation for their definition.

**Theorem 2.5. (Regularity theorem)** *Let $\psi \in {}^\circ\mathcal{C}$. The Eisenstein integral $E^\circ(\psi\colon \lambda)$ is meromorphic in $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ with a singular locus disjoint from $i\mathfrak{a}_{\mathrm{q}}^*$.*

The above result is due to [8], p. 537, Thm. 2. A different proof of the regularity theorem has been given by [5]. The latter approach was generalized to arbitrary $\sigma\theta$-stable parabolic subgroups by J. Carmona and P. Delorme, yielding the regularity theorem for Eisenstein integrals as a consequence of the Maass-Selberg relations in that setting; see [19], Thm. 3 (i).

# 3 Fourier inversion

The regularity theorem allows us to define a Fourier transform that is regular for imaginary values of the spectral parameter $\lambda$. For its definition it is convenient to define $E^\circ(\lambda\colon x) \in \mathrm{Hom}({}^\circ\mathcal{C}, V_\tau)$ by

$$E^\circ(\lambda\colon x)\psi := E^\circ(\psi\colon \lambda\colon x).$$

In addition, we define the dualized Eisenstein integral by conjugation,

$$E^*(\lambda\colon x) := E^\circ(-\bar{\lambda}\colon x)^* \in \mathrm{Hom}(V_\tau, {}^\circ\mathcal{C}), \tag{3.1}$$

for $x \in \mathrm{X}$, as a meromorphic function of $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. We now define the (most-continuous) **Fourier transform** $\mathcal{F}_\emptyset f$ of a function $f \in C_c^\infty(\mathrm{X}\colon \tau)$ to be the meromorphic function in $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*) \otimes {}^\circ\mathcal{C}$ given by

$$\mathcal{F}_\emptyset f(\lambda) := \int_{\mathrm{X}} E^*(\lambda\colon x) f(x)\, dx, \qquad (\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*). \tag{3.2}$$

It follows from (2.4) combined with the definition of $E^*(\lambda\colon x)$ and the Maass-Selberg relations that, for each $s \in W$,

$$\mathcal{F}_\emptyset f(s\lambda) = C^\circ(s\colon \lambda)\mathcal{F}_\emptyset f(\lambda). \tag{3.3}$$

It follows from the regularity theorem that the Fourier transform $\mathcal{F}_\emptyset f$ is a regular function on $i\mathfrak{a}_{\mathrm{q}}^*$. The following theorem is one of the main results of [9], see loc. cit., Thm. 15.1.

**Theorem 3.1.** *The Fourier transform $\mathcal{F}_\emptyset$ is injective on $C_c^\infty(\mathrm{X}:\tau)$.*

There exists a notion of **Schwartz space** $\mathcal{C}(\mathrm{X}:\tau)$, which is the proper generalization of Harish-Chandra's Schwartz space for the group, see [3], Sect. 17. It has the property that $\mathcal{F}_\emptyset$ extends to a continuous linear map from $\mathcal{C}(\mathrm{X}:\tau)$ into the Euclidean Schwartz space $\mathcal{S}(i\mathfrak{a}_{\mathrm{q}}^*)\otimes{}^\circ\mathcal{C}$, see [8], p. 573, Cor. 4. We emphasize that the extended Fourier transform is in general not injective on the Schwartz space. More precisely, there is a continuous linear **wave packet transform** $\mathcal{J}_\emptyset\colon \mathcal{S}(i\mathfrak{a}_{\mathrm{q}}^*)\otimes{}^\circ\mathcal{C}\to\mathcal{C}(\mathrm{X}:\tau)$, defined by the formula

$$\mathcal{J}_\emptyset\varphi(x)=\int_{i\mathfrak{a}_{\mathrm{q}}^*}E^\circ(\lambda:x)\varphi(\lambda)\,d\lambda,\qquad (x\in\mathrm{X}),\tag{3.4}$$

for $\varphi\in\mathcal{S}(i\mathfrak{a}_{\mathrm{q}}^*)\otimes{}^\circ\mathcal{C}$, see [5], Thm. 1. Here $d\lambda$ denotes Lebesgue measure on $i\mathfrak{a}_{\mathrm{q}}^*$, suitably normalized. Furthermore, in [9], Sect. 14, it is shown that there exists an invariant differential operator $D$ on X, depending on $\tau$, whose principal symbol is sufficiently generic, such that

$$D\mathcal{J}_\emptyset\mathcal{F}_\emptyset=D\tag{3.5}$$

on the Schwartz space $\mathcal{C}(\mathrm{X}:\tau)$. The idea is that $D$ annihilates the contributions of the discrete and intermediate series to the Plancherel decomposition of $L^2(\mathrm{X}:\tau)$. Accordingly, $\mathcal{J}_\emptyset\mathcal{F}_\emptyset$ corresponds to the projection onto the most continuous part of this decomposition at the $K$-type $\tau$.

By an application of **Holmgren's uniqueness theorem** the above mentioned genericity of the principal symbol of the differential operator $D$ implies that it is injective on $C_c^\infty(\mathrm{X}:\tau)$, see [6], Thm. 2. The injectivity of $\mathcal{F}_\emptyset$ asserted in Theorem 3.1 follows from the injectivity of $D$ on $C_c^\infty(\mathrm{X}:\tau)$ combined with (3.5).

In the case of the group Theorem 3.1 is a straightforward consequence of the subrepresentation theorem of [21]. For indeed, if $f$ belongs to the kernel of $\mathcal{F}_\emptyset$, then by the subrepresentation theorem, $f$ is annihilated when integrated against any $K$-finite matrix coefficient. This is not a valid argument in the general setting. A priori there might be a $K$-finite right $H$-fixed generalized matrix coefficient that cannot be produced from the Eisenstein integrals of the minimal principal series.

We shall now describe the solution to the problem of Fourier inversion mentioned in the introduction. For this we need the concept of partial Eisenstein integral. It follows from the simplifying assumption made in the beginning of Section 2 that

$$\mathrm{X}_+:=KA_{\mathrm{q}}^+H.\tag{3.6}$$

is an open dense subset of X. In the situation without the simplifying assumption the definition of $\mathrm{X}_+$ should be adapted by replacing the set on the right-hand side of (3.6) by a finite disjoint union of open sets of the form $KA_{\mathrm{q}}^+vH$, with $v$ running through a set $\mathcal{W}\subset N_K(\mathfrak{a}_{\mathrm{q}})$ of representatives for $W/W_{K\cap H}$.

In the obvious manner we define $C^\infty(\mathrm{X}_+:\tau)$ as the space of $\tau$-spherical smooth functions $\mathrm{X}_+\to V_\tau$. Via restriction, the space (1.1) may naturally be

identified with the subspace of functions in $C^\infty(\mathrm{X}_+ : \tau)$ that extend smoothly to the full space X.

For $s \in W$ and $\psi \in {}^\circ\mathcal{C}$ we define the **partial Eisenstein integral** $E_{+,s}(\lambda : \cdot)\psi$ to be the meromorphic function of $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ with values in $C^\infty(\mathrm{X}_+ : \tau)$, given by

$$E_{+,s}(\lambda : kaH)\psi = a^{s\lambda-\rho}\,\tau(k)\,\Phi_{s\lambda}(a)\,[C^\circ(s : \lambda)\psi](e), \qquad (a \in A_{\mathrm{q}}^+, k \in K),$$

for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$; here $\Phi_\lambda$ is as in Proposition 2.2. We agree to write $E_+ = E_{+,1}$. Then, clearly,

$$E_{+,s}(\lambda : x)\psi = E_+(s\lambda : x)C^\circ(s : \lambda)\psi, \qquad (\psi \in {}^\circ\mathcal{C}),$$

for $s \in W$, $x \in \mathrm{X}_+$ and generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. The following result describes the singular set of the functions involved in the formulation of the inversion theorem. We use the notation (2.1).

