An Autoparametric System with Dry-friction

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Abstract

We consider an autoparametric system containing a dry-friction oscillator characterised by a small parameter. One interesting aspect of the analysis of the semitrivial (planar) solution is the possibility of calculating the boundary value for the existence of a non-smooth periodic solution. Moreover, the application of the software package SlideCont shows the phenomenon of the *sliding* bifurcation of the periodic solution of the system . The study of the 4-dimensional system is treated qualitatively by asymptotic analysis, enhanced with numerical simulations.

Key words. Autoparametric, dry-friction, bifurcation.

1 Introduction

During the past few decades, there have been many studies on non-smooth dynamical systems or systems with a discontinuity. A number of papers are devoted to the fundamentals (e.g. Aizerman and Gantmakher [2], Filippov [9], Aubin and Cellina [3], Kunze [15], Kunze and Küpper [14], Kuznetsov [16]). Other papers emphasise the mechanical engineering viewpoint (e.g. Tondl [25], Popp and Stelter [22], Bothe [5], Babitsky and Krupenin [4]). These studies clearly point out that dry friction in dynamical systems is an important topic in both mathematical and engineering research.

We will start with some observations regarding non-smooth systems with a small parameter. In particular we consider the relevance of Poincaré expansion and averaging. We illustrate this by briefly discussing a classical problem involving a small parameter. Then, we study non-smooth solutions of an autoparametric system containing a dry friction oscillator. We are particularly interested in non-smooth solutions of the system like periodic solutions and other invariant sets. A combination of analytical (asymptotic) and numerical techniques enables us to obtain results on the presence of a non-smooth invariant manifold in our system.

Recently, in [1] the author studied a Rayleigh-type oscillator embedded in an autoparametric system. For more references on autoparametric systems the reader may consult [26], or [7] for a more recent one.

2 Non-smooth equations with a small parameter

There are many results dealing with systems containing discontinuities by using asymptotic methods. One of the most common method that has been used to analyse such a problem is the averaging method. We mention results by Matveev et al [19], Plotnikov [20, 21], Samoilenko and Pere'styuk [23, 24], Fidlin [8] and the references therein. We shall discuss another possibility which is the use of Poincaré expansion to analyse such a problem.

2.1 Poincaré expansion

A discussion of the Poincaré expansion or continuation method for periodic solutions is presented in [27]. In most studies, this method requires the analyticity of the equation with respect to variables and parameters. The Poincaré expansion method in a weaker and more practical form is based on two steps, first expansion with respect to a small parameter to a certain order and then applying a periodicity condition.

We consider a system with a small discontinuous right-hand side in the following form

$$\dot{\mathbf{x}} = \mathbf{f_0}(t, \mathbf{x}) + \varepsilon \mathbf{f_1}(t, \mathbf{x}) + \mathcal{O}(\varepsilon^2) \tag{2.1}$$

with $\mathbf{x} \in \mathfrak{D} \subset \mathbb{R}^n$ and ε is a small positive parameter. We have initial value $\mathbf{x}(0) = \eta$, and $0 < t < h$, $\mathbf{x}$ is in a compact set $\mathfrak{D} \subset \mathbb{R}^n$, $0 \leq \varepsilon \leq \varepsilon_0$. $\mathbf{f_0}$ is a continuous function in t, continuous differentiable vector function in $\mathbf{x}$. $\mathbf{f_1}$ is a continuous function in t, a piecewise differentiable vector function in $\mathbf{x}$, with a finite number of discontinuities. For the sake of simplicity, we assume in the sequel that $\mathbf{f_1}$ has a single surface of discontinuity, Σ_1 in $\mathbb{R}^n$.

We formally expand $\mathbf{x}(t) = \mathbf{x_0}(t) + \varepsilon \mathbf{x_1}(t) + \mathcal{O}(\varepsilon^2)$, where $\mathbf{x_0}(t)$ is smooth. After applying the concept of differential inclusion to equation (2.1) (see [9] or [3] for the theory), then substituting the expansion into the corresponding differential inclusion, expansion to lowest order and equating coefficients of equal powers of ε yields the following equations:

$$\dot{\mathbf{x}}_\mathbf{0} = \mathbf{f_0}(t, \mathbf{x_0}), \ \mathbf{x_0}(0) = \eta,$$

$$\dot{\mathbf{x}}_\mathbf{1} \in \mathbf{F_1}(t, \mathbf{x_0}) + \frac{\partial \mathbf{f_0}(t, \mathbf{x_0})}{\partial \mathbf{x}} \mathbf{x_1}, \ \mathbf{x_1}(0) = 0,$$

with

$$\mathbf{F_1}(t, \mathbf{x_0}) = \overline{\text{co}}\{\mathbf{h_0} \in \mathfrak{D} \subset \mathbb{R}^n \mid \mathbf{h_0} = \lim_{\mathbf{k_0}^* \to \mathbf{x_0}} \mathbf{f_1}(t, \mathbf{k_0}^*), \mathbf{k_0}^* \in \mathfrak{D} \backslash \Sigma_1\}.$$

It is straightforward to show that the expansion which we propose, yields an asymptotic approximation of the solution of the system. One can follow the idea of the proof of Theorem 9.1 in [27] which is based on Gronwall's lemma. The result is that $\|\mathbf{x}(t) - (\mathbf{x_0}(t) + \varepsilon \mathbf{x_1}(t))\| = \mathcal{O}(\varepsilon^2)$ on the time-scale 1.

The assumptions on the boundedness of the derivatives of the solution in the vicinity of the discontinuity surface and on the smallness in time of the discontinuity are crucial in the analysis above. Note also that the integration process defined above, produces a *smooth solution*, at least to the order of approximation which has been taken. This is connected with the choice of continuation of the solution when applying the equivalent integral equation. It restricts the use of the method severely.

2.2 A classical example

A simple example of a dry-friction oscillator is:

$$\ddot{x} + x = \varepsilon \text{ sgn } (v - \dot{x}), \tag{2.2}$$

where sgn is the signum function and v is a positive constant so that

$$x(t) = \pm\varepsilon + C\cos(t + \varphi),$$

where C and φ are the constants of integration that can be determined from the initial conditions.

Let $I_0(a_0, b_0)$ be an initial point. Taking I_0 in the domain where $v > \dot{x}$, such that $(a_0 - \varepsilon)^2 + b_0^2 \leq v^2$, we have a set of smooth periodic solutions (circles). On the other hand, taking I_0 in the domain where $v < \dot{x}$, we obtain a set of non-smooth solutions with the circle

$$(x - \varepsilon)^2 + \dot{x}^2 = v^2$$

as the limiting case. Figure 1 describes the phenomenon in the phase plane.

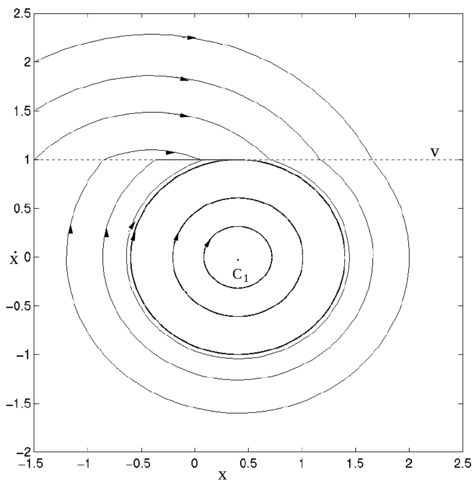


Figure 1: **The circles and the non-smooth solutions**.

The smooth solutions (circles) arise if the solutions do not approach Σ_1, given by $\dot{x} = v$, whereas for the non-smooth ones Σ_1 is crossed or contained in the orbit. We may have a sliding solution along the line $\dot{x} = v$, starting from the left before it slips into the limiting circle.