**Proposition 3.2.** *The functions $\lambda \mapsto E^*(\lambda : \cdot)$ and $\lambda \mapsto E_+(\lambda : \cdot)$ are meromorphic functions on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ with a singular set consisting of a locally finite union of hyperplanes of the form $\lambda_0 + (\alpha^\perp)_\mathbb{C}$, with $\lambda_0 \in \mathfrak{a}_{\mathrm{q}}^*$ (real) and with $\alpha \in \Sigma$. For every $R \in \mathbb{R}$ the set $\mathfrak{a}_{\mathrm{q}}^*(P, R)$ meets only finitely many of these hyperplanes.*

A proof of this proposition can be found in [11], Sect. 3. Let $\mathcal{H}$ be the collection of singular hyperplanes of $\lambda \mapsto E^*(\lambda : \cdot)$. Then in view of (3.2) the Fourier transform $\mathcal{F}_\emptyset f$ is meromorphic on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ with singular locus contained in $\cup\mathcal{H}$, for every $f \in C_c^\infty(\mathrm{X} : \tau)$.

The solution to the inversion problem is provided by the following theorem. A sketch of proof will be given in Section 5.

**Theorem 3.3. (Fourier inversion theorem)** *There exists a constant $R < 0$ such that the functions $\lambda \mapsto E_+(\lambda : \cdot)$ and $\lambda \mapsto E^*(\lambda : \cdot)$ are holomorphic in the region $\mathfrak{a}_{\mathrm{q}}^*(P_\emptyset, R)$. Moreover, let $\eta \in \mathfrak{a}_{\mathrm{q}}^*(P_\emptyset, R)$. Then, for every $f \in C^\infty(\mathrm{X} : \tau)$,*

$$f(x) = |W| \int_{\eta + i\mathfrak{a}_{\mathrm{q}}^*} E_+(\lambda : x)\mathcal{F}_\emptyset f(\lambda)\, d\lambda, \qquad \text{for} \quad x \in \mathrm{X}_+. \tag{3.7}$$

The integral converges absolutely, with local uniformity in $x$, since the partial Eisenstein integral grows at most of order $(1 + \|\lambda\|)^N$ along $\eta + i\mathfrak{a}_{\mathrm{q}}^*$, for some $N \in \mathbb{N}$, whereas the Fourier transform decreases faster than $C_N(1 + \|\lambda\|)^{-N}$, for any $N \in \mathbb{N}$. Moreover, by Cauchy's integral theorem, the integral on the right-hand side of (3.7) is independent of $\eta$ in the mentioned region. Details can be found in [11]. The proof of Theorem 3.3, given in the same paper, involves shifting $\eta$ to 0. If no singularities would be encountered during the shift, then in view of (3.3) the integral would become equal to $\mathcal{J}_\emptyset\mathcal{F}_\emptyset f$. However, in general singularities are encountered, due to the presence of representations from the discrete and intermediate series for X. This results in residues that can be handled by a calculus that we developed in [10]. These residues can be encoded in terms of the concept of Laurent functional, introduced in the next section. Their contribution to the Fourier inversion can be analyzed by means of the principle of induction of relations, also discussed in the next section.

# 4 Laurent functionals and induction of relations

For the formulation of the principle of induction of relations it is convenient to introduce the following concept of Laurent functional.

Let $V$ be a finite dimensional real linear space and let $X$ be a finite subset of $V^* \setminus \{0\}$. Given a point $a \in V_{\mathbb{C}}$ we define the polynomial function $\pi_a$ on $V_{\mathbb{C}}$ by

$$\pi_a := \prod_{\xi \in X} (\xi - \xi(a)).$$

We denote the ring of germs of meromorphic functions at $a$ by $\mathcal{M}(V_{\mathbb{C}}, a)$, and the subring of germs of holomorphic functions by $\mathcal{O}_a$. In addition, we define the subring

$$\mathcal{M}(V_{\mathbb{C}}, a, X) := \cup_{N \in \mathbb{N}} \ \pi_a^{-N} \mathcal{O}_a.$$

We now define an $X$-**Laurent functional** at $a \in V_{\mathbb{C}}$ to be any linear functional $\mathcal{L} \in \mathcal{M}(V_{\mathbb{C}}, a, X)^*$ such that for every $N \in \mathbb{N}$ there exists a $u_N$ in $S(V)$, the symmetric algebra of $V_{\mathbb{C}}$, such that

$$\mathcal{L} = \mathrm{ev}_a \circ u_N \circ \pi_a^N \qquad \text{on} \quad \pi_a^{-N} \mathcal{O}_a. \tag{4.1}$$

Here $S(V)$ is identified with the algebra of translation invariant holomorphic differential operators on $V_{\mathbb{C}}$ and $\mathrm{ev}_a$ denotes evaluation of a function at the point $a$. Finally, the space of all Laurent functionals on $V_{\mathbb{C}}$, relative to $X$, is defined by

$$\mathcal{M}(V_{\mathbb{C}}, X)^*_{\mathrm{laur}} := \bigoplus_{a \in V_{\mathbb{C}}} \ \mathcal{M}(V_{\mathbb{C}}, X, a)^*_{\mathrm{laur}}. \tag{4.2}$$

Given a Laurent functional $\mathcal{L}$ from the space on the left-hand side of (4.2), the finite set of $a \in V_{\mathbb{C}}$ for which the component $\mathcal{L}_a$ is non-zero, is called the support of $\mathcal{L}$ and denoted by $\operatorname{supp} \mathcal{L}$. Accordingly,

$$\mathcal{L} = \sum_{a \in \operatorname{supp} \mathcal{L}} \mathcal{L}_a.$$

Let now $\mathcal{M}(V_{\mathbb{C}}, X)$ be the space of meromorphic functions $\varphi$ on $V_{\mathbb{C}}$ with the property that the germ $\varphi_a$ belongs to $\mathcal{M}(V_{\mathbb{C}}, a, X)$, for every $a \in V_{\mathbb{C}}$. Then the natural bilinear map $(\mathcal{L}, \varphi) \mapsto \mathcal{L}\varphi$, $\mathcal{M}(V_{\mathbb{C}}, X)^*_{\mathrm{laur}} \times \mathcal{M}(V_{\mathbb{C}}, X) \to \mathbb{C}$, defined by

$$\mathcal{L}\varphi = \sum_{a \in \operatorname{supp} \mathcal{L}} \mathcal{L}_a \varphi_a, \tag{4.3}$$

induces a linear embedding of $\mathcal{M}(V_{\mathbb{C}}, X)^*_{\mathrm{laur}}$ into the dual space $\mathcal{M}(V_{\mathbb{C}}, X)^*$. More details concerning Laurent functionals can be found in [12], Sect. 10.

We end this section with the formulation of the principle of induction of relations for the partial Eisenstein integrals $E_{+,s}(\lambda : \cdot)$. In the proof of the Paley-Wiener theorem, the use of this principle replaces the use in [1] of a lifting principle due to W. Casselman, a proof of which has not appeared in the literature. Our induction principle does not seem to imply Casselman's lifting principle for the group. However, it does imply a version of the lifting principle for normalized Eisenstein integrals, see [12], Thm. 16.10.

Let $F \subset \Delta$ be a subset of simple roots, let $\Sigma_F := \Sigma \cap \mathbb{Z}F$ be the associated subsystem of $\Sigma$, and $W_F$ its Weyl group. Then $\Sigma_F$ and $W_F$ are the analogues

of $\Sigma$ and $W$ for the symmetric space $\mathrm{X}_F = M_F/M_F \cap H$. Let $W^F \subset W$ be the set of minimal length coset representatives for $W/W_F$. Then the multiplication map of $W$ induces a bijection $W^F \times W_F \to W$.

The group $W_F$ equals the centralizer of $\mathfrak{a}_{F\mathrm{q}}$ in $W$. The orthocomplement ${}^*\mathfrak{a}_{F\mathrm{q}}$ of $\mathfrak{a}_{F\mathrm{q}}$ in $\mathfrak{a}_{\mathrm{q}}$ is the analogue of $\mathfrak{a}_{\mathrm{q}}$ for the space $\mathrm{X}_F$. Let $K_F = K \cap M_F$ and $\tau_F := \tau|_{K_F}$. For $t \in W_F$ we denote by

$$E_{+,t}(\mathrm{X}_F : \mu : m) \in \mathrm{Hom}({}^\circ\mathcal{C}(\tau), V_\tau), \qquad (\mu \in {}^*\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}},\, m \in \mathrm{X}_F),$$

the analogue for the pair $\mathrm{X}_F, \tau_F$ of the partial Eisenstein integral $E_{+,t}(\mathrm{X} : \lambda : x)$. Here we note that the space ${}^\circ\mathcal{C}(\tau)$ for X coincides with the similar space ${}^\circ\mathcal{C}(\tau_F)$ for $\mathrm{X}_F$.

**Theorem 4.1. (Induction of relations)** *Let, for each $t \in W_F$, a Laurent functional $\mathcal{L}_t \in \mathcal{M}({}^*\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}} \otimes {}^\circ\mathcal{C}$ be given and assume that*

$$\sum_{t \in W_F} \mathcal{L}_t[E_{+,t}(\mathrm{X}_F : \cdot : m)] = 0, \qquad (m \in \mathrm{X}_{F+}). \tag{4.4}$$

*Then for each $s \in W^F$ the following meromorphic identity in the variable $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$ is valid,*

$$\sum_{t \in W_F} \mathcal{L}_t[E_{+,st}(\mathrm{X} : \cdot + \nu : x)] = 0, \qquad (x \in \mathrm{X}_+). \tag{4.5}$$

*Conversely, if (4.5) holds for a fixed $s \in W^F$ and all $\nu$ in a non-empty open subset of $\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$, then (4.4) holds.*

This result is proved in our paper [12], Thm. 16.1. The proof relies on a more general **vanishing theorem**, see [12], Thm. 12.10. This vanishing theorem asserts that a suitably restricted meromorphic family $\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}} \ni \nu \mapsto f_\nu \in C^\infty(\mathrm{X}_+ : \tau)$ of eigenfunctions for $\mathbb{D}(\mathrm{X})$ is completely determined by the coefficient of $a^{\nu-\rho_F}$ in its asymptotic expansion towards infinity along $A^+_{F\mathrm{q}}$, the positive chamber determined by $P_F$. In particular, if the mentioned coefficient is zero, then $f_\nu = 0$ for all $\nu$; whence the name vanishing theorem. Part of the mentioned restriction on families in the vanishing theorem is a so called **asymptotic globality condition**. It requires that certain asymptotic coefficients in the expansions of $f_\nu$ along certain codimension one walls should have smooth behavior as functions in the variables transversal to these walls. The precise condition is given in [12], Def. 9.5.

Let $f^s_\nu$, for $s \in W^F$, denote the expression on the left-hand side of (4.5). Then the sum $f_\nu = \sum_{s \in W^F} f^s_\nu$ defines a family for which the vanishing theorem holds; the summation over $W^F$ is needed for the family to satisfy the asymptotic globality condition. The expression on the left-hand side of (4.4) is the coefficient of $a^{\nu-\rho_F}$ of the asymptotic expansion of $f_\nu$ along $A^+_{F\mathrm{q}}$. Its vanishing implies that $f = 0$. From the fact that the sets of the asymptotic exponents of $f^s_\nu$ along $A^+_{F\mathrm{q}}$ are mutually disjoint for distinct $s \in W^F$ and generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$, it follows that each individual function $f^s_\nu$ vanishes. This implies the validity of (4.5).

For the proof of the converse statement it is first shown that the vanishing of an individual term $f^s_\nu$ implies that of $f_\nu$. Here the condition of asymptotic globality once more plays an essential role. The validity of (4.4) then follows by taking the coefficient of $a^{\nu-\rho_F}$ in the asymptotic expansion along $A^+_{F\mathrm{q}}$.

# 5 Induction of relations and the inversion formula

In this section we shall discuss the role of induction of relations, as formulated in Theorem 4.1, in the proof of the inversion formula. Details can be found in [11].

**Sketch of proof of Theorem 3.3** Let us denote the integral on the right-hand side of (3.7) by $\mathcal{T}_\eta(\mathcal{F}_\emptyset f)(x)$. The main difficulty in the proof is to show that the function $\mathcal{T}_\eta\mathcal{F}_\emptyset f \in C^\infty(\mathrm{X}_+ : \tau)$ extends smoothly from $\mathrm{X}_+$ to X. By applying a Paley-Wiener shift argument, with $\eta \to \infty$ in $-\mathfrak{a}_{\mathrm{q}}^{*+}$, it then follows that $\mathcal{T}_\eta\mathcal{F}_\emptyset f \in C_c^\infty(\mathrm{X} : \tau)$. There exists a differential operator $D$ as in (3.5), such that $D\mathcal{T}_\eta\mathcal{F}_\emptyset f$ is free of singularities during a shift of the integral with $\eta$ moving to 0. In view of Cauchy's theorem this leads to $D\mathcal{T}_\eta\mathcal{F}_\emptyset f = D\mathcal{T}_0\mathcal{F}_\emptyset f = D\mathcal{J}_\emptyset\mathcal{F}_\emptyset f = Df$. From the injectivity of $D$ on $C_c^\infty(\mathrm{X} : \tau)$ we then obtain (3.7).

The most difficult part of the proof concerns the smooth extension of $\mathcal{T}_\eta\mathcal{F}_\emptyset f$. This involves a shift of integration applied to $\mathcal{T}_\eta(\mathcal{F}_\emptyset f)(x)$ with $\eta$ moving to 0. According to the residue calculus developed in [10], the process of picking up residues is governed by any choice of a so-called **residue weight** on $\Sigma$. We fix such a weight, which is by definition a map $t\colon \mathcal{P}_\sigma \to [0,1]$ with the property that, for every $Q \in \mathcal{P}_\sigma$,

$$\sum_{\substack{P\in\mathcal{P}_\sigma \\ \mathfrak{a}_{P\mathrm{q}}=\mathfrak{a}_{Q\mathrm{q}}}} t(P) = 1.$$

Moreover, we choose $t$ to be $W$-invariant and even. The latter condition means that $t(P) = t(\bar{P})$ for all $P \in \mathcal{P}_\sigma$. The encountered residues can be encoded by means of a finite set of Laurent functionals $\mathcal{R}_F^t \in \mathcal{M}({}^*\mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*, \Sigma_F)_{\mathrm{laur}}^*$, for $F \subset \Delta$, depending only on the root system $\Sigma$, the choice of the residue weight $t$ and the locally finite union of hyperplanes which forms the union of the singular sets of $\lambda \mapsto E_+(\lambda : \cdot)$ and $\lambda \mapsto E^*(\lambda : \cdot)$.

The shift results in the formula

$$\begin{aligned} &\mathcal{T}_\eta(\mathcal{F}_\emptyset f)(x) = \qquad (5.1)\\ &= |W| \sum_{F\subset\Delta} t(P_F) \int_{\varepsilon_F + i\mathfrak{a}_{F\mathrm{q}}^*} \mathcal{R}_F^t \left( \sum_{s\in W^F} E_{+,s}(\nu + \cdot : x)\mathcal{F}_\emptyset f(\nu + \cdot) \right) d\mu_F(\nu). \end{aligned}$$

where $\varepsilon_F$ is any choice of elements sufficiently close to zero in $\mathfrak{a}_{F\mathrm{q}}^{*+}$, the positive chamber associated with $P_F$. Moreover, $d\mu_F$ is the translate by $\varepsilon_F$ of suitably normalized Lebesgue measure on $i\mathfrak{a}_{F\mathrm{q}}^*$.

From the fact that the singular set of the integrand is real in the sense of Proposition 3.2, it follows that the Laurent functionals $\mathcal{R}_F^t$ are real in the following sense. Their support is a set of real points $a \in {}^*\mathfrak{a}_{F\mathrm{q}}^*$ and at each such point the functional is defined by a string $\{u_N\} \subset S({}^*\mathfrak{a}_{F\mathrm{q}}^*)$ as in (4.1) with $u_N$ real for all $N$.