A similar example as equation (2.2) is considered in the monograph of Klotter [13] and in Popp and Stelter [22] and Galvanetto and Bishop [11]. They did not give a correct description of the phase portrait to illustrate the phenomenon as given in Figure 1 above. They conjectured that the limit of the set of non-smooth solutions is a periodic solution with a sliding segment along the discontinuity line which is symmetric with respect to the tangent point, instead of a circle which is tangent to the line. This is impossible, because the discontinuity line to the right of the tangent point is simply a crossing segment.

Therefore, the solution must cross the discontinuity line whenever it arrives at the line. The discussion in the next subsection will confirm this in a different way; by using a smoothing procedure.
We note that the Poincaré expansion method does not apply to the non-smooth solution with sliding.

2.3 A remark on smoothing

In a number of cases it is possible to consider non-smooth systems as the limit of a smooth system. Consider the smooth equation

$$\ddot{x} + x = \varepsilon \frac{2}{\pi} \arctan(\frac{v - \dot{x}}{\delta(\varepsilon)}). \tag{2.3}$$

The order function $\delta(\varepsilon)$ is positive if $\varepsilon > 0$ and has the properties that $\delta(\varepsilon) \to 0$ if $\varepsilon \to 0$ and $\delta(\varepsilon) = o(\varepsilon^n), n = 1, 2, \cdots$.
If $\varepsilon \to 0$, equation (2.3) tends to equation (2.2). A difference between the two equations is that, except if $\dot{x} = v$, the smooth equation (2.3) is slightly dissipative. We can correct this locally by introducing exponentially small terms. The resulting equation is still not conservative like (2.2) but locally it will look conservative, especially as $\delta(\varepsilon)$ is exponentially small and can even be chosen smaller. We note that this inconsistency does not arise when discussing more realistic equations as in section 3.2.

Equations like (2.3) are studied in singular perturbation theory using slow manifolds; for an introduction see Jones [12]. In this case it would be natural to introduce the variable y by

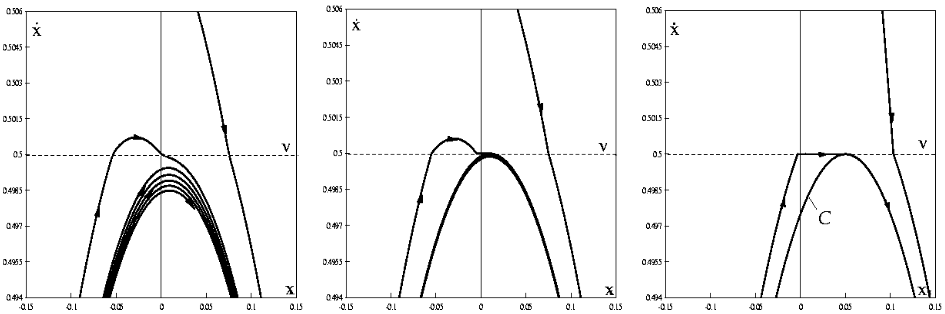
$$\delta(\varepsilon) y = v - \dot{x},$$

which exhibits very fast motion except in and near the slow manifold given by $\dot{x} = v$. The transition between fast and slow motion is described by boundary layers, the size of which are proportional to $\delta(\varepsilon)$. So here the boundary layer contribution is smaller than any power of ε. In this way the solutions of (2.3) are expected to mimic the behaviour of the solutions of the non-smooth equation (2.2) very accurately.

The slow manifold is not (normally) hyperbolic so we cannot apply Fenichel theory. This is typical for equations resulting from smoothing. In general we have an n-dimensional problem with Σ_1 a single k-dimensional surface of discontinuity ($k < n$). We assume that we can introduce local coordinates to describe Σ_1 after which we propose smoothing as in equation(2.3). Again Σ_1 will correspond with a slow manifold which is not (normally) hyperbolic.
It is easy to obtain approximations of the solutions of equation (2.3). Take for instance initial conditions $(x(0), \dot{x}(0)) = (x_0, 0)$. Away from the slow manifold i.e. at $O(1)$ distance, and if $\dot{x}(t) < v$, $x(t)$ is approximated by $\varepsilon + (x_0 - \varepsilon)\cos t$ plus exponentially small terms. If $\dot{x}(t)$ is ε-close to the slow manifold this approximation is still valid, if $\dot{x}(t)$ gets still nearer it helps to formulate the equivalent integral equation:

$$x(t) = x_0 \cos t + \varepsilon \frac{2}{\pi} \int_0^t \arctan(\frac{v - \dot{x}(\tau)}{\delta(\varepsilon)}) \sin(t - \tau) d\tau.$$

Partial integration of the integral expression produces local approximations near the slow manifold. A number of solutions of equation (2.3) are illustrated in Figure 2. Considering equation (2.2) as a limiting case of equation (2.3) we note that smoothing confirms our analysis of the non-smooth system in the preceding subsection.



(i) (ii) (iii)

Figure 2: **Some solutions of equation (2.3)**. (i) for $\delta(\varepsilon) = \varepsilon^2$ with $\varepsilon = 0.07$, (ii) for $\delta(\varepsilon) = \varepsilon^3$ with $\varepsilon = 0.07$, (iii) for $\delta(\varepsilon) = e^{-\frac{1}{\varepsilon}}$ with $\varepsilon = 0.07$. $\mathcal{C}$ is the limiting circle and $\dot{x} = v = 0.5$ is the discontinuity boundary.

In Figure 2 we observe the phase portrait of the solution near the discontinuity boundary $\dot{x} = v = 0.5$; $\{(x, \dot{x}); -0.15 \leq x \leq 0.15, 0.494 \leq \dot{x} \leq 0.506\}$. Taking different functions for $\delta(\varepsilon)$, we can compare the closeness of the approximation of the solution of (2.3). For $\delta(\varepsilon) = \varepsilon^2$ with $\varepsilon = 0.07$ we obtain a smooth approximate solution of (2.3) which is, clearly, dissipative (Figure 2 (i)). However, by taking $\delta(\varepsilon) = \varepsilon^3$ with the same value of ε, we obtain a similar smooth solution but with small dissipation. This can be seen in Figure 2(ii) shown by a thick curve describing that by taking a longer time integration, the solution will collapse to the equilibrium of the equation. Furthermore, taking $\delta(\varepsilon) = e^{-\frac{1}{\varepsilon}}$, the approximate solution mimics the solution of equation (2.2) very accurately. This is described in Figure 2(iii) where we find a limiting circle $\mathcal{C}$, which corresponds with the limiting circle of equation (2.2) as shown in Figure 1.

We note that a numerical study of the relation between non-smooth and smooth oscillators can also be found in [18].

2.4 The averaging method

For the sake of completeness we consider briefly averaging as, with some modifications in the theorems, we may still apply averaging to a system containing a discontinuity.

In [20, 21], Plotnikov considered a perturbation problem which is in the following standard form

$$\dot{\mathbf{y}} = \varepsilon\tilde{\mathbf{f}}(t, \mathbf{y}), \ \mathbf{y}(0) = y_0 \tag{2.4}$$

with $\mathbf{y}(t) \in \mathbf{D} \subset \mathbb{R}^n$, ε is a small positive parameter, and $\tilde{\mathbf{f}}$ is a vector function containing a discontinuity with respect to $\mathbf{y}$.

To approximate the solution of the initial value problem (2.4), it is sufficient to approximate the solution of its corresponding differential inclusion and the corresponding the averaged inclusion.

Independently, Matveev et.al [19] constructed a similar result as above to show the applicability of the averaging method to approximate solutions of differential inclusions.

On the other hand, in [8] Fidlin treated a discontinuous system of the following form:

$$\dot{x} = \varepsilon X(x,t) + Z(z,t)[E(g(t) + \varepsilon f(x,t)) - E(g(t))], \; x(0) = x_0, \tag{2.5}$$

where E is a one-step function, and all the functions are continuous and T-periodic.

In [23] [24], Samoilenko and Pere'styuk considered differential equations with impulsive action.

Some important assumptions have been made in [8, 19, 20, 21, 23, 24], in order to maintain the applicability of the averaging method. That is, one should know a priori when the discontinuity jumps occur and one should assume that the size of the jump is small in time. These agree with the assumptions made when applying the Poincaré expansion method explained earlier. However, to obtain the standard forms to be analysed, one still needs to construct the generalised Green's function to invert the operator.

The problem remains how to treat systems with discontinuities, for which we do not know a priori when the jumps occur.

3 Solutions of systems with dry friction

To start with, we consider first a planar system of the Filippov type which has two subspaces of continuity $\mathcal{V}_i, \; i = 1, 2$, i.e.

$$\dot{x} = \begin{cases} f_1(x), \; x \in \mathcal{V}_1 \\ f_2(x), \; x \in \mathcal{V}_2 \end{cases} \tag{3.1}$$

where f_i are smooth and C^1 on $\mathcal{V}_i \cup \Sigma$, for Σ is a discontinuity boundary, separating the two subspaces;

$$\Sigma = \{x \in \mathbb{R}^2 \mid h(x) = 0\}$$

where h is a smooth scalar function with $\nabla(h(x)) \neq 0$ on Σ. Therefore, the subspaces $\mathcal{V}_i$ can be formulated as follows:

$$\mathcal{V}_1 = \{x \in \mathbb{R}^2 \mid h(x) > 0\} \text{ and } \mathcal{V}_2 = \{x \in \mathbb{R}^2 \mid h(x) < 0\}.$$

With regards to the systems considered later in this paper, we restrict ourselves to only considering the case that f_i are non-zero vectors and only one of them can be tangent to Σ. Such a point is called a *singular point* or a *tangent point*.

3.1 Periodic solutions in the planar case

There are some methods for the construction of a periodic solution of system (3.1). A discussion on the topic can be found in [9], or in [16] for a more recent one. In our case we have the following scenario.

Consider the planar system $\dot{x} = f_1(x)$ in $\mathcal{V}_1$. Assume that the discontinuity boundary Σ is a horizontal line in the phase plane and that the system has a hyperbolic unstable equilibrium P_c of focus type.

We are going to continue the solution from T^* where the solution hits Σ. Since P_c is an unstable focus, right after the solution is tangent to the discontinuity line at point T, the next hit of the solution to the discontinuity line, at point T^*, cannot be a tangent point in $\mathcal{V}_1$ again. Thus, we cannot have a closed orbit which entirely lies in $\mathcal{V}_1$.

Continuation from point T^* results in three possibilities: crossing into $\mathcal{V}_2$ and subsequently returning into $\mathcal{V}_1$ exactly at tangent point T, crossing into $\mathcal{V}_2$ and subsequently sliding along Σ until tangent point T is reached, immediate sliding along Σ until tangent point T is reached. The three possibilities are illustrated in Figure 3.

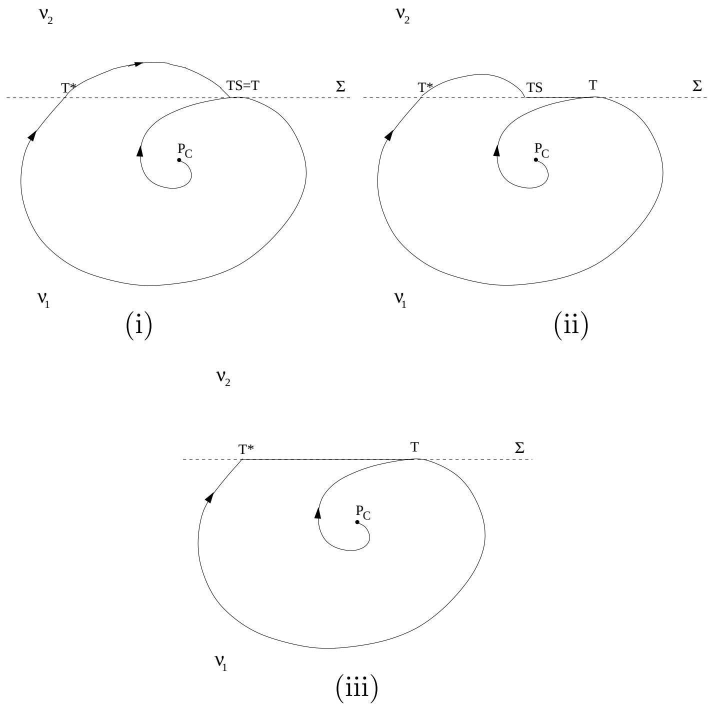


Figure 3: **Three possibilities of the non-smooth periodic solution**. At point T^* the line Σ is hit, point T is tangent point.

The presence of the sliding segment as a part of the periodic solution of the system prevents us to apply any of the asymptotic methods explained in section 2. Obviously, we cannot apply the Poincaré expansion method to approximate the periodic solution of the system (a similar situation happened when treating Example 1 of section 2). Also, we cannot apply the averaging method as in [19, 20, 21, 23, 24], since we do not know explicitly when the jumps due to the discontinuity in the system occur.

3.2 The 4-dimensional autoparametric system

Having some insight into the periodic solutions of systems with dry friction, we are now ready to consider our autoparametric system. We begin with a well-known prototype of an oscillator with self-excited oscillations induced by relative dry friction: Consider a mass m resting on a continuous conveyor belt moving at constant speed v, the mass being held near the equilibrium position by a linear spring with stiffness c. Let the mass be acted upon by an absolute dry friction damper. See Figure 4 for an illustration.

When the belt moves, the equilibrium position is unstable. When the static friction

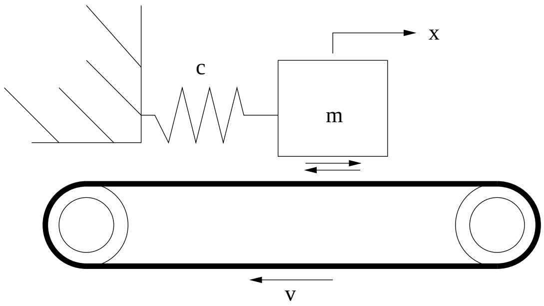


Figure 4: **The oscillator with dry friction**.

force, which is a function of the spring force cx, reaches its maximum value, the mass oscillates on the belt. Then, self-excited vibrations of the mass arise when the absolute magnitude of the 'kinetic' friction force is a decreasing function of the relative velocity $v - \dot{x}$, where $\dot{x}$ is the velocity of the mass.

Such a motion described above will act as an oscillator in the self-excited autoparametric system considered in this paper. To this purpose, we suppose the oscillator (x mode) on the belt is nonlinearly coupled to a damped oscillator (y mode). The system is represented by the following differential equations, after transforming the equations into dimensionless form,

$$\begin{aligned} x'' + x &= F(v - x') - \gamma_1 y^2 \\ y'' + \frac{1}{4}y &= -(\kappa y' + \sigma y + \gamma_2 xy), \end{aligned} \tag{3.1}$$

where $F(v - x')$ has a discontinuity when $v = x'$. The simplest case is by considering $F(v - x')$ as a function of the following form.

$$F(v - x') = \alpha_0 \text{ sgn } (v - x') - \alpha_1(v - x'),\ \alpha_0,\ \alpha_1 > 0,$$

where sgn is the signum function.