We now define the **kernel functions**

$$K_F(\nu : x : y) := \mathcal{R}_F^t \left( \sum_{s\in W^F} E_{+,s}(\nu + \cdot : x)E^*(\nu + \cdot : y) \right). \qquad (5.2)$$

Then by using the definition (3.2) of $\mathcal{F}_\emptyset$, we may rewrite the equation (5.1) as

$$\mathcal{T}_\eta(\mathcal{F}_\emptyset f)(x) = |W| \sum_{F\subset\Delta} t(P_F) \int_{\varepsilon_F + i\mathfrak{a}^*_{F\mathrm{q}}} \left[\int_X K^t_F(\nu : x : y) f(y)\, dy\right] d\mu_F(\nu). \tag{5.3}$$

For fixed generic $\nu$, the kernel functions $K^t_F(\nu : \cdot : \cdot) \in C^\infty(\mathrm{X}_+ \times \mathrm{X} : \tau \otimes \tau^*)$ are spherical and $\mathbb{D}(\mathrm{X})$-finite in both variables. It follows that they belong to a tensor product of the form ${}^1E_\nu \otimes {}^2E_\nu$, with ${}^1E_\nu$ and ${}^2E_\nu$ finite dimensional subspaces of $C^\infty(\mathrm{X}_+ : \tau)$ and $C^\infty(\mathrm{X}_+ : \tau^*)$, respectively. Let ${}^jE'_\nu$ be the subspace of functions in ${}^jE_\nu$ extending smoothly to X, for $j = 1, 2$. Then by the symmetry formulated in Proposition 5.1 below it follows that the kernel $K^t_F(\nu : \cdot : \cdot)$ belongs to ${}^1E_\nu \otimes {}^2E'_\nu \cap {}^1E'_\nu \otimes {}^2E_\nu = {}^1E'_\nu \otimes {}^2E'_\nu$. This shows that the kernel functions extend smoothly to $\mathrm{X} \times \mathrm{X}$ and finishes the proof. □

**Proposition 5.1.** *Let $x, y \in \mathrm{X}_+$. Then*

$$K^t_F(\nu : x : y) = K^t_F(-\bar{\nu} : y : x)^* \tag{5.4}$$

*as a meromorphic identity in the variable $\nu \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$.*

Before giving a sketch of the proof we observe that, due to the fact that $\mathcal{R}^t_F$ is scalar and real in the sense mentioned in the proof of Theorem 3.3 above, the adjoint of the kernel is given by

$$K^t_F(-\bar{\nu} : y : x)^* = \mathcal{R}^t_F \left( \sum_{s\in W^F} E^\circ(\nu - \cdot : x) E^*_{+,s}(\nu - \cdot : y) \right), \tag{5.5}$$

where the dual partial Eisenstein integrals are defined by

$$E^*_{+,s}(\lambda : x) := E_{+,s}(-\bar{\lambda} : x)^*.$$

**Sketch of proof of Proposition 5.1** The final part of the proof of Theorem 3.3 can be modified in such a way that (5.4) is only needed for $F \subset \Delta$ with $F \neq \Delta$. The validity of (5.4) for $F = \Delta$ is derived in the course of the modified argument. For details, we refer the reader to [11], Sect. 9.

Thus, we may restrict ourselves to proving (5.4) for $F \subsetneq \Delta$. This in turn is achieved by using induction of relations in order to reduce to the lower dimensional space $\mathrm{X}_F$. More precisely, the residue weight $t$ naturally induces a residue weight ${}^*t$ on $\Sigma_F$, the analogue of $\Sigma$ for $\mathrm{X}_F$. The set $F$ is a simple system for $\Sigma_F$. Let $K^{{}^*t}_F(\mathrm{X}_F : \cdot : \cdot)$ be the analogue of $K^t_\Delta$ for the space $\mathrm{X}_F$. Then by induction, $K^{{}^*t}_F(\mathrm{X}_F : \cdot : \cdot)$ is a smooth function on $\mathrm{X}_F \times \mathrm{X}_F$ and satisfies the symmetry condition

$$K^{{}^*t}_F(\mathrm{X}_F : m : m') = K^{{}^*t}_F(\mathrm{X}_F : m' : m)^*, \tag{5.6}$$

for $m, m' \in \mathrm{X}_F$. Here we have suppressed the analogue of the parameter $\nu$, which is zero dimensional in the present setting.

The residue calculus behaves well with respect to induction. In particular, let $\mathcal{R}^{{}^*t}_F \in \mathcal{M}({}^*\mathfrak{a}^*_{F\mathrm{q}}, \Sigma_F)^*_{\mathrm{laur}}$ be the analogue of $\mathcal{R}^t_\Delta$ for the data $\mathrm{X}_F, \Sigma_F, F, {}^*t$. Then $\mathcal{R}^{{}^*t}_F = \mathcal{R}^t_F$; for obvious reasons, we have called this result **transivity of**

**residues**, see [10], Sect. 3.6. Using (5.2) and (5.5) for $K_F^{*t}(\mathrm{X}_F)$, taking into account that $(W_F)^F = \{1\}$, we thus see that (5.6) is equivalent to

$$\begin{aligned} &\mathcal{R}_F^t\left(E_+(\mathrm{X}_F : \cdot : m)E^*(\mathrm{X}_F : \cdot : m')\right) \\ &\qquad = \mathcal{R}_F^t\left(E^\circ(\mathrm{X}_F : -\cdot : m)E_+^*(\mathrm{X}_F : -\cdot : m')\right), \end{aligned} \tag{5.7}$$

where $E_+^* := E_{+,1}^*$. In view of (5.2) and (5.5), the relation (5.4) can now be derived from (5.7), by applying induction of relations, first with respect to the variable $x$ and then a second time with respect to the variable $y$. For details we refer the reader to [11], Sect. 8. □

# 6 Arthur-Campoli relations

In this section we describe the so called Arthur-Campoli relations, needed for the formulation of the Paley-Wiener theorem in the next section. We start with the definition of an Arthur-Campoli functional.

**Definition 6.1.** *An* **Arthur-Campoli functional** *for* $\mathrm{X}, \tau$ *is a Laurent functional* $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^* \otimes {}^\circ\mathcal{C}(\tau)$ *with the property that*

$$\mathcal{L}E^*(\,\cdot : x) = 0 \quad \textit{for all} \quad x \in \mathrm{X}.$$

*The linear space of such functionals is denoted by* $\mathrm{AC}\,(\mathrm{X} : \tau)$.

From the principle of induction of relations as formulated in Theorem 4.1, the following result follows in a straightforward manner. See [15] for details.

**Lemma 6.2. (Induction of AC relations)** *Let* $F \subset \Delta$ *and* $\mathcal{L} \in \mathrm{AC}\,(\mathrm{X}_F : \tau_F)$. *Then for generic* $\nu \in \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$, *the Laurent functional*

$$\mathcal{L}_\nu : \varphi \mapsto \mathcal{L}[\varphi(\nu + \,\cdot\,)]$$

*belongs to* $\mathrm{AC}\,(\mathrm{X} : \tau)$.

In this result, 'generic' can be made more precise as follows. There exists a locally finite collection $\mathcal{H}_S$ of hyperplanes in $\mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$, specified explicitly in terms of the support $S$ of $\mathcal{L}$, such that the statement is valid for $\nu \in \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^* \setminus \cup\mathcal{H}_S$.

# 7 The Paley-Wiener theorem

In this section we shall formulate the Paley-Wiener theorem, and indicate how induction of relations enters its proof. Our first objective is to define a space of Paley-Wiener functions. The first step is to define a suitable space of meromorphic functions that takes the singularities of the Fourier transform into account.

Let $\mathcal{H} = \mathcal{H}(X, \tau)$ be the smallest collection of hyperplanes of the form $\lambda_0 + (\alpha^\perp)_\mathbb{C}$, with $\lambda_0 \in \mathfrak{a}_\mathrm{q}^*$ and $\alpha \in \Sigma$, such that the $C^\infty(\mathrm{X}) \otimes \mathrm{Hom}(V_\tau, {}^\circ\mathcal{C}(\tau))$-valued meromorphic function $\lambda \mapsto E^*(\lambda : \,\cdot\,)$ is regular on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \setminus \cup\mathcal{H}$. By the requirement of minimality, the collection $\mathcal{H}$ has the properties of Proposition 3.2.

If $H \in \mathcal{H}$ we select $\alpha_H \in \Sigma$ and $s_H \in \mathbb{R}$ such that $H$ is given by the equation $\langle \lambda\,, \alpha_H \rangle = s_H$. Let $d(H)$ denote the order of the singularity of $\lambda \mapsto E^*(\lambda)$ along $H$. Thus, $d(H)$ is the smallest natural number for which $\lambda \mapsto$

$(\langle\lambda\,,\,\alpha_H\rangle - s_H)^{d(H)}E^*(\lambda)$ is regular at the points of $H$ that are not contained in any hyperplane from $\mathcal{H}\setminus\{H\}$.

If $\omega\subset\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ is a bounded subset, then in view of the mentioned properties of $\mathcal{H}$ we may define a polynomial function $\pi_\omega\colon\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*\to\mathbb{C}$ by

$$\pi_\omega(\lambda)=\prod_{H\in\mathcal{H},H\cap\omega\neq\emptyset}(\langle\lambda\,,\,\alpha_H\rangle - s_H)^{d(H)}.$$

We define $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*,\mathcal{H},d)$ to be the space of meromorphic functions $\varphi\colon\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*\to\mathbb{C}$ such that, for every bounded open set $\omega\subset\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, the function $\pi_\omega\varphi$ is regular on $\omega$. Taking into account that the $\alpha_H$ and $s_H$ are real for $H\in\mathcal{H}$, we readily see that for each function $\varphi\in\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*,\mathcal{H},d)$ and every bounded open subset $\omega\subset\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, the function $\pi_\omega\varphi$ is in fact regular on $\omega+i\mathfrak{a}_{\mathrm{q}}^*$.

In view of the definitions just given, the function $\lambda\mapsto E^*(\lambda\colon x)$ belongs to the space $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*,\mathcal{H},d)\otimes\mathrm{Hom}(V_\tau,{}^\circ\mathcal{C}(\tau))$, for every $x\in\mathrm{X}$. Moreover, $\mathcal{F}_\emptyset$ maps $C_c^\infty(\mathrm{X}\colon\tau)$ into $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*,\mathcal{H},d)\otimes{}^\circ\mathcal{C}(\tau)$.

It follows from Proposition 3.2 that the set $\mathcal{H}_0$ of $H\in\mathcal{H}$ having empty intersection with $\mathrm{cl}\,\mathfrak{a}_{\mathrm{q}}^*(P_\emptyset,0)$ is finite. We define the polynomial function $\pi\colon\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*\to\mathbb{C}$ by

$$\pi(\lambda)=\prod_{H\in\mathcal{H}_0}(\langle\lambda\,,\,\alpha_H\rangle - s_H)^{d(H)}.$$

Then there exists a constant $\varepsilon>0$ such that $\lambda\mapsto\pi(\lambda)E^*(\lambda)$ is regular on $\mathfrak{a}_{\mathrm{q}}^*(P_\emptyset,\varepsilon)$. It follows that for every $f\in C_c^\infty(\mathrm{X}\colon\tau)$ the ${}^\circ\mathcal{C}(\tau)$-valued meromorphic function $\lambda\mapsto\pi(\lambda)\mathcal{F}_\emptyset f(\lambda)$ is regular on $\mathfrak{a}_{\mathrm{q}}^*(P_\emptyset,\varepsilon)$.

We now define $\mathcal{P}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*,\mathcal{H},d)$ as the subspace of $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*,\mathcal{H},d)$ consisting of functions $\varphi$ which satisfy the following condition of decay in the imaginary directions

$$\sup_{\lambda\in\omega+i\mathfrak{a}_{\mathrm{q}}^*}(1+|\lambda|)^n\,|\pi_\omega(\lambda)\varphi(\lambda)|<\infty,$$

for every compact set $\omega\subset\mathfrak{a}_{\mathrm{q}}^*$ and all $n\in\mathbb{N}$. Equipped with the suggested seminorms, the space $\mathcal{P}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*,\mathcal{H},d)$ is a Fréchet space. Moreover, via (4.3) the space of Laurent functionals $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*,\Sigma)_{\mathrm{laur}}^*$ naturally embeds into the continuous linear dual of $\mathcal{P}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*,\mathcal{H},d)$. It follows that the following subspace of $\mathcal{P}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*,\mathcal{H},d)\otimes{}^\circ\mathcal{C}(\tau)$ is closed, hence Fréchet,

$$\mathcal{P}_{\mathrm{AC}}(\mathrm{X}\colon\tau):=\{\varphi\in\mathcal{P}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*,\mathcal{H},d)\otimes{}^\circ\mathcal{C}(\tau)\mid\mathcal{L}\varphi=0,\ \ \forall\,\mathcal{L}\in\mathrm{AC}(\mathrm{X}\colon\tau)\}.$$

Finally, we define the Paley-Wiener space by incorporating a condition of exponential growth along a closed cone.

**Definition 7.1.** *The* **Paley-Wiener space** $\mathrm{PW}(\mathrm{X}\colon\tau)$ *is defined to be the space of functions* $\varphi\in\mathcal{P}_{\mathrm{AC}}(\mathrm{X}\colon\tau)$ *for which there exists a constant* $M>0$ *such that, for all* $n\in\mathbb{N}$,

$$\sup_{\lambda\in\,\mathrm{cl}\,\mathfrak{a}_{\mathrm{q}}^*(P_\emptyset,0)}(1+|\lambda|)^n\,e^{-M|\mathrm{Re}\,\lambda|}\,\|\pi(\lambda)\varphi(\lambda)\|<\infty.$$

*The subspace of functions satisfying this estimate with a fixed* $M>0$ *and all* $n\in\mathbb{N}$ *is denoted by* $\mathrm{PW}_M(\mathrm{X}\colon\tau)$.

By using Euclidean Fourier analysis, it can be shown that $\mathrm{PW}_M(\mathrm{X}:\tau)$ is a closed subspace of $\mathcal{P}_{\mathrm{AC}}(\mathrm{X}:\tau)$, for each $M>0$, hence a Fréchet space for the restriction topology. For details we refer the reader to [15]. Accordingly, for $M<M'$ we have a continuous linear embedding of $\mathrm{PW}_M(\mathrm{X}:\tau)$ onto a closed subspace of $\mathrm{PW}_{M'}(\mathrm{X}:\tau)$. The space $\mathrm{PW}(\mathrm{X}:\tau)$, being the union of the spaces $\mathrm{PW}_M(\mathrm{X}:\tau)$, is equipped with the associated direct limit topology. Thus, it becomes a strict LF-space.

For $M>0$ we denote by $B_M$ the closed ball in $\mathfrak{a}_{\mathrm{q}}$ of center 0 and radius $M$. Moreover, we denote by $C_M^\infty(\mathrm{X}:\tau)$ the space of functions in $C^\infty(\mathrm{X}:\tau)$ with compact support contained in $K\exp B_M H$.

**Theorem 7.2. (Paley-Wiener theorem)** *The Fourier transform $\mathcal{F}_\emptyset$ is a topological linear isomorphism from $C_c^\infty(\mathrm{X}:\tau)$ onto $\mathrm{PW}(\mathrm{X}:\tau)$. More precisely, for each $M>0$ it maps $C_M^\infty(\mathrm{X}:\tau)$ homeomorphically onto $\mathrm{PW}_M(\mathrm{X}:\tau)$.*

In the Riemannian case $H=K$ and $\tau=1$, this result is equivalent to the Paley-Wiener theorem of S. Helgason and R. Gangolli, see [32], Thm. IV, 7.1. In the case of the group our Paley-Wiener theorem can be shown to be equivalent to the one of J. Arthur, [1], which in turn generalizes the result of O.A. Campoli, [16], for groups of split rank one. Arthur's proof relies on Harish-Chandra's Plancherel theorem and the lifting principle mentioned in Section 5, due to W. Casselman. It also makes use of ideas from the residue calculus appearing in the work of R.P. Langlands, [33]. In [22], P. Delorme used a different method to obtain a Paley-Wiener theorem for semisimple groups with one conjugacy class of Cartan subgroups, with explicit symmetry conditions instead of the Arthur-Campoli relations. This work in turn generalized work of Zhelobenko, [38], for the complex groups.