Assuming that the parameters are small, we rescale all the parameters of the system: $\alpha_0 = \varepsilon\bar{\alpha}_0,\ \alpha_1 = \mu\bar{\alpha}_1,\ \kappa = \delta\bar{\kappa},\ \sigma = \delta\bar{\sigma}, \gamma_1 = \delta\bar{\gamma}_1,\ \gamma_2 = \delta\bar{\gamma}_2$, where $\varepsilon,\ \mu,$ and δ are small positive parameters which may be different in order. Substituting these into (3.1), then, after dropping the bars, we obtain the following:

$$\begin{aligned} x'' + x &= \varepsilon\alpha_0 \text{ sgn } (v - x') - \mu\alpha_1(v - x') - \delta\gamma_1 y^2 \\ y'' + \frac{1}{4}y &= -\delta(\kappa y' + \sigma y + \gamma_2 xy). \end{aligned} \tag{3.2}$$

4 The planar subsystem

The results in this section are to some extent related to results in Galvanetto and Bishop [11] but our analysis is new. It is based on two quantitative approaches: asymptotic and numerical bifurcation analysis.

The semitrivial solution of system (3.2) is defined by putting $y = 0$. Thus, we analyse the equation:

$$x'' + x = \varepsilon\alpha_0 \text{ sgn } (v - x') - \mu\alpha_1(v - x'). \tag{4.3}$$

Assume first that $v > x'$ holds. So, we are dealing with the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 + \varepsilon\alpha_0 - \mu\alpha_1(v - x_2). \end{aligned} \tag{4.4}$$

Since we are interested in solutions that can undergo self-excitation, we investigate the existence of periodic solutions of the system. Thus, we need to analyse the vector fields in both domains $\mathcal{V}_1$ and $\mathcal{V}_2$ where $v > x_2$ and $v < x_2$, respectively.

A straightforward calculation gives the result that system (4.4) has a critical point $P_1 = (\varepsilon\alpha_0 - \mu\alpha_1 v, 0)$. Linearisation in a neighbourhood of point P_1 leads to the eigenvalues

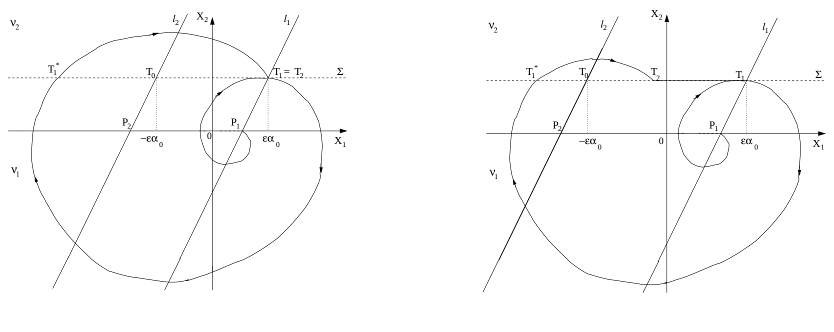
$$\lambda_{1,2} = \frac{\mu\alpha_1 \pm \sqrt{\mu^2\alpha_1^2 - 4}}{2}. \tag{4.5}$$

From (4.5) we conclude that P_1 is an unstable focus. By taking $x_2' = 0$, we have the line l_1 passing points P_1 and $T_1(\varepsilon\alpha_0, v)$, where trajectories in $\mathcal{V}_1$ have zero derivatives.

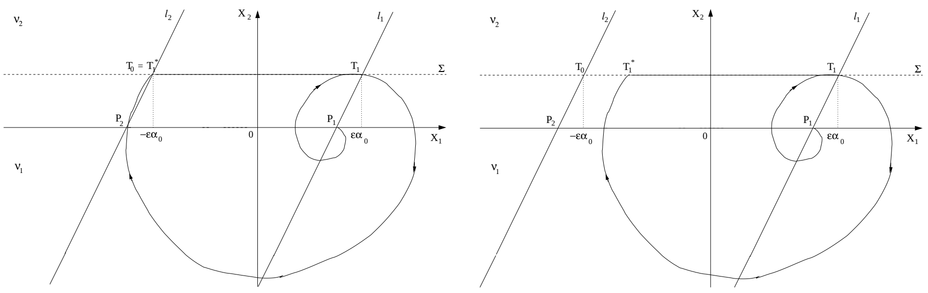
Similarly, assuming that $v < x_2$ holds, we obtain the corresponding critical point $P_2 = (-\varepsilon\alpha_0 - \mu\alpha_1 v, 0)$, having the same stability characteristic as point P_1. Also, we have the line l_2 passing points P_2 and $T_0(-\varepsilon\alpha_0, v)$, where trajectories in $\mathcal{V}_2$ have zero derivatives. Note that by assuming $\alpha_0 > 0,\ \alpha_1 > 0,\ v > 0$, the positions of point P_2 and line l_2 are to the left of point P_1 and line l_1, respectively. (See Figure 5).

The trajectory started from point P_1 is winding away from the point until it arrives at its tangent point T_1 with the discontinuity line $\Sigma \equiv v - x_2 = 0$ and at its next hit T_1^* to the line. On the other hand, in domain $\mathcal{V}_2$ the presence of point P_2 and line l_2 play a role to the flow in the domain, and hence to the continuation of the solution from point T_1^* entering $\mathcal{V}_2$.

The continuation of the solution starting from point T_1^* depends upon the position of the point with respect to the lines l_1 and l_2. Figure 5 describes all possibilities for having periodic solutions of the system regarding the position of point T_1^*.



(i) (ii)



(iii) (iv)

Figure 5: **The possible construction of periodic solutions** . (i) The case when T_2 coincides with T_1. (ii) The case when T_2 is between T_0 and T_1. (iii) The case when T_1^* coincides with T_0. (iv) The case when T_1^* is between T_0 and T_1. l_1 and l_2 are the lines where the trajectories in $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively, have zero derivatives.

Figure 5 illustrates the construction of all possible periodic solutions described above.

The constructions of non-smooth periodic solutions described in Figure 5 leads us to the bifurcation analysis of the periodic solution of system 4.3. One can notice that the cases described in Figure 5(i) and Figure 5(iii) are critical cases where the periodic solution bifurcate. Thus, we are studying so-called *sliding* bifurcations.

4.1 The boundary case of the sliding periodic solution

The boundary case of equation 4.3 for having a sliding periodic solution is described by Figure 5(i). In the following, we derive such a boundary by calculating the critical value of parameters from which we are able to conclude the existence of the periodic solution consisting of a sliding segment.

Solving $\mathbf{x}' = f_1(\mathbf{x})$ with point $T_1(\varepsilon\alpha_0, v)$ as the initial condition, we obtain

$$\begin{aligned} x_1(t) = \mu\alpha_1 v e^{\frac{1}{2}\mu\alpha_1 t}\cos(\frac{1}{2}\sqrt{4-\mu^2\alpha_1^2}t) + \frac{(2-\mu^2\alpha_1^2)v}{\sqrt{4-\mu^2\alpha_1^2}}e^{\frac{1}{2}\mu\alpha_1 t}\times \\ \sin(\frac{1}{2}\sqrt{4-\mu^2\alpha_1^2}t) + \varepsilon\alpha_0 - \mu\alpha_1 v. \end{aligned} \tag{4.6}$$

We know that $x_1'(t) = v$, for some $t = t_1^*$. Suppose that $t_1^* = 2\pi - \triangle t$, then Taylor expanding with respect to μ and $\triangle t$, we have for $\triangle t$:

$$\triangle t = \frac{1}{8}\mu^2\alpha_1^2(2\pi - 1) \pm \sqrt{2\mu\alpha_1\pi + \frac{5}{2}\alpha_1^2\pi^2\mu^2 + \mathcal{O}(\mu^3)}. \tag{4.7}$$