We conjectured the present Paley-Wiener theorem in slightly different but equivalent form in [9], where we proved it under the assumption that $\dim\mathfrak{a}_{\mathrm{q}}=1$. The proof of Theorem 7.2 is given in the paper [15]. It relies on the inversion theorem, Theorem 3.3, and on the principle of induction of relations, see Theorem 4.1. In particular, our proof is independent of the theory of the discrete series and the existing proofs of the Plancherel theorem (in [25], [13] and [14]). The precise relation with the Plancherel decomposition will be described in Section 8.

In the following sketch we will indicate the main ideas of our proof of the Paley-Wiener theorem.

**Sketch of proof of Theorem 7.2** As usual, the proof that $\mathcal{F}_\emptyset$ maps $C_M^\infty(\mathrm{X}:\tau)$ continuously into $\mathrm{PW}_M(\mathrm{X}:\tau)$ is rather straightforward. For details, see [9]. The injectivity of $\mathcal{F}_\emptyset$ was already asserted in Theorem 3.1. By the open mapping theorem for Fréchet spaces, it remains to establish the surjectivity of $\mathcal{F}_\emptyset$. Let $\varphi\in\mathrm{PW}_M(\mathrm{X}:\tau)$. In view of the inversion theorem the only possible candidate for a function $f\in C_M^\infty(\mathrm{X}:\tau)$ with Fourier transform equal to $\varphi$ is given by the formula

$$f(x)=|W|\int_{\eta+i\mathfrak{a}_{\mathrm{q}}^*}E_+(\lambda:x)\varphi(\lambda)\,d\lambda,$$

for $x\in\mathrm{X}_+$ and for $\eta\in\mathfrak{a}_{\mathrm{q}}^*$ sufficiently $\bar{P}_\emptyset$-dominant. The problem with this formula is that it only defines a smooth function $f$ on the open dense subset

$\mathrm{X}_+$ of X. By a standard shift argument of Paley-Wiener type, with $\eta$ moving to infinity in $-\mathfrak{a}_{\mathrm{q}}^{*+}$, it follows that the support of $f$ is contained in $K \exp B_M H$. Therefore, it suffices to show that the function $f$ has a smooth extension to all of $\mathrm{X}_+$. This is the central theme of the proof.

We will actually show that $f$ has a smooth extension under the weaker assumption that $\varphi \in \mathcal{P}_{\mathrm{AC}}(\mathrm{X}:\tau)$. As in the proof of Theorem 3.3 the idea is to write the integral differently by application of a contour shift, with $\eta$ moving to 0, and by organizing the residual integrals according to the calculus described in the mentioned proof. This leads to the formula

$$f(x) = \sum_{F \subset \Delta} \mathcal{T}_F^t \varphi(x), \qquad (x \in \mathrm{X}_+), \tag{7.1}$$

with

$$\mathcal{T}_F^t \varphi(x) := |W|\, t(P_F) \int_{\varepsilon_F + i\mathfrak{a}_{F\mathrm{q}}^*} \mathcal{R}_F^t \left( \sum_{s \in W^F} E_{+,s}(\nu + \cdot : x) \varphi(\nu + \cdot) \right) d\mu_F(\nu). \tag{7.2}$$

The problem now is to show that each of the individual terms $\mathcal{T}_F \varphi$ extends smoothly to all of X. This is done by writing $\mathcal{T}_F \varphi$ as a superposition of certain generalized Eisenstein integrals.

These were defined in [11] by using the symmetry property of the kernels $K_F^t$, as formulated in Proposition 5.1. As in the proof of Theorem 3.3 let $K_F^{*t}(\mathrm{X}_F) \in C^\infty(\mathrm{X}_F \times \mathrm{X}_F) \otimes \mathrm{End}(V_\tau)$ be the analogue for $\mathrm{X}_F$ and $\tau_F$ of the kernel $K_\Delta^t$ for X and $\tau$. We recall that $K_F^{*t}(\mathrm{X}_F)$ does not depend on a spectral parameter, since the analogue of $\mathfrak{a}_{\Delta\mathrm{q}}$ for $\mathrm{X}_F$ is the zero space. We define the following subspace of $C^\infty(\mathrm{X}_F : \tau_F)$,

$$\mathcal{A}_F = \mathcal{A}^{*t}(\mathrm{X}_F : \tau_F) := \mathrm{span}\,\{K_F^{*t}(\mathrm{X}_F : \cdot : m')u \mid m' \in \mathrm{X}_{F+}, u \in V_\tau\}.$$

Being annihilated by a cofinite ideal of $\mathbb{D}(\mathrm{X}_F)$, this space is finite dimensional. It can be shown that $\mathcal{A}_F$ is the discrete series subspace $L^2_d(\mathrm{X}_F : \tau_F)$ of $L^2(\mathrm{X}_F : \tau_F)$, see [13], Lemma 12.6 and Thm. 21.2, but this fact is not needed for the proof of the Paley-Wiener theorem.

For $\psi \in \mathcal{A}_F$ we define the **generalized Eisenstein integral** $E_F^\circ(\psi : \nu)$ as a meromorphic $C^\infty(\mathrm{X} : \tau)$-valued function of $\nu \in \mathfrak{a}_{F\mathrm{qc}}^*$, as follows. If

$$\psi = \sum_i K_F^{*t}(\mathrm{X}_F : \cdot : m_i')u_i, \tag{7.3}$$

with $m_i' \in \mathrm{X}_{F+}$ and $u_i \in V_\tau$, then

$$E_F^\circ(\psi : \nu : x) := \sum_i \mathcal{R}_F^t[E^\circ(\nu - \cdot : x) E_+^*(\mathrm{X}_F : -\cdot : m_i')u_i]. \tag{7.4}$$

It follows by induction of relations, Theorem 4.1, that the expression (7.4) is independent of the particular representation of $\psi \in \mathcal{A}_F$ given in (7.3). It also follows by induction of relations, combined with the symmetry of the kernel $K_F^{*t}$, that for $\psi \in \mathcal{A}_F$ given by (7.3),

$$E_F^\circ(\psi : \nu : x) = \sum_i \mathcal{R}_F^t[\sum_{s \in W^F} E_{+,s}(\nu + \cdot : x) E^*(\mathrm{X}_F : \cdot : m_i')u_i], \tag{7.5}$$

for generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$ and all $x \in \mathrm{X}_+$. Let

$$T_F(\mathrm{X}_F : \cdot\,) = T_F^{*t}(\mathrm{X}_F : \cdot\,) : C_c^\infty(\mathrm{X}_F : \tau_F) \to \mathcal{A}_F$$

be the analogue for $\mathrm{X}_F$ of the operator $T_\Delta^t$ occurring in (7.1). Then it follows from (7.4) and (7.5), essentially by integration with respect to the variable $m'$ that, for all $f \in C_c^\infty(\mathrm{X}_F : \tau_F)$,

$$|W_F|^{-1}\, E_F^\circ(T_F(\mathrm{X}_F : f) : \nu : x) = \mathcal{R}_F^t[\sum_{s\in W^F} E_{+,s}(\nu + \cdot : x)\mathcal{F}_\emptyset(\mathrm{X}_F : f)(\cdot\,)]. \tag{7.6}$$

Here $\mathcal{F}_\emptyset(\mathrm{X}_F : \cdot\,)$ denotes the analogue of $\mathcal{F}_\emptyset$ for $\mathrm{X}_F$.

The next step in the proof of the Paley-Wiener theorem consists of the following result, which follows from the Arthur-Campoli relations and their inductive property described in Lemma 6.2, essentially by application of linear algebra.

**Proposition 7.3.** *Let $F \subset \Delta$. There exists a finite dimensional complex linear subspace $V \subset C_c^\infty(\mathrm{X}_F : \tau_F)$ and a Laurent functional $\mathcal{L}' \in \mathcal{M}({}^*\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}} \otimes \mathrm{Hom}({}^\circ\mathcal{C}(\tau), V)$ such that, for generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$, the map $\varphi \mapsto f_{\nu,\varphi}$, $\mathcal{P}_{\mathrm{AC}}(\mathrm{X} : \tau) \to V$, defined by*

$$f_{\nu,\varphi} = \mathcal{L}'[\varphi(\nu + \cdot\,)],$$

*has the following property, for all $x \in \mathrm{X}_+$,*

$$\mathcal{R}_F^t\left[\sum_{s\in W^F} E_{+,s}(\nu + \cdot : x)\varphi(\nu + \cdot\,)\right] = \mathcal{R}_F^t\left[\sum_{s\in W^F} E_{+,s}(\nu + \cdot : x)\mathcal{F}_\emptyset(\mathrm{X}_F : f_{\nu,\varphi})(\cdot\,)\right].$$

The final step in the proof is the following result, which follows by combining Proposition 7.3 with (7.6).

**Proposition 7.4.** *There exists a $\mathcal{L}_F \in \mathcal{M}({}^*\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}} \otimes \mathrm{Hom}({}^\circ\mathcal{C}(\tau), \mathcal{A}_F)$ such that*

$$\mathcal{R}_F^t\left[\sum_{s\in W^F} E_{+,s}(\nu + \cdot : x)\varphi(\nu + \cdot\,)\right] = E_F^\circ(\mathcal{L}_F[\varphi(\nu + \cdot\,)] : \nu : x),$$

*for all $\varphi \in \mathcal{P}_{\mathrm{AC}}(\mathrm{X} : \tau)$, $x \in \mathrm{X}_+$ and generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$.*

It follows from combining this proposition with (7.2) that, for $\varphi \in \mathcal{P}_{\mathrm{AC}}(\mathrm{X} : \tau)$,

$$\mathcal{T}_F\varphi(x) = |W|\, t(P_F) \int_{\varepsilon_F + i\mathfrak{a}^*_{F\mathrm{q}}} E_F^\circ(\mathcal{L}_F[\varphi(\nu + \cdot\,)] : \nu : x)\, d\mu_F(\nu), \tag{7.7}$$

for all $x \in \mathrm{X}_+$. From this expression it is readily seen that $\mathcal{T}_F$ extends to a continuous linear map $\mathcal{P}_{\mathrm{AC}}(\mathrm{X} : \tau) \to C^\infty(\mathrm{X} : \tau)$. □

# 8 Relation with the Plancherel decomposition

In this section we briefly discuss the relation between the Paley-Wiener theorem and the Plancherel theorem, obtained by P. Delorme [25] and, independently, by H. Schlichtkrull and myself in [13] and [14]. Earlier, a Plancherel theorem had been announced by T. Oshima, [34], p. 32, but the details have not appeared. For the case of the group, the Plancherel theorem is due to Harish-Chandra, [29], [30], [31]. For the case of a complex reductive group modulo a real form, the Plancherel theorem has been obtained by P. Harinck, [27].

The starting point of our proof of the Plancherel theorem is the Fourier inversion formula

$$f(x) = |W| \sum_{F\subset\Delta} t(P_F) \int_{\varepsilon_F + i\mathfrak{a}^*_{F\mathrm{q}}} \int_{\mathrm{X}} K^t_F(\nu : x : y) f(y)\, dy\, d\mu_F(\nu), \qquad (x \in \mathrm{X}), \tag{8.1}$$

which follows from Theorem 3.3 and (5.3). The crucial part of the proof of the Plancherel theorem consists of showing that this formula, which is valid for $\varepsilon_F$ sufficiently close to zero in $\mathfrak{a}^{*+}_{F\mathrm{q}}$, remains valid with $\varepsilon_F = 0$ for all $F \subset \Delta$. This in turn is achieved by showing that the kernel functions $K^t_F$ are regular for $\nu \in i\mathfrak{a}^*_{F\mathrm{q}}$.

The regularity is achieved in a long inductive argument in [13]. It is in this argument that we need the theory of the **discrete series** for X initiated by M. Flensted-Jensen [26] and further developed in the fundamental paper [35] by T. Oshima and T. Matsuki. Of the latter paper two results on the discrete series are indispensable. The crucial results needed are the necessity and sufficiency of the rank condition for the discrete series to be non-empty as well as the fact that representations from the discrete series have real and regular $\mathbb{D}(\mathrm{X})$-characters; see [13] for details.

In the course of the inductive argument, it is is shown that $K^t_F$ is independent of the choice of the residue weight $t$; moreover, $\mathcal{A}_F = L^2_d(\mathrm{X}_F : \tau_F)$ and the generalized Eisenstein integral $E^\circ_F$ is independent of $t$ as well. It is then shown that

$$K_F(\nu : x : y) = |W_F|^{-1} E^\circ_F(\nu : x) E^*_F(\nu : y), \tag{8.2}$$

with $E^*_F(\nu : y) := E^\circ_F(-\bar{\nu} : y)^*$. At this point we note that if we define the Fourier transform $\mathcal{F}_F : C^\infty_c(\mathrm{X} : \tau_F) \to \mathcal{M}(\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}) \otimes \mathcal{A}_F$ as $\mathcal{F}_\emptyset$ in (3.2) with $E^*_F$ in place of $E^*$, then (8.1) becomes

$$f(x) = \sum_{F\subset\Delta} [W : W_F]\, t(P_F) \int_{\varepsilon_F + i\mathfrak{a}^*_{F\mathrm{q}}} E^\circ_F(\mathcal{F}_F f(\nu) : \nu : x)\, d\mu_F(\nu). \tag{8.3}$$

The relation of this formula with (7.1) and (7.7) for $\varphi = \mathcal{F}_\emptyset f$ is given by

$$E^\circ_F(\mathcal{F}_F f(\nu) : \nu : x) = |W_F|^{-1}\, E^\circ_F(\mathcal{L}_F[\mathcal{F}_\emptyset f](\nu + \cdot\,) : \nu : x),$$

for every $x \in \mathrm{X}$, as an identity of meromorphic functions in the variable $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. From taking coefficients of $a^{\nu-\rho_F}$ in the asymptotic expansions of both members along $M_F A^+_{F\mathrm{q}}$ it follows that

$$\mathcal{F}_F f(\nu) = |W_F|^{-1}\, \mathcal{L}_F[\mathcal{F}_\emptyset f](\nu + \cdot\,),$$

for every $f \in C_c^\infty(\mathrm{X}:\tau)$, as an identity of meromorphic functions in the variable $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. This in turn leads to the meromorphic identity $E_F^*(\nu : x) = |W_F|^{-1}\,\mathcal{L}_F[E^*(\nu + \cdot : x)]$, for all $x \in \mathrm{X}$.

In view of (8.2), the regularity result for the kernel is reduced to the similar result for the generalized Eisenstein integral $E_F^\circ(\nu : \cdot)$, namely its regularity for $\nu \in i\mathfrak{a}^*_{F\mathrm{q}}$. This is the analogue of Theorem 2.5. By the work of J. Carmona on the theory of the **constant term** for X, which in turn generalizes Harish-Chandra's work [29] for the case of the group, we can define generalized $c$-functions, which are the analogues of the $c$-functions in Proposition 2.2. A key step in the proof of the regularity theorem is then to prove the **Maass-Selberg relations** for these generalized $c$-functions, see Theorem 2.3. It should be said that at the time of the announcement of our proof of the Plancherel theorem we had to rely on the Maass-Selberg relations proved by Delorme in [24]. Since then we have found a way to derive the generalized Maass-Selberg relations from those associated with a minimal $\sigma\theta$-stable parabolic subgroup, as formulated in Theorem 2.3; see [13], Thm. 18.3.

From the regularity theorem it follows that (8.3) holds with $\varepsilon_F = 0$. Defining the wave packet transform $\mathcal{J}_F$ as $\mathcal{J}_\emptyset$ in (3.4) with $E_F^\circ$ instead of $E^\circ$ we now obtain that

$$f = \sum_{F \subset \Delta} [W : W_F]\, t(P_F) \mathcal{J}_F \mathcal{F}_F f. \tag{8.4}$$

In [13] we establish uniform tempered estimates for the generalized Eisenstein integrals. These allow to show that the formula (8.4) extends continuously to the Schwartz space $\mathcal{C}(\mathrm{X}:\tau)$. It can be shown that $\mathcal{J}_F \circ \mathcal{F}_F$ depends on $F$ through its class for the equivalence relation $\sim$ on the powerset $2^\Delta$ defined by $F \sim F' \iff \exists w \in W\colon w(\mathfrak{a}_{F\mathrm{q}}) = \mathfrak{a}_{F'\mathrm{q}}$. By a simple counting argument it then follows that

$$I = \sum_{[F] \in 2^\Delta/\sim} [W : W_F^*]\, \mathcal{J}_F \mathcal{F}_F \qquad \text{on} \quad \mathcal{C}(\mathrm{X}:\tau); \tag{8.5}$$

here $W_F^*$ denotes the normalizer of $\mathfrak{a}_{F\mathrm{q}}$ in $W$. In particular, in this **Plancherel formula** for $\tau$-spherical functions the residue weight $t$ has disappeared.