By taking the positive sign in (4.7) and plugging $t_1^* = 2\pi - \triangle t$ into (4.6), Taylor expansion of the equation with respect to μ and $\triangle t$, results in:

$$x_1(t_1^*) = -v\sqrt{2\mu\alpha_1\pi} + \varepsilon\alpha_0 - \mu\alpha_1 v\pi\sqrt{2\mu\alpha_1\pi} - \frac{5}{4}\mu^2\alpha_1^2 v\pi + \text{ h.o.t.} \tag{4.8}$$

Having the coordinate of point T_1^*, we continue by concatenating the obtained solution with the solution in the other domain. Thus, we switch to domain $\mathcal{V}_2$ and solve $\mathbf{x}' = f_2(\mathbf{x})$ with point $T_1^*(x_1(t_1^*), v)$ as the initial condition. Therefore, we proceed the calculation to obtain:

$$\begin{aligned} x_2(t) = {} & C_1 e^{\frac{1}{2}\mu\alpha_1 t}\cos(\frac{1}{2}\sqrt{4-\mu^2\alpha_1^2}t) + C_2 e^{\frac{1}{2}\mu\alpha_1 t}\sin(\frac{1}{2}\sqrt{4-\mu^2\alpha_1^2}t) \\ & - \varepsilon\alpha_0 - \mu\alpha_1 v, \end{aligned} \tag{4.9}$$

where $C_1 = x_1(t_1^*) + \varepsilon\alpha_0 + \mu\alpha_1 v$, and $C_2 = \frac{2v-\mu\alpha_1 C_1}{\sqrt{4-\mu^2\alpha_1^2}}$.

Similarly as before, we are looking for some $t = t_1$ such that $x_2'(t_1) = v$. Thus, after Taylor expansion with respect to μ and neglecting the higher order terms, we solve for t_1; $t_1 = 0$ or

$$t_1 = \frac{2\varepsilon\alpha_0 - v\sqrt{2\mu\alpha_1\pi} - \mu\alpha_1 v\pi\sqrt{2\mu\alpha_1\pi} - \frac{5}{4}\mu^2\alpha_1^2 v\pi}{-\frac{1}{2}v - \varepsilon\mu\alpha_0\alpha_1 + \frac{1}{2}\mu\alpha_1 v\sqrt{2\mu\alpha_1\pi} - \frac{1}{2}\mu^2\alpha_1^2 v}. \tag{4.10}$$

Since we are interested in periodic solutions of system (4.3), the condition that

$$x_2(t) \leq \varepsilon\alpha_0,\ t_1^* < t < t_1 \tag{4.11}$$

must hold. Substituting the latter value of t_1 in (4.10) into (4.9) and applying the inequality (4.11), we have the following relation:

$$\mu \leq \mu_0, \tag{4.12}$$

where $\mu_0 = \frac{-v+\sqrt{v^2+64\alpha_0^2\varepsilon^2}}{4\alpha_1 v\pi}$ which gives to $O(\varepsilon)$ the boundary value for parameter μ where a periodic solution containing a sliding segment of system (4.3) may take place.

4.2 Bifurcation analysis using SlideCont

To perform bifurcation analysis for the solutions of system of the form (4.3), which consists of discontinuities, we apply SlideCont, a recent software package by Dercole and Kuznetsov [17]. SlideCont is a suite of routines accompanying AUTO97 (see [6]) for sliding bifurcation analysis of discontinuous piecewise-smooth autonomous systems.

We start the computation using SlideCont by providing a data file which specifies numerically the starting solution. We take the non-smooth periodic solution of Figure 5(ii) type, as our starting solution, by fixing the parameter values: $\varepsilon = 0.1$, $\mu = 0.07$, $\alpha_0 = 4$, $\alpha_1 = 4, v = 0.5$, with the initial point $T_1(0.4, 0.5)$. At these parameter values, in fact, the periodic solution is coexisted with an exponentially unstable crossing cycle which will be continued separately. After providing the suitable constants file required by SlideCont for such a problem, we continue our starting periodic solution for increasing values of μ (`PAR(2)`). We get the following output

```
BR   PT   TY LAB    PAR(2)         PAR(11)        PAR(12)        PAR(25)        PAR(26)
1    40        2  7.577569E-02  5.093463E+00  9.284519E-01 -5.375930E-01  2.624070E-01
1    80        3  8.477924E-02  5.047204E+00  1.308431E+00 -3.868472E-01  4.131528E-01
1   120        4  9.479559E-02  5.002518E+00  1.673162E+00 -1.834290E-01  6.165710E-01
1   148   UZ   5  1.023117E-01  4.973173E+00  1.903980E+00 -2.747659E-10  8.000000E-01
1   160        6  1.057707E-01  4.960783E+00  1.998549E+00  9.477795E-02  8.947780E-01
1   200   EP   7  1.175286E-01  4.923548E+00  2.272102E+00  4.743172E-01  1.274317E+00
```

with the corresponding continuation output for the unstable crossing cycle

BR	PT	TY	LAB	PAR(2)	PAR(11)	PAR(12)	PAR(21)	PAR(22)
1	40		2	7.650734E-02	3.886142E+00	2.508395E+00	9.353822E-01	1.735382E+00
1	80		3	8.721882E-02	4.060236E+00	2.392258E+00	6.606936E-01	1.460694E+00
1	120		4	9.729280E-02	4.292873E+00	2.252847E+00	4.142413E-01	1.214241E+00
1	160		5	1.035993E-01	4.577489E+00	2.100710E+00	2.079071E-01	1.007907E+00
1	180		6	1.044549E-01	4.730361E+00	2.024133E+00	1.205938E-01	9.205938E-01
1	200		7	1.035279E-01	4.890144E+00	1.945318E+00	3.947016E-02	8.394702E-01
1	211	UZ	8	1.023117E-01	4.973173E+00	1.903980E+00	-1.028633E-11	8.000000E-01
1	240		9	9.613130E-02	5.206056E+00	1.781395E+00	-1.060742E-01	6.939258E-01
1	250	EP	10	9.307145E-02	5.284681E+00	1.735776E+00	-1.416944E-01	6.583056E-01

Note that `UZ` and `EP` indicate, respectively, a bifurcation point and an endpoint, `PAR(11)` and `PAR(12)` are the time length needed for the passage through, respectively, domains $\mathcal{V}_1$ and $\mathcal{V}_2$, `PAR(20+`k`)` assigns the test function k. (See [17] for all details about SlideCont).

Label 5 (in accordance with label 8) indicates a *crossing* bifurcation [16], i.e. a crossing orbit of vector fields in $\mathcal{V}_1$ and $\mathcal{V}_2$ connecting two tangent points of the same vector field in $\mathcal{V}_1$. (zero of test function 5 (`PAR(25)`)).

One can notice that the value of μ (`PAR(2)`) at label 5 corresponds with the boundary value $\mu = \mu_0 \approx 0.1089306$ obtained previously, when the critical case described by Figure 5(i) occurs. Figure 6 clearly illustrates the phenomenon of the existence of the periodic solution. Taking $\mu < \mu_0$ there exists a sliding cycle coexisting with an exponentially unstable crossing cycle (Figure 6 (i)). Figure 6(ii) is the boundary case $\mu = \mu_0$, when the two cycles appear in Figure 6 (i) collide and a single crossing cycle takes place. While taking $\mu > \mu_0$, there is no periodic solution obtained (Figure 6(iii)).

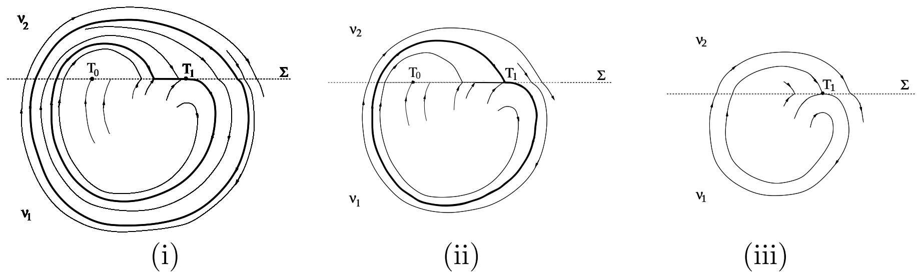


Figure 6: **The crossing bifurcation**.

Since SlideCont does not support the automatic switch for such a problem, we must provide another data file specifying numerically the solution to be continued for decreasing values of μ (`PAR(2)`). By taking the same values of parameters and the initial point as before, and after providing the suitable constant for SlideCont, we get the following output.

BR	PT	TY	LAB	PAR(2)	...	PAR(11)	PAR(23)	PAR(24)
1	20		2	6.408450E-02	...	5.163572E+00	-8.967644E-01	-9.676436E-02
1	28	UZ	3	5.573552E-02	...	5.221891E+00	-8.000000E-01	2.504543E-10
1	40		4	4.394584E-02	...	5.319004E+00	-6.674277E-01	1.325723E-01
1	60		5	2.759741E-02	...	5.494567E+00	-4.850706E-01	3.149294E-01
1	80		6	1.540296E-02	...	5.679865E+00	-3.394173E-01	4.605827E-01
1	100	EP	7	6.992006E-03	...	5.869885E+00	-2.183029E-01	5.816971E-01

and the corresponding output of the continuation of the unstable crossing cycle

BR	PT	TY	LAB	PAR(2)	PAR(11)	PAR(12)	PAR(21)	PAR(22)
1	40		2	6.410177E-02	3.725900E+00	2.625593E+00	1.323278E+00	2.123278E+00
1	80		3	5.673113E-02	3.643200E+00	2.690173E+00	1.616141E+00	2.416141E+00
1	120		4	5.071809E-02	3.580562E+00	2.740972E+00	1.909066E+00	2.709066E+00
1	160		5	4.577884E-02	3.531709E+00	2.781720E+00	2.201174E+00	3.001174E+00
1	200		6	4.167504E-02	3.492618E+00	2.815033E+00	2.492262E+00	3.292262E+00
1	240		7	3.822340E-02	3.460656E+00	2.842735E+00	2.782363E+00	3.582363E+00
1	250	EP	8	3.744541E-02	3.453557E+00	2.848944E+00	2.854746E+00	3.654746E+00

Label 3 indicates a *switching* (*buckling*) bifurcation [16], i.e. the presence of an orbit of vector field in $\mathcal{V}_1$ connecting a tangent point of $\mathcal{V}_1$ with a tangent point of $\mathcal{V}_2$. (zero of test function 4 (`PAR(24)`)). This corresponds to the critical case described by Figure 5(iii).

On the other hand, continuation of the unstable crossing cycle for decreasing values of μ only shows the persistence of the unstable cycle coexisting with the sliding cycle due to the buckling bifurcation explained previously.

Figure 7 illustrates the bifurcation of the sliding cycle and the persistence of the unstable crossing cycle.

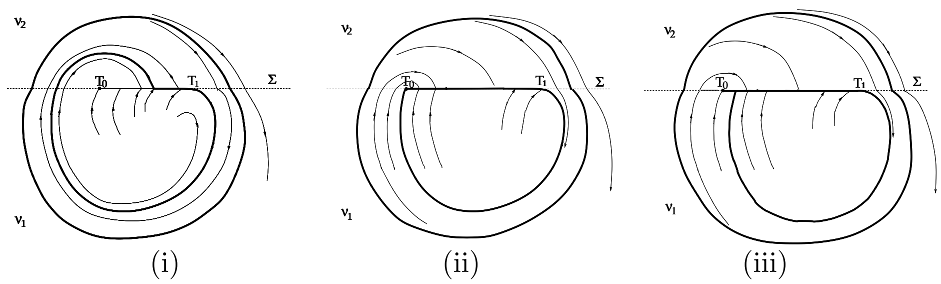


Figure 7: **The switching (buckling) bifurcation**.

For the analysis of co-dimension 1 sliding bifurcations of planar systems, see [16].

5 The 4-dimensional system

As in the planar case, we cannot apply the standard techniques of Poincarè expansion or averaging explained in section 2. And unfortunately, neither can we apply SlideCont to perform bifurcation analysis for the nontrivial case. This is because of the difficulty in providing the starting solution that is required by SlideCont; we could do this in the semitrivial case. However, it is still possible to perform a qualitative analysis regarding the solutions of the full system. In addition, a numerical integration consisting of a routine which allows an automatic switch between the two domains might also be performed to provide us a confirmation for the analytical calculation.

From the previous calculation, for having a solution with a sliding segment, we may take $\mu = \varepsilon^2 < \mu_0$. Thus, system (3.2) becomes:

$$\begin{aligned} x'' + x &= \varepsilon\alpha_0 \text{ sgn } (v - x') - \varepsilon^2\alpha_1(v - x') - \delta\gamma_1 y^2 \\ y'' + \frac{1}{4}y &= -\delta(\kappa y' + \sigma y + \gamma_2 xy). \end{aligned} \tag{5.13}$$

Since we do not know the order of δ, to study (5.13) we consider two cases; the case of $\delta = \varepsilon^2$ and the case of $\delta = \varepsilon$.

5.0.1 The case $\delta = \varepsilon^2$

Taking $\delta = \varepsilon^2$, we have:

$$\begin{aligned} x'' + x &= \varepsilon\alpha_0 \text{ sgn } (v - x') - \varepsilon^2(\alpha_1(v - x') + \gamma_1 y^2) \\ y'' + \frac{1}{4}y &= -\varepsilon^2(\kappa y' + \sigma y + \gamma_2 xy). \end{aligned} \tag{5.14}$$

Assuming that we are in the domain where $v > x'$, then we can approximate the solution of the system in the domain. First, let $x = \varepsilon\alpha_0 - \varepsilon^2\alpha_1 v + r_1\cos(\tau + \psi_1),\ x' = -r_1\sin(\tau + \psi_1),\ y = r_2\cos(\frac{1}{2}\tau + \psi_2),\ y' = -\frac{1}{2}r_2\sin(\frac{1}{2}\tau + \psi_2)$. Then, applying averaging over τ to second order in ε, we obtain the following averaged system (after introducing the phase-difference $\phi = \psi_1 - 2\psi_2$)

$$\begin{aligned} r_1' &= \varepsilon^2(\frac{1}{2}\alpha_1 r_1 + \frac{1}{4}\gamma_1 r_2^2 \sin\phi) \\ r_2' &= \varepsilon^2(-\frac{1}{2}\kappa r_2 - \frac{1}{2}\gamma_2 r_1 r_2 \sin\phi) \\ \phi' &= \varepsilon^2((\frac{1}{4}\gamma_1\frac{r_2^2}{r_1} - \gamma_2 r_1)\cos\phi - 2\sigma), \end{aligned} \tag{5.15}$$

which shows that higher order terms play an important part.

For the construction of solutions of the 4d system, we need to determine whether or not we have an “unstable focus” in the domain, to guarantee that, after some time, the solution will intersect the discontinuity curve (or surface in the full system). Thus, a non-smooth solution can be obtained. Hence, in the analysis which follows, we study the behaviour of the amplitudes r_1 and r_2 in (5.15) and their role in the construction of the nontrivial solution of the system.