In [14] it is shown that the Eisenstein integrals $E_F^\circ(\nu)$, for $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$, are essentially sums of generalized matrix coefficients of parabolically induced representations of the form $\mathrm{Ind}_{P_F}^G(\sigma \otimes \nu \otimes 1)$ with $\sigma$ a discrete series representation of $\mathrm{X}_F = M_F/M_F \cap H$. Here a key role is played by the **automatic continuity theorem** due to W. Casselman and N. Wallach, [20] and [37]. This allows to conclude that (8.5) is the $\tau$-spherical part of the Plancherel formula in the sense of representation theory. Moreover, the Eisenstein integrals $E_F^\circ(\nu)$ and the associated Fourier and wave packet transforms can be identified with those introduced in [19] by Carmona and Delorme.

# References

[1] *J. Arthur,* A Paley-Wiener theorem for real reductive groups. Acta Math. **150** (1983), 1-89.

[2] *E. P. van den Ban,* The principal series for a reductive symmetric space, I. $H$-fixed distribution vectors. Ann. scient. Éc. Norm. Sup. **21** (1988), 359-412.

[3] *E. P. van den Ban,* The principal series for a reductive symmetric space II. Eisenstein integrals. J. Funct. Anal. **109** (1992), 331-441.

[4] *E. P. van den Ban,* The action of intertwining operators on spherical vectors in the minimal principal series of a reductive symmetric space. Indag. Math. **145** (1997), 317-347.

[5] *E. P. van den Ban, J. Carmona and P. Delorme,* Paquets d' ondes dans l'espace de Schwartz d'un espace symétrique réductif. J. Funct. Anal. **139** (1996), 225-243.

[6] *E. P. van den Ban and H. Schlichtkrull,* Convexity for invariant differential operators on a semisimple symmetric space. Compos. Math. **89** (1993), 301-313.

[7] *E. P. van den Ban and H. Schlichtkrull,* Expansions for Eisenstein integrals on semisimple symmetric spaces. Ark. Mat. **35** (1997), 59-86.

[8] *E. P. van den Ban and H. Schlichtkrull,* Fourier transforms on a semisimple symmetric space. Invent. Math. **130** (1997), 517-574.

[9] *E. P. van den Ban and H. Schlichtkrull,* The most continuous part of the Plancherel decomposition for a reductive symmetric space. Annals Math. **145** (1997), 267-364.

[10] *E. P. van den Ban and H. Schlichtkrull,* A residue calculus for root systems. Compositio Math. **123** (2000), 27-72.

[11] *E. P. van den Ban and H. Schlichtkrull,* Fourier inversion on a reductive symmetric space. Acta Math. **182** (1999), 25-85.

[12] *E. P. van den Ban and H. Schlichtkrull,* Analytic families of eigenfunctions on a reductive symmetric space. Represent. Theory **5** (2001), 615-712.

[13] *E. P. van den Ban and H. Schlichtkrull,* The Plancherel decomposition for a reductive symmetric space, I. Spherical functions. arXiv.math.RT/0107063 v2.

[14] *E. P. van den Ban and H. Schlichtkrull,* The Plancherel decomposition for a reductive symmetric space, II. Representation theory. arXiv.math.RT/0111304 v2.

[15] *E. P. van den Ban and H. Schlichtkrull,* A Paley-Wiener theorem for reductive symmetric spaces. arXiv.math.RT/0302232 v1.

[16] *O. A. Campoli,* Paley-Wiener type theorems for rank-1 semisimple Lie groups, Rev. Union Mat. Argent. **29** (1980), 197-221.

[17] *J. Carmona,* Terme constant des fonctions tempérées sur un espace symétrique réductif. J. reine angew. Math. **491** (1997), 17-63.

[18] *J. Carmona and P. Delorme,* Base méromorphe de vecteurs distributions H-invariants pour les séries principales généralisées d'espaces symétriques réductifs: Equation fontionelle. J. Funct. Anal. **122** (1994), 152-221.

[19] *J. Carmona and P. Delorme,* Transformation de Fourier sur l'espace de Schwartz d'un espace symétrique réductif. Invent. math. **134** (1998), 59-99.

[20] *W. Casselman,* Canonical extensions of Harish-Chandra modules to representations of $G$. Canad. J. Math. **41** (1989), 385-438.

[21] *W. Casselman and D. Miličić,* Asymptotic behavior of matrix coefficients of admissible representations. Duke Math. J. 49 (1982), 869-930.

[22] *P. Delorme,* Téorème de type Paley-Wiener pour les groupes de Lie semisimples réels avec une seule classe de conjugaison de sous groupes de Cartan. J. Funct. Anal. **47** (1982), 26–63.

[23] *P. Delorme,* Intégrales d'Eisenstein pour les espaces symétriques réductifs: tempérance, majorations. Petite matrice B. J. Funct. Anal. **136** (1994), 422-509.

[24] *P. Delorme,* Troncature pour les espaces symétriques réductifs. Acta Math. **179** (1997), 41-77.

[25] *P. Delorme,* Formule de Plancherel pour les espaces symétriques réductifs. Annals Math. **147** (1998), 417-452.

[26] *M. Flensted-Jensen,* Discrete series for semisimple symmetric spaces. Annals Math. **111** (1980), 253-311.

[27] *P. Harinck,* Fonctions orbitales sur $G_{\mathbb{C}}/G_{\mathbb{R}}$. Formule d'inversion des intégrales orbitales et formule de Plancherel. J. Funct. Anal. **153** (1998), 52-107.

[28] *Harish-Chandra,* On the theory of the Eisenstein integral. Lecture Notes in Math. **266**, 123-149, Springer-Verlag, New York, 1972. Also: Collected papers, Vol 4, pp. 47-73, Springer-Verlag, New York, 1984.

[29] *Harish-Chandra,* Harmonic analysis on real reductive groups I. The theory of the constant term. J. Funct. Anal. **19** (1975), 104-204. Also: Collected papers, Vol 4, pp. 102-202, Springer-Verlag, New York, 1984.

[30] *Harish-Chandra,* Harmonic analysis on real reductive groups II. Wave packets in the Schwartz space. Invent. math. **36** (1976), 1-55. Also: Collected papers, Vol 4, pp. 203-257, Springer-Verlag, New York, 1984.

[31] *Harish-Chandra,* Harmonic analysis on real reductive groups III. The Maass-Selberg relations and the Plancherel formula. Annals of Math. **104** (1976), 117-201. Also: Collected papers, Vol 4, pp. 259-343, Springer-Verlag, New York, 1984.

[32] *S. Helgason,* Groups and Geometric Analysis. Academic Press, Orlando, FL, 1984.

[33] *R. P. Langlands,* On the functional equations satisfied by Eisenstein series. Springer Lecture Notes **544**, Springer-Verlag, Berlin, 1976.

[34] *T. Oshima,* A realization of semisimple symmetric spaces and construction of boundary value maps. Adv. Studies in Pure Math. **14** (1988), 603-650.

[35] *T. Oshima and T. Matsuki,* A description of discrete series for semisimple symmetric spaces. Adv. Stud. Pure Math. **4** (1984), 331-390.

[36] *N. R. Wallach,* Real Reductive Groups I. Academic Press, Inc., San Diego, 1988.

[37] *N. R. Wallach,* Real Reductive Groups II. Academic Press, Inc., San Diego, 1992.

[38] *D. P. Zhelobenko,* Harmonic Analysis of functions on semisimple Lie groups. II. Izv. Akad. Nauk SSSR Ser. Mat. **33** (1969), 1255-1295.

E. P. van den Ban
Mathematisch Instituut
Universiteit Utrecht
PO Box 80 010
3508 TA Utrecht
Netherlands
E-mail: ban@math.uu.nl