The simplest case is by taking $\sigma = 0$, meaning that we are dealing with the exact resonance case. Then, we look for phase-locked solutions in the manifold where $\cos\phi = 0$. This implies that $\sin\phi = \pm 1$. Thus, taking $\sin\phi = 1$ and after absorbing the rescaling factor ε^2, we have:

$$\begin{aligned} r_1' &= \frac{1}{2}\alpha_1 r_1 + \frac{1}{4}\gamma_1 r_2^2 \\ r_2' &= -\frac{1}{2}\kappa r_2 - \frac{1}{2}\gamma_2 r_1 r_2. \end{aligned} \tag{5.16}$$

A phase plane study for equation (5.16) gives the information about the behaviour of r_1

and r_2, as shown in Figure 8 (taking $\alpha_1 = 4$ and $\kappa = 1$).

In the domain where $v < x'$, similar phase portraits as shown in Figure 8 are obtained. As in the planar case, a non-smooth solution in the full system can be obtained, if a solution in one of the domains crosses the discontinuity surface and then passes through the other domain. Or else, the solution slides along the discontinuity surface and then goes back into the same domain where it comes from. Due to the assumption that $\mu = \varepsilon^2$, the latter case of solution (which is periodic) is under consideration.

To read Figure 8, one should associate each of the phase portraits with the corresponding phase portraits (which are, in fact, identical) of the other domain. Hence, as in the planar case, the non-smooth periodic solution with a sliding segment that is constructed is mainly subject to the influence of the flows in both domains to a solution, especially when the solution hits the discontinuity surface.

In the case of Figure 8(i) (associated with that of the other domain), r_1 grows at any time, but $r_2 = 0$ (corresponding with $y = 0$) is a stable manifold. Thus, after some time the non-smooth nontrivial solution so constructed will collapse to the planar solution. Figure 8(ii) demonstrates a similar situation as Figure 8(i) does, if we take an initial point to the right of the curve where $r_1 = -\frac{1}{2}\frac{\gamma_1}{\alpha_1}r_2^2$. In Figure 8(iii) both r_1 and r_2 grow to infinity as time increases, while in Figure 8(iv) both amplitudes may also grow, depending on where their initial points are in the phase plane. One should pay attention to the presence of the unstable focus in the domain, when taking an initial point to produce a non-smooth 4d solution.

The last two cases imply the possibility of the existence of a non-smooth nontrivial solution which will *not* tend to the planar solution.

By performing a numerical simulation, we obtain an attracting non-smooth 4d solution of the full system. Figure 9 describes the solution projected in the $x{-}x'{-}y$ phase space for the case of Figure 8(iii), which is located on a non-smooth cylinder.

More information about the solutions of the full system can be drawn if we proceed as follows. Consider again for $v > x'$ the averaged system (5.15). We may define the following transformation in order to remove the singularity of the system:

$$u = r_1 \cos\phi, \;\; w = r_1 \sin\phi, \;\; \rho = r_2^2. \tag{5.17}$$

And, after absorbing the rescaling factor, we obtain:

$$\begin{aligned} u' &= \frac{1}{2}\alpha_1 u + \gamma_2 u w + 2\sigma w \\ w' &= \frac{1}{2}\alpha_1 w + \frac{1}{4}\gamma_1 \rho - \gamma_2 u^2 - 2\sigma u \\ \rho' &= -\kappa\rho - \gamma_2 \rho w. \end{aligned} \tag{5.18}$$

Taking $\sigma = 0$, we immediately see that $u = 0$ and $\rho = 0$ are invariant manifolds of the system. Considering $r_1 \neq 0$ (otherwise we end up with the trivial solution of the system), the manifold $u = 0$ simply corresponds with the solutions where $\cos\phi = 0$ which we have analysed previously. The manifold $\rho = 0$ corresponds with the semitrivial solution of the system. By taking certain values of the parameters, we may conclude that the manifolds are attracting. For example, taking $\alpha_1 = 4, \;\; \kappa = 1, \gamma_1 = 1, \;\; \gamma_2 = 3,$ and $\sigma = 0$ (the case of Figure 8(i)), the solution starting from outside the manifolds, will be expanding

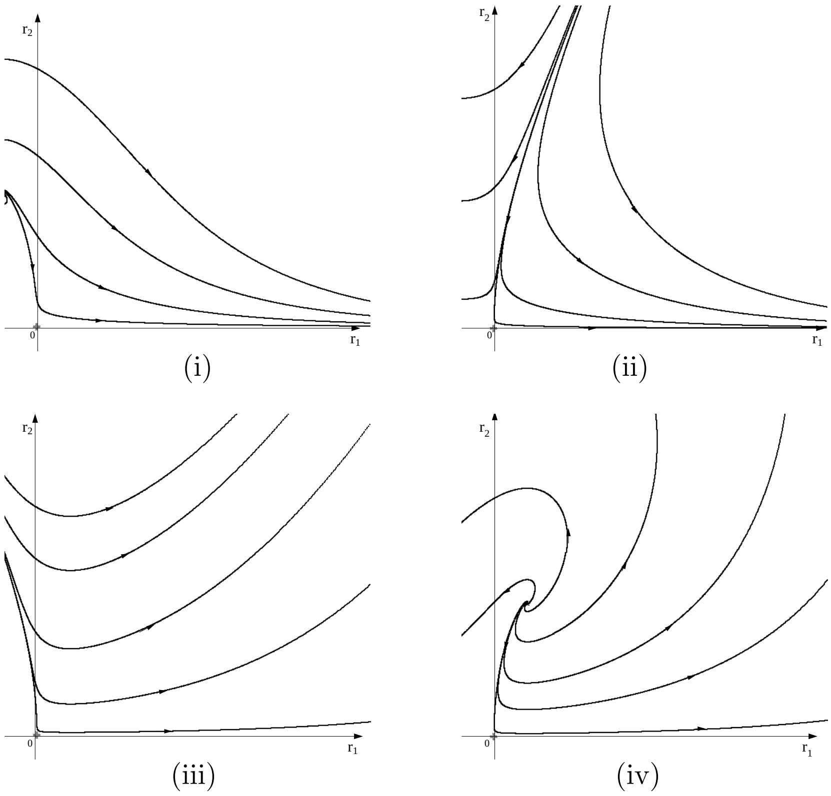


Figure 8: **The phase plane in the domain where** $v > x'$. (i) the case of $\gamma_1 > 0,\ \gamma_2 > 0$, (ii) the case of $\gamma_1 < 0,\ \gamma_2 > 0$, (iii) the case of $\gamma_1 > 0,\ \gamma_2 < 0$, (iv) the case of $\gamma_1 < 0,\ \gamma_2 < 0$.

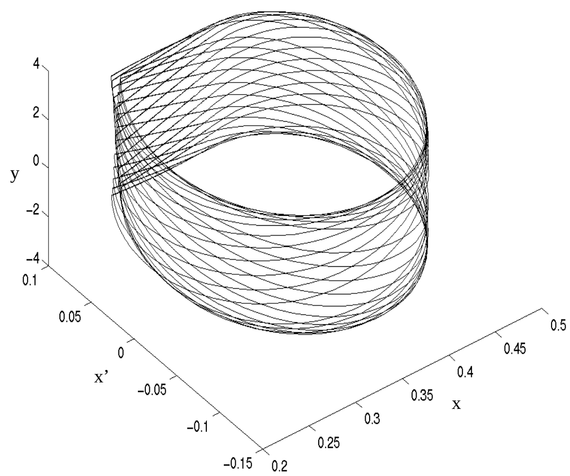


Figure 9: **The non-smooth cylinder**. Taking $\gamma_1 = 1,\ \gamma_2 = -3,$ and $v = 0.1$

and attracted to plane $u = 0$ and plane $\rho = 0$ alternately, before it finally collapses into plane $\rho = 0$. The observations of the other cases follow similarly, and we omit them for brevity.

5.0.2 The case $\delta = \varepsilon$

In the case of $\delta = \varepsilon$, a similar procedure as in the previous case can be applied. Then, we analyse the corresponding averaged system, taking $\sigma = 0$ and look for special solutions where $\sin\phi = 1$. From the analysis we conclude that qualitatively similar results as the previous case are obtained, except that the attraction of the manifold $r_2 = 0$ is now stronger. So, in this case, the 4d solution so constructed, which follows from similar descriptions given by Figure 8(i) and 8(ii), will collapse to the planar solution faster than that in the case of $\delta = \varepsilon^2$. The other two cases follows similarly as the previous corresponding cases. Therefore, we have a similar conclusion with regard to the possibility for having an attracting non-smooth manifold in the phase space, as a 4d solution of the system.

6 Concluding remarks

The implementation of the dry friction oscillator in an autoparametric system features some interesting phenomena. The periodic solution, which is typical in self-excited autoparametric systems, is non-smooth. The solution is constructed in such a way that stick-slip conditions as explained in section 3 are met.

The study of the planar solutions of the system is very interesting. The boundary value μ_0 is important in the application, especially if we expect a periodic solution arising from self excitation. Moreover, the bifurcation analysis of the non-smooth periodic solution using SlideCont gives more insight in the behaviour of the solution to be expected when we vary the value of one of the parameters of the system. One may undertake a continuation by varying the value of another parameter and similar bifurcation results as presented in subsection 4.2 might follow.

The study of the 4d solutions of the system is far from complete. There are many complications in the analysis due to the lack of analytical and numerical techniques to analyse such a system. However, by doing a qualitative asymptotic analysis and performing numerical simulations, even in the manifold $\cos\phi = 0$ we obtain some interesting results already. That is, an attracting non-smooth nontrivial solution may exist, depending on the assumption on the parameter values. A further study may be carried out to find out the boundary value for the existence of the non-smooth 4d solution, related to the boundary value for the existence of the non-smooth periodic solution in the planar case. Moreover, the study of the case when $\sigma \neq 0$ (the near resonance case) is also interesting from the application point of view.

We note finally that a different 4-dimensional non-smooth system was studied by Galvanetto, Bishop, and Briseghella [10]. In this paper two masses can be separately or simultaneously in a stick-slip phase. The dynamics is again complex but of a different nature.

Acknowledgement
The formulation of the 4d autoparametric system was obtained from A. Tondl. The numerical results using the software package SlideCont was done with kind assistances of Yu. A. Kuznetsov. There were many discussions with F. Verhulst regarding this research. The research was conducted in Mathematisch Instituut, Utrecht Universiteit, The Netherlands, and it was supported by the PGSM grant from Indonesia (IBRD Loan No. 3979 - IND) and CICAT TUDelft, the Netherlands.

References

[1] Abadi, On Self-Excited Autoparametric System, *Nonlinear Dynamics*, 24, 2001, 147–166.

[2] Aizerman, M. A. and Gantmakher, F. R., Stability in Linear Approximation of the Periodic Solution of System of Differential Equations with Discontinuous Right Hand Sides, *Prikl.Mat.Mekh*, 21, 5 1957, 658–669.

[3] Aubin, J. P. and Cellina, A.,*Differential Inclusions*, Springer-Verlag, Berlin, 1984.

[4] Babitsky, V. I. and Krupenin, V. L.,*Vibration of Strongly Nonlinear Discontinuous Systems*, Springer-Verlag, Berlin, 2001.

[5] Bothe, D., Periodic Solutions of Non-smooth Friction Oscillators, *Z. Angew. Math.Phys*, 50, 1999, 779–808.

[6] Doedel, E., Champneys, A., Fairgrieve, T., Kuznetsov Yu. A., Sanstede, B., Wang, X., AUTO97: Continuation and Bifurcation Software for Ordinary Differential Equations (with HomCont), User's Guide, Concordia University, Canada, 1997.

[7] Fatimah, S. and Ruijgrok, M., Bifurcation in an Auto-parametric System in 1:1 Internal Resonance with Parametric Excitation, *Int. J. Nonlinear Mechanics*, 37, 2002, 297–308.

[8] Fidlin, A., On the Asymptotic Analysis of Discontinuous Systems, *Z. Angew. Math. Mech*, 82, No. 2, 2002, 75–88.

[9] Filippov, A. F.,*Differential Equations with Discontinuous Right-hand Side*, Nauka, Moscow, 1970.

[10] Galvanetto, U., Bishop, S. R., and Briseghella, L., Mechanical Stick-slip Vibrations, *International Journal of Bifurcation and Chaos*, 5, No. 3, 1995, 637–651.

[11] Galvanetto, U. and Bishop, S. R., Dynamics of a Simple Damped Oscillator Undergoing Stick-slip Vibrations, *Meccanica*, 34, 1999, 337–347.

[12] Jones, C. K. R. T., Geometric Singular Perturbation Theory, in *Dynamical Systems*, Montecatini Terme 1994 (R. Johnson, ed.), Lecture Notes in Mathematics 1609, 44–118, Springer, Berlin, 1995.

[13] Klotter, K.,*Technische Schwingungslehre*, Springer, Berlin, 1978.

[14] Kunze, M. and Küpper, T., Qualitative Bifurcation Analysis of a Non-smooth Friction-oscillator Model, *Z. Angew. Math.Phys*, 48, 1997, 87–101.

[15] Kunze, M., *Non-smooth Dynamical Systems*, Springer, Berlin, 2000.

[16] Kuznetsov, Yu. A., Rinaldi, S., Gragnani, A., One-parameter Bifurcations in Planar Filippov Systems, *Int. J. Bifurcation and Chaos*, [to appear], 2002.

[17] Dercole, F., and Kuznetsov, Yu. A., SlideCont: An AUTO97 Driver for Sliding Bifurcation Analysis, *Preprint 1261*, Universiteit Utrecht, 2002.

[18] Leine, R. I., van Campen, D. H., de Kraker, A., and van den Steen, L., Stick-slip Vibrations Induced by Alternate Friction Models, *Nonlinear Dynamics*, 16(1), 1998, 41–54.

[19] Matveev, N. M., Nosov, S. L., and Pokrovskii, A. N., The averaging-principle question, *Differential Equations*, 14, 1978, 261–263.

[20] Plotnikov, V. A., Averaging of Differential Inclusions, *Ukrainian Math. J.*, 31, 1979, 454–457.

[21] Plotnikov, V. A., The averaging method for differential inclusions and its application to optimal-control problems, *Differential Equations*, 15, 1979, 1013–1018.

[22] Popp, K. and Stelter, P., Stick-slip vibrations and chaos, *Phil. Trans. Roy. Soc. Lond.* A 332, 1990, 89–105.

[23] Samoilenko, A. M. and Pere'styuk, N. A., The method of averaging in systems with an impulsive action, *Ukrainian Math. J.*, 26, 1974, 342–347.

[24] Samoilenko, A. M. and Pere'styuk, N. A., N. N. Bogolyubov's second theorem for differential equation systems with impulse action, *Differential Equations*, 10, 1974, 1543–1550.

[25] Tondl, A. *Self-excited Vibrations*, Monograph and Memoranda No. 9, National Research Institute for Machine Design, Běchovice, 1970.

[26] Tondl, A., Ruijgrok, M., Verhulst, F., and Nabergoj, R., *Autoparametric Resonance in Mechanical Systems*, Cambridge University Press, New York, 2000.

[27] Verhulst, F., *Nonlinear Differential Equations and Dynamical Systems*, Springer, Berlin, 1996